

*THE CHARACTER ALGEBRA FOR  
MODULE CATEGORIES OVER HOPF ALGEBRAS*

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**Abstract.** Given a finite-dimensional Hopf algebra  $H$  and an exact indecomposable module category  $\mathcal{M}$  over  $\text{Rep}(H)$ , we explicitly compute the adjoint algebra  $\mathcal{A}_{\mathcal{M}}$  as an object in the category of Yetter–Drinfeld modules over  $H$ , and the space of class functions  $\text{CF}(\mathcal{M})$  associated to  $\mathcal{M}$ , as introduced by K. Shimizu (2020). We use our construction to describe these algebras when  $H$  is a group algebra and a dual group algebra. This result allows us to compute the adjoint algebra for certain group-theoretical fusion categories.

**Introduction.** In [13], K. Shimizu introduced the notion of adjoint algebra  $\mathcal{A}_{\mathcal{C}}$  and the space of class functions  $\text{CF}(\mathcal{C})$  for an arbitrary finite tensor category  $\mathcal{C}$ . The adjoint algebra is defined as the end  $\int_{X \in \mathcal{C}} X \otimes X^*$ . The dual object  $\mathcal{A}_{\mathcal{C}}^*$  is a crucial ingredient in Lyubashenko’s theory of the modular group action in non-semisimple tensor categories [8], [9].

Both the adjoint algebra and the space of class functions are interesting objects that generalize the well known adjoint representation and the character algebra of a finite group. In [13] many results concerning the table of characters, conjugacy classes, and orthogonality relations of characters in finite group theory have been generalized to the setting of fusion categories. Also, in [15] the adjoint algebra was used to develop a theory of integrals for finite tensor categories.

Assume  $\mathcal{M}$  is an arbitrary module category over a finite tensor category  $\mathcal{C}$ . In [14], K. Shimizu introduced the notion of adjoint algebra  $\mathcal{A}_{\mathcal{M}}$  and the space of class functions  $\text{CF}(\mathcal{M})$  associated to  $\mathcal{M}$ , generalizing the definitions given in [13]. The main task of this paper is the explicit computation of those objects in the particular case of  $\mathcal{C}$  being the representation category of a finite-dimensional Hopf algebra.

Assume that  $\mathcal{C}$  is a finite tensor category, and  $\mathcal{M}$  an exact left  $\mathcal{C}$ -module category with action functor  $\bar{\otimes} : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ . Then we can consider the

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functor  $\rho_{\mathcal{M}} : \mathcal{C} \rightarrow \text{Rex}(\mathcal{M})$ ,  $\rho_{\mathcal{M}}(X)(M) = X \overline{\otimes} M$ ,  $X \in \mathcal{C}$ ,  $M \in \mathcal{M}$ . Here  $\text{Rex}(\mathcal{M})$  denotes the category of right exact endofunctors of  $\mathcal{M}$ . The right adjoint of the action functor is  $\rho_{\mathcal{M}}^{\text{ra}} : \text{Rex}(\mathcal{M}) \rightarrow \mathcal{C}$ , explicitly described as

$$\rho_{\mathcal{M}}^{\text{ra}}(F) = \int_{M \in \mathcal{M}} \underline{\text{Hom}}(M, F(M))$$

for  $F \in \text{Rex}(\mathcal{M})$  [14, Thm. 3.4]. Here for any  $M \in \mathcal{M}$ ,  $\underline{\text{Hom}}(M, -)$  is the right adjoint of the functor  $\mathcal{C} \rightarrow \mathcal{M}$ ,  $X \mapsto X \overline{\otimes} M$ . It is called the *internal Hom* of the module category  $\mathcal{M}$ . The adjoint algebra is defined as  $\mathcal{A}_{\mathcal{M}} = \rho_{\mathcal{M}}^{\text{ra}}(\text{Id}_{\mathcal{M}})$ . This object has a half-braiding  $\sigma_{\mathcal{M}}(X) : \mathcal{A}_{\mathcal{M}} \otimes X \rightarrow X \otimes \mathcal{A}_{\mathcal{M}}$  defined as the unique morphism in  $\mathcal{C}$  such that the diagram

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{M}} \otimes X & \xrightarrow{\pi_{\mathcal{M}}(X \overline{\otimes} M) \otimes \text{id}_V} & \underline{\text{End}}(X \overline{\otimes} M) \otimes X \\ \sigma_{\mathcal{M}}(X) \downarrow & & \downarrow \simeq \\ X \otimes \mathcal{A}_{\mathcal{M}} & \xrightarrow{\text{id}_X \otimes \pi_{\mathcal{M}}} & X \otimes \underline{\text{Hom}}(M, M) \xrightarrow{\simeq} \underline{\text{Hom}}(X \overline{\otimes} M, M) \end{array}$$

is commutative. Here  $\pi_{\mathcal{M}}(M) : \mathcal{A}_{\mathcal{M}} \rightarrow \underline{\text{Hom}}(M, M)$  is the dinatural transformation of the end  $\mathcal{A}_{\mathcal{M}}$ . It turns out that  $(\mathcal{A}_{\mathcal{M}}, \sigma_{\mathcal{M}})$  is a commutative algebra in the Drinfeld center  $\mathcal{Z}(\mathcal{C})$ . Although this description of the half-braiding of  $\mathcal{A}_{\mathcal{M}}$  is rather clear, for us it was complicated to use it to make calculations in particular examples. However, there is another way of describing this structure.

If  $\mathcal{B}$  is a  $\mathcal{C}$ -bimodule category, one can consider the *relative center*  $\mathcal{Z}_{\mathcal{C}}(\mathcal{B})$ . When  $\mathcal{C}$  is considered as a bimodule over itself, the relative center coincides with the Drinfeld center. The correspondence  $\mathcal{B} \mapsto \mathcal{Z}_{\mathcal{C}}(\mathcal{B})$  is in fact part of a 2-functor

$$\mathcal{Z}_{\mathcal{C}} : {}_{\mathcal{C}} \text{Bimod} \rightarrow \text{Ab}_{\mathbb{k}},$$

where  ${}_{\mathcal{C}} \text{Bimod}$  is the 2-category of finite  $\mathcal{C}$ -bimodule categories, bimodule functors and bimodule natural transformations, and  $\text{Ab}_{\mathbb{k}}$  is the 2-category of finite abelian  $\mathbb{k}$ -linear categories. Both  $\text{Rex}(\mathcal{M})$  and  $\mathcal{C}$  are  $\mathcal{C}$ -bimodule categories. It turns out that  $\rho_{\mathcal{M}}^{\text{ra}}$  has a  $\mathcal{C}$ -bimodule structure [14, Section 3.4]. Applying the 2-functor  $\mathcal{Z}_{\mathcal{C}}$  one obtains a functor  $\mathcal{Z}_{\mathcal{C}}(\rho_{\mathcal{M}}^{\text{ra}}) : \mathcal{Z}_{\mathcal{C}}(\text{Rex}(\mathcal{M})) \simeq \mathcal{C}_{\mathcal{M}}^* \rightarrow \mathcal{Z}(\mathcal{C})$ . Hence  $(\mathcal{A}_{\mathcal{M}}, \sigma_{\mathcal{M}}) = \mathcal{Z}_{\mathcal{C}}(\rho_{\mathcal{M}}^{\text{ra}})(\text{Id}_{\mathcal{M}})$ .

Assume  $H$  is a finite-dimensional Hopf algebra. If  $\mathcal{M}$  is an exact indecomposable module category over  $\text{Rep}(H)$ , we explicitly describe the adjoint algebra  $(\mathcal{A}_{\mathcal{M}}, \sigma_{\mathcal{M}})$  and the space of class functions  $\text{CF}(\mathcal{M})$ . For this purpose, we need to explain all ingredients in the construction of those objects. Our description of both algebras relies heavily on the explicit description of module categories over Hopf algebras. In Section 3 we embark on this task. Module categories over  $\text{Rep}(H)$  are categories  ${}_K \mathcal{M}$  of finite-dimensional left

$K$ -modules, where  $K$  are certain  $H$ -comodule algebras. We also recall how to describe module functor categories, and that there is a monoidal equivalence  $\text{Rep}(H)_{K\mathcal{M}}^* \simeq {}^H_K\mathcal{M}_K$ . This equivalence will be used when explaining the functor  $\mathcal{Z}_{\mathcal{C}}(\rho_{\mathcal{M}}^{\text{ra}})$ . Another ingredient is the internal Hom. In this section we also describe, in a precise way, the internal Hom of the module category  ${}^H_K\mathcal{M}_K$ . In Section 4, after recalling the definitions of [14], for an object  $P \in {}^H_K\mathcal{M}_K$  representing a module functor in  $F_P \in \text{End}_{\text{Rep}(H)}(K\mathcal{M})$ , we explicitly give the structure of the functor

$$F_P \mapsto \int_{M \in \mathcal{M}} \underline{\text{Hom}}(M, F_P(M)).$$

For this we compute, in an explicit way, the end  $\int_{M \in \mathcal{M}} \underline{\text{Hom}}(M, F_P(M))$  as an object in the category  ${}^H_H\mathcal{YD}$  of Yetter–Drinfeld modules over  $H$ . In Section 5, we illustrate this description in the particular cases when  $H$  is a group algebra or its dual. As a direct consequence, we compute the adjoint algebra and the space of class functions for certain group-theoretical fusion categories.

**1. Preliminaries.** Let  $\mathbb{k}$  be an algebraically closed field. All algebras are assumed to be over  $\mathbb{k}$ . If  $A$  is an algebra, we shall denote by  ${}_A\mathcal{M}$  (respectively  $\mathcal{M}_A$ ) the category of finite-dimensional left  $A$ -modules (respectively right  $A$ -modules). If  $A, B$  are two algebras, we shall denote by  ${}_B\mathcal{M}_A$  the category of finite-dimensional  $(B, A)$ -modules. From now on, all categories are assumed to be abelian  $\mathbb{k}$ -linear, and all functors are  $\mathbb{k}$ -linear.

**1.1. Hopf algebras.** For a Hopf algebra  $H$ , we shall denote by  $\Delta : H \rightarrow H \otimes_{\mathbb{k}} H$  the comultiplication,  $\mathcal{S} : H \rightarrow H$  the antipode, and  $\epsilon : H \rightarrow \mathbb{k}$  the counit. We shall use Sweedler’s notation:  $\Delta(h) = h_{(1)} \otimes h_{(2)}$ ,  $h \in H$ . The category  ${}_H\mathcal{M}$  has a canonical structure of tensor category with monoidal product given by  $\otimes_{\mathbb{k}}$ . We shall denote this tensor category by  $\text{Rep}(H)$ .

For a finite-dimensional Hopf algebra  $H$ , we shall denote by  ${}^H_H\mathcal{YD}$  the category of finite-dimensional *Yetter–Drinfeld modules*. An object  $V \in {}^H_H\mathcal{YD}$  is a left  $H$ -module  $\cdot : H \otimes_{\mathbb{k}} V \rightarrow V$ , and a left  $H$ -comodule  $\lambda : V \rightarrow H \otimes_{\mathbb{k}} V$  such that

$$(1.1) \quad \lambda(h \cdot v) = h_{(1)}v_{(-1)}\mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)},$$

for any  $h \in H$  and  $v \in V$ . If  $V \in {}^H_H\mathcal{YD}$ , the map  $\sigma_X : V \otimes_{\mathbb{k}} X \rightarrow X \otimes_{\mathbb{k}} V$  given by  $\sigma_X(v \otimes x) = v_{(-1)} \cdot x \otimes v_{(0)}$  is a half-braiding for  $V$ , and this correspondence establishes a monoidal equivalence  ${}^H_H\mathcal{YD} \simeq \mathcal{Z}(\text{Rep}(H))$ .

**1.2. Finite categories.** A category  $\mathcal{C}$  is *finite* [4] if

- it has finitely many simple objects;
- each simple object  $X$  has a projective cover  $P(X)$ ;

- the Hom spaces are finite-dimensional;
- each object has finite length.

Equivalently, a category is finite if it is equivalent to a category  ${}_A\mathcal{M}$  for some finite-dimensional algebra  $A$ .

If  $\mathcal{M}, \mathcal{N}$  are two finite categories, and  $F : \mathcal{M} \rightarrow \mathcal{N}$  is a functor, we shall denote by  $F^{\text{la}}, F^{\text{ra}} : \mathcal{N} \rightarrow \mathcal{M}$  its left and right adjoints, if they exist. We shall also denote by  $\text{Rex}(\mathcal{M}, \mathcal{N})$  the category of right exact functors from  $\mathcal{M}$  to  $\mathcal{N}$ .

**1.3. Ends and coends.** We briefly recall the notion of end and coend. The reader is referred to [10] for the details. Let  $\mathcal{C}, \mathcal{D}$  be categories, and let  $S, T : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$  be functors. A *dinatural transformation*  $\xi : S \rightrightarrows T$  is a collection of morphisms in  $\mathcal{D}$ ,

$$\xi_X : S(X, X) \rightarrow T(X, X), \quad X \in \mathcal{C},$$

such that for any morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ ,

$$(1.2) \quad T(\text{id}_X, f) \circ \xi_X \circ S(f, \text{id}_X) = T(f, \text{id}_Y) \circ \xi_Y \circ S(\text{id}_Y, f).$$

An *end* of  $S$  is a pair  $(E, p)$  consisting of an object  $E \in \mathcal{D}$  and a dinatural transformation  $p : E \rightrightarrows S$  with the following universal property. For any pair  $(D, q)$  consisting of an object  $D \in \mathcal{D}$  and a dinatural transformation  $q : D \rightrightarrows S$ , there exists a *unique* morphism  $h : D \rightarrow E$  in  $\mathcal{D}$  such that  $q_X = p_X \circ h$  for any  $X \in \mathcal{C}$ . A *coend* of  $S$  is the dual notion: it is a pair  $(C, \pi)$  consisting of an object  $C \in \mathcal{D}$  and a dinatural transformation  $\pi : S \rightrightarrows C$  with the following universal property. For any pair  $(B, t)$ , where  $B \in \mathcal{D}$  is an object and  $t : S \rightrightarrows B$  is a dinatural transformation, there exists a unique morphism  $h : C \rightarrow B$  such that  $h \circ \pi_X = t_X$  for any  $X \in \mathcal{C}$ .

