THE EFFECT OF THE SMOOTHNESS OF FRACTIONAL TYPE OPERATORS OVER THEIR COMMUTATORS WITH LIPSCHITZ SYMBOLS ON WEIGHTED SPACES

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Abstract

We prove boundedness results for integral operators of fractional type and their higher order commutators between weighted spaces, including L^p-L^q, L^p-BMO and L^p-Lipschitz estimates. The kernels of such operators satisfy certain size condition and a Lipschitz type regularity, and the symbol of the commutator belongs to a Lipschitz class. We also deal with commutators of fractional type operators with less regular kernels satisfying a Hörmander's type inequality. As far as we know, these last results are new even in the unweighted case. Moreover, we give a characterization result involving symbols of the commutators and continuity results for extreme values of p.

1. Introduction

There is a close relationship between the theory of Partial Differential Equations and Harmonic Analysis. This is evidenced, for instance, by the existence of a mechanism that provides us with regular solutions of *PDE's* when we equip this machinery with continuity properties of certain related operators (see, for example, [4], [5], [8], [9], [12], [33]). Therefore, it seems appropriate to explore the boundedness properties of the mentioned operators and, particularly, we shall be concerned with commutators of integral operators of fractional type.

It is well known that the fractional integral operator of order α, $0 < \alpha < n$, is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy,$$

whenever this integral is finite. There is a vast amount of information about the behavior of the operator above (see for example [18], [20], [26], [30] and [35]). In [6], Chanillo introduced the first order commutator of I_α with a symbol $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, formally defined by

$$[b, I_\alpha](f) = b \, I_\alpha f - I_\alpha(bf).$$

Particularly, if $b \in BMO$, the space of bounded mean oscillation, the author proved that, for $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, the operator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Some topics related to properties of boundedness for commutators of fractional integral operators for extreme values of p can be found in [19]. (For more information about BMO spaces see [21]).

The continuity properties of the commutator of the fractional integral operator acting on different spaces were studied by several authors contributing, in this way, to the development of $PDE's$. Some related articles are given by [2], [10], [14], [17], [22], [27], [28], [29], [32], [34]. In [25] the authors consider the commutators of certain fractional type operators with Lipschitz symbols and prove the boundedness between Lebesgue spaces, including the boundedness from Lebesgue spaces into BMO and Lipschitz spaces on non-homogeneous spaces. (See also [31] in the context of variable Lebesgue spaces).

Nevertheless, there is not enough information about the behavior of the commutators acting between weighted Lebesgue spaces, even less for extreme values of p, that is, the weighted L^p-BMO or L^p-$Lipschitz$ boundedness. Hence, one of our main aim is, precisely, to give sufficient conditions on the weights in order to obtain these continuity properties. Some previous results in this direction were given in [1] where the authors study the boundedness between Lebesgue spaces with variable exponent for commutators of fractional type operators with BMO symbols (see also [11] in the framework of Orlicz spaces).

We shall first consider fractional type operators, and their commutators, which kernels satisfy certain size condition and a Lipschitz type regularity. For this type of operators we prove boundedness results of the type described above. Particularly we prove a characterization result involving symbols of the commutators and continuity results for extreme values of p.

Later, we study commutators of fractional type operators with less regular kernels. These type of operators include a great variety of operators and were introduced in [3]. See §2.2 for examples and more information.

The paper is organized as follows. In §2 we give the preliminaries definitions in order to state the main results of the article, which are also included in this section. Then, in §3 we give some auxiliary results which allow us to prove the main results in §4.

2. Preliminaries and main results

In this section we give the definitions of the operators we shall be dealing with and the functional class of the symbols in order to define the commutators.

We shall consider fractional operators of convolution type T_α, $0 < \alpha < n$, defined by

$$T_\alpha f(x) = \int_{\mathbb{R}^n} K_\alpha(x - y) f(y) dy, \tag{2.1}$$

where the kernel K_α is not identically zero and verifies certain size and smoothness conditions. We will consider the following size conditions: we say that K_α satisfies the condition S_α^* if the following pointwise inequality

$$|K_\alpha(x)| \leq \frac{C}{|x|^{n-\alpha}}, \quad |x| \neq 0$$

holds for some positive constant C. In this case, we write $K_\alpha \in S_\alpha^*$.

The examples of fractional type operators of the form (2.1) (see [3]) show that we can have kernels satisfying a weaker size condition, which was introduced in [3]: we say that K_α satisfies the condition S_α if it verifies the following integral inequality

$$\int_{s < |x| \leq 2s} |K_\alpha(x)| \, dx \leq C s^\alpha,$$

for every $s > 0$ and some positive constant C. We write it as $K_\alpha \in S_\alpha$. Clearly, condition S_α^* implies condition S_α.

Let $0 < \delta < 1$. We say that a function b belongs to the space $\Lambda(\delta)$ if there exists a positive constant C such that, for every $x, y \in \mathbb{R}^n$

$$|b(x) - b(y)| \leq C|x - y|^\delta.$$

The smallest of such constants will be denoted by $\|b\|_{\Lambda(\delta)}$. We will be dealing with commutators with symbols belonging to this class of functions.

Given a weight w, that is, a non-negative and locally integrable function, we say that a measurable function f belongs to $L_w^p(\mathbb{R}^n)$ for some $1 < p < \infty$ if $fw \in L^p(\mathbb{R}^n)$.

Given two functions f and g, by $\lesssim$ and $\gtrsim$ we will mean that there exists a positive constant c such that $f \leq cg$ and $cf \geq g$, respectively. When both inequalities hold, that is, $f \lesssim g \lesssim f$, we will write it as $f \approx g$.

We will now classify the operators into two different types, according to the regularity conditions satisfied by their kernels.

2.1. Fractional integral operators with Lipschitz regularity. We shall assume that K_α satisfies the smoothness condition $H^*_{\alpha,\infty}$, that is, there exist a positive constant C and $0 < \eta \leq 1$ such that

$$|K_\alpha(x-y) - K_\alpha(x'-y)| + |K_\alpha(y-x) - K_\alpha(y-x')| \leq C\frac{|x-x'|^\eta}{|x-y|^{n-\alpha+\eta}},$$

whenever $|x-y| \geq 2|x-x'|$.

A typical example is the fractional integral operator I_α, whose kernel $K_\alpha(x) = |x|^{\alpha-n}$ satisfies condition $H^*_{\alpha,\infty}$, as it can be easily checked.

Related with the fractional integral operators T_α, we can formally define the higher order commutators with symbol $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, by

$$T^m_{\alpha,b}f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m K_\alpha(x-y)f(y)dy,$$

where $m \in \mathbb{N}_0$ is the order of the commutator. Clearly, $T^0_{\alpha,b} = T_\alpha$.

