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RESEARCH ARTICLE

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Two different fractional Stefan problems that are convergent to the same classical Stefan problem

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Two fractional Stefan problems are considered by using Riemann-Liouville and Caputo derivatives of order $\alpha \in (0, 1)$ such that, in the limit case ($\alpha = 1$), both problems coincide with the same classical Stefan problem. For the one and the other problem, explicit solutions in terms of the Wright functions are presented. We prove that these solutions are different even though they converge, when $\alpha \nearrow 1$, to the same classical solution. This result also shows that some limits are not commutative when fractional derivatives are used.

KEYWORDS
Caputo derivative, explicit solutions, fractional Stefan problem, Riemann–Liouville derivative, Wright functions

1 | INTRODUCTION

In this paper, two fractional Stefan problems are considered. These kind of problems are free boundary problems where the governed equation is a fractional diffusion equation in the temporal variable t.

A one-phase classical Stefan problem for a semi-infinite material with initial and boundary conditions can be formulated as

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t}u(x,t) = \lambda \frac{\partial^2}{\partial x^2}u(x,t), & 0 < x < s(t), 0 < t < T, \\ (ii) & u(x,0) = f(x), & 0 \le x \le b = s(0), \\ (iii) & u(0,t) = g(t), & 0 < t \le T, \\ (iv) & u(s(t),t) = 0, & 0 < t \le T, \\ (v) & \frac{\mathrm{d}}{\mathrm{d}t}s(t) = -k\frac{\partial}{\partial x}u(s(t),t), & 0 < t \le T, \end{array} \tag{1}$$

where λ is the diffusivity and k is the conductivity of the material. This kind of problems have been widely studied (see other works[1-3]).

The fractional Caputo derivative[4] in the t variable is defined by

$$ {}_0^C D_t^\alpha u(x,t) = {}_0 I_t^{1-\alpha} u_t(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial}{\partial t}u(x,\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau, \tag{2}$$

where ${}_0I_t^{\beta} f(x,t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(x,\tau)}{(t-\tau)^{1-\beta}} \mathrm{d}\tau$ is the fractional Riemann-Liouville integral defined for every $\beta > 0$, and Γ is the Gamma function.

If we replace in problem (1) the time derivative by the Caputo derivative (2), then the following fractional one-phase Stefan problem is obtained:

$$\begin{array}{rll} (i) & {}_0^CD_t^{\alpha} u(x,t) = \lambda^2 \frac{\partial}{\partial x^2} u(x,t), & 0 < x < s(t), 0 < t < T, \\ (ii) & u(x,0) = f(x), & 0 \leq x \leq b = s(0), \\ (iii) & u(0,t) = g(t), & 0 < t \leq T, \\ (iv) & u(s(t),t) = 0, & 0 < t \leq T, \\ (v) & {}_0^CD^{\alpha} s(t) = -u_x(s(t),t), & 0 < t \leq T. \end{array} \tag{3}$$

Some works[5-11] focused in problems like (3).

Let us aboard now the physical approach. The classical mathematical model for heat flux is through the Fourier law, which says that the heat flux is proportional to the temperature gradient

$$q^l(x,t) = -k \frac{\partial}{\partial x} u(x,t). \tag{4}$$

However, in the last 40 years, many generalizations of the Fourier law has been proposed,[12-16] giving rise to the emergence of new models. In particular, Gurtin and Pipkin[17] proposed the following law for the heat conduction, characterized by the nonlocality given by:

$$q = -k \int_0^t K(t-\tau) \nabla u(\tau) \mathrm{d}\tau,$$

and different theories can be developed from the consideration of different kernels of convolution. For example, in the works of Povstenko[18] and Voller et al,[19] a nonlocal flow given by

$$q = -k \, {}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} u(x,t) \tag{5}$$

is considered, where the fractional derivative is the Riemann-Liouville derivative respect on time of order $1-\alpha (\alpha \in (0,1))$ defined by

$${}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} u(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{\frac{\partial}{\partial x} u(x,\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \qquad \alpha \in (0,1).$$

Note that the nonlocal flux coincide with the Fourier flux for $\alpha = 1$ because ${}_0^{RL}D_t^0 = Id$.