The end and coend of the functor  $S$  are denoted, respectively, as

$$\int_{X \in \mathcal{C}} S(X, X) \quad \text{and} \quad \int^{X \in \mathcal{C}} S(X, X).$$

**2. Representations of tensor categories.** For basic notions on finite tensor categories we refer to [2], [4]. Let  $\mathcal{C}$  be a finite tensor category over  $\mathbb{k}$ . A (left) *module* over  $\mathcal{C}$  is a finite category  $\mathcal{M}$  together with a  $\mathbb{k}$ -bilinear bifunctor  $\bar{\otimes} : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ , exact in each variable, endowed with natural associativity and unit isomorphisms

$$m_{X,Y,M} : (X \otimes Y) \bar{\otimes} M \rightarrow X \bar{\otimes} (Y \bar{\otimes} M), \quad \ell_M : \mathbf{1} \bar{\otimes} M \rightarrow M.$$

These isomorphisms are subject to the following conditions:

$$(2.1) \quad m_{X,Y,Z \bar{\otimes} M} m_{X \otimes Y, Z, M} = (\text{id}_X \bar{\otimes} m_{Y, Z, M}) m_{X, Y \otimes Z, M} (\alpha_{X, Y, Z} \bar{\otimes} \text{id}_M),$$

$$(2.2) \quad (\text{id}_X \bar{\otimes} l_M) m_{X, \mathbf{1}, M} = r_X \bar{\otimes} \text{id}_M,$$

for any  $X, Y, Z \in \mathcal{C}, M \in \mathcal{M}$ . Here  $\alpha$  is the associativity constraint of  $\mathcal{C}$ . Sometimes we shall also say that  $\mathcal{M}$  is a  $\mathcal{C}$ -module or a  $\mathcal{C}$ -module category.

Let  $\mathcal{M}$  and  $\mathcal{M}'$  be a pair of  $\mathcal{C}$ -modules. A *module functor* is a pair  $(F, c)$ , where  $F : \mathcal{M} \rightarrow \mathcal{M}'$  is a functor equipped with natural isomorphisms

$$c_{X,M} : F(X \otimes M) \rightarrow X \otimes F(M), \quad X \in \mathcal{C}, M \in \mathcal{M},$$

such that for any  $X, Y \in \mathcal{C}, M \in \mathcal{M}$ ,

$$(2.3) \quad (\text{id}_X \otimes c_{Y,M})c_{X,Y \otimes M}F(m_{X,Y,M}) = m_{X,Y,F(M)}c_{X \otimes Y,M},$$

$$(2.4) \quad \ell_{F(M)}c_{\mathbf{1},M} = F(\ell_M).$$

There is a composition of module functors: if  $\mathcal{M}''$  is another  $\mathcal{C}$ -module and  $(G, d) : \mathcal{M}' \rightarrow \mathcal{M}''$  is another module functor then the composition

$$(2.5) \quad (G \circ F, e) : \mathcal{M} \rightarrow \mathcal{M}'', \quad e_{X,M} = d_{X,F(M)} \circ G(c_{X,M}),$$

is also a module functor.

A *natural module transformation* between module functors  $(F, c)$  and  $(G, d)$  is a natural transformation  $\theta : F \rightarrow G$  such that for any  $X \in \mathcal{C}$  and  $M \in \mathcal{M}$ ,

$$(2.6) \quad d_{X,M}\theta_{X \otimes M} = (\text{id}_X \otimes \theta_M)c_{X,M}.$$

Two module functors  $F, G$  are *equivalent* if there exists a natural module isomorphism  $\theta : F \rightarrow G$ . We denote by  $\text{Func}(\mathcal{M}, \mathcal{M}')$  the category whose objects are module functors  $(F, c)$  from  $\mathcal{M}$  to  $\mathcal{M}'$  and arrows are module natural transformations.

Two  $\mathcal{C}$ -modules  $\mathcal{M}$  and  $\mathcal{M}'$  are *equivalent* if there exist module functors  $F : \mathcal{M} \rightarrow \mathcal{M}'$  and  $G : \mathcal{M}' \rightarrow \mathcal{M}$  and natural module isomorphisms  $\text{Id}_{\mathcal{M}'} \rightarrow F \circ G$  and  $\text{Id}_{\mathcal{M}} \rightarrow G \circ F$ .

A module is *indecomposable* if it is not equivalent to a direct sum of two non-trivial modules. Recall from [4], that a module  $\mathcal{M}$  is *exact* if for any projective object  $P \in \mathcal{C}$  the object  $P \otimes M$  is projective in  $\mathcal{M}$ , for all  $M \in \mathcal{M}$ . If  $\mathcal{M}$  is an exact indecomposable module category over  $\mathcal{C}$ , the dual category  $\mathcal{C}_{\mathcal{M}}^* = \text{End}_{\mathcal{C}}(\mathcal{M})$  is a finite tensor category [4]. The tensor product is the composition of module functors.

A *right module category* over  $\mathcal{C}$  is a finite category  $\mathcal{M}$  equipped with an exact bifunctor  $\otimes : \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{M}$  and natural isomorphisms

$$\tilde{m}_{M,X,Y} : M \otimes (X \otimes Y) \rightarrow (M \otimes X) \otimes Y, \quad r_M : M \otimes \mathbf{1} \rightarrow M$$

such that

$$(2.7) \quad \tilde{m}_{M \otimes X, Y, Z} \tilde{m}_{M, X, Y \otimes Z} (\text{id}_M \otimes a_{X, Y, Z}) = (\tilde{m}_{M, X, Y} \otimes \text{id}_Z) \tilde{m}_{M, X \otimes Y, Z},$$

$$(2.8) \quad (r_M \otimes \text{id}_X) \tilde{m}_{M, \mathbf{1}, X} = \text{id}_M \otimes l_X.$$

If  $\mathcal{M}, \mathcal{M}'$  are right  $\mathcal{C}$ -modules, a *module functor* from  $\mathcal{M}$  to  $\mathcal{M}'$  is a pair  $(T, d)$  where  $T : \mathcal{M} \rightarrow \mathcal{M}'$  is a functor and  $d_{M,X} : T(M \otimes X) \rightarrow T(M) \otimes X$

are natural isomorphisms such that for any  $X, Y \in \mathcal{C}$ ,  $M \in \mathcal{M}$ :

$$(2.9) \quad (d_{M,X} \otimes \text{id}_Y)d_{M \otimes X, Y}T(m_{M, X, Y}) = m_{T(M), X, Y}d_{M, X \otimes Y},$$

$$(2.10) \quad r_{T(M)}d_{M, \mathbf{1}} = T(r_M).$$

Assume that  $\mathcal{M}, \mathcal{N}$  are categories,  $F : \mathcal{M} \rightarrow \mathcal{N}$  is a functor with right adjoint  $G : \mathcal{N} \rightarrow \mathcal{M}$ . We shall denote by  $\epsilon : F \circ G \rightarrow \text{Id}_{\mathcal{N}}$ ,  $\eta : \text{Id}_{\mathcal{M}} \rightarrow G \circ F$  the counit and unit of the adjunction. The next result will be needed later.

LEMMA 2.1 ([5, Lemma 2.11]). *The following holds.*

(i) *If  $\mathcal{M}, \mathcal{N}$  are left  $\mathcal{C}$ -module categories and  $(F, c) : \mathcal{M} \rightarrow \mathcal{N}$  is a module functor then  $G$  has a module functor structure given by*

$$e_{X, N}^{-1} = G(\text{id}_X \otimes \epsilon_N)G(c_{X, G(N)})\eta_{X \otimes G(N)}, \quad X \in \mathcal{C} \text{ and } N \in \mathcal{N}.$$

(ii) *If  $\mathcal{M}, \mathcal{N}$  are right  $\mathcal{C}$ -module categories and  $(F, d) : \mathcal{M} \rightarrow \mathcal{N}$  is a module functor then  $G$  has a module functor structure given by*

$$h_{N, X}^{-1} = G(\epsilon_N \otimes \text{id}_X)G(d_{G(N), X})\eta_{G(N) \otimes X}, \quad X \in \mathcal{C}, N \in \mathcal{N}. \blacksquare$$

**2.1. Bimodule categories.** Assume  $\mathcal{D}$  is another finite tensor category. A  $(\mathcal{C}, \mathcal{D})$ -bimodule category is a category  $\mathcal{M}$  with left  $\mathcal{C}$ -module category and right  $\mathcal{D}$ -module category structure together with natural isomorphisms

$$(2.11) \quad \gamma_{X, M, Y} : (X \otimes M) \otimes Y \rightarrow X \otimes (M \otimes Y), \quad X \in \mathcal{C}, Y \in \mathcal{D}, M \in \mathcal{M},$$

satisfying

$$(2.12) \quad \begin{array}{ccc} ((X \otimes Y) \otimes M) \otimes Z & \xrightarrow{\gamma} & (X \otimes Y) \otimes (M \otimes Z) \\ m^l \otimes \text{id} \downarrow & & \downarrow m^l \\ (X \otimes (Y \otimes M)) \otimes Z & & \\ \gamma \downarrow & & \\ X \otimes ((Y \otimes M) \otimes Z) & \xrightarrow{\text{id} \otimes \gamma} & X \otimes (Y \otimes (M \otimes Z)) \end{array}$$

$$(2.13) \quad \begin{array}{ccc} (X \otimes M) \otimes (Y \otimes Z) & \xrightarrow{\gamma} & X \otimes (M \otimes (Y \otimes Z)) \\ m^r \downarrow & & \downarrow \text{id} \otimes m^r \\ ((X \otimes M) \otimes Y) \otimes Z & & \\ \gamma \otimes \text{id} \downarrow & & \\ (X \otimes (M \otimes Y)) \otimes Z & \xrightarrow{\gamma} & X \otimes ((M \otimes Y) \otimes Z) \end{array}$$

$$(2.14) \quad \begin{array}{ccc} (\mathbf{1} \overline{\otimes} M) \overline{\otimes} \mathbf{1} & \xrightarrow{\gamma_{\mathbf{1},M,\mathbf{1}}} & \mathbf{1} \overline{\otimes} (M \overline{\otimes} \mathbf{1}) \\ \downarrow l_M \otimes \text{id}_{\mathbf{1}} & & \downarrow r_M \\ M \overline{\otimes} \mathbf{1} & & \mathbf{1} \overline{\otimes} M \\ \downarrow \text{id}_{\mathbf{1}} \otimes r_M & \xleftarrow{l_M} & \downarrow \\ M & & \end{array}$$

where  $m^l$  and  $m^r$  are the associativity isomorphisms of the left, respectively right, module category. If  $\mathcal{M}, \mathcal{N}$  are  $(\mathcal{C}, \mathcal{D})$ -bimodule categories, a *bimodule functor* is a triple  $(F, c, d) : \mathcal{M} \rightarrow \mathcal{N}$ , where  $(F, c)$  is a  $\mathcal{C}$ -module functor,  $(F, d)$  is a  $\mathcal{D}$ -module functor and the equality

$$(2.15) \quad \gamma_{X,F(M),Y}(c_{X,M} \overline{\otimes} \text{id}_Y) d_{X \overline{\otimes} M, Y} = (\text{id}_X \overline{\otimes} d_{M,Y}) c_{X, M \overline{\otimes} Y} F(\gamma_{X,M,Y})$$

holds for all  $X \in \mathcal{C}, Y \in \mathcal{D}$ , and  $M \in \mathcal{M}$ .

It is known that  $\mathcal{M}$  is a  $(\mathcal{C}, \mathcal{D})$ -bimodule category if and only if it is a left  $\mathcal{C} \boxtimes \mathcal{D}^{\text{op}}$ -module category, and a bimodule functor is the same as a  $\mathcal{C} \boxtimes \mathcal{D}^{\text{op}}$ -module functor. See for example [7].

If  $\mathcal{M}, \mathcal{N}$  are left  $\mathcal{C}$ -module categories, then  $\text{Rex}(\mathcal{M}, \mathcal{N})$  is a  $\mathcal{C}$ -bimodule category as follows. If  $X \in \mathcal{C}, F \in \text{Rex}(\mathcal{M}, \mathcal{N})$  and  $M \in \mathcal{M}$ , then

$$(2.16) \quad (X \overline{\otimes} F)(M) = X \overline{\otimes} F(M), \quad (F \overline{\otimes} X)(M) = F(X \overline{\otimes} M).$$

**2.2. The internal Hom.** Let  $\mathcal{C}$  be a tensor category and  $\mathcal{M}$  a left  $\mathcal{C}$ -module category. For any pair of objects  $M, N \in \mathcal{M}$ , the *internal Hom* is an object  $\underline{\text{Hom}}(M, N) \in \mathcal{C}$  representing the functor  $\text{Hom}_{\mathcal{M}}(- \overline{\otimes} M, N) : \mathcal{C} \rightarrow \text{vect}_{\mathbb{k}}$ . This means that there are natural isomorphisms, inverse to each other,

$$(2.17) \quad \begin{aligned} \phi_{M,N}^X &: \text{Hom}_{\mathcal{C}}(X, \underline{\text{Hom}}(M, N)) \rightarrow \text{Hom}_{\mathcal{M}}(X \overline{\otimes} M, N), \\ \psi_{M,N}^X &: \text{Hom}_{\mathcal{M}}(X \overline{\otimes} M, N) \rightarrow \text{Hom}_{\mathcal{C}}(X, \underline{\text{Hom}}(M, N)), \end{aligned}$$

for all  $M, N \in \mathcal{M}$ , and  $X \in \mathcal{C}$ . Sometimes we shall denote the internal Hom of the module category  $\mathcal{M}$  by  $\underline{\text{Hom}}_{\mathcal{M}}$  to emphasize that it is related to this module category.