As we have said, we are interested in studying the boundedness properties of the commutators $T^m_{\alpha,b}$, with symbol $b \in \Lambda(\delta)$, on weighted spaces. We shall first consider their continuity on weighted Lebesgue spaces of the type defined previously. We shall also analyze the boundedness of $T^m_{\alpha,b}$ from weighted Lebesgue spaces into certain weighted version of Lipschitz spaces. For $0 < \delta < 1$ and w a weight, these spaces are denoted by $\mathbb{L}_w(\delta)$ and collect the functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ that satisfy

$$\|f\|_{\mathbb{L}_w(\delta)} := \sup_B \frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B |f(x) - f_B|dx < \infty,$$

where $\|g\|_\infty$ denotes the essential supremum of a measurable function g. The case $\delta = 0$ of the space above was introduced in [26] as a weighted version of the space of functions with bounded mean oscillation.

The classes of weights we will be dealing with are the well-known $A_{p,q}$ classes of Muckenhoupt and Wheeden ([26]). For $1 \leq p, q < \infty$ these classes are defined as the weights w such that

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)^q dx\right)^{1/q} \left(\frac{1}{|B|} \int_B w(x)^{-p'} dx\right)^{1/p'} < \infty.$$

When $q = \infty$, we understand that $w \in A_{p,\infty}$ as $w^{-p'} \in A_1$.

In this subsection we shall assume that the operator T_α has a convolution kernel K_α that verifies condition $H^*_{\alpha,\infty}$ with $0 < \eta \le 1$. In order to simplify the hypotheses we shall suppose that $m \in \mathbb{N} \cup \{0\}$ with the convention that $\beta/0 = \infty$ if $\beta > 0$.

We now give the boundedness result between weighted Lebesgue spaces for the higher order commutators of T_α with Lipschitz symbols.

THEOREM 2.1.   Let $0 < \alpha < n$ and $0 < \delta < \min\{\eta, (n - \alpha)/m\}$. Let $1 < p < n/(m\delta + \alpha)$, $1/q = 1/p - (m\delta + \alpha)/n$ and $b \in \Lambda(\delta)$. If $K_\alpha \in S_\alpha$ and $w \in A_{p,q}$, then there exists a positive constant C such that

$$\left(\int_{\mathbb{R}^n} |T^m_{\alpha,b} f(x)|^q w(x)^q dx \right)^{1/q} \le C\|b\|^m_{\Lambda(\delta)} \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p dx \right)^{1/p}$$

for every $f \in L^p_w(\mathbb{R}^n)$.

REMARK 2.1.   When $m = 0$ and $T_\alpha = I_\alpha$, the result above was proved in [26]. Notice that there are no symbols or parameters δ in the hypotheses when this is the case.

The next result gives the continuity properties of $T^m_{\alpha,b}$ between weighted Lebesgue spaces and $\mathbb{L}_w(\widetilde{\delta})$ spaces. By β^+ we understand β if $\beta > 0$ and 0 if $\beta \le 0$.

THEOREM 2.2.   Let $0 < \alpha < n$ and $0 < \delta < \min\{\eta, (n - \alpha)/m\}$. Let $n/(m\delta + \alpha) \le r < n/(\alpha + (m - 1)\delta)$, if $m \ge 1$ or $n/\alpha \le r < n/(\alpha - \eta)^+$, if $m = 0$. Let $\widetilde{\delta} = m\delta + \alpha - \frac{n}{r}$ and $b \in \Lambda(\delta)$. If $K_\alpha \in S^*_\alpha$ and $w \in A_{r,\infty}$, then there exists a positive constant C such that

$$\|T^m_{\alpha,b} f\|_{\mathbb{L}_w(\widetilde{\delta})} \le C\|b\|^m_{\Lambda(\delta)} \|fw\|_{L^r}$$

for every $f \in L^r_w(\mathbb{R}^n)$.

REMARK 2.2.   When $m = 1$, $w = 1$ and $T_\alpha = I_\alpha$, the result above was proved in [25] in the general context of non-doubling measures.

REMARK 2.3.   If $r = n/(m\delta + \alpha)$, then $\widetilde{\delta} = 0$ and the space $\mathbb{L}_w(\widetilde{\delta})$ is the weighted version of the BMO spaces introduced in [26]. By taking into account the range of p in Theorem 2.1, it is clear that this is the endpoint value from which the Lebesgue spaces change into BMO and Lipschitz spaces when the commutator acts. Particularly, if $m = 0$ and $T_\alpha = I_\alpha$,

this is the well-known result proved in [26]. Notice again that there are no parameters δ or symbols in this case.

On the other hand, if $m = 0$, $T_\alpha = I_\alpha$ and $n/\alpha \le r < n/(\alpha - 1)^+$, that is, $\eta = 1$ in the definition of the class $H^*_{\alpha,\infty}$, the result above was proved in [30].

For the extreme value $r = n/((m-1)\delta + \alpha)$, $m \in \mathbb{N}$ and $0 < \delta < \eta \le 1$, we obtain the following endpoint result in order to characterize the symbol b in $\Lambda(\delta)$ in terms of the boundedness of $T^m_{\alpha,b}$ in the sense of Theorem 2.2. In order to give this result we introduce some previous notation. For $k = 0, 1, \ldots, m$ we denote $c_k = m!/(k!(m-k)!)$. If also $x, u \in \mathbb{R}^n$, we denote $S(x, u, k) = (b(x) - b_B)^{m-k} T_\alpha((b - b_B)^k f_2)(u)$, where $f_2 = f\chi_{\mathbb{R}^n \setminus B}$ for a given ball B and a locally integrable function f.

THEOREM 2.3.   Let $m \in \mathbb{N}$, $0 < \delta < \min\{\eta, (n - \alpha)/m\}$ and $r = n/((m-1)\delta + \alpha)$. If $K_\alpha \in S^*_\alpha$, $w \in A_{n/(m\delta + \alpha),\infty}$ and $b \in \Lambda(\delta)$, the following statements are equivalent.

(i) $T^m_{\alpha,b} : L^r_w(\mathbb{R}^n) \hookrightarrow \mathbb{L}_w(\delta)$;
(ii) There exists a positive constant C such that

$$\frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B \left| \sum_{k=0}^m c_k \left[S(x, u, k) - (S(\cdot, u, k))_B \right] \right| dx \le C\|fw\|_r, \qquad (2.2)$$

for every ball $B \subset \mathbb{R}^n$, $x, u \in B$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

REMARK 2.4.   When $m = 1$, $w = 1$ and $T_\alpha = I_\alpha$ the result above was proved in [25] in a more general context of non-homogeneous spaces. Certainly, their result was inspired in the article of [19], where the same result is proved for $m = 1$, $w = 1$, $T_\alpha = I_\alpha$ and $b \in BMO$.