Therefore, we consider this nonlocal flux. If (5) is replaced in the heat balance equation, then a fractional diffusion equation for the fractional Riemann-Liouville derivative is obtained

$$\frac{\partial}{\partial t} u(x,t) = \lambda \frac{\partial}{\partial x} \left({}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} u(x,t) \right). \tag{6}$$

Recalling that ${}_0^{RL}D_t^{1-\alpha}$ is the left inverse operator of ${}_0I_t^{1-\alpha}$, we can apply ${}_0^{RL}D_t^{1-\alpha}$ to both sides of equation (3-i) obtaining, under certain hypothesis, the fractional diffusion equation (6).

Fractional diffusion equations for Caputo derivatives, like (3-i), are linked to the modeling of diffusive processes in heterogeneous media, such called sub or superdiffusive processes (see related works[20-23]).

Now, let us focus in the Stefan condition. The classical Stefan condition derived in a one-phase Stefan problem is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} s(t) = q^l(x,t)\Big|_{(s(t)^-,t)}, \quad 0 < t \leq T, \tag{7}$$

where q^l is the local flux given by (4). Thus, replacing the nonlocal flux (5) in (7), we obtain the following "fractional Stefan condition":

$$\frac{\mathrm{d}}{\mathrm{d}t} s(t) = - {}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} u(x,t)\Big|_{(s(t),t)}, \quad 0 < t \leq T.$$

Therefore, the second fractional Stefan problem that we can consider is given by

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t}u(x,t)=\lambda\frac{\partial}{\partial x}\left({}^{RL}_{0}D^{1-\alpha}_{t}\frac{\partial}{\partial x}u(x,t)\right), & 0<x<s(t),\ 0<t<T, \\ (ii) & u(x,0)=f(x), & 0\leq x\leq b=s(0), \\ (iii) & u(0,t)=g(t), & 0<t\leq T, \\ (iv) & u(s(t),t)=0, & 0<t\leq T, \\ (v) & \frac{\mathrm{d}}{\mathrm{d}t}s(t)=-{}^{RL}_{0}D^{1-\alpha}_{t}\frac{\partial}{\partial x}u(x,t)\Big|_{(s(t),t)}, & 0<t\leq T. \end{array} \tag{8}$$

The last formulation is not usually considered because of the singularity of the Riemann-Liouville derivative and also because the Caputo derivative is a better choice for posing fractional initial-boundary problems for fractional parabolic operators.

We have seen that equations (8-i) and (3-i) are closely linked. However, what happen with the fractional Stefan conditions (8-v) and (3-v)?

For example, if we apply ${}^{RL}_{0}D^{1-\alpha}_{t}$ to both sides of the Stefan condition (3-v), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}s(t)=-{}^{RL}_{0}D^{1-\alpha}_{t}\frac{\partial}{\partial x}u(s(t),t),$$

which is not exactly as condition (8-v), unless $\alpha = 1$. In fact, the right side of (8-v) is

$$\begin{aligned} -{}^{RL}_{0}D^{1-\alpha}_{t}\frac{\partial}{\partial x}u(x,t)\Big|_{(s(t),t)} &= -\lim_{x\to s(t)}{}^{RL}_{0}D^{1-\alpha}_{t}\frac{\partial}{\partial x}u(x,t) \\ &= -\lim_{x\to s(t)}\frac{\partial}{\partial t}\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\frac{\partial}{\partial x}u(x,\tau)\mathrm{d}\tau. \end{aligned} \tag{9}$$

The aim of this paper is to show explicit solutions to problems (3) and (8), respectively, and prove that they are different, which clearly implies that the "fractional Stefan conditions" (8-v) and (3-v) are different and that for fractional derivatives some limits like (9) are not commutative.

2 | PREVIOUS RESULTS

Definition 1. For every $x \in \mathbb{R}$, Wright function is defined by

$$W(x;\rho;\beta)=\sum_{k=0}^{\infty}\frac{x^k}{k!\Gamma(\rho k+\beta)},\quad \rho>-1 \text{ and } \beta\in\mathbb{R}. \tag{10}$$

An important particular case of a Wright function is the Mainardi function defined by

$$M_\rho(x)=W(-x,-\rho,1-\rho)=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!\Gamma(-\rho n+1-\rho)},\quad 0<\rho<1.$$