For any  $X \in \mathcal{C}$  and  $M, N \in \mathcal{M}$  define

$$\begin{aligned} \text{coev}_{X,M}^{\mathcal{M}} &: X \rightarrow \underline{\text{Hom}}(M, X \overline{\otimes} M), & \text{ev}_{M,N}^{\mathcal{M}} &: \underline{\text{Hom}}(M, N) \overline{\otimes} M \rightarrow N, \\ \text{coev}_{X,M}^{\mathcal{M}} &= \psi_{M, X \overline{\otimes} M}^X(\text{id}_{X \overline{\otimes} M}), & \text{ev}_{M,N}^{\mathcal{M}} &= \phi_{M,N}^{\underline{\text{Hom}}(M,N)}(\text{id}_{\underline{\text{Hom}}(M,N)}). \end{aligned}$$

Define also  $f_M = \text{ev}_{M,M}^{\mathcal{M}}(\text{id}_{\underline{\text{Hom}}(M,M)}) \overline{\otimes} \text{ev}_{M,M}^{\mathcal{M}}$ , and

$$\begin{aligned} \text{comp}_M^{\mathcal{M}} &: \underline{\text{Hom}}(M, M) \otimes \underline{\text{Hom}}(M, M) \rightarrow \underline{\text{Hom}}(M, M), \\ \text{comp}_M^{\mathcal{M}} &= \psi_{M,M}^{\underline{\text{Hom}}(M,M) \otimes \underline{\text{Hom}}(M,M)}(f_M). \end{aligned}$$

It is known (see [4]), that  $\underline{\text{Hom}}(M, M)$  is an algebra in the category  $\mathcal{C}$  with product given by  $\text{comp}_M^M$ .

**2.3. The relative center.** Let  $\mathcal{C}$  be a tensor category and  $\mathcal{M}$  a  $\mathcal{C}$ -bimodule category. The *relative center* of  $\mathcal{M}$  is the category of  $\mathcal{C}$ -bimodule functors from  $\mathcal{C}$  to  $\mathcal{M}$ . We denote the relative center of  $\mathcal{M}$  by  $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ . Explicitly, objects of  $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$  are pairs  $(M, \sigma)$ , where  $M$  is an object of  $\mathcal{M}$  and

$$\sigma_X : M \bar{\otimes} X \xrightarrow{\sim} X \bar{\otimes} M$$

is a family of natural isomorphisms such that

$$(2.18) \quad m_{X,Y,M}^l \sigma_{X \otimes Y} = (\text{id}_X \bar{\otimes} \sigma_Y) \gamma_{X,M,Y} (\sigma_X \bar{\otimes} \text{id}_Y) m_{M,X,Y}^r,$$

where  $\gamma_{X,M,Y} : (X \bar{\otimes} M) \bar{\otimes} Y \rightarrow X \bar{\otimes} (M \bar{\otimes} Y)$  are the associativity constraints of the left and right actions on  $\mathcal{M}$  (see (2.11)). The isomorphism  $\sigma$  is called the *half-braiding* for  $M$ .

As explained in [14, Section 3.6], the relative center can be thought of as a 2-functor

$$\mathcal{Z}_{\mathcal{C}} : {}_{\mathcal{C}}\mathcal{Bimod} \rightarrow \text{Ab}_{\mathbb{k}},$$

where  ${}_{\mathcal{C}}\mathcal{Bimod}$  is the 2-category whose 0-cells are  $\mathcal{C}$ -bimodule categories, 1-cells are bimodule functors and 2-cells are bimodule natural transformations. Also  $\text{Ab}_{\mathbb{k}}$  is the 2-category of finite  $\mathbb{k}$ -linear abelian categories. If  $\mathcal{M}, \mathcal{N}$  are  $\mathcal{C}$ -bimodule categories, then  $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$  is the relative center. If  $(F, c, d) : \mathcal{M} \rightarrow \mathcal{N}$  is a bimodule functor, then  $\mathcal{Z}_{\mathcal{C}}(F) : \mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \rightarrow \mathcal{Z}_{\mathcal{C}}(\mathcal{N})$  is the functor  $\mathcal{Z}_{\mathcal{C}}(F)(M, \sigma) = (F(M), \tilde{\sigma})$ , where  $\tilde{\sigma}_X : F(M) \bar{\otimes} X \rightarrow X \bar{\otimes} F(M)$  is defined as

$$(2.19) \quad \tilde{\sigma}_X = c_{X,M} F(\sigma_X) d_{M,X}^{-1}, \quad X \in \mathcal{C}.$$

The following is [14, Example 3.11].

**EXAMPLE 2.2.** If  $\mathcal{M}, \mathcal{N}$  are exact  $\mathcal{C}$ -module categories, then  $\text{Rex}(\mathcal{M}, \mathcal{N})$  is a  $\mathcal{C}$ -bimodule category (see (2.16)). In this case there exists an equivalence  $\mathcal{Z}_{\mathcal{C}}(\text{Rex}(\mathcal{M}, \mathcal{N})) \simeq \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ .

**EXAMPLE 2.3.** When  $\mathcal{C}$  is considered as a  $\mathcal{C}$ -bimodule category, then  $\mathcal{Z}_{\mathcal{C}}(\mathcal{C}) = \mathcal{Z}(\mathcal{C})$  is the usual center of the category  $\mathcal{C}$ .

**REMARK 2.4.** If  $(X, \sigma) \in \mathcal{Z}(\mathcal{C})$  and  $\mathcal{M}$  is a left  $\mathcal{C}$ -module category, then the functor  $L_{(X,\sigma)} : \mathcal{M} \rightarrow \mathcal{M}$  given by  $L_{(X,\sigma)}(M) = X \bar{\otimes} M$  is a  $\mathcal{C}$ -module functor. The module structure is given by

$$\begin{aligned} c_{Y,M}^{(X,\sigma)} &: X \bar{\otimes} (Y \bar{\otimes} M) \rightarrow Y \bar{\otimes} (X \bar{\otimes} M), \\ c_{Y,M}^{(X,\sigma)} &= m_{Y,X,M} (\sigma_Y \bar{\otimes} \text{id}_M) m_{X,Y,M}^{-1}, \end{aligned}$$

for any  $X, Y \in \mathcal{C}, M \in \mathcal{M}$ .



DEFINITION 2.5. For any exact indecomposable left  $\mathcal{C}$ -module category  $\mathcal{M}$ , we shall denote by

$$\mathcal{F}_{\mathcal{M}} : \mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*) \rightarrow \mathcal{C}_{\mathcal{M}}^*, \quad (V, \sigma) \mapsto V,$$

the forgetful functor. In particular  $\mathcal{F}_{\mathcal{C}} : \mathcal{Z}_{\mathcal{C}}(\mathcal{C}) \rightarrow \mathcal{C}$  is the usual forgetful functor.

**2.4. Morita invariance of the Drinfeld center.** Let  $\mathcal{M}$  be an exact indecomposable module category over  $\mathcal{C}$ . Using results of P. Schauenburg [12], K. Shimizu [14, Section 3.7] proved that there exists a braided monoidal equivalence

$$\theta_{\mathcal{M}} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*).$$

For later use, we shall recall the definition of this equivalence. Let  $(V, \sigma) \in \mathcal{Z}(\mathcal{C})$ . Then  $\theta_{\mathcal{M}}(V, \sigma) : \mathcal{M} \rightarrow \mathcal{M}$  is the functor defined as  $\theta_{\mathcal{M}}(V, \sigma)(M) = V \overline{\otimes} M$ , for any  $M \in \mathcal{M}$ . The module structure of the functor  $\theta_{\mathcal{M}}(V, \sigma)$  is

$$c_{X, M} : \theta_{\mathcal{M}}(V, \sigma)(X \overline{\otimes} M) = V \overline{\otimes} (X \overline{\otimes} M) \rightarrow X \overline{\otimes} (V \overline{\otimes} M),$$

given by the composition

$$V \overline{\otimes} (X \overline{\otimes} M) \xrightarrow{m_{V, X, M}^{-1}} (V \overline{\otimes} X) \overline{\otimes} M \xrightarrow{\sigma_X \overline{\otimes} \text{id}} (X \overline{\otimes} V) \overline{\otimes} M \xrightarrow{m_{X, V, M}} X \overline{\otimes} (V \overline{\otimes} M).$$

Then  $\theta_{\mathcal{M}}(V, \sigma)$  becomes a  $\mathcal{C}$ -module functor. It remains to explain how the functor  $\theta_{\mathcal{M}}(V, \sigma)$  is an object in the center of  $\mathcal{C}_{\mathcal{M}}^*$ . For any  $(F, d) \in \mathcal{C}_{\mathcal{M}}^*$  we have to define a half-braiding  $\tau_{(F, d)} : \theta_{\mathcal{M}}(V, \sigma) \circ (F, d) \rightarrow (F, d) \circ \theta_{\mathcal{M}}(V, \sigma)$ . This is the module natural transformation defined by

$$(2.20) \quad (\tau_{(F, d)})_M : V \overline{\otimes} F(M) \rightarrow F(V \overline{\otimes} M), \quad (\tau_{(F, d)})_M = d_{V, M}^{-1},$$

for any  $M \in \mathcal{M}$ .

**3. Module categories over Hopf algebras.** Throughout this section,  $H$  will denote a finite-dimensional Hopf algebra. We shall present families of module categories over  $\text{Rep}(H)$ , and compute explicitly its internal Hom and their module functor categories.

If  $\lambda : K \rightarrow H \otimes_{\mathbb{k}} K$  is a left  $H$ -comodule algebra then the category of finite-dimensional left  $K$ -modules  ${}_K\mathcal{M}$  is a module category over  $\text{Rep}(H)$  with action  $\overline{\otimes} : \text{Rep}(H) \times {}_K\mathcal{M} \rightarrow {}_K\mathcal{M}$ ,  $X \overline{\otimes} M = X \otimes_{\mathbb{k}} M$ , for all  $X \in \text{Rep}(H)$ ,  $M \in {}_K\mathcal{M}$ . The left  $K$ -module structure on  $X \otimes_{\mathbb{k}} M$  is given by  $\lambda$ , that is, if  $k \in K$ ,  $x \in X$ ,  $m \in M$  then

$$k \cdot (x \otimes m) = \lambda(k)(x \otimes m) = k_{(-1)} \cdot x \otimes k_{(0)} \cdot m.$$

THEOREM 3.1 ([1, Prop. 1.20]). *If  $K$  is right  $H$ -simple then  ${}_K\mathcal{M}$  is an exact indecomposable module category over  $\text{Rep}(H)$ . Moreover, if  $\mathcal{M}$  is an exact indecomposable module category over  $\text{Rep}(H)$ , then there exists a*

right  $H$ -simple left  $H$ -comodule algebra  $K$  with trivial coinvariants such that  $\mathcal{M} \simeq_K \mathcal{M}$ . ■

REMARK 3.2. If  $K, S$  are isomorphic  $H$ -comodule algebras, then the categories  ${}_K\mathcal{M}, {}_S\mathcal{M}$  are equivalent as  $\text{Rep}(H)$ -module categories. The converse is not always true.

**3.1. The internal Hom.** We shall explicitly compute the internal Hom of module categories over  $\text{Rep}(H)$ .

If  $M, N$  are left  $K$ -modules, then the space  $\text{Hom}_K(H \otimes_{\mathbb{k}} M, N)$  has a left  $H$ -action given by

$$(h \cdot \alpha)(t \otimes m) = \alpha(th \otimes m)$$

for any  $h, t \in H$ ,  $\alpha \in \text{Hom}_K(H \otimes_{\mathbb{k}} M, N)$  and  $m \in M$ . We can identify the space  $\text{Hom}_K(H \otimes_{\mathbb{k}} M, N)$  with the subspace of  $H^* \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(M, N)$  consisting of elements  $\sum_i f_i \otimes T_i \in H^* \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(M, N)$  such that

$$(3.1) \quad \sum_i \langle f_i, k_{(-1)}h \rangle T_i(k_{(0)} \cdot m) = \sum_i \langle f_i, h \rangle k \cdot T_i(m)$$

for any  $h \in H$ ,  $k \in K$  and  $m \in M$ . An element  $\sum_i f_i \otimes T_i$  is seen as a map from  $H \otimes_{\mathbb{k}} M$  to  $N$ , sending  $h \otimes m$  to  $\sum_i \langle f_i, h \rangle T_i(m)$ . We shall freely use this identification from now on. Condition (3.1) says that this morphism is a  $K$ -module map.

For any  $K$ -module  $M$ , the space  $\text{Hom}_K(H \otimes_{\mathbb{k}} M, M)$  has an algebra structure as follows. If  $\sum_i f_i \otimes T_i, \sum_j g_j \otimes U_j$  are elements in  $\text{Hom}_K(H \otimes_{\mathbb{k}} M, M)$ , their product is defined by

$$(3.2) \quad \left( \sum_i f_i \otimes T_i \right) \left( \sum_j g_j \otimes U_j \right) = \sum_{i,j} f_i g_j \otimes T_i \circ U_j.$$

The proof of the next result is straightforward.

LEMMA 3.3. *With the product described in (3.2),  $\text{Hom}_K(H \otimes_{\mathbb{k}} M, M)$  becomes an  $H$ -module algebra. ■*

LEMMA 3.4. *Let  $M, N \in {}_K\mathcal{M}$ , and  $\underline{\text{Hom}}(M, N)$  the internal Hom of the module category  ${}_K\mathcal{M}$ . There is an isomorphism of  $H$ -modules*

$$\underline{\text{Hom}}(M, N) \simeq \text{Hom}_K(H \otimes_{\mathbb{k}} M, N).$$

*When  $M = N$  this isomorphism is an  $H$ -module algebra isomorphism.*

*Proof.* Let  $X \in \text{Rep}(H)$ . The maps

$$\begin{aligned} \phi &: \text{Hom}_H(X, \text{Hom}_K(H \otimes_{\mathbb{k}} M, N)) \rightarrow \text{Hom}_K(X \otimes_{\mathbb{k}} M, N), \\ \psi &: \text{Hom}_K(X \otimes_{\mathbb{k}} M, N) \rightarrow \text{Hom}_H(X, \text{Hom}_K(H \otimes_{\mathbb{k}} M, N)), \end{aligned}$$

defined by  $\phi(\alpha)(x \otimes m) = \alpha(x)(1 \otimes m)$  and  $\psi(\beta)(x)(h \otimes m) = \beta(h \cdot x \otimes m)$  for any  $h \in H$ ,  $x \in X$ ,  $m \in M$ , are well-defined maps, inverse to each other. It

follows straightforwardly that when  $M = N$ , this isomorphism is an algebra map. ■

**3.2. Module functors.** Given two  $H$ -comodule algebras  $K, S$ , we shall explicitly describe the category of  $\text{Rep}(H)$ -module functors between the associated module categories.