REMARK 2.5.   In [19] the authors also have obtained that, in the case $T_\alpha = I_\alpha$, $b \in BMO$, $m = 1$ and $w = 1$, the boundedness of the commutator $I_{\alpha,b}$ from $L^{n/\alpha}$ into BMO can only occur if b is constant. In our case, if $b \in \Lambda(\delta)$, when $T_\alpha = I_\alpha$, $m = 1$ and $w = 1$, we can deduce, by (2.2), that, if $I_{\alpha,b}$ is bounded from $L^{n/\alpha}$ into $\mathbb{L}(\delta)$, then

$$\frac{1}{|B|^{1+\frac{\delta}{n}}} \int_B \left| \sum_{k=0}^1 c_k \left[S(x, u, k) - (S(\cdot, u, k))_B \right] \right| dx \le C\|f\|_{n/\alpha}.$$

Since it is easy to see that

$$\frac{1}{|B|^{1+\frac{\delta}{n}}} \int_B \left| \sum_{k=0}^{1} c_k \left[S(x,u,k) - (S(\cdot,u,k))_B \right] \right| dx$$

$$= \frac{1}{|B|^{1+\frac{\delta}{n}}} \int_B |b(x) - b_B| dx \left| \int_{(2B)^c} \frac{f(y)}{|u-y|^{n-\alpha}} dy \right|,$$

we have

$$\frac{1}{|B|^{1+\frac{\delta}{n}}} \int_B |b(x) - b_B| dx \left| \int_{(2B)^c} \frac{f(y)}{|u-y|^{n-\alpha}} dy \right| \leq C \|f\|_{n/\alpha}.$$

Following the same guidelines of [19] with $f_N(y) = |u-y|^{-\alpha} \chi_{B(0,N)}(u-y) \chi_{(2B)^c}(y)$ for $N \in \mathbb{N}$, we obtain that

$$\frac{1}{|B|^{1+\frac{\delta}{n}}} \int_B |b(x) - b_B| dx \left| \int_{(2B)^c \cup \{|u-y|<N\}} \frac{dy}{|u-y|^n} \right|^{1-\alpha/n} \leq C.$$

Due to the fact that $\left| \int_{(2B)^c \cup \{|u-y|<N\}} \frac{dy}{|u-y|^n} \right|^{1-\alpha/n} \to \infty$ when $N \to \infty$, we have $b(x) = b_B$ almost everywhere, for every ball B, which yields that b is essentially constant.

2.2. Fractional integral operators with Hörmander type regularity.

We now introduce the conditions on the kernel that will be considered in this section. First, we must give some notation.

It is well-known that the commutators of fractional integral operators can be controlled, in some sense, by maximal type operators associated to Young functions. By a Young function we mean a function $\Phi : [0,\infty) \to [0,\infty)$ that is increasing, convex and verifies $\Phi(0) = 0$ and $\Phi(t) \to \infty$ when $t \to \infty$. The Φ-Luxemburg average over a ball B is defined, for a locally integrable function f, by

$$\|f\|_{\Phi,B} = \inf \left\{ \lambda > 0 : \frac{1}{|B|} \int_B \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

The maximal type operators that control the commutators involve these averages. More precisely, if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $0 < \alpha < n$, we define the fractional type maximal operator associated to a Young function Φ, by

$$M_{\alpha,\Phi} f(x) = \sup_{B \ni x} |B|^{\alpha/n} \|f\|_{\Phi,B},$$

where the supremum is taken over every ball B that contains $x \in \mathbb{R}^n$. When $\Phi(t) = t$, $M_{\alpha,\Phi} = M_\alpha$, the classical fractional maximal operator.

Given a Young function Φ, the following Hölder's type inequality holds for every pair of measurable functions f, g

$$\frac{1}{|B|} \int_B |f(x)g(x)| dx \leq 2\|f\|_{\Phi,B}\|g\|_{\widetilde{\Phi},B},$$

where $\widetilde{\Phi}$ is the complementary Young function of Φ, defined by

$$\widetilde{\Phi}(t) = \sup_{s>0}\{st - \Phi(s)\}.$$

It is easy to see that $t \leq \Phi^{-1}(t)\widetilde{\Phi}^{-1}(t) \leq 2t$ for every $t > 0$.

Moreover, given Φ, Ψ and Θ Young functions verifying, for every $t > 0$, that $\Phi^{-1}(t)\Psi^{-1}(t) \lesssim \Theta^{-1}(t)$, the following generalization holds

$$\|fg\|_{\Theta,B} \lesssim \|f\|_{\Phi,B}\|g\|_{\Psi,B}.$$

We are now in position to define the smoothness condition on K_α.

We say that $K_\alpha \in H_{\alpha,\Phi}$ if there exist $c \geq 1$ and $C > 0$ such that for every $y \in \mathbb{R}^n$ and $R > c|y|$

$$\sum_{j=1}^{\infty} (2^j R)^{n-\alpha}\| \left(K_\alpha(\cdot - y) - K_\alpha(\cdot)\right) \chi_{|\cdot| \sim 2^j R}\|_{\Phi,B(0,2^{j+1}R)} \leq C,$$

where $|\cdot| \sim s$ means the set $\{x \in \mathbb{R}^n : s < |x| \leq 2s\}$.

When $\Phi(t) = t^q$, $1 \leq q < \infty$, we denote this class by $H_{\alpha,q}$ and it can be written as

$$\sum_{j=1}^{\infty} (2^j R)^{n-\alpha} \left(\frac{1}{(2^j R)^n} \int_{|x| \sim 2^j R} |K_\alpha(x-y) - K_\alpha(x)|^q dx\right)^{1/q} \leq C.$$

The kernels given above are, a priori, less regular than the kernel of the fractional integral operator I_α and they have been studied by several authors. For example, in [23], the author studied fractional integrals given by a multiplier. If $m : \mathbb{R}^n \to \mathbb{R}$ is a function, the multiplier operator T_m is defined, through the Fourier transform, as $\widehat{T_m f}(\zeta) = m(\zeta)\widehat{f}(\zeta)$ for f in the Schwartz class. Under certain conditions on the derivatives of m, the multiplier operator T_m can be seen as the limit of convolution operators T_m^N, having a simpler form. Their corresponding kernels K_α^N belong to the class $S_\alpha \cap H_{\alpha,r}$ with constant independent of N, for certain values of $r > 1$ given by the regularity of the function m (see [23]).