Proposition 1. *Let $\rho \in (0,1)$ be. Then, the next assertions follows.*

1. *Let $\beta\in\mathbb{R}$ be. For every $x\in\mathbb{R}$, we have*

$$\frac{\partial}{\partial x}W(x,\rho,\beta)=W(x,\rho,\rho+\beta).$$

2. *If $\beta \geq 0$, then $W(-x,-\rho,\beta)$ is a positive and strictly decreasing function in $\mathbb{R}^+$.*
3. *Let $\alpha > 0$ and $\beta\in\mathbb{R}$ be. For every $x > 0$ and $c > 0$,*

$${}_0I^{\alpha}_{x}\, x^{\beta-1}W\left(-cx^{-\rho},-\rho,\beta\right)=x^{\beta+\alpha-1}W\left(-cx^{-\rho},-\rho,\beta+\alpha\right). \tag{11}$$

Proof. See the work of Wright[24] for 1. Item 2 follows from theorem 8 in the work of Stankovic[25] and the chain rule. Item 3 is a particular case of corollary 5 in the work of Pskhu.[26] □

Lemma 1. *For every $n \in \mathbb{N}$, it holds that*[27]

1. $(2n)! = 2^n n!(2n-1)!!$.
2. $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$,

where the definition $(2n-1)!! = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$ *is used for compactness expression.*

Proposition 2. *Let $x \in \mathbb{R}_0^+$ be. Then, the following limits hold:*

$$\lim_{\alpha\nearrow 1} M_{\alpha/2}(2x) = \lim_{\alpha\nearrow 1} W\left(-2x,-\frac{\alpha}{2},1-\frac{\alpha}{2}\right) = M_{1/2}(2x) = \frac{e^{-x^2}}{\sqrt{\pi}}, \tag{12}$$

$$\lim_{\alpha\nearrow 1} W\left(-2x,-\frac{\alpha}{2},\frac{\alpha}{2}\right) = \frac{e^{-x^2}}{\sqrt{\pi}}, \tag{13}$$

$$\lim_{\alpha\nearrow 1}\left[1-W\left(-2x,-\frac{\alpha}{2},1\right)\right] = \operatorname{erf}(x), \tag{14}$$

and

$$\lim_{\alpha\nearrow 1}\left[W\left(-2x,-\frac{\alpha}{2},1\right)\right] = \operatorname{erfc}(x), \tag{15}$$

where $\operatorname{erf}(\cdot)$ *is the error function defined by* $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\mathrm{d}z$ *and* $\operatorname{erfc}(\cdot)$ *is the complementary error function defined by* $\operatorname{erfc}(x) = 1-\operatorname{erf}(x)$. *Moreover, the convergence is uniform over compact sets.*

Proof. See the work of Roscani and Santillan Marcus[9] for (12) and (14). Now, for proving (13), let α be such that $0<\alpha<1$. From (10),

$$W\left(-2x;-\frac{\alpha}{2};\frac{\alpha}{2}\right) = \sum_{k=0}^{\infty}\frac{(-2x)^k}{k!\Gamma\left(-\frac{\alpha}{2}k+\frac{\alpha}{2}\right)}. \tag{16}$$

Let us limit the series by a convergent series, which not depend on α, so we can apply the Weierstrass M-test and interchange the series and the limit. Recall that, for all $x \in \mathbb{R}$,[28]

$$\frac{1}{\Gamma(x)\Gamma(1-x)} = \frac{\sin(\pi x)}{\pi}, \tag{17}$$

and for every $k \in \mathbb{N}$,

$$\Gamma(k+1) = k\Gamma(k). \tag{18}$$

Then,

$$\begin{aligned}\left|\frac{1}{k!\Gamma\left(-\frac{\alpha}{2}k+\frac{\alpha}{2}\right)}\right| &= \left|\frac{1}{\Gamma(k+1)\Gamma\left(1-\frac{\alpha}{2}k+\frac{\alpha}{2}-1\right)}\right| \\ &= \left|\frac{\Gamma\left(\frac{\alpha}{2}k-\frac{\alpha}{2}+1\right)\sin\left(\pi\left(\frac{\alpha}{2}k-\frac{\alpha}{2}+1\right)\right)}{\pi\Gamma(k+1)}\right| \\ &\le \left|\frac{\Gamma\left(\frac{\alpha}{2}(k-1)+1\right)}{\pi\Gamma(k+1)}\right|.\end{aligned} \tag{19}$$