Under these hypotheses, we shall denote by  ${}^H_S\mathcal{M}_K$  the category of finite-dimensional  $(S, K)$ -bimodules that are also left  $H$ -comodules, with comodule structure a morphism of  $(S, K)$ -bimodules.

PROPOSITION 3.5. *Assume  $K, S$  are right  $H$ -simple left  $H$ -comodule algebras, and  ${}_K\mathcal{M}, {}_S\mathcal{M}$  the corresponding  $\text{Rep}(H)$ -module categories. There are equivalences*

$$\text{Rex}({}_K\mathcal{M}, {}_S\mathcal{M}) \simeq {}_S\mathcal{M}_K, \quad \text{Fun}_{\text{Rep}(H)}({}_K\mathcal{M}, {}_S\mathcal{M}) \simeq {}^H_S\mathcal{M}_K.$$

*Proof.* We shall only explain the definition of the equivalences. For the complete proof see [1, Prop. 1.23]. The first equivalence is a consequence of a theorem of Watts [16]. The functor  $\Phi : {}_S\mathcal{M}_K \rightarrow \text{Rex}({}_K\mathcal{M}, {}_S\mathcal{M})$ ,  $\Phi(B)(M) = B \otimes_K M$ , is an equivalence of categories.

If  $P \in {}_S\mathcal{M}_K$ , define the functor  $F_P : {}_K\mathcal{M} \rightarrow {}_S\mathcal{M}$  by  $F_P(M) = P \otimes_K M$ . The correspondence  $P \mapsto F_P$  is an equivalence of categories.

If  $P \in {}^H_S\mathcal{M}_K$ ,  $X \in \text{Rep}(H)$ , and  $M \in {}_K\mathcal{M}$ , the functor  $F_P$  has a module structure:

$$\begin{aligned} c_{X,M} : P \otimes_K (X \otimes_{\mathbb{k}} M) &\rightarrow X \otimes_{\mathbb{k}} (P \otimes_K M), \\ c_{X,M}(p \otimes x \otimes m) &= p_{(-1)} \cdot x \otimes p_{(0)} \otimes m, \quad p \in P, x \in X, m \in M. \end{aligned}$$

Here the map  $\lambda : P \rightarrow H \otimes P$ ,  $\lambda(p) = p_{(-1)} \otimes p_{(0)}$ , is the left  $H$ -coaction of  $P$ . ■

The next result is a direct consequence of Proposition 3.5.

COROLLARY 3.6. *Let  $K$  be a right  $H$ -simple left  $H$ -comodule algebra. There is a monoidal equivalence*

$$\text{Rep}(H)_{{}_K\mathcal{M}}^* \simeq {}^H_K\mathcal{M}_K. \quad \blacksquare$$

Assume  $K, S$  are  $H$ -comodule algebras. The category  ${}_S\mathcal{M}_K$  has a  $\text{Rep}(H)$ -bimodule structure as follows. If  $P \in {}_S\mathcal{M}_K$ ,  $X \in \text{Rep}(H)$ , then

$$X \overline{\otimes} P = X \otimes_{\mathbb{k}} P, \quad P \overline{\otimes} X = P \otimes_K (X \otimes_{\mathbb{k}} K).$$

The  $(S, K)$ -actions on the spaces  $X \overline{\otimes} P$  and  $P \overline{\otimes} X$  are

$$\begin{aligned} s \cdot (x \otimes p) \cdot k &= s_{(-1)} \cdot x \otimes s_{(0)} \cdot p \cdot k, \\ s \cdot (p \otimes (x \otimes l)) \cdot k &= s \cdot p \otimes (x \otimes lk), \quad s \in S, k, l \in K, p \in P, x \in X. \end{aligned}$$

The natural isomorphisms relating both actions are given by

$$(3.3) \quad \begin{aligned} \gamma_{X,P,Y} &: (X \overline{\otimes} P) \overline{\otimes} Y \rightarrow X \overline{\otimes} (P \overline{\otimes} Y), \\ \gamma_{X,P,Y}((x \otimes p) \otimes (y \otimes k)) &= x \otimes (p \otimes y \otimes k), \end{aligned}$$

for any  $X, Y \in \text{Rep}(H), P \in {}_S\mathcal{M}_K, x \in X, y \in Y, p \in P$  and  $k \in K$ . It follows by a straightforward computation that the maps  $\gamma_{X,P,Y}$  satisfy (2.12)–(2.14).

Recall that  $\text{Rex}({}_K\mathcal{M}, {}_S\mathcal{M})$  has a  $\text{Rep}(H)$ -bimodule category structure (see (2.16)). The next lemma is straightforward.

LEMMA 3.7. *The equivalence  $\text{Rex}({}_K\mathcal{M}, {}_S\mathcal{M}) \simeq {}_S\mathcal{M}_K$  of Proposition 3.5 is an equivalence of  $\text{Rep}(H)$ -bimodule categories. ■*

**3.3. The center of dual tensor categories.** In Section 2.4, for any exact  $\mathcal{C}$ -module category  $\mathcal{M}$  we presented an equivalence of braided tensor categories  $\theta_{\mathcal{M}} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*)$ . In this section, we shall explicitly give this equivalence in the case  $\mathcal{C} = \text{Rep}(H)$  and  $\mathcal{M} = {}_K\mathcal{M}$  for a right  $H$ -simple left  $H$ -comodule algebra  $K$ . For this, we shall use the monoidal equivalences  $\mathcal{Z}(\text{Rep}(H)) \simeq {}^H_H\mathcal{YD}$  and  $\text{Rep}(H)_{K\mathcal{M}}^* \simeq {}^H_K\mathcal{M}_K$ . The latter is given in Corollary 3.6.

Set  $\theta_K = \theta_{K\mathcal{M}} : {}^H_H\mathcal{YD} \rightarrow \mathcal{Z}({}^H_K\mathcal{M}_K)$ . If  $V \in {}^H_H\mathcal{YD}$  then  $\theta_K(V) = V \otimes_{\mathbb{k}} K$ . The  $K$ -bimodule and left  $H$ -comodule structures are given by

$$\begin{aligned} k \cdot (v \otimes t) \cdot s &= k_{(-1)} \cdot v \otimes k_{(0)}ts, \\ \lambda(v \otimes t) &= v_{(-1)}t_{(-1)} \otimes v_{(0)} \otimes t_{(0)}, \quad v \in V, t, k, s \in K. \end{aligned}$$

The half braiding of the object  $V \otimes_{\mathbb{k}} K$  is given by

$$\begin{aligned} \sigma_P^V &: (V \otimes_{\mathbb{k}} K) \otimes_K P \rightarrow P \otimes_K (V \otimes_{\mathbb{k}} K), \\ \sigma_P^V(v \otimes t \otimes p) &= (t \cdot p)_{(0)} \otimes \mathcal{S}^{-1}((t \cdot p)_{(-1)}) \cdot v \otimes 1 \end{aligned}$$

for any  $P \in {}^H_K\mathcal{M}_K, v \in V, p \in P,$  and  $t \in K$ . This formula comes from (2.20).

**4. The character algebra for representations of  $\text{Rep}(H)$ .** Given a finite-dimensional Hopf algebra  $H,$  and  $\mathcal{M}$  a representation of the tensor category  $\text{Rep}(H)$ . We aim to compute the adjoint algebra  $\mathcal{A}_{\mathcal{M}}$  and the corresponding space of class functions as introduced by K. Shimizu [13], [14].

**4.1. The adjoint algebra and the space of class functions.** Let  $\mathcal{C}$  be a finite tensor category, and  $\mathcal{M}$  an exact indecomposable left module category over  $\mathcal{C}$ . We shall further assume that  $\mathcal{M}$  is strict. First, we shall recall the definition of the algebra  $\mathcal{A}_{\mathcal{M}} \in \mathcal{Z}(\mathcal{C})$ .

The action functor  $\rho_{\mathcal{M}} : \mathcal{C} \rightarrow \text{Rex}(\mathcal{M})$  is

$$\rho_{\mathcal{M}}(X)(M) = X \overline{\otimes} M, \quad X \in \mathcal{C}, M \in \mathcal{M}.$$

It was proven in [14, Thm. 3.4] that the right adjoint of  $\rho_{\mathcal{M}}$  is the functor  $\rho_{\mathcal{M}}^{\text{ra}} : \text{Rex}(\mathcal{M}) \rightarrow \mathcal{C}$  such that for any  $F \in \text{Rex}(\mathcal{M})$ ,

$$\rho_{\mathcal{M}}^{\text{ra}}(F) = \int_{M \in \mathcal{M}} \underline{\text{Hom}}(M, F(M)).$$

The counit and unit of the adjunction  $(\rho_{\mathcal{M}}, \rho_{\mathcal{M}}^{\text{ra}})$ , will be denoted by

$$\epsilon : \rho_{\mathcal{M}} \circ \rho_{\mathcal{M}}^{\text{ra}} \rightarrow \text{Id}_{\text{Rex}(\mathcal{M})}, \quad \eta : \text{Id}_{\mathcal{C}} \rightarrow \rho_{\mathcal{M}}^{\text{ra}} \circ \rho_{\mathcal{M}}.$$

According to Lemma 2.1 the functor  $\rho_{\mathcal{M}}^{\text{ra}}$  has the structure of a  $\mathcal{C}$ -bimodule functor as follows. The left and right module structures of  $\rho_{\mathcal{M}}^{\text{ra}}$  are

$$(4.1) \quad \begin{aligned} \xi_{X,F}^l : \rho_{\mathcal{M}}^{\text{ra}}(X \otimes F) &\rightarrow X \otimes \rho_{\mathcal{M}}^{\text{ra}}(F), \\ (\xi_{X,F}^l)^{-1} &= \rho_{\mathcal{M}}^{\text{ra}}(\text{id}_X \otimes \epsilon_F) \eta_{X \otimes \rho_{\mathcal{M}}^{\text{ra}}(F)}, \end{aligned}$$

$$(4.2) \quad \begin{aligned} \xi_{X,F}^r : \rho_{\mathcal{M}}^{\text{ra}}(F \otimes X) &\rightarrow \rho_{\mathcal{M}}^{\text{ra}}(F) \otimes X, \\ (\xi_{X,F}^r)^{-1} &= \rho_{\mathcal{M}}^{\text{ra}}(\epsilon_F \otimes \text{id}_X) \eta_{\rho_{\mathcal{M}}^{\text{ra}}(F) \otimes X}, \quad X \in \mathcal{C}, F \in \text{Rex}(\mathcal{M}). \end{aligned}$$

This description appears in [14, Equation A.9].

Since  $\rho_{\mathcal{M}}^{\text{ra}} : \text{Rex}(\mathcal{M}) \rightarrow \mathcal{C}$  is a  $\mathcal{C}$ -bimodule functor, we can consider the functor  $\mathcal{Z}_{\mathcal{C}}(\rho_{\mathcal{M}}^{\text{ra}}) : \text{End}_{\mathcal{C}}(\mathcal{M}) \rightarrow \mathcal{Z}(\mathcal{C})$ . Here  $\mathcal{Z}_{\mathcal{C}}$  is the 2-functor described in Section 2.3.

DEFINITION 4.1 ([14, Subsection 4.2]). The *adjoint algebra* of the module category  $\mathcal{M}$  is the algebra  $\mathcal{A}_{\mathcal{M}} := \mathcal{Z}(\rho_{\mathcal{M}}^{\text{ra}})(\text{Id}_{\mathcal{M}})$  in the center of  $\mathcal{C}$ . The *adjoint algebra of the tensor category  $\mathcal{C}$*  is the algebra  $\mathcal{A}_{\mathcal{C}}$  of the regular module category  $\mathcal{C}$ .

It was explained in [14, Subection 4.2] that the algebra structure of  $\mathcal{A}_{\mathcal{M}}$  is given as follows. Let  $\pi_{\mathcal{M}} : \mathcal{A}_{\mathcal{M}} \rightarrow \underline{\text{Hom}}(-, -)$  denote the dinatural transformation of the end  $\mathcal{A}_{\mathcal{M}}$ . The product and the unit of  $\mathcal{A}_{\mathcal{M}}$  are

$$m_{\mathcal{M}} : \mathcal{A}_{\mathcal{M}} \otimes \mathcal{A}_{\mathcal{M}} \rightarrow \mathcal{A}_{\mathcal{M}}, \quad u_{\mathcal{M}} : \mathbf{1} \rightarrow \mathcal{A}_{\mathcal{M}},$$

defined to be the unique morphisms that satisfy

$$(4.3) \quad \begin{aligned} \pi_{\mathcal{M}}(M) \circ m_{\mathcal{M}} &= \text{comp}_M^{\mathcal{M}} \circ (\pi_{\mathcal{M}}(M) \otimes \pi_{\mathcal{M}}(M)), \\ \pi_{\mathcal{M}}(M) \circ u_{\mathcal{M}} &= \text{coev}_{\mathbf{1}, M}^{\mathcal{M}}, \quad M \in \mathcal{M}. \end{aligned}$$

For the definition of  $\text{coev}^{\mathcal{M}}$  and  $\text{comp}^{\mathcal{M}}$  see Section 2.2.