Other examples of this type of operators are fractional integrals with rough kernels, that is, with kernel $K_\alpha(x) = \Omega(x)|x|^{\alpha-n}$ where Ω is a function defined on the unit sphere S^{n-1} of $\mathbb{R}^n$, extended to $\mathbb{R}^n \setminus \{0\}$ radially. The function Ω is an homogeneous function of degree 0. In [3, Proposition 4.2], the authors showed that $K_\alpha \in S_\alpha \cap H_{\alpha,\Phi}$, for certain Young

function Φ, provided that $\Omega \in L^\Phi(S^{n-1})$ with

$$\int_0^1 \omega_\Phi(t)\frac{dt}{t} < \infty,$$

where ω_Φ is the L^Φ-modulus of continuity given by

$$\omega_\Phi(t) = \sup_{|y| \le t} ||\Omega(\cdot + y) - \Omega(\cdot)||_{\Phi, S^{n-1}} < \infty,$$

for every $t \ge 0$. This type of operators where also studied in [7] and [13].

As we said previously, we are interested in studying the higher order commutators of T_α. Since we are dealing with symbols of Lipschitz type, the smoothness condition associated to these commutators will depend on the Lipschitz parameter δ when the L^p-Lipschitz continuity is analyzed. This condition is defined as follows.

DEFINITION 2.1.   Let $m \in \mathbb{N}_0$, $0 < \alpha < n$, $0 \le \delta < \min\{1, (n-\alpha)/m\}$ and let Φ be a Young function. We say that $K_\alpha \in H_{\alpha,\Phi,m}(\delta)$ if

$$\sum_{j=1}^{\infty}(2^j)^{m\delta}(2^j R)^{n-\alpha}||\left(K_\alpha(\cdot - y) - K_\alpha(\cdot)\right)\chi_{|\cdot|\sim 2^j R}||_{\Phi, B(0, 2^{j+1}R)} \le C$$

for some constants $c \ge 1$ and $C > 0$ and for every $y \in \mathbb{R}^n$ with $R > c|y|$.

Clearly, when $\delta = 0$ or $m = 0$, $H_{\alpha,\Phi,m}(\delta) = H_{\alpha,\Phi}$.

REMARK 2.6.   It is easy to see that $H_{\alpha,\Phi,m}(\delta_2) \subset H_{\alpha,\Phi,m}(\delta_1) \subset H_{\alpha,\Phi}$ whenever $0 \le \delta_1 < \delta_2 < \min\{1, (n-\alpha)/m\}$.

Recall that Fourier multipliers and fractional integrals with rough kernels are examples of fractional integral operators with $K_\alpha \in H_{\alpha,\Phi}$ for certain Young function. By assuming adequate conditions depending on δ, on the multiplier m, or on the L^Φ-modulus of continuity ω_Φ, one can obtain kernels $K_\alpha \in H_{\alpha,\Phi,m}(\delta)$. This fact can be proved by adapting Proposition 4.2 and Corollary 4.3 given in [3] (see also [24]).

We shall also deal with a class of Young functions that arises in connection with the boundedness of the fractional maximal operator $M_{\alpha,\Psi}$ on weighted Lebesgue spaces (see §3). Given $0 < \alpha < n$, $1 \le \beta < p < n/\alpha$ and a Young function Ψ, we shall say that $\Psi \in \mathcal{B}_{\alpha,\beta}$ if $t^{-\alpha/n}\Psi^{-1}(t)$ is the inverse of a Young function and $\Psi^{1+\frac{\rho\alpha}{n}} \in B_\rho$ for every $\rho > n\beta/(n - \alpha\beta)$,

that is, there exists a positive constant c such that

$$\int_c^\infty \frac{\Psi^{1+\frac{\rho\alpha}{n}}(t)}{t^\rho} \frac{dt}{t} < \infty$$

for each of those values of ρ.

We now state the following generalizations of Theorems 2.1 and 2.2. We shall consider again $m \in \mathbb{N}_0$.

THEOREM 2.4. *Let $0 < \alpha < n$ and $0 < \delta < \min\{1, (n-\alpha)/m\}$. Let $1 < p < n/(m\delta + \alpha)$, $1/q = 1/p - (m\delta + \alpha)/n$ and $b \in \Lambda(\delta)$. Assume that T_α has a kernel $K_\alpha \in S_\alpha \cap H_{\alpha,\Phi}$ for a Young function Φ such that its complementary function $\widetilde{\Phi} \in \mathcal{B}_{m\delta+\alpha,\beta}$ for some $1 \leq \beta < p$. Then, if w is a weight verifying $w^\beta \in A_{\frac{p}{\beta},\frac{q}{\beta}}$, there exists a positive constant C such that*

$$\left(\int_{\mathbb{R}^n} |T_{\alpha,b}^m f(x)|^q w(x)^q dx\right)^{1/q} \leq C\|b\|_{\Lambda(\delta)}^m \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p dx\right)^{1/p}$$

for every $f \in L_w^p(\mathbb{R}^n)$.

REMARK 2.7. If we consider, for example, $\Phi(t) = e^{t^{1/\gamma}} - e$ with $\gamma > 0$, then $\widetilde{\Phi}(t) \approx t(1 + \log^+ t)^\gamma$ and this function verifies condition $\mathcal{B}_{m\delta+\alpha,1}$. Thus, Φ satisfies the hypotheses of the theorem above and, in this case, we can take $w \in A_{p,q}$. As we have mentioned before, this condition $\mathcal{B}_{m\delta+\alpha,\beta}$ is related with the boundedness of the corresponding fractional maximal operator $M_{m\delta+\alpha,\widetilde{\Phi}}$ between L_w^p and L_w^q when $w^\beta \in A_{\frac{p}{\beta},\frac{q}{\beta}}$ (see Theorems 3.1 and 3.2 below). When $\beta > 1$, a typical example is $\widetilde{\Phi}(t) = t^\beta(1 + \log^+ t)^\gamma$ for $\gamma \geq 0$. In this case, the Young function Φ related with the smoothness condition on the kernel K_α given in the theorem above is $\Phi(t) = t^{\beta'}(1 + \log^+ t)^{-\gamma/(\beta-1)}$, where $\beta' = \beta/(\beta - 1)$.

THEOREM 2.5. *Let $0 < \alpha < n$, $0 < \delta < \min\{1, (n-\alpha)/m\}$ and $n/(m\delta + \alpha) \leq r < n/((m-1)\delta + \alpha)$ such that $\widetilde{\delta} = m\delta + \alpha - \frac{n}{r}$. Let w be a weight such that $w^\beta \in A_{r/\beta,\infty}$ for some $1 < \beta < r$. Assume that T_α has a kernel $K_\alpha \in S_\alpha^* \cap H_{\alpha,\Phi,m}(\delta)$ for a Young function Φ such that $\Phi^{-1}(t) \lesssim t^{\frac{\beta-1}{r}}$ for every $t > 0$. If $b \in \Lambda(\delta)$, then there exists a positive constant C such that*

$$\|T_{\alpha,b}^m f\|_{\mathbb{L}_w(\widetilde{\delta})} \leq C\|b\|_{\Lambda(\delta)}^m \|fw\|_{L^r}$$

for every $f \in L_w^r(\mathbb{R}^n)$.