Now, let $x^* > 0$ be the abscissa of the minimum of the Gamma function and let k_0 such that $\frac{\alpha}{2}(k_0-1)+1 > x^*$. Applying that the Gamma function is an increasing function in $(x^*,+\infty)$, it yields

$$\left|\frac{\Gamma\left(\frac{\alpha}{2}(k-1)+1\right)}{\Gamma(k+1)}\right| \le \frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}{\Gamma(k+1)}, \quad \text{for all } k \ge k_0. \tag{20}$$

Let us separate in even and odd terms. If $k=2n$, $n\in\mathbb{N}$, then applying Lemma 1, it results that

$$\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}{\Gamma(k+1)} = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)} = \frac{(2n-1)!!\sqrt{\pi}}{2^n(2n)!} < \frac{1}{2^n} = \frac{1}{\sqrt{2}^k}. \tag{21}$$

If $k = 2n + 1$, $n \in \mathbb{N}$, from Lemma 1, we have

$$\begin{aligned}\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}{\Gamma(k+1)} &= \frac{\Gamma(n+1)}{\Gamma(2n+2)} = \frac{n!}{(2n+1)!} = \frac{n!}{(2n+1)2^n n!(2n-1)!!} = \\ &= \frac{1}{(2n+1)2^n(2n-1)!!} \leq \frac{1}{2^{n+1}} = \frac{1}{2^{\frac{k+1}{2}}} < \frac{1}{\sqrt{2}^k}.\end{aligned} \tag{22}$$

From (21) and (22), we can state that

$$\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}{\Gamma(k+1)} \leq \frac{1}{\sqrt{2}^k}, \quad \text{for all } k \geq k_0. \tag{23}$$

From (19), (20), and (23), it results that the series (16) is bounded by a convergent series that not depend on α. Taking the limit when $\alpha \nearrow 1$, using (17) and Lemma 1, the limit (13) holds

$$\begin{aligned}&\lim_{\alpha\nearrow 1} W\left(-2x; -\frac{\alpha}{2}; \frac{\alpha}{2}\right) = \\ &= \sum_{k=0}^{\infty} \lim_{\alpha\nearrow 1} \frac{x^{2k}}{(2k)!\Gamma\left(-\frac{\alpha}{2}2k+1-\frac{\alpha}{2}\right)} + \sum_{k=0}^{\infty} \lim_{\alpha\nearrow 1} \frac{-x^{2k+1}}{(2k+1)!\Gamma(1-\alpha(k+1))} \\ &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!\Gamma\left(-k+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{x^{2k}\Gamma\left(k+\frac{1}{2}\right)\sin(\pi((-k+1/2)))}{\pi(2k)!} \\ &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(-x^2\right)^k}{4^k k!} = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{4}}.\end{aligned}$$

□

Remark 1. Proposition 2 shows that two different Wright functions $\Gamma\left(1-\frac{\alpha}{2}\right)M_{\alpha/2}(2x)$ and $\Gamma\left(\frac{\alpha}{2}\right)W\left(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right)$ are convergent to the Gaussian function $G(x) = e^{-x^2}$. A graphic for a particular value is given in Figure 1 and the key of this article is to prove that these functions does not intersect for any positive real value.

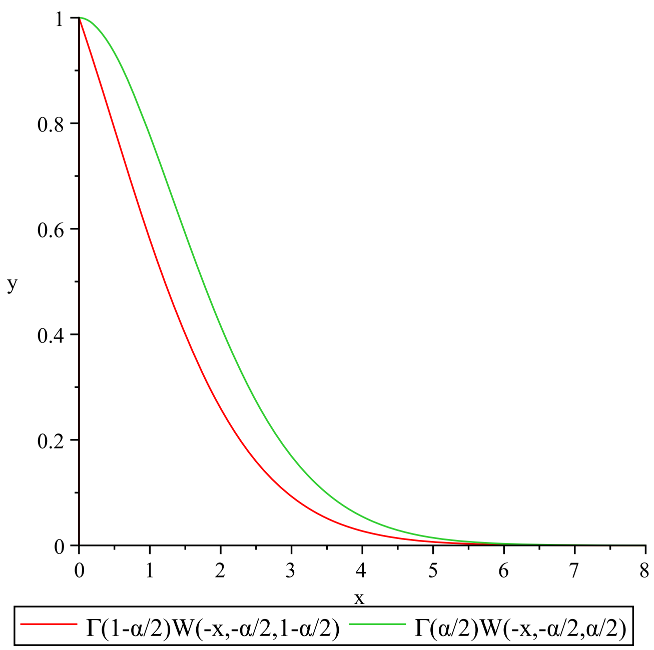

FIGURE 1 Functions $\Gamma\left(1-\frac{\alpha}{2}\right)M_{\alpha/2}(2x)$ and $\Gamma\left(\frac{\alpha}{2}\right)W\left(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right)$, for $\alpha = \frac{3}{4}$ [Colour figure can be viewed at wileyonlinelibrary.com]