DEFINITION 4.2 ([14, Definition 5.1]). The *space of class functions* of  $\mathcal{M}$  is  $\text{CF}(\mathcal{M}) := \text{Hom}_{\mathcal{C}}(\mathcal{F}_{\mathcal{C}}(\mathcal{A}_{\mathcal{M}}), \mathbf{1}) = \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathcal{A}_{\mathcal{M}}, \mathcal{A}_{\mathcal{C}})$ .

The following result will be useful when computing the adjoint algebra in particular examples. The first three statements are contained in [13], [14].

LEMMA 4.3. *Let  $\mathcal{M}$  be an exact indecomposable  $\mathcal{C}$ -module category.*

- (i) *If  $I : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$  is a right adjoint to the forgetful functor  $\mathcal{F} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ , then  $\mathcal{A}_{\mathcal{C}} \simeq I(\mathbf{1})$ .*

- (ii) *There exists an isomorphism  $\mathcal{A}_{\mathcal{C}_{\mathcal{M}}^*} \simeq \theta_{\mathcal{M}}(\mathcal{A}_{\mathcal{M}})$  as algebra objects in  $\mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*)$ .*
- (iii)  $\text{CF}(\mathcal{C}_{\mathcal{M}}^*) \simeq \text{End}_{\mathcal{Z}(\mathcal{C})}(\mathcal{A}_{\mathcal{M}})$ .
- (iv)  $\text{FPdim}(\mathcal{A}_{\mathcal{M}}) = \text{FPdim}(\mathcal{C})$ .

*Proof.* Recall that  $\theta_{\mathcal{M}} : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*)$  is the braided equivalence presented in Section 2.4.

It is proven in [14, Corollary 3.15] that the functor  $\theta_{\mathcal{M}} \circ \mathcal{Z}(\rho_{\mathcal{M}}^{\text{ra}})$  is the right adjoint of the forgetful functor  $\mathcal{F}_{\mathcal{C}_{\mathcal{M}}^*}$ . Taking  $\mathcal{M} = \mathcal{C}$  implies (i), and taking an arbitrary  $\mathcal{M}$  yields (ii).

(iii) is [14, Theorem 5.12].

(iv) Let  $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$  be the forgetful functor, and  $I : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$  its right adjoint. It was proven in [2, Proposition 7.16.5] that  $\text{FPdim}(I(\mathbf{1})) = \text{FPdim}(\mathcal{C})$ . Hence  $\text{FPdim}(\mathcal{A}_{\mathcal{C}}) = \text{FPdim}(\mathcal{C})$  for any finite tensor category  $\mathcal{C}$ . Applying this result to  $\mathcal{C}_{\mathcal{M}}^*$  we obtain

$$\text{FPdim}(\mathcal{A}_{\mathcal{M}}) = \text{FPdim}(\mathcal{A}_{\mathcal{C}_{\mathcal{M}}^*}) = \text{FPdim}(\mathcal{C}_{\mathcal{M}}^*) = \text{FPdim}(\mathcal{C}).$$

The first equality follows from (ii), and the last one is [4, Corollary 3.43]. ■

**4.2. The adjoint algebra for module categories over Hopf algebras.** Let  $H$  be a finite-dimensional Hopf algebra. Let  $K$  be a finite-dimensional left  $H$ -comodule algebra. The category  ${}_K\mathcal{M}$  is a left  $\text{Rep}(H)$ -module category (see Section 3). We aim to compute  $\mathcal{A}_K = \mathcal{A}_{{}_K\mathcal{M}}$  as an algebra in the category  ${}^H_H\mathcal{YD}$  of Yetter–Drinfeld modules over  $H$ . For this, we shall explicitly give a description of the functor  $\mathcal{Z}(\rho_{\mathcal{M}}^{\text{ra}})$ .

Identifying  $\text{Rex}({}_K\mathcal{M}) = {}_K\mathcal{M}_K$ , we shall denote by  $\rho_K : \text{Rep}(H) \rightarrow {}_K\mathcal{M}_K$  the action functor. Explicitly, if  $X \in \text{Rep}(H)$  then

$$\rho_K(X) = X \otimes_{\mathbb{k}} K.$$

The left and right  $K$ -actions on  $X \otimes_{\mathbb{k}} K$  are given by

$$s \cdot (x \otimes k) \cdot t = s_{(-1)} \cdot x \otimes s_{(0)}kt, \quad x \in X, s, t, k \in K.$$

DEFINITION 4.4. For any  $P \in {}^H_K\mathcal{M}_K$ , define  $S^K(H, P)$  as the space of left  $K$ -linear morphisms  $\alpha \in \text{Hom}_K(H \otimes_{\mathbb{k}} K, P)$  such that for any  $k \in K$  and  $h \in H$ ,

$$(4.4) \quad \alpha(h \otimes k) = \alpha(h \otimes 1) \cdot k.$$

The space  $S^K(H, P)$  has a left  $H$ -module structure  $\cdot : H \otimes_{\mathbb{k}} S^K(H, P) \rightarrow S^K(H, P)$  and a left  $H$ -comodule structure  $\lambda : S^K(H, P) \rightarrow H \otimes_{\mathbb{k}} S^K(H, P)$ , defined by

$$(4.5) \quad (h \cdot \alpha)(x \otimes k) = \alpha(xh \otimes k),$$

$$(4.6) \quad \lambda : S^K(H, P) \rightarrow H \otimes_{\mathbb{k}} S^K(H, P), \quad \lambda(\alpha) = \alpha^{-1} \otimes \alpha^0,$$

for any  $\alpha \in S^K(H, P)$ ,  $h, x \in H$  and  $k \in K$ . Here for any  $h \in H$  and  $k \in K$ ,

$$(4.7) \quad \alpha^{-1} \otimes \alpha^0(h \otimes k) = \mathcal{S}(h_{(1)})\alpha(h_{(2)} \otimes 1)_{(-1)}h_{(3)} \otimes \alpha(h_{(2)} \otimes 1)_{(0)} \cdot k.$$

When  $P = K$ , we shall denote  $S(H, K) := S^K(H, K)$ . It follows by a straightforward computation that (4.5) and (4.6) are well defined maps, and they define an  $H$ -action and a  $H$ -coaction.

LEMMA 4.5. *The space  $S^K(H, P)$  is an object in the category  ${}^H_H\mathcal{YD}$ .*

*Proof.* We must prove the compatibility condition (1.1), that is,

$$(4.8) \quad \lambda(x \cdot \alpha) = x_{(1)}\alpha^{-1}S(x_{(3)}) \otimes x_{(2)} \cdot \alpha^0$$

for all  $x \in H$  and  $\alpha \in S^K(H, P)$ . Take  $\phi \in H^*$ ,  $h, x \in H$  and  $k \in K$ . Evaluating the right hand side of (4.8) at  $\phi \otimes h \otimes k$  gives

$$\begin{aligned} & \langle \phi, x_{(1)}\alpha^{-1}S(x_{(3)}) \rangle (x_{(2)} \cdot \alpha^0)(h \otimes k) \\ &= \langle \phi_{(1)}, x_{(1)} \rangle \langle \phi_{(3)}, S(x_{(3)}) \rangle \langle \phi_{(2)}, \alpha^{-1} \rangle x_{(2)} \cdot \alpha^0(h \otimes k) \\ &= \langle \phi_{(1)}, x_{(1)} \rangle \langle \phi_{(3)}, S(x_{(3)}) \rangle \langle \phi_{(2)}, \alpha^{-1} \rangle \alpha^0(hx_{(2)} \otimes k) \\ &= \langle \phi_{(1)}, x_{(1)} \rangle \langle \phi_{(3)}, S(x_{(3)}) \rangle \\ & \quad \langle \phi_{(2)}, S((hx_{(2)})_{(1)})\alpha((hx_{(2)})_{(2)} \otimes 1)_{(-1)}(hx_{(2)})_{(3)} \rangle \alpha((hx_{(2)})_{(2)} \otimes 1)_{(0)} \cdot k \\ &= \langle \phi, x_{(1)}S(x_{(2)})S(h_{(1)})\alpha(h_{(2)}x_{(3)} \otimes 1)_{(-1)}h_{(3)}x_{(4)}S(x_{(5)}) \rangle \\ & \quad \alpha(h_{(2)}x_{(3)} \otimes 1)_{(0)} \cdot k \\ &= \langle \phi, S(h_{(1)})\alpha(h_{(2)}x \otimes 1)_{(-1)}h_{(3)} \rangle \alpha(h_{(2)}x \otimes 1)_{(0)} \cdot k \\ &= \langle \phi, S(h_{(1)})(x \cdot \alpha)(h_{(2)} \otimes 1)_{(-1)}h_{(3)} \rangle (x \cdot \alpha)(h_{(2)} \otimes 1)_{(0)} \cdot k \\ &= \langle \phi, (x \cdot \alpha)^{-1} \rangle (x \cdot \alpha)^0(h \otimes k). \blacksquare \end{aligned}$$

LEMMA 4.6. *The space  $S(H, K)$  is identified with the subspace of elements  $\sum_i f_i \otimes k_i \in H^* \otimes_{\mathbb{k}} K$  such that for any  $t \in K$  and  $h \in H$ ,*

$$(4.9) \quad \sum_i \langle f_i, t_{(-1)}h \rangle k_i t_{(0)} = \sum_i \langle f_i, h \rangle t k_i.$$

*Proof.* We explained in Section 3.1 that the space  $\text{Hom}_K(H \otimes_{\mathbb{k}} K, K)$  can be identified with elements  $\sum_i f_i \otimes T_i \in H^* \otimes_{\mathbb{k}} \text{End}(K)$  that satisfy (3.1). An element  $\sum_i f_i \otimes T_i \in H^* \otimes_{\mathbb{k}} \text{End}(K)$  with  $\{f_i\}_i$  linearly independent belongs to  $S(H, K)$  if it satisfies (4.4). This means that  $T_i(k) = T_i(1)k$  for all  $i$  and all  $k \in K$ . Thus, we can identify  $T_i$  with left multiplication by  $T_i(1)$ . Under this identification, (3.1) is equivalent to (4.9).  $\blacksquare$

REMARK 4.7. If  $K \subseteq H$  is a left coideal subalgebra, then elements of the space  $(H/K^+H)^* \otimes_{\mathbb{k}} Z(K)$  are in  $S(H, K)$ . Here  $Z(K)$  is the center of  $K$ . This follows from the fact that  $(H/K^+H)^*$  can be identified with the elements  $f \in H^*$  such that  $\langle f, kh \rangle = \langle \epsilon, k \rangle \langle f, h \rangle$  for any  $k \in K$  and  $h \in H$ .

**THEOREM 4.8.** *Let  $K$  be a finite-dimensional left  $H$ -comodule algebra and  $P \in {}^H_K\mathcal{M}_K$ . Then  $\mathcal{M} = {}_K\mathcal{M}$  is a left  $\text{Rep}(H)$ -module category. There is an isomorphism of  $H$ -modules*

$$S^K(H, P) \simeq \int_{M \in \mathcal{M}} \underline{\text{Hom}}(M, P \otimes_K M).$$

When  $P = K$ , this isomorphism is an algebra map.

*Proof.* We shall use the description of the internal Hom of the module category  ${}_K\mathcal{M}$  given in Lemma 3.4. First, we shall prove that  $S^K(H, P) = \int_{M \in \mathcal{M}} \underline{\text{Hom}}(M, P \otimes_K M)$  as objects in  $\text{Rep}(H)$ . Observe that if  $M, M', N, N'$  are objects in  ${}_K\mathcal{M}$ , and  $f : M \rightarrow M'$  and  $g : N \rightarrow N'$  are  $K$ -module morphisms, then the functor  $\underline{\text{Hom}} : \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \text{Rep}(H)$  is defined on morphisms by

$$\begin{aligned} \underline{\text{Hom}}(f, g) : \text{Hom}_K(H \otimes_{\mathbb{k}} M', N) &\rightarrow \text{Hom}_K(H \otimes_{\mathbb{k}} M, N'), \\ \alpha &\mapsto g \circ \alpha \circ (\text{id}_H \otimes f). \end{aligned}$$

For any  $M \in {}_K\mathcal{M}$  define  $\pi_M^P : S^K(H, P) \rightarrow \text{Hom}_K(H \otimes_{\mathbb{k}} M, P \otimes_K M)$ , by

$$\pi_M^P(\alpha)(h \otimes m) = \alpha(h \otimes 1) \otimes_K m, \quad h \in H, m \in M.$$

Equation (4.4) implies that  $\pi_M^P(\alpha)$  is a  $K$ -module morphism. It follows directly that  $\pi_M^P$  is an  $H$ -module map and that it is dinatural.

Assume that  $E \in \text{Rep}(H)$  and

$$d : E \rightrightarrows \underline{\text{Hom}}(-, P \otimes_K -)$$

is a dinatural transformation. Dinaturality, in this case, implies that for any  $K$ -modules  $M, N$  and any  $K$ -module map  $f : M \rightarrow N$ , we have

$$(4.10) \quad (\text{id}_P \otimes f) \circ d_M(e) = d_N(e)(\text{id}_H \otimes f),$$

for any  $e \in E$ . In particular, if  $N$  is any  $K$ -module and  $n \in N$ , define  $f_n : K \rightarrow N$  by  $f_n(k) = k \cdot n$ . Hence  $f_n$  is a  $K$ -module map, so (4.10) implies that  $(\text{id}_P \otimes f_n) \circ d_K(e) = d_N(e)(\text{id}_H \otimes f_n)$ . Evaluating this equality at  $h \otimes 1 \in H \otimes_{\mathbb{k}} K$  we obtain

$$(4.11) \quad d_K(e)(h \otimes 1) \otimes n = d_N(e)(h \otimes n),$$

for any  $h \in H$ . Taking  $N = K$  implies that  $d_K(e) \in S^K(H, P)$ , that is,  $d_K(e)$  satisfies (4.4).