THEOREM 2.6. Let $m \in \mathbb{N}$, $0 < \delta < \min\{1, (n-\alpha)/m\}$ and $r = n/((m-1)\delta + \alpha)$. Let w be a weight such that $w^\beta \in A_{r/\beta,\infty}$ for some $1 < \beta < r$. Let T_α be a fractional integral operator with kernel $K_\alpha \in S_\alpha^* \cap H_{\alpha,\Phi,m}(\delta)$ where Φ is a Young function verifying $\Phi^{-1}(t) \lesssim t^{\frac{\beta-1}{r}}$ for every $t > 0$, and $\widetilde{\Phi} \in \mathcal{B}_{m\delta+\alpha,\beta}$. If $b \in \Lambda(\delta)$, the following statements are equivalent.

  (i) $T_{\alpha,b}^m : L_w^r(\mathbb{R}^n) \hookrightarrow \mathbb{L}_w(\delta)$;
  (ii) There exists a positive constant C such that

$$\frac{\|w\chi_B\|_\infty}{|B|^{1+\frac{\delta}{n}}} \int_B \left| \sum_{k=0}^m c_k \left[S(x,u,k) - (S(\cdot,u,k))_B \right] \right| dx \leq C\|fw\|_r,$$

for every ball $B \subset \mathbb{R}^n$, $x, u \in B$ and $f \in L_{\mathrm{loc}}^1(\mathbb{R}^n)$.

3. Auxiliary results

In this section we give some previous results. We begin with an inequality involving functions in $\Lambda(\delta)$ which is easy to check.

LEMMA 3.1. Let $0 < \delta < 1$ and $B \subset \mathbb{R}^n$ a ball. If $b \in \Lambda(\delta)$, then for every $y \in \lambda B$, $\lambda \geq 1$,

$$|b(y) - b_B| \leq C\|b\|_{\Lambda(\delta)}|\lambda B|^{\frac{\delta}{n}}.$$

The following lemma is an easy consequence of condition S_α.

LEMMA 3.2. Let K_α be a kernel verifying condition S_α with $0 < \alpha < n$. Then, for any ball $B = B(x_B, r_B)$, we have

$$\int_B |K_\alpha(x - x_B)| dx \lesssim r_B^\alpha.$$

In order to obtain the boundedness result between Lebesgue spaces, we prove the following key estimate, which shows how can we control the higher order commutators of T_α by a fractional maximal function via the sharp maximal operator $M_{0,\gamma}^\sharp$, $0 < \gamma < 1$, given by

$$M_0^\sharp f(x) = \sup_{B \ni x} \inf_{a \in \mathbb{R}} \frac{1}{|B|} \int_B |f(y) - a| \, dy,$$

where $M_{0,\gamma}^\sharp f := M_0^\sharp(|f|^\gamma)^{1/\gamma}$. In the following, for $0 \leq \alpha < n$ and $0 < \gamma < 1$ we shall denote $\mathcal{M}_{\alpha,\gamma} f := M_{\alpha\gamma}(|f|^\gamma)^{1/\gamma}$.

LEMMA 3.3. Let $m \in \mathbb{N}$, $0 < \gamma < 1/m$, $0 < \alpha < n$ and $0 < \delta < \min\{1, (n-\alpha)/m\}$. Let $b \in \Lambda(\delta)$ and T_α a fractional integral operator with kernel $K_\alpha \in S_\alpha$. Then, there exists a positive constant C such that

(i) if $K_\alpha \in H^*_{\alpha,\infty}$,

$$M^\sharp_{0,\gamma}(T^m_{\alpha,b}f)(x) \leq C\|b\|^m_{\Lambda(\delta)} \left(\sum_{j=0}^{m-1} \mathcal{M}_{\theta_j,\gamma}(|T^j_{\alpha,b}f|)(x) + M_{\theta_0+\alpha}f(x) \right),$$

where $\theta_j = \delta(m-j)$, $j = 0, \ldots, m$.

(ii) if $K_\alpha \in H_{\alpha,\Phi}$ for some Young function Φ,

$$M^\sharp_{0,\gamma}(T^m_{\alpha,b}f)(x) \leq C\|b\|^m_{\Lambda(\delta)} \left(\sum_{j=0}^{m-1} \mathcal{M}_{\theta_j,\gamma}(|T^j_{\alpha,b}f|)(x) + M_{\theta_0+\alpha,\widetilde{\Phi}}f(x) \right),$$

where $\theta_j = \delta(m-j)$, $j = 0, \ldots, m$, and $\widetilde{\Phi}$ is the complementary function of Φ.

REMARK 3.1. For $0 < \delta < 1$, $m = 1$ and $K_\alpha \in H^*_{\alpha,\infty}$ and homogeneous of degree $\alpha - n$, the proof of (i) can be found in [31] for a larger class of Lipschitz spaces with variable parameter.

P r o o f o f L e m m a 3.3. Fix B a ball containing x, and decompose the commutator in the following way (see, for instance, [16] or [27])

$$T^m_{\alpha,b}f(x) = \sum_{j=0}^{m-1} C_{j,m}(b(x) - b_{2B})^{m-j}T^j_{\alpha,b}f(x) + T_\alpha((b - b_{2B})^m f)(x).$$

If we split $f = f_1 + f_2$ where $f_1 = f\chi_{2B}$, it is sufficient to estimate, for $0 < \gamma < 1/m$, the average

$$\left(\frac{1}{|B|} \int_B |T^m_{\alpha,b}f(y) - T_\alpha((b - b_{2B})^m f_2)(x_B)|^\gamma dy \right)^{1/\gamma} \leq I + II + III,$$

where x_B denotes the center of B, and

$$I = \sum_{j=0}^{m-1} \left(\frac{1}{|B|} \int_B (b(y) - b_{2B})^{(m-j)\gamma}|T^j_{\alpha,b}f(y)|^\gamma dy \right)^{\frac{1}{\gamma}},$$

$$II = \left(\frac{1}{|B|} \int_B |T_\alpha((b - b_{2B})^m f_1)(y)|^\gamma dy \right)^{\frac{1}{\gamma}},$$

$$III = \left(\frac{1}{|B|} \int_B |T_\alpha((b - b_{2B})^m f_2)(y) - T_\alpha((b - b_{2B})^m f_2)(x_B)|^\gamma dy \right)^{\frac{1}{\gamma}}.$$

For simplicity, we will assume $\|b\|_{\Lambda(\delta)} = 1$. We shall first estimate I. From Lemma 3.1 we have

$$I \lesssim \sum_{j=0}^{m-1} \|b\|_{\Lambda(\delta)}^{m-j} |B|^{\frac{\delta(m-j)}{n}} \left(\frac{1}{|B|} \int_B |T_{\alpha,b}^j f(y)|^\gamma dy \right)^{\frac{1}{\gamma}}$$

$$= \sum_{j=0}^{m-1} \left(\frac{1}{|B|^{1-\frac{(m-j)\delta\gamma}{n}}} \int_B |T_{\alpha,b}^j f(y)|^\gamma dy \right)^{1/\gamma} \lesssim \sum_{j=0}^{m-1} \mathcal{M}_{\theta_j,\gamma}(|T_{\alpha,b}^j f|)(x)$$

where $\theta_j = (m-j)\delta$. Note that the last maximal operator is of fractional-type since $0 < \theta_j < (m-j)(n-\alpha)/m \le n$ for every $0 \le j \le m-1$.