Proposition 3. *Let $x > 0$ be and let $0 < \rho \leq \mu < \delta$. Then,*

$$\Gamma(\delta)W(-x,-\rho,\delta) < \Gamma(\mu)W(-x,-\rho,\mu).$$

Proof. Consider $\alpha = \delta - \mu$ and $\beta = \mu$ in (11). Then,

$$ {}_0I_y^{\delta-\mu}\, y^{\mu-1}W\left(-cy^{-\rho},-\rho,\mu\right) = y^{\delta-1}W\left(-cy^{-\rho},-\rho,\delta\right). \tag{24}$$

Making the substitution $y = x^{-1/\rho}$, using (24) and Proposition 1, it yields that

$$\begin{aligned}
W(-x,-\rho,\delta) = W(-y^{-\rho},-\rho,\delta) &= y^{-\delta+1}\,{}_0I_y^{\delta-\mu}\, y^{\mu-1}W(-y^{-\rho},-\rho,\mu) = \\
&= y^{-\delta+1}\frac{1}{\Gamma(\delta-\mu)}\int_0^y t^{\mu-1}W\left(-t^{-\rho},-\rho,\mu\right)(y-t)^{\delta-\mu-1}\mathrm{d}t \\
&< y^{-\delta+1}\frac{1}{\Gamma(\delta-\mu)}W(-y^{-\rho},-\rho,\mu)\int_0^y t^{\mu-1}(y-t)^{\delta-\mu-1}\mathrm{d}t \\
&= y^{-\delta+1}\frac{1}{\Gamma(\delta-\mu)}W(-y^{-\rho},-\rho,\mu)\frac{\Gamma(\mu-1+1)\Gamma(\delta-\mu)}{\Gamma(\delta)}y^{\delta-1} \\
&= \frac{\Gamma(\mu)}{\Gamma(\delta)}W(-y^{-\rho},-\rho,\mu) = \frac{\Gamma(\mu)}{\Gamma(\delta)}W(-x,-\rho,\mu).
\end{aligned}$$

□

3 | TWO DIFFERENT EXPLICIT SOLUTIONS

We consider two particular fractional Stefan problems

$$\begin{array}{lll}
(i) & {}_0^CD_t^\alpha u(x,t) = \frac{\partial}{\partial x^2}u(x,t), & 0<x<s(t),\ 0<t<T, \\
(ii) & s(0)=0, & \\
(iii) & u(0,t)=1, & 0<t\leq T, \\
(iv) & u(s(t),t)=0, & 0<t\leq T, \\
(v) & {}_0^CD^\alpha s(t) = -u_x(s(t),t), & 0<t\leq T,
\end{array} \tag{25}$$

and

$$\begin{array}{lll}
(i) & \frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\left({}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)\right), & 0<x<s(t),\ 0<t<T, \\
(ii) & s(0)=0, & \\
(iii) & u(0,t)=1, & 0<t\leq T, \\
(iv) & u(s(t),t)=0, & 0<t\leq T, \\
(v) & \frac{\mathrm{d}}{\mathrm{d}t}s(t) = -{}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)\Big|_{(s(t),t)}, & 0<t\leq T.
\end{array} \tag{26}$$

It was proved in the work of Roscani and Santillan Marcus[9] that the pair $\{w_\alpha, r_\alpha\}$ is a solution to problem (25), where

$$\begin{aligned}
w_\alpha(x,t) =& 1-\frac{1}{1-W\left(-2\eta_\alpha,-\frac{\alpha}{2},1\right)}\left[1-W\left(-\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1\right)\right], \\
r_\alpha(t) =& 2\eta_\alpha t^{\alpha/2},
\end{aligned} \tag{27}$$

and η_α is the unique solution to the equation

$$2x\left[1-W\left(-2x,-\frac{\alpha}{2},1\right)\right] = M_{\alpha/2}(2x)\frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)}, \quad x>0. \tag{28}$$