Define  $\phi : E \rightarrow S^K(H, P)$  as  $\phi = d_K$ . Then (4.11) implies that  $d_N = \pi_N^P \circ \phi$  for any  $K$ -module  $N$ . This proves that the object  $S^K(H, P)$  together with the dinatural transformations  $\pi^P$  has the universal property of the end. Thus  $S^K(H, P) = \int_{M \in \mathcal{M}} \underline{\text{Hom}}(M, P \otimes_K M)$ .

When  $P = K$ , it is not difficult to verify that the product of the adjoint algebra defined in terms of the dinatural transformation (see (4.3)), coincides with the product described in (3.2). ■



So far, we have described the structure of the end  $\bigvee_{M \in \mathcal{M}} \underline{\text{Hom}}(M, P \otimes_K M)$  as an object in  $\text{Rep}(H)$ . It remains to describe the structure as an object in the category of Yetter–Drinfeld modules over  $H$ . The next two results will be the initial steps towards this objective.

Define the functor  $\bar{\rho}_K : {}_K\mathcal{M}_K \rightarrow \text{Rep}(H)$  by  $\bar{\rho}_K(P) = S^K(H, P)$  for any  $P \in {}_K\mathcal{M}_K$ . If  $P, Q \in {}_K\mathcal{M}_K$  and  $f : P \rightarrow Q$  is a morphism of  $K$ -bimodules, then

$$\bar{\rho}_K(f) : S^K(H, P) \rightarrow S^K(H, Q), \quad \bar{\rho}_K(f)(\alpha) = f \circ \alpha.$$

PROPOSITION 4.9. *The functor  $\bar{\rho}_K : {}_K\mathcal{M}_K \rightarrow \text{Rep}(H)$  is the right adjoint of  $\rho_K$ . The unit and counit of the adjunction  $(\rho_K, \bar{\rho}_K)$  are given by*

$$\begin{aligned} \eta : \text{Id}_{\text{Rep}(H)} &\rightarrow \bar{\rho}_K \circ \rho_K, & \epsilon : \rho_K \circ \bar{\rho}_K &\rightarrow \text{Id}_{{}_K\mathcal{M}_K}, \\ \eta_X(x)(h \otimes k) &= h \cdot x \otimes k, & \epsilon_P(\alpha \otimes k) &= \alpha(1 \otimes k), \end{aligned}$$

for any  $X \in \text{Rep}(H)$ ,  $P \in {}_K\mathcal{M}_K$ ,  $x \in X$ ,  $h \in H$ ,  $\alpha \in S^K(H, P)$  and  $k \in K$ .

*Proof.* For any  $X \in \text{Rep}(H)$ ,  $P \in {}_K\mathcal{M}_K$ ,  $x \in X$ ,  $k \in K$  and  $h \in H$  define

$$(4.12) \quad \begin{aligned} \phi_{X,P} : \text{Hom}_H(X, S^K(H, P)) &\rightarrow \text{Hom}_{(K,K)}(X \otimes_{\mathbb{k}} K, P), \\ \phi_{X,P}(\alpha)(x \otimes k) &= \alpha(x)(1 \otimes k), \end{aligned}$$

$$(4.13) \quad \begin{aligned} \psi_{X,P} : \text{Hom}_{(K,K)}(X \otimes_{\mathbb{k}} K, P) &\rightarrow \text{Hom}_H(X, S^K(H, P)), \\ \psi_{X,P}(\beta)(x)(h \otimes k) &= \beta(h \cdot x \otimes k). \end{aligned}$$

It follows by a straightforward computation that the maps  $\phi_{X,P}, \psi_{X,P}$  are natural morphisms, and are inverses of each other. The unit and counit of the adjunction are given by

$$\eta_X = \psi_{X, X \otimes_{\mathbb{k}} K}(\text{id}_{X \otimes_{\mathbb{k}} K}), \quad \epsilon_P = \phi_{S^K(H,P), P}(\text{id}_{S^K(H,P)}). \quad \blacksquare$$

The next result is a particular case of Example 2.2. Since we need an explicit equivalence, we write out the proof.

LEMMA 4.10. *There is an equivalence of categories  $\mathcal{Z}_{\text{Rep}(H)}({}_K\mathcal{M}_K) \simeq {}^H_K\mathcal{M}_K$ .*

*Proof.* Let  $(M, \sigma) \in \mathcal{Z}_{\text{Rep}(H)}({}_K\mathcal{M}_K)$ . This means that  $M \in {}_K\mathcal{M}_K$ , and the half-braiding is given by  $\sigma_X : M \otimes_K (X \otimes_{\mathbb{k}} K) \rightarrow X \otimes_{\mathbb{k}} M$  for any  $X \in \text{Rep}(H)$ . Define  $\lambda : M \rightarrow H \otimes_{\mathbb{k}} M$  by  $\lambda(m) = \sigma_H(m \otimes 1_H \otimes 1_K)$  for any  $m \in M$ . This establishes a functor

$$\Phi : \mathcal{Z}_{\text{Rep}(H)}({}_K\mathcal{M}_K) \rightarrow {}^H_K\mathcal{M}_K, \quad \Phi(M, \sigma) = (M, \lambda).$$

If  $(M, \lambda) \in {}^H_K\mathcal{M}_K$ , define  $\sigma_X^\lambda : M \otimes_K (X \otimes_{\mathbb{k}} K) \rightarrow X \otimes_{\mathbb{k}} M$  by

$$\sigma_X^\lambda(m \otimes x \otimes k) = m_{(-1)} \cdot x \otimes m_{(0)} \cdot k, \quad X \in \text{Rep}(H), m \in M, k \in K.$$

It follows by a simple computation that  $\sigma_X^\lambda$  is a well-defined isomorphism, it is a  $K$ -bimodule map and it satisfies (2.18). This defines a functor  $\Psi : {}^H_K\mathcal{M}_K \rightarrow \mathcal{Z}_{\text{Rep}(H)}({}_K\mathcal{M}_K)$  by  $\Psi(M, \lambda) = (M, \sigma^\lambda)$ . ■

For any  $P \in {}^H_K\mathcal{M}_K$  recall the structure of Yetter–Drinfeld module over  $H$  of  $S^K(H, P)$  given by (4.5), (4.6).

**THEOREM 4.11.** *For any  $P \in {}^H_K\mathcal{M}_K$  there is an isomorphism  $S^K(H, P) \simeq \mathcal{Z}(\bar{\rho}_K)(P)$  as objects in  ${}^H_H\mathcal{YD}$ .*

*Proof.* If  $P \in {}^H_K\mathcal{M}_K$ , then  $\mathcal{Z}(\bar{\rho}_K)(P, \sigma^\lambda) = (S^K(H, P), \sigma^P)$ , where, according to (2.19), the half-braiding for  $\bar{\rho}_K(P) = S^K(H, P)$  is the morphism  $\sigma_X^P : S^K(H, P) \otimes_{\mathbb{k}} X \rightarrow X \otimes_{\mathbb{k}} S^K(H, P)$  given by the composition

$$S^K(H, P) \otimes_{\mathbb{k}} X \xrightarrow{(\xi_{X,P}^r)^{-1}} S^K(H, P \otimes_K (X \otimes_{\mathbb{k}} K)) \xrightarrow{\bar{\rho}_K(\sigma_X^\lambda)} S^K(H, X \otimes_{\mathbb{k}} P) \xrightarrow{\xi_{X,P}^l} X \otimes_{\mathbb{k}} S^K(H, P)$$

for any  $X \in \text{Rep}(H)$ . Recall that  $\sigma^\lambda$  is the half-braiding associated to  $P$  explained in Lemma 4.10. To compute  $\sigma^P$ , we need to compute the bimodule structure of the functor  $\bar{\rho}_K$ . Both structures are given by (4.1) and (4.2).

Using the formula for the unit and counit of the adjunction  $(\rho_K, \bar{\rho}_K)$  given in Proposition 4.9 we obtain

$$\begin{aligned} (\xi_{X,P}^l)^{-1}(x \otimes \alpha)(h \otimes k) &= \bar{\rho}_K(\text{id}_X \otimes \epsilon_P)\eta_{X \otimes_{\mathbb{k}} S^K(H,P)}(x \otimes \alpha)(h \otimes k) \\ &= (\text{id}_X \otimes \epsilon_P)\eta_{X \otimes_{\mathbb{k}} S^K(H,P)}(x \otimes \alpha)(h \otimes k) \\ &= (\text{id}_X \otimes \epsilon_P)(h_{(1)} \cdot x \otimes h_{(2)} \cdot \alpha \otimes k) \\ &= h_{(1)} \cdot x \otimes h_{(2)} \cdot \alpha(1 \otimes k) \\ &= h_{(1)} \cdot x \otimes \alpha(h_{(2)} \otimes k), \end{aligned}$$

and

$$\begin{aligned} (\xi_{X,P}^r)^{-1}(\alpha \otimes x)(h \otimes k) &= (\epsilon_P \otimes \text{id}_{X \otimes_{\mathbb{k}} K})\eta_{S^K(H,P) \otimes_{\mathbb{k}} X}(\alpha \otimes x)(h \otimes k) \\ &= (\epsilon_P \otimes \text{id}_{X \otimes_{\mathbb{k}} K})(h_{(1)} \cdot \alpha \otimes 1 \otimes h_{(2)} \cdot x \otimes k) \\ &= (h_{(1)} \cdot \alpha)(1 \otimes 1) \otimes h_{(2)} \cdot x \otimes k \\ &= \alpha(h_{(1)} \otimes 1) \otimes h_{(2)} \cdot x \otimes k \end{aligned}$$

for any  $\alpha \in S^K(H, P)$ ,  $x \in X$ ,  $h \in H$ ,  $k \in K$ .

Now, the  $H$ -coaction of  $\mathcal{Z}(\bar{\rho}_K)(P, \sigma^\lambda)$  associated with the half-braiding  $\sigma^P$  is

$$\lambda^P : S^K(H, P) \rightarrow H \otimes_{\mathbb{k}} S^K(H, P), \quad \lambda^P(\alpha) = \sigma_H^P(\alpha \otimes 1_H).$$

Let us denote  $\lambda^P(\alpha) = \alpha^{-1} \otimes \alpha^0$ . Using the formula for  $\sigma_X^P$ , we know that  $(\xi_{H,P}^l)^{-1}\sigma_H^P(\alpha \otimes 1_H) = \bar{\rho}_K(\sigma_H^\lambda)(\xi_{H,P}^r)^{-1}(\alpha \otimes 1_H)$ . Evaluating this equality

at  $h \otimes k \in H \otimes_{\mathbb{k}} K$  we find that

$$(\xi_{H,P}^l)^{-1} \sigma_H^P(\alpha \otimes 1_H)(h \otimes k) = h_{(1)} \alpha^{-1} \otimes \alpha^0(h_{(2)} \otimes k)$$

is equal to

$$\begin{aligned} \bar{\rho}_K(\sigma_H^\lambda) \xi_{H,P}^r(\alpha \otimes 1_H)(h \otimes k) &= \sigma_H^\lambda(\xi_{H,P}^r)^{-1}(\alpha \otimes 1_H)(h \otimes k) \\ &= \sigma_H^\lambda(\alpha(h_{(1)} \otimes 1) \otimes h_{(2)} \otimes k) \\ &= \alpha(h_{(1)} \otimes 1)_{(-1)} h_{(2)} \otimes \alpha(h_{(1)} \otimes 1)_{(0)} \cdot k. \end{aligned}$$

Thus

$$(4.14) \quad h_{(1)} \alpha^{-1} \otimes \alpha^0(h_{(2)} \otimes k) = \alpha(h_{(1)} \otimes 1)_{(-1)} h_{(2)} \otimes \alpha(h_{(1)} \otimes 1)_{(0)} \cdot k$$

for any  $\alpha \in S^K(H, P)$  and  $h \otimes k \in H \otimes_{\mathbb{k}} K$ . Hence

$$\begin{aligned} \alpha^{-1} \otimes \alpha^0(h \otimes k) &= \alpha^{-1} \otimes \alpha^0(\epsilon_H(h_{(1)}) h_{(2)} \otimes k) \\ &= \mathcal{S}(h_{(1)}) h_{(2)} \alpha^{-1} \otimes \alpha^0(h_{(3)} \otimes k) \\ &= \mathcal{S}(h_{(1)}) \alpha(h_{(2)} \otimes 1)_{(-1)} h_{(3)} \otimes \alpha(h_{(2)} \otimes 1)_{(0)} \cdot k. \end{aligned}$$

The last equality follows from (4.14). This formula coincides with (4.7). ■

As a consequence of Theorems 4.8 and 4.11 we obtain the next result.

**COROLLARY 4.12.** *Let  $K$  be a finite-dimensional right  $H$ -simple left  $H$ -comodule with trivial coinvariants. There exists an isomorphism of algebras*

$$S(H, K) \simeq \mathcal{A}_K$$

*in the category  ${}^H_H\mathcal{YD}$ . ■*

**EXAMPLE 4.13** (Case  $K = H$ ). We denote by  $H_{\text{ad}}$  the algebra in the category  ${}^H_H\mathcal{YD}$  whose underlying algebra is  $H$ , with  $H$ -coaction given by the coproduct and  $H$ -action given by the adjoint action, that is,  $h \triangleright x = h_{(1)} x \mathcal{S}(h_{(2)})$  for  $h, x \in H$ . Since  $H$  is an  $H$ -comodule algebra with coproduct, we can consider  $S^H(H, H)$ . The map  $\phi : S(H, H) \rightarrow H_{\text{ad}}$ ,  $\phi(\alpha) = \alpha(1 \otimes 1)$ , is an isomorphism of algebras in  ${}^H_H\mathcal{YD}$ . Indeed, it is an  $H$ -module map. Take  $\alpha \in S(H, H)$  and  $h, t \in H$ . Then

$$(4.15) \quad \begin{aligned} \alpha(h \otimes t) &= \alpha(h_1 \otimes h_2 \mathcal{S}(h_3) t) \\ &= h_{(1)} \alpha(1 \otimes \mathcal{S}(h_2) t) = h_{(1)} \alpha(1 \otimes 1) \mathcal{S}(h_2) t. \end{aligned}$$

The second equality holds because  $\alpha$  is an  $H$ -module map, and the last equality follows from (4.4). Then

$$\phi(h \cdot \alpha) = (h \cdot \alpha)(1 \otimes 1) = \alpha(h \otimes 1) = h_{(1)} \alpha(1 \otimes 1) \mathcal{S}(h_2) = h \triangleright \phi(\alpha).$$

It follows by a straightforward computation that,  $\phi$  is an algebra map and an  $H$ -comodule map. Using (4.15), it follows that the map  $\psi : H_{\text{ad}} \rightarrow S(H, H)$ ,  $\psi(x)(h \otimes t) = h_{(1)} x \mathcal{S}(h_2) t$ ,  $x, h, t \in H$ , is the inverse of  $\phi$ .