We will now estimate II. If $y \in B$ and $z \in 2B$, then $|y - z| < 3R$, where R is the radius of B. We have, by Lemma 3.2, that

$$II \le \frac{1}{|B|} \int_B \left(\int_{2B} |K_\alpha(y-z)(b(z) - b_{2B})^m f(z)| dz \right) dy$$

$$\le \frac{1}{|B|} \int_{2B} |b(z) - b_{2B}|^m |f(z)| \left(\int_{B(z,3R)} |K_\alpha(y-z)| dy \right) dz$$

$$\lesssim \frac{|3B|^{\frac{\alpha}{n}}}{|B|} \int_{2B} |b(z) - b_{2B}|^m |f(z)| dy.$$

From Lemma 3.1 , we can estimate $|b(z) - b_{2B}|^m \le \|b\|_{\Lambda(\delta)}^m |2B|^{\frac{m\delta}{n}} = \|b\|_{\Lambda(\delta)}^m |2B|^{\frac{\theta_0}{n}}$ to obtain

$$II \lesssim \|b\|_{\Lambda(\delta)}^m \frac{1}{|2B|^{1-\frac{\theta_0+\alpha}{n}}} \int_{2B} |f(z)| dz \le \|b\|_{\Lambda(\delta)}^m M_{\theta_0+\alpha} f(x).$$

Since $0 < \delta < (n-\alpha)/m$, it is clear that $0 < \theta_0 + \alpha < n$, so $M_{\theta_0+\alpha}$ is a fractional-type maximal operator.

In order to estimate III, we first observe that, by Jensen's inequality

$$III \le \frac{1}{|B|} \int_B |T_{\alpha,b}^m \left((b - b_{2B})^m f_2 \right)(y) - T_\alpha((b - b_{2B})^m f_2)(x_B)| dy,$$

and, setting $B_j = 2^j B$, the integrand can be estimated, using Lemma 3.1 , as follows

$$|T_{\alpha,b}^m \left((b - b_{2B})^m f_2 \right)(y) - T_\alpha((b - b_{2B})^m f_2)(x_B)| \qquad (3.3)$$

$$\le \sum_{j=1}^\infty \int_{B_{j+1} \setminus B_j} |K_\alpha(y-z) - K_\alpha(x_B - z)||b(z) - b_{2B}|^m |f(z)| dz$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^\infty |B_{j+1}|^{\frac{m\delta}{n}} \int_{B_{j+1} \setminus B_j} |K_\alpha(y-z) - K_\alpha(x_B - z)||f(z)| dz.$$

Here, we must distinguish the cases $K_\alpha \in H^*_{\alpha,\infty}$ and $K_\alpha \in H_{\alpha,\Phi}$. If $K_\alpha \in H^*_{\alpha,\infty}$,

$$|T^m_{\alpha,b}((b - b_{2B})^m f_2)(y) - T_\alpha((b - b_{2B})^m f_2)(x_B)|$$

$$\lesssim \|b\|^m_{\Lambda(\delta)} \sum_{j=1}^\infty |B_{j+1}|^{\frac{\delta m}{n}} \int_{B_{j+1} \setminus B_j} \frac{|y - x_B|^\eta}{|y - z|^{n-\alpha+\eta}} |f(z)| dz$$

$$\lesssim \|b\|^m_{\Lambda(\delta)} \sum_{j=1}^\infty |B_{j+1}|^{\frac{m\delta+\alpha}{n}} 2^{-j\eta} \frac{1}{|B_{j+1}|} \int_{B_{j+1}} |f(z)| dz$$

$$= \|b\|^m_{\Lambda(\delta)} \sum_{j=1}^\infty 2^{-j\eta} \frac{1}{|B_{j+1}|^{1-\frac{\theta_0+\alpha}{n}}} \int_{B_{j+1}} |f(z)| dz$$

$$\leq \|b\|^m_{\Lambda(\delta)} M_{\theta_0+\alpha} f(x) \sum_{j=1}^\infty 2^{-j\eta} \lesssim \|b\|^m_{\Lambda(\delta)} M_{\theta_0+\alpha} f(x),$$

since $\eta > 0$. Therefore

$$III \lesssim \|b\|^m_{\Lambda(\delta)} M_{\theta_0+\alpha} f(x).$$

Let us now consider the case $K_\alpha \in H_{\alpha,\Phi}$. Applying Hölder's inequality with Φ and $\widetilde{\Phi}$ in (3.3), we obtain

$$|T^m_{\alpha,b}((b - b_{2B})^m f_2)(y) - T_\alpha((b - b_{2B})^m f_2)(x_B)|$$

$$\lesssim \|b\|^m_{\Lambda(\delta)} \sum_{j=1}^\infty |B_{j+1}|^{\frac{m\delta}{n}+1} \| (K_\alpha(\cdot - (y - x_B)) - K_\alpha(\cdot)) \chi_{|\cdot| \sim 2^j R} \|_{\Phi, B_{j+1}}$$

$$\times \|f\|_{\widetilde{\Phi}, B_{j+1}}$$

$$\lesssim \|b\|^m_{\Lambda(\delta)} \sum_{j=1}^\infty |B_{j+1}|^{1-\frac{\alpha}{n}} \| (K_\alpha(\cdot - (y - x_B)) - K_\alpha(\cdot)) \chi_{|\cdot| \sim 2^j R} \|_{\Phi, B_{j+1}}$$

$$\times |B_{j+1}|^{\frac{m\delta+\alpha}{n}} \|f\|_{\widetilde{\Phi}, B_{j+1}}$$

$$\lesssim \|b\|^m_{\Lambda(\delta)} M_{\theta_0+\alpha,\widetilde{\Phi}} f(x)$$

$$\times \sum_{j=1}^\infty (2^j R)^{n-\alpha} \| (K_\alpha(\cdot - (y - x_B)) - K_\alpha(\cdot)) \chi_{|\cdot| \sim 2^j R} \|_{\Phi, B_{j+1}}$$

$$\lesssim \|b\|^m_{\Lambda(\delta)} M_{\theta_0+\alpha,\widetilde{\Phi}} f(x).$$

Therefore,

$$III \lesssim \|b\|^m_{\Lambda(\delta)} M_{\theta_0+\alpha,\widetilde{\Phi}} f(x).$$

Combining all these estimates, we obtain the desired pointwise inequalities. $\qquad \square$

The following result is a variant of the well-known Fefferman-Stein's inequality (see [15]) and it will be a key estimate to prove Theorem 2.1. In this result, the A_∞ class of weights means the union of the Muckenhoupt A_p classes for every $1 \leq p < \infty$. According to our definitions, $w \in A_p$ is the same as $w^{1/p} \in A_{p,p}$.