By the other side, it was proved in the work of Roscani and Tarzia[29] that the pair $\{u_\alpha, s_\alpha\}$ is a solution to problem (26), where

$$\begin{aligned} u_\alpha(x,t) &= 1 - \frac{1}{1 - W\left(-2\xi_\alpha, -\frac{\alpha}{2}, 1\right)}\left[1 - W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right] \\ s_\alpha(t) &= 2\xi_\alpha t^{\alpha/2}, \end{aligned} \tag{29}$$

and ξ_α is the unique solution to the equation

$$2x\left[1 - W\left(-2x, -\frac{\alpha}{2}, 1\right)\right] = 2xW\left(-2x, -\frac{\alpha}{2}, 1\right) + W\left(-2x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right), x > 0. \tag{30}$$

Looking at the similarity between solutions (27) and (29), it is natural to ask whether both are the same solution or not.

Theorem 1. *The explicit solutions (27) to problem (25) and (29) to problem (26) are different.*

Proof. From the work of Wright,[24] we know that, for every $x \in \mathbb{R}$, the next equality holds

$$xW\left(-x, -\frac{\alpha}{2}, 1\right) + W\left(-x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right) = \frac{2}{\alpha}W\left(-x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right). \tag{31}$$

Replacing equality (31) into (30), we can say that the parameter ξ_α appearing in solution (29) is the unique solution to the equation

$$2x\left[1 - W\left(-2x, -\frac{\alpha}{2}, 1\right)\right] = \frac{2}{\alpha}W\left(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right), \quad x > 0. \tag{32}$$

By the other side, we know that the parameter η_α, which is part of the solution (27) to problem (25), is the unique solution to Equation (28).

Therefore, if we suppose that solutions (27) and (29) coincides, from (28) and (32), we can conclude that there exists $\nu_\alpha > 0$ such that

$$M_{\alpha/2}(2\nu_\alpha)\frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)} = \frac{2}{\alpha}W\left(-2\nu_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2}\right)$$

or equivalently,

$$M_{\alpha/2}(2\nu_\alpha)\Gamma(1-\alpha/2) = \Gamma(\alpha/2)W\left(-2\nu_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2}\right).$$

However, this is a contradiction from Proposition 3 and then the thesis holds. □

Theorem 2. *If we take the limit when $\alpha \nearrow 1$, the solutions (29) and (27) converge to the unique solution $\{u, s\}$ to the classical Stefan problem*

$$\begin{array}{lll} (i) & u_t(x,t) = \dfrac{\partial}{\partial x^2}u(x,t), & 0 < x < s(t), 0 < t < T, \\ (ii) & u(0,t) = 1, & 0 < t \le T, \\ (iii) & u(s(t),t) = 0, & 0 < t \le T, s(0) = 0, \\ (iv) & s'(t) = -u_x(s(t),t), & 0 < t < T. \end{array} \tag{33}$$

Proof. The unique solution to problem (33) is given by (see, eg, the works of Cannon[2] and Tarzia[3])

$$\begin{aligned} w(x,t) &= 1 - \frac{1}{\text{erf }(\eta_1)}\text{erf}\left(\frac{x}{2\sqrt{t}}\right), \\ s(t) &= 2\eta_1\sqrt{t}, \end{aligned} \tag{34}$$

where η_1 is the unique solution to the equation

$$\eta_1 \,\text{erf}(\eta_1) = \frac{e^{-\eta_1^2}}{\sqrt{\pi}}. \tag{35}$$

Note that, if we take $\alpha = 1$ in Equation (28), we recover Equation (35). Now, let the sequence $\{\eta_\alpha\}_\alpha$ be, where η_α is the unique positive solution to Equation (28). Then,

$$2\eta_\alpha = M_{\alpha/2}(2\eta_\alpha)\frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)} + 2\eta_\alpha W\left(-2\eta_\alpha, -\frac{\alpha}{2}, 1\right).$$

If we define the following functions for every $x \in \mathbb{R}^+$ and $0 < \alpha < 1$:

$$f_\alpha(x) = M_{\alpha/2}(2x)\frac{\Gamma(1-\alpha/2)}{2\Gamma(1+\alpha/2)} + xW\left(-2x, -\frac{\alpha}{2}, 1\right)$$

and

$$f_1(x) = \frac{e^{-x^2}}{\sqrt{\pi}} + x\mathrm{erfc}(x),$$

we have that $f_\alpha(\eta_\alpha) = \eta_\alpha, f_1(\eta_1) = \eta_1$. Let us prove that

$$\lim_{\alpha \nearrow 1} \eta_\alpha = \eta_1. \tag{36}$$

Firstly, from Proposition 2, it holds that

$$\lim_{\alpha \nearrow 1} f_\alpha(x) = f_1(x), \tag{37}$$

where the convergence is uniform over compact sets.