EXAMPLE 4.14 (Case  $K = \mathbb{k}$ ). We denote by  $H_{\text{ad}}^*$  the following algebra in the category  ${}^H_H\mathcal{YD}$ . The underlying algebra is  $H^*$ . The  $H$ -action and  $H$ -coaction are  $\cdot : H \otimes_{\mathbb{k}} H_{\text{ad}}^* \rightarrow H_{\text{ad}}^*$ ,  $\lambda : H_{\text{ad}}^* \rightarrow H \otimes_{\mathbb{k}} H_{\text{ad}}^*$ ,  $\lambda(f) = f_{(-1)} \otimes f_{(0)}$ , where

$$(h \cdot f)(x) = f(xh), \quad \langle g, f_{(-1)} \rangle f_{(0)} = S(g_{(1)})fg_{(2)},$$

for any  $h, x \in H$  and  $g \in H^*$ . It follows that  $S(H, \mathbb{k}) = H_{\text{ad}}^*$ .

**5. Some explicit calculations.** In this section we shall explicitly compute the adjoint algebra for the representation categories of group algebras and their duals. We shall use the identification of  $S(H, K)$  with the elements in  $H^* \otimes_{\mathbb{k}} K$  that satisfy (4.9). First we recall the classification of exact indecomposable module categories over group algebras and their duals.

**5.1. Module categories over the tensor categories  $\text{Rep}(\mathbb{k}^G)$  and  $\text{Rep}(\mathbb{k}G)$ .** Assume  $G$  is a finite group. We shall recall the classification of exact indecomposable module categories over  $\text{Rep}(\mathbb{k}^G)$  and  $\text{Rep}(\mathbb{k}G)$ . For this, we shall give families of simple left  $H$ -comodule algebras, where  $H = \mathbb{k}^G, \mathbb{k}G$ .

Assume  $F \subseteq G$  is a subgroup and  $\psi \in Z^2(F, \mathbb{k}^\times)$  a 2-cocycle. We denote by  $\mathbb{k}_\psi F$  the twisted group algebra. We can choose  $\psi$  (in a cohomology class) such that

$$\psi(f, g)\psi(g^{-1}, f^{-1}) = 1, \quad \psi(f, 1) = \psi(1, f) = 1,$$

for any  $f, g \in F$ . In that case we shall say that  $\psi$  is *normalized*.

The twisted group algebra  $\mathbb{k}_\psi F$  is a left  $\mathbb{k}G$ -comodule algebra as follows. Elements in  $\mathbb{k}_\psi F$  are linear combinations of  $e_f$ ,  $f \in F$ . The product and left  $\mathbb{k}G$ -coaction are

$$e_f e_h = \psi(f, h)e_{fh}, \quad \lambda(e_f) = f \otimes e_f,$$

for any  $f, h \in F$ . If  $V$  is a simple  $\mathbb{k}_\psi F$ -module, we can form the following algebra. The endomorphism algebra  $\text{End}(V)$  is a right  $\mathbb{k}F$ -module with action given by

$$(T \cdot f)(v) = f^{-1} \cdot T(f \cdot v), \quad f \in F, v \in V, T \in \text{End}(V).$$

Define  $\mathcal{K}(F, \psi, V) = \text{End}(V) \otimes_{\mathbb{k}F} \mathbb{k}G$ . Let  $S \subseteq G$  be a set of representative elements of cosets  $F \backslash G$ .

Any element in  $\mathcal{K}(F, \psi, V)$  is of the form  $\overline{T \otimes s}$ , for some  $s \in S, T \in \text{End}(V)$ . Here  $\bar{z}$  denotes the class of  $z \in \mathbb{k}G \otimes_{\mathbb{k}} \text{End}(V)$  in the quotient  $\mathbb{k}G \otimes_{\mathbb{k}F} \text{End}(V)$ . The product in  $\mathcal{K}(F, \psi, V)$  is defined as follows:

$$(\overline{T \otimes x})(\overline{U \otimes y}) = \overline{\delta_{x,y} T \circ U \otimes x}, \quad T, U \in \text{End}(V), x, y \in S.$$

The unit is  $\sum_{s \in S} \overline{\text{Id} \otimes s}$ . The vector space  $\mathcal{K}(F, \psi, V)$  has the structure of a

right  $\mathbb{k}G$ -module that makes it into a module algebra. The right action is

$$(\overline{T \otimes x}) \cdot g = \overline{T \otimes xg}, \quad g \in G.$$

With this action  $\mathcal{K}(F, \psi, V)$  is a right  $\mathbb{k}G$ -module algebra, hence it is a left  $\mathbb{k}G$ -comodule algebra with coaction

$$\lambda : \mathcal{K}(F, \psi, V) \rightarrow \mathbb{k}^G \otimes_{\mathbb{k}} \mathcal{K}(F, \psi, V), \lambda(k) = k_{(-1)} \otimes k_{(0)}$$

such that for any  $g \in G$ ,  $\langle k_{(-1)}, g \rangle k_{(0)} = k \cdot g$ .

The next result is part of the folklore of representations of tensor categories. See, for example, [4, Proposition 4.1, Lemma 4.3]. It also follows from Theorem 3.1.

**THEOREM 5.1.** *Let  $G$  be a finite group.*

- (i) *If  $\mathcal{M}$  is an exact indecomposable module category over  $\text{Rep}(\mathbb{k}G)$ , then there exists a subgroup  $F \subseteq G$  and a normalized 2-cocycle  $\psi \in Z^2(F, \mathbb{k}^\times)$  such that  $\mathcal{M} \simeq_{\mathbb{k}_\psi F} \mathcal{M}$  as module categories.*
- (ii) *If  $\mathcal{M}$  is an exact indecomposable module category over  $\text{Rep}(\mathbb{k}^G)$ , then there exists a subgroup  $F \subseteq G$ , a normalized 2-cocycle  $\psi \in Z^2(F, \mathbb{k}^\times)$  and a simple  $\mathbb{k}_\psi F$ -module  $V$  such that  $\mathcal{M} \simeq_{\mathcal{K}(F, \psi, V)} \mathcal{M}$  as module categories. ■*

**REMARK 5.2.** The equivalence class of the module category  $_{\mathcal{K}(F, \psi, V)} \mathcal{M}$  does not depend on the choice of the simple  $\mathbb{k}_\psi F$ -module  $V$ . The twisted group algebra  $\mathbb{k}_\psi F$  is an algebra in the category  $\text{Rep}(\mathbb{k}^G)$ . One can prove that regardless of the choice of  $V$ , the module category  $_{\mathcal{K}(F, \psi, V)} \mathcal{M}$  is equivalent to  $\text{Rep}(\mathbb{k}^G)_{\mathbb{k}_\psi F}$ .

**REMARK 5.3.** Let  $F = \{1\}$  be the trivial subgroup of  $G$ ,  $\psi = 1$  and  $V = \mathbb{k}$  with the trivial action. Denote  $K = \mathcal{K}(F, \psi, V)$ . It is not difficult to see that  $K \simeq \mathbb{k}^G$  as left  $\mathbb{k}^G$ -comodule algebras. Hence  $_{\mathcal{K}(\{1\}, 1, \mathbb{k})} \mathcal{M} \simeq \text{Rep}(\mathbb{k}^G)$  as  $\text{Rep}(\mathbb{k}^G)$ -module categories.

**5.2. Case  $H = \mathbb{k}G$ .** Let  $F \subset G$  be a subgroup, and take a normalized 2-cocycle  $\psi \in Z^2(F, \mathbb{k}^\times)$ . Let  $K = \mathbb{k}_\psi F$  be the twisted group algebra. We shall denote by  $\{e_f\}_{f \in F}$  the canonical basis of  $\mathbb{k}_\psi F$ . The product in this algebra is then  $e_f e_l = \psi(f, l) e_{fl}$  for any  $f, l \in F$ .

Let  $S \subseteq G$  be a set of representatives of right cosets  $F \backslash G$  such that  $1 \in S$ . Define  $b : F \times F \rightarrow \mathbb{k}^\times$  as

$$b(l, f) = \frac{\psi(l, l^{-1}fl)}{\psi(f, l)}.$$

Also, for any  $l \in F$ , set  $C_l = \{(g, f) \in F \times F : g^{-1}fg = l\}$ . For any  $s \in S$ ,  $l \in F$  define

$$\alpha_{s, l} = \sum_{(g, f) \in C_l} b(g, f) \delta_{gs} \otimes e_f \in \mathbb{k}^G \otimes_{\mathbb{k}} K.$$

Using the identification explained in Subsection 3.1, the element  $\alpha_{s,l}$  can be seen as an element in  $\text{Hom}(\mathbb{k}G \otimes_{\mathbb{k}} K, K)$ , where

$$(5.1) \quad \alpha_{s,l}(x \otimes e_h) = \delta_{s,t} b(f, flf^{-1})\psi(flf^{-1}, h) e_{flf^{-1}h}$$

if  $x = ft \in G$  with  $t \in S$  and  $h \in F$ .

LEMMA 5.4. *The set  $B = \{\alpha_{s,l} \in \mathbb{k}^G \otimes_{\mathbb{k}} \mathbb{k}_{\psi}F : s \in S, l \in F\}$  is a basis of  $S(\mathbb{k}G, \mathbb{k}_{\psi}F)$ .*

*Proof.* Clearly  $B$  is a set of linearly independent elements. Let  $z$  be an arbitrary element of  $\mathbb{k}^G \otimes_{\mathbb{k}} \mathbb{k}_{\psi}F$ . Thus  $z = \sum_{x \in G, f \in F} \xi_{x,f} \delta_x \otimes e_f$  for certain scalars  $\xi_{x,f} \in \mathbb{k}$ . If  $z \in S(\mathbb{k}G, \mathbb{k}_{\psi}F)$ , equation (4.9) implies that

$$\sum_{x \in G, f \in F} \xi_{x,f} \delta_x(l y) e_f e_l = \sum_{x \in G, f \in F} \xi_{x,f} \delta_x(y) e_l e_f$$

for any  $y \in G$  and  $l \in F$ . This implies that

$$\sum_{f \in F} \xi_{ly,f} \psi(f, l) e_{fl} = \sum_{f \in F} \xi_{y,f} \psi(l, f) e_{lf}.$$

This equality implies, by looking at the coefficient of  $e_{lf}$ , that

$$\xi_{ly,f} = \xi_{y,l^{-1}fl} b(l, f)$$

for any  $l, f \in F$  and  $y \in G$ . Hence

$$\begin{aligned} z &= \sum_{x \in G, f \in F} \xi_{x,f} \delta_x \otimes e_f = \sum_{s \in S, g, f \in F} \xi_{gs,f} \delta_{gs} \otimes e_f \\ &= \sum_{s \in S, g, f \in F} \xi_{s,g^{-1}fg} b(g, f) \delta_{gs} \otimes e_f \\ &= \sum_{s \in S, l \in F} \sum_{(g,f) \in C_l} \xi_{s,l} b(g, f) \delta_{gs} \otimes e_f = \sum_{s \in S, l \in F} \xi_{s,l} \alpha_{s,l}. \end{aligned}$$

Thus  $z$  is a linear combination of elements of  $B$ . ■

The proof of the next result follows by a straightforward computation.

LEMMA 5.5. *The  $\mathbb{k}G$ -coaction of  $S(\mathbb{k}G, \mathbb{k}_{\psi}F)$ , given in (4.7), is determined by*

$$\lambda(\alpha_{s,g}) = s^{-1}gs \otimes \alpha_{s,g}$$

for any  $g \in F$  and  $s \in S$ . The  $\mathbb{k}G$ -action on  $S(\mathbb{k}G, \mathbb{k}_{\psi}F)$ , given in (4.5), is determined by

$$x \cdot \alpha_{s,g} = b(h^{-1}, h^{-1}gh) \alpha_{r,h^{-1}gh}$$

if  $x = ft$  and  $st^{-1}f^{-1} = hr$ , where  $f, h \in F$  and  $t, r \in S$ . ■

For any subgroup  $F \subset G$  and a normalized 2-cocycle  $\psi \in Z^2(F, \mathbb{k}^{\times})$ , define  $C_{\psi}(G, F)$  as the subspace of  $\text{Hom}_{\mathbb{k}}(\mathbb{k}[S \times F], \mathbb{k})$  generated by functions

$\phi : S \times F \rightarrow \mathbb{k}$  such that

$$(5.2) \quad b(x, xs^{-1}gsx^{-1})\phi(s, g) = b(h^{-1}, h^{-1}gh)\phi(r, h^{-1}gh),$$

for any  $x \in G$  such that  $sx^{-1} = hr$  with  $r \in S$  and  $h \in F$ . Observe that if  $F = G$  and  $\psi = 1$  then  $C_\psi(G, F)$  is the space of class functions on  $G$ .