LEMMA 3.4 ([28]). *Let $0 < p < \infty$ and $0 < \gamma < 1$. Let w be a weight in the A_∞ class. Then, there exists a positive constant C such that*

$$\int_{\mathbb{R}^n} M_\gamma f(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} M_{0,\gamma}^\sharp f(x)^p w(x) dx \qquad (3.4)$$

for every measurable function f.

We shall also need two results involving the boundedness of fractional maximal operators associated with Young functions (see [1]).

THEOREM 3.1 ([1]). *Let $0 < \alpha < n$, $1 \leq \beta < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Let w be a weight such that $w^\beta \in A_{p/\beta,q/\beta}$. Let Ψ be a Young function that satisfies $\Psi \in \mathcal{B}_{\alpha,\beta}$. Then, $M_{\alpha,\Psi}$ is bounded from $L^p(w^p, \mathbb{R}^n)$ into $L^q(w^q, \mathbb{R}^n)$.*

THEOREM 3.2 ([1]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Let w be a weight and $\Psi(t) = t^\beta (1 + \log^+ t)^\gamma$ where $1 \leq \beta < p$ and $\gamma \geq 0$. Then, $M_{\alpha,\Psi}$ is bounded from $L^p(w^p, \mathbb{R}^n)$ into $L^q(w^q, \mathbb{R}^n)$ if and only if $w^\beta \in A_{p/\beta,q/\beta}$.*

In order to prove Theorem 2.3, we shall need the following estimate.

LEMMA 3.5. *Let $0 < \delta < \min\{\eta, (n-\alpha)/(m-1)\}$, for $0 < \eta \leq 1$. Let $r = n/((m-1)\delta + \alpha)$, $w \in A_{r,\infty}$, $b \in \Lambda(\delta)$ and $f \in L_w^r(\mathbb{R}^n)$. Let $B \subset \mathbb{R}^n$ be a ball and $f_2 = f\chi_{\mathbb{R}^n \setminus 2B}$. If T_α is a fractional integral operator with kernel $K_\alpha \in H_{\alpha,\infty}^*$, then, for every $x, u \in B$,*

$$|T_\alpha((b - b_B)^k f_2)(x) - T_\alpha((b - b_B)^k f_2)(u)| \lesssim \frac{\|b\|_{\Lambda(\delta)}^k \|fw\|_r |B|^{\frac{\delta(k-m+1)}{n}}}{\|w\chi_B\|_\infty}$$

for each $k = 0, \ldots, m$.

P r o o f o f L e m m a 3.5. Since $K_\alpha \in H_{\alpha,\infty}^*$, by taking $x, u \in B$, and $0 \leq k \leq m$, and setting $B_j = 2^j B$, we have from Lemma 3.1 that

$$|T_\alpha((b - b_B)^k f_2)(x) - T_\alpha((b - b_B)^k f_2)(u)|$$

$$\leq \int_{(2B)^c} |K_\alpha(x-y) - K_\alpha(u-y)||b(y) - b_B|^k |f(y)| dy$$

$$\lesssim \|b\|_{\Lambda(\delta)}^k \sum_{j=1}^{\infty} |B_{j+1}|^{\frac{\delta k}{n}} \int_{B_{j+1}\backslash B_j} \frac{|x-u|^\eta}{|y-u|^{n+\eta-\alpha}} |f(y)| dy$$

$$\lesssim \|b\|_{\Lambda(\delta)}^k \sum_{j=1}^{\infty} \frac{|B_{j+1}|^{\frac{\delta k}{n}} |B|^{\frac{\eta}{n}}}{|B_{j+1}|^{\frac{n+\eta-\alpha}{n}}} \int_{B_{j+1}\backslash B_j} |f(y)| dy.$$

Now by Hölder's inequality and the fact that $w \in A_{r,\infty}$ with $r = n/(\alpha + (m-1)\delta)$, we get

$$|T_\alpha((b-b_B)^k f_2)(x) - T_\alpha((b-b_B)^k f_2)(u)|$$

$$\lesssim \|b\|_{\Lambda(\delta)}^k \|fw\|_r \sum_{j=1}^{\infty} \frac{|B_{j+1}|^{\frac{\delta k}{n}}}{|B_{j+1}|^{1-\frac{\alpha}{n}} 2^{j\eta}} \|w^{-1}\chi_{B_{j+1}}\|_{r'}$$

$$\lesssim \|b\|_{\Lambda(\delta)}^k \|fw\|_r \|w\chi_B\|_{\infty}^{-1} \sum_{j=1}^{\infty} \frac{|B_{j+1}|^{\frac{\delta k}{n}-1+\frac{\alpha}{n}+1-\frac{\alpha+(m-1)\delta}{n}}}{2^{j\eta}}$$

$$\lesssim \|b\|_{\Lambda(\delta)}^k \|fw\|_r \|w\chi_B\|_{\infty}^{-1} |B|^{\frac{\delta(k-m+1)}{n}} \sum_{j=1}^{\infty} 2^{j(\delta(k-m+1)-\eta)}$$

$$\lesssim \|b\|_{\Lambda(\delta)}^k \|fw\|_r \|w\chi_B\|_{\infty}^{-1} |B|^{\frac{\delta(k-m+1)}{n}}$$

where the series is summable since $0 \leq k \leq m$ and $\delta < \eta$. □

LEMMA 3.6.   Let $m \in \mathbb{N}$, $0 < \alpha < n$, $0 < \delta < \min\{1, (n-\alpha)/(m-1)\}$, $r = n/((m-1)\delta+\alpha)$, $b \in \Lambda(\delta)$ and $f \in L_w^r(\mathbb{R}^n)$ where w is a weight such that $w^\beta \in A_{r/\beta,\infty}$ for some $1 < \beta < r$. Let $B \subset \mathbb{R}^n$ be a ball and $f_2 = f\chi_{\mathbb{R}^n\backslash 2B}$. If T_α is a fractional integral operator with kernel $K_\alpha \in H_{\alpha,\Phi,m}(\delta)$, where Φ is a Young function verifying $\Phi^{-1}(t) \lesssim t^{\frac{\beta-1}{r}}$ for every $t > 0$, then, for every $x, u \in B$,

$$|T_\alpha((b-b_B)^k f_2)(x) - T_\alpha((b-b_B)^k f_2)(u)| \lesssim \frac{\|fw\|_r \|b\|_{\Lambda(\delta)}^k |B|^{\frac{\delta(k-m+1)}{n}}}{\|w\chi_B\|_{\infty}}$$

for each $k = 0, \ldots, m$.