Secondly, analyzing f_1', we have that $f_1'(0) = 1$, $f_1'(+\infty) = 0^-$, there exists a unique $\eta_0 \approx 0.3195$ such that $f_1'(\eta_0) = 0$ and $f_1'(x) < 0$, for all $x > \eta_0$. In fact, η_0 is the unique positive solution to equation $\sqrt{\pi}xe^{x^2}\mathrm{erfc}(x) = 4x^2$. Being $\eta_1 \approx 0.6201$, it follows that $f_1'(\eta_1) < 0$. Then, there exists an interval $[\eta_1 - \rho, \eta_1 + \rho]$, for some $\rho > 0$, where f_1 is decreasing.

Thirdly, let $\varepsilon > 0$ be ($\varepsilon < \rho$) and let be the line of equation $y = x$. Clearly, $P_1(\eta_1, \eta_1) \in r$ and we can take $P_a(a, a)$ and $P_b(b, b)$ in r ($a < \eta_1 < b$) such that

$$d(P_1, P_a) < \varepsilon, \quad d(P_1, P_b) < \varepsilon \quad \text{and} \quad f_1 \quad \text{is decreasing in} [a, b]. \tag{38}$$

Being $f_1(\eta_1) = \eta_1$, it holds that $f_1(a) - a > 0$ and $f_1(b) - b < 0$.

Now, let $h_0 = \min\{f_1(a) - a, b - f_1(b)\} > 0$. From (37), it results that there exists $\alpha_0 \in (0, 1)$ such that

$$|f_\alpha(x) - f_1(x)| < h_0 \quad \text{for all } x \in [a, b], \text{ for all } \alpha \in (\alpha_0, 1].$$

Then, if $\alpha \in (\alpha_0, 1]$, we have that

$$f_\alpha(a) > f_1(a) - h_0 > a \quad \text{and} \quad f_\alpha(b) < f_1(b) + h_0 < b.$$

Applying Bolzano's theorem (f_α is continuous in $\mathbb{R}^+$ for all $\alpha \in (0, 1]$), it holds that the unique solution η_α to equation $f_\alpha(x) = x$ belongs to (a, b). From (38) and calling $P_\alpha(\eta_\alpha, \eta_\alpha)$, we get that $|\eta_\alpha - \eta_1| < d(P_\alpha, P_1) < d(P_a, P_1) < \varepsilon$, for all $\alpha \in (\alpha_0, 1]$, and (36) holds.

Finally, applying Propositions 1 and 2, we get that

$$\begin{aligned}\lim_{\alpha \nearrow 1} w_\alpha(x, t) &= \lim_{\alpha \nearrow 1} 1 - \frac{1}{1 - W\left(-2\eta_\alpha, -\frac{\alpha}{2}, 1\right)}\left[1 - W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right] \\ &= 1 - \frac{1}{\mathrm{erf}(\eta_1)}\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\end{aligned}$$

and

$$\lim_{\alpha \nearrow 1} r_\alpha = \lim_{\alpha \nearrow 1} 2\eta_\alpha t^{\alpha/2} = 2\eta_1\sqrt{t},$$

which proves that solution (27) of problem (25) converges to solution (34) of problem (33) as we wanted to see. The second part of the proof is analogous. □

4 | CONCLUSIONS

We have considered two fractional Stefan problems involving Riemann-Liouville and Caputo derivatives of order $\alpha \in (0, 1)$ such that in the limit case ($\alpha = 1$) both problems coincide with the same classical Stefan problem, and the relation between the governed equations and the Stefan conditions is analyzed. For both problems, explicit solutions were presented and it has been proved that these solutions are different, and therefore, the fractional Stefan conditions are different (unless $\alpha = 1$). Finally, the convergence when $\alpha \nearrow 1$ was computed obtaining for both problems the same classical solution.