PROPOSITION 5.6. *Let  $F \subset G$  be a subgroup, and  $\psi \in Z^2(F, \mathbb{k}^\times)$  a normalized 2-cocycle. There exists a linear isomorphism  $CF(\mathbb{k}_\psi F\mathcal{M}) \simeq C_\psi(G, F)$ .*

*Proof.* As before, let  $S \subseteq G$  be a set of representatives of elements of  $F \backslash G$ . Let  $\phi \in CF(\mathbb{k}_\psi F\mathcal{M})$ . This means that  $\phi : S(\mathbb{k}G, \mathbb{k}_\psi F) \rightarrow S(\mathbb{k}G, \mathbb{k}G)$  is a morphism of  $\mathbb{k}G$ -Yetter–Drinfeld modules. Elements of the basis described in Lemma 5.4 for  $S(\mathbb{k}G, \mathbb{k}G)$  are of the form  $\alpha_{1,g} \in \mathbb{k}^G \otimes_{\mathbb{k}} \mathbb{k}G$  for any  $g \in G$ .

Using Lemma 5.5, since  $\phi$  is a  $\mathbb{k}G$ -comodule map, we observe that

$$\phi(\alpha_{s,g}) = \phi_{s,g}\alpha_{1,s^{-1}gs}$$

for any  $s \in S$  and  $g \in F$ . Here  $\phi_{s,g} \in \mathbb{k}$ . This implies that  $\phi$  is determined by the scalars  $\phi_{s,g}$ . It remains to prove that these scalars satisfy (5.2). Take  $x \in G$ , and write it as  $x = ft$ , where  $f \in F$  and  $t \in S$ . Assume that  $st^{-1}f^{-1} = hr$ , where  $h \in F$  and  $r \in S$ . Since  $\phi$  is a  $\mathbb{k}G$ -module map, we have

$$\begin{aligned} \phi(x \cdot \alpha_{s,g}) &= b(h^{-1}, h^{-1}gh)\phi(\alpha_{r,h^{-1}gh}) = b(h^{-1}, h^{-1}gh)\phi_{r,h^{-1}gh}\alpha_{1,r^{-1}h^{-1}ghr} \\ &= x \cdot \phi(\alpha_{s,g}) = \phi_{s,g}x \cdot \alpha_{1,s^{-1}gs} = \phi_{s,g}b(x, xs^{-1}gsx^{-1})\alpha_{1,xs^{-1}gsx^{-1}}. \end{aligned}$$

This implies that  $b(h^{-1}, h^{-1}gh)\phi_{r,h^{-1}gh} = b(x, xs^{-1}gsx^{-1})\phi_{s,g}$ . ■

**5.3. Case  $H = \mathbb{k}^G$ .** Let  $F \subset G$  be a subgroup, and  $\psi \in Z^2(F, \mathbb{k}^\times)$  a normalized 2-cocycle. Let also  $V$  be a simple  $\mathbb{k}_\psi F$ -module. Recall the definition of the left  $\mathbb{k}^G$ -comodule algebra  $\mathcal{K}(F, \psi, V)$  from Subsection 5.1. Again, let  $S \subseteq G$  be a set of representatives of right cosets  $F \backslash G$  such that  $1 \in S$ . The following technical result will be needed later.

LEMMA 5.7. *Let  $f \in F$ . The vector space consisting of all  $T \in \text{End}(V)$  such that*

$$(5.3) \quad U \circ T = T \circ (U \cdot f),$$

*for any  $U \in \text{End}(V)$  is 1-dimensional.*

*Proof.* Since the group  $F$  is finite, the linear operator  $f \cdot : V \rightarrow V$  is diagonalizable. Let  $\{v_i\}_{i=1}^n$  be a basis of  $V$  such that  $f \cdot v_i = q_i v_i$  with  $q_i \in \mathbb{k}^\times$  for any  $i = 1, \dots, n$ . Let  $T \in \text{End}(V)$  be a linear transformation that satisfies (5.3). For any  $j, k = 1, \dots, n$  define  $U_{j,k} : V \rightarrow V$  the operator  $U_{j,k}(v_i) = \delta_{j,i}v_k$  for any  $i = 1, \dots, n$ . Assume that  $T(v_i) = \sum_l t_{i,l}v_l$ . On the one hand, for any  $i = 1, \dots, n$  we have

$$(U_{j,k} \circ T)(v_i) = t_{i,j}v_k.$$

On the other hand,

$$T \circ (U_{j,k} \cdot f)(v_i) = q_i q_k^{-1} \delta_{j,i} \sum_l t_{k,l} v_l.$$

Hence, (5.3) implies that if  $i \neq j$  then  $t_{i,j} = 0$ , and if  $i = j$  then  $q_i q_k^{-1} t_{k,k} = t_{i,i}$ . This implies the lemma. ■

For any  $f \in F$  denote by  $T_f \in \text{End}(V)$  the unique (up to scalar) non-zero linear operator that fulfils condition (5.3) of Lemma 5.7. For any  $(f, s) \in F \times S$ , define  $\alpha_{(f,s)} \in \mathbb{k}G \otimes_{\mathbb{k}} \mathcal{K}(F, \psi, V)$  by

$$\alpha_{(f,s)} = s^{-1} f s \otimes \overline{T_f \otimes s}.$$

When  $f = 1$ , we can choose  $T_f = \text{Id}_V$ .

PROPOSITION 5.8. *The linearly independent set  $\{\alpha_{(f,s)} : (f, s) \in F \times S\}$  is a basis for  $S(\mathbb{k}^G, \mathcal{K}(F, \psi, V))$ .*

*Proof.* It is straightforward that for any  $(f, s) \in F \times S$  the element  $\alpha_{(f,s)}$  satisfies condition (4.9). It follows from Lemma 4.3(iv) that

$$\dim(S(\mathbb{k}^G, \mathcal{K}(F, \psi, V))) = \dim(\mathbb{k}^G) = |G|.$$

Since the set  $\{\alpha_{(f,s)} : (f, s) \in F \times S\}$  has cardinality  $|G|$ , it must be a basis. ■

For any  $(f, s) \in F \times S$  define  $I(f, s) = \{(h, a) \in G \times G : a h a^{-1} = s^{-1} f s\}$ .

LEMMA 5.9. *The  $\mathbb{k}^G$ -coaction of  $S(\mathbb{k}^G, \mathcal{K}(F, \psi, V))$ , given by (4.7), is determined by*

$$\lambda(\alpha_{(f,s)}) = \sum_{(a,h) \in I(f,s)} \delta_a \otimes h \otimes \overline{T_f \otimes s a}$$

for any  $(f, s) \in F \times S$ . The  $\mathbb{k}^G$ -action on  $S(\mathbb{k}^G, \mathcal{K}(F, \psi, V))$ , given by (4.5), is determined by

$$\delta_g \cdot \alpha_{(f,s)} = \begin{cases} 0 & \text{if } g \neq s^{-1} f s, \\ \alpha_{(f,s)} & \text{if } g = s^{-1} f s. \end{cases} \blacksquare$$

PROPOSITION 5.10. *Let  $F \subset G$  be a subgroup, and  $\psi \in Z^2(F, \mathbb{k}^\times)$  a normalized 2-cocycle. Let  $V$  be a simple  $\mathbb{k}_\psi F$ -module. There exists a linear isomorphism  $\text{CF}(\mathcal{K}(F, \psi, V) \mathcal{M}) \simeq \mathbb{k}^S$ .*

*Proof.* Recall that the comodule algebra representing the regular module category  $\text{Rep}(\mathbb{k}^G)$  is the algebra  $\mathcal{K}(\{1\}, 1, \mathbb{k})$  (see Remark 5.3). Let  $\phi \in \text{CF}(\mathcal{K}(F, \psi, V) \mathcal{M})$ . Thus  $\phi : S(\mathbb{k}^G, \mathcal{K}(F, \psi, V)) \rightarrow S(\mathbb{k}^G, \mathcal{K}(\{1\}, 1, \mathbb{k}))$  is a  $\mathbb{k}^G$ -module morphism and  $\mathbb{k}^G$ -comodule morphism.



Elements in the basis of  $S(\mathbb{k}^G, \mathcal{K}(\{1\}, 1, \mathbb{k}))$  are  $\alpha_{(1,g)}$ , for any  $g \in G$ . Hence

$$\phi(\alpha_{(f,s)}) = \phi(\delta_{sf s^{-1}} \cdot \alpha_{(f,s)}) = \delta_{sf s^{-1}} \cdot \phi(\alpha_{(f,s)}) = \begin{cases} 0 & \text{if } f \neq 1, \\ \phi(\alpha_{(f,s)}) & \text{if } f = 1. \end{cases}$$

for any  $f \in F$  and  $s \in S$ . We deduce that  $\phi$  is determined at the values of  $\alpha_{(1,s)}$  for any  $s \in S$ . Assume that

$$\phi(\alpha_{(1,s)}) = \sum_{g \in G} \phi_{s,g} \alpha_{(1,g)}$$

for certain  $\phi_{s,g} \in \mathbb{k}$ . Since  $I(1, s) = \{(1, a) \in G \times G\}$ , and  $\phi$  is a  $\mathbb{k}^G$ -comodule map, we have that  $\phi_{sa,g} = \phi_{s,ga^{-1}}$  for any  $s \in S$  and  $g, a \in G$ . Here we are abusing notation, since  $sa$  denotes the element  $t \in S$  that represents the class to which  $sa$  belongs. Hence the scalars  $\phi_{s,g}$  are determined by a function  $f : F \setminus G \rightarrow \mathbb{k}$  as follows:

$$\phi_{s,g} = f(\overline{sg^{-1}})$$

for any  $s \in S$  and  $g \in G$ . ■

**5.4. The adjoint algebra for tensor categories  $\mathcal{C}(G, 1, F, \psi)$ .** Let  $G$  be a finite group and  $\omega \in Z^3(G, \mathbb{k}^\times)$  a 3-cocycle. Then  $\mathcal{C}(G, \omega)$  stands for the category of finite-dimensional  $G$ -graded vector spaces with associativity constraint defined by

$$a_{X,Y,Z}((x \otimes y) \otimes z) = \omega(g, h, f)x \otimes (y \otimes z)$$

for any  $G$ -graded vector spaces  $X, Y, Z$ , and any homogeneous elements  $x \in X_g, y \in Y_h$  and  $z \in Z_f$ . Note that if  $\omega = 1$ , then there is a monoidal equivalence  $\mathcal{C}(G, \omega) \simeq \text{Rep}(\mathbb{k}^G)$ .

If  $F \subseteq G$  is a subgroup, and  $\psi \in Z^2(F, \mathbb{k}^\times)$  is a 2-cocycle such that  $d\psi \omega = 1$ , then the twisted group algebra  $\mathbb{k}_\psi F$  is an algebra in  $\mathcal{C}(G, \omega)$ . Then  $\mathcal{C}(G, \omega, F, \psi)$  is the category  ${}_{\mathbb{k}_\psi F} \mathcal{C}(G, \omega)_{\mathbb{k}_\psi F}$  of  $\mathbb{k}_\psi F$ -bimodules in  $\mathcal{C}(G, \omega)$ . These categories are called *group-theoretical fusion categories*.

We shall describe the adjoint algebra  $\mathcal{A}_{\mathcal{D}}$  and the space of class functions  $\text{CF}(\mathcal{D})$  when  $\mathcal{D} = \mathcal{C}(G, 1, F, \psi)$ . We shall keep the notation of Section 5.2.

It follows from Corollary 3.6 that there is a monoidal equivalence

$$\mathcal{C}(G, 1, F, \psi) \simeq \text{Rep}(G)_{\mathbb{k}_\psi F}^* \mathcal{M}.$$

Using this equivalence and Lemma 4.3(ii), it follows that  $\mathcal{A}_{\mathcal{C}(G,1,F,\psi)}$  is isomorphic to  $\theta_{\mathbb{k}_\psi F}(\mathcal{A}_{\mathbb{k}_\psi F})$ , where  $\mathcal{A}_{\mathbb{k}_\psi F}$  is the adjoint algebra corresponding to the  $\text{Rep}(G)$ -module category  ${}_{\mathbb{k}_\psi F} \mathcal{M}$ .

Using the explicit description of the functor  $\theta_{\mathbb{k}_\psi F}$  given in Subsection 3.3, we find that  $\mathcal{A}_{\mathcal{C}(G,1,F,\psi)}$  is isomorphic to  $S(\mathbb{k}G, \mathbb{k}_\psi F) \otimes_{\mathbb{k}} \mathbb{k}_\psi F$ . Recall that  $S(\mathbb{k}G, \mathbb{k}_\psi F)$  has a basis consisting of the elements  $\alpha_{s,l} \in \mathbb{k}^G \otimes_{\mathbb{k}} \mathbb{k}_\psi F$  for

$s \in S$ , and  $l \in F$ . The vector space  $S(\mathbb{k}G, \mathbb{k}_\psi F) \otimes_{\mathbb{k}} \mathbb{k}_\psi F$  is the following object in the category  $\mathcal{Z}(\mathbb{k}_\psi F \mathcal{C}(G, \omega)_{\mathbb{k}_\psi F})$ . The left  $\mathbb{k}G$ -coaction is given by

$$\begin{aligned} \lambda : S(\mathbb{k}G, \mathbb{k}_\psi F) \otimes_{\mathbb{k}} \mathbb{k}_\psi F &\rightarrow \mathbb{k}G \otimes_{\mathbb{k}} S(\mathbb{k}G, \mathbb{k}_\psi F) \otimes_{\mathbb{k}} \mathbb{k}_\psi F, \\ \lambda(\alpha_{s,f} \otimes e_h) &= s^{-1} fsh \otimes \alpha_{s,f} \otimes e_h, \quad f, h \in F, s \in S. \end{aligned}$$

The  $\mathbb{k}_\psi F$ -bimodule structure is given by

$$e_g \cdot (\alpha_{s,f} \otimes e_h) \cdot e_l = b(d^{-1}, d^{-1}fd)\psi(g, h)\psi(gh, l) \alpha_{r,d^{-1}fd} \otimes e_{ghl}$$

if  $sg^{-1} = df$  where  $g, f, l, d \in F$  and  $r, s \in S$ .

The next result is a direct consequence of Lemma 4.3 and (the proof of) Proposition 5.6.

LEMMA 5.11. *The space of class functions  $\text{CF}(\mathcal{C}(G, 1, F, \psi))$  is isomorphic to  $C_1(G, F)$ . ■*

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