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REFERENCES

1. Alexiades V, Solomon AD. Mathematical modelling of melting and freezing processes; Taylor and Francis, Washington: Hemisphere;1993.
2. Cannon JR. *The One-Dimensional Heat Equation*. Cambridge, UK: Cambridge University Press; 1984.
3. Tarzia DA. Explicit and approximated solutions for heat and mass transfer problems with a moving interface. In: Mohamed El-Amin, ed. *Advanced Topics in Mass Transfer*. Rijeka, Croatia: InTech; 2011;439-484.
4. Caputo M. Linear models of dissipation whose Q is almost frequency independent II. *Geophys J Int*. 1967;13:529-539.
5. Atkinson C. Moving boundary problems for time fractional and composition dependent diffusion. *Fract Calc Appl Anal*. 2012;15(2):207-221.
6. Błasik M. Numerical scheme for one-phase 1D fractional Stefan problem using the similarity variable technique. *J Appl Math Comput Mech*. 2014;13(1):13-21.
7. Junyi L, Mingyu X. Some exact solutions to stefan problems with fractional differential equations. *J Math Anal Appl*. 2009;351:536-542.
8. Li X, Sun X. Similarity solutions for phase change problems with fractional governing equations. *Appl Math Lett*. 2015;45:7-11.
9. Roscani S, Santillan Marcus E. Two equivalen Stefan's problems for the time–fractional diffusion equation. *Fract Calc Appl Anal*. 2013;16(4):802-815.
10. Roscani S, Tarzia D. A generalized Neumann solution for the two–phase fractional Lamé–Clapeyron–Stefan problem. *Adv Math Sci Appl*. 2014;24(2):237-249.
11. Voller VR. Fractional Stefan problems. *Int J Heat Mass Transf* . 2014;74:269-277.
12. Chandrasekharaiah DS. Thermoelasticity with second sound: a review. *Appl Mech Rev*. 1986;39:355-376.
13. Hetnarski RB, Ignaczak J. Generalized thermoelasticity. *J Therm Stresses*. 1999;22:451-476.
14. Ignaczak J, Ostoja-Starzewski M. *Thermoelasticity With Finite Wave Speeds*. Oxford, UK: Oxford University Press; 2010.
15. Joseph DD, Preziosi L. Heat waves. *Rev Mod Phys*. 1989;61:41-73.
16. Podstrigach YS, Kolyano YM. *Generalized Thermomechanics (in Russian)*. Kiev, Ukraine: Naukova Dumka; 1976.
17. Gurtin ME, Pipkin AC. A general theory of heat conduction with finite wave speeds. *Arch Ration Mech Anal*. 1968;31(2):113-126.
18. Povstenko Y. *Linear Fractional Diffusion–Wave Equation for Scientists and Engineers*. Cham, Switzerland: Springer; 2015.
19. Voller VR, Falcini F, Garra R. Fractional Stefan problems exhibing lumped and distributed latent–heat memory effects. *Phys Rev E*. 2013;87(042401).
20. Banks D, Fradin C. Anomalous diffusion of proteins due to molecular crowding. *Biophys J*. 2005;89(5):2960-2971.
21. Gusev AA, Suter UW. Dynamics of small molecules in dense polymers subjet to thermal motion. *J Chem Phys*. 1993;99:2228-2234.
22. Metzler R, Klafter J. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys Rep*. 2000;339:1-77.
23. Yuste SB, Abad E, Lindenberg K. Reaction-subdiffusion model of morphogen gradient formation. *Phys Rev E*. 2010;82(061123).
24. Wright EM. The generalized Bessel function of order greater than one. *Q J Math*. 1940;11:36-48.
25. Stankovic B. On the function of E. M. Wright. *Publ Inst math, Nouvelle sér tome*. 1970;10(24):113-124.
26. Pskhu AV. *Partial Differential Equations of Fractional Order (In Russian)*. Moscow, Russia: Nauka; 2005.
27. Bourkrouche M, Tarzia D. Non-classical heat conduction problem with nonlocal source. *Bound Value Probl*. 2017;2017:51.
28. Erdélyi A. *Higher Transcendental Functions*, Vol. 1. New York, NY: McGraw-Hill; 1955.
29. Roscani S, Tarzia D. An integral relationship for a fractional one-phase Stefan problem. Available in https://arxiv.org/abs/1609.05169.pdf; 2017.

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