P r o o f.   Fix $x, u \in B$ and $0 \leq k \leq m$. Setting $B_u = B(u, R)$ which satisfies $B \subset 2B_u \subset 4B$, and using Lemma 3.1 we have

$$|T_\alpha((b-b_B)^k f_2)(x) - T_\alpha((b-b_B)^k f_2)(u)|$$

$$\lesssim \int_{\mathbb{R}^n \setminus B_u} |b(y) - b_{B_u}|^k |K_\alpha(x - y) - K_\alpha(u - y)||f(y)|dy$$

$$\lesssim \sum_{j=0}^{\infty} \int_{2^{j+1}B_u \setminus 2^j B_u} |b(y) - b_{B_u}|^k |K_\alpha(x - y) - K_\alpha(u - y)||f(y)|dy$$

$$\lesssim \|b\|_{\Lambda(\delta)}^k \sum_{j=0}^{\infty} |2^{j+1}B_u|^{\frac{\delta k}{n}} \int_{2^{j+1}B_u \setminus 2^j B_u} |K_\alpha(x - y) - K_\alpha(u - y)||f(y)|dy.$$

Since $1/r + 1/(r/\beta)' = 1 - (\beta - 1)/r$, we can use Hölder's inequality with $\Phi^{-1}(t)t^{1/r}t^{1/(r/\beta)'} \lesssim t$ and the fact that $w^\beta \in A_{r/\beta,\infty}$, to get

$$|T_\alpha((b - b_B)^k f_2)(x) - T_\alpha((b - b_B)^k f_2)(u)| \tag{3.5}$$

$$\lesssim \|b\|_{\Lambda(\delta)}^k \|fw\|_r \sum_{j=0}^{\infty} |2^{j+1}B_u|^{\frac{\delta k}{n} + 1 - \frac{1}{r}} \frac{\|w^{-1}\chi_{2^{j+1}B_u}\|_{(r/\beta)'}}{|2^{j+1}B_u|^{1/(r/\beta)'}}$$

$$\times \||K_\alpha(\cdot - (u - x)) - K_\alpha(\cdot)|\chi_{|\cdot| \sim 2^j R}\|_{\Phi, 2^{j+1}B_u}$$

$$\lesssim \frac{\|b\|_{\Lambda(\delta)}^k |B|^{\frac{\delta k + \alpha}{n} - \frac{1}{r}} \|fw\|_r}{\|w\chi_{2B_u}\|_\infty}$$

$$\times \sum_{j=1}^{\infty} (2^j R)^{n - \alpha} 2^{j(\delta k + \alpha - \frac{n}{r})} \||K_\alpha(\cdot - (u - x)) - K_\alpha(\cdot)|\chi_{|\cdot| \sim 2^j R}\|_{\Phi, 2^{j+1}B_u}$$

$$\leq \frac{\|b\|_{\Lambda(\delta)}^k |B|^{\frac{\delta(k - m + 1)}{n}} \|fw\|_r}{\|w\chi_B\|_\infty}$$

$$\times \sum_{j=1}^{\infty} 2^{jm\delta} (2^j R)^{n - \alpha} \||K_\alpha(\cdot - (u - x)) - K_\alpha(\cdot)|\chi_{|\cdot| \sim 2^j R}\|_{\Phi, 2^{j+1}B_u}$$

$$\lesssim \frac{\|b\|_{\Lambda(\delta)}^k |B|^{\frac{\delta(k - m + 1)}{n}} \|fw\|_r}{\|w\chi_B\|_\infty},$$

where we have used that $\delta k + \alpha - n/r \leq m\delta$, for $m \in \mathbb{N}$, and that the kernel $K_\alpha \in H_{\alpha, \Phi, m}(\delta)$. $\qquad \square$

4. **Proofs of main results**

P r o o f o f T h e o r e m 2.1. The proof will be done by induction and, without loss of generality, we shall assume $\|b\|_{\Lambda(\delta)} = 1$. Notice that when $m = 0$, $1/q = 1/p - \alpha/n$ and the boundedness result is known to be true for $A_{p,q}$ weights (see [1] in the more general setting of variable Lebesgue spaces).

Fix $m \in \mathbb{N}$ and define the following auxiliary exponents

$$\frac{1}{p_j} = \frac{1}{q} + \frac{\delta(m-j)}{n} = \frac{1}{p} - \frac{j\delta + \alpha}{n}, \quad j = 0, \ldots, m.$$

Clearly, $p_m = q$ and, if $\theta_j = (m-j)\delta$, we have that

$$\frac{1}{p_j} = \frac{1}{q} + \frac{\theta_j}{n} = \frac{1}{p} - \frac{j\delta + \alpha}{n}, \quad j = 0, \ldots, m.$$

Notice also that $p \le p_j \le p_l \le q$ for every $0 \le j \le l \le m$.

It is easy to see that $w \in A_{p,q}$ yields $w^\gamma \in A_{\frac{p}{\gamma}, \frac{q}{\gamma}}$ for every $0 < \gamma < 1$. Moreover, from the properties of these classes, we have that $w^\gamma \in A_{\frac{p_j}{\gamma}, \frac{p_l}{\gamma}}$ for every $0 \le j \le l \le m$.

By applying Fefferman-Stein's inequality (3.4) with $w^q \in A_{1+q/p'} \subset A_\infty$, we get

$$\|wT_{\alpha,b}^m f\|_q \le \|wM_\gamma(T_{\alpha,b}^m f)\|_q \lesssim \|wM_{0,\gamma}^\sharp(T_{\alpha,b}^m f)\|_q.$$

Now, since $K_\alpha \in S_\alpha \cap H_{\alpha,\infty}^*$, from Lemma 3.3 we have that

$$\|wT_{\alpha,b}^m f\|_q \lesssim \sum_{j=0}^{m-1} \|w\mathcal{M}_{\theta_j,\gamma}(|T_{\alpha,b}^j f|)\|_q + \|wM_{\theta_0+\alpha}f\|_q.$$

Since $w \in A_{p,q}$ and $1/q = 1/p - (\theta_0 + \alpha)/n$, we have that

$$\|wT_{\alpha,b}^m f\|_q \lesssim \sum_{j=0}^{m-1} \|w\mathcal{M}_{\theta_j,\gamma}(|T_{\alpha,b}^j f|)\|_q + \|wf\|_p.$$

On the other hand, since $w^\gamma \in A_{\frac{p_j}{\gamma}, \frac{q}{\gamma}}$ for every $j = 1, \ldots, m-1$, then the fractional maximal operator $M_{\theta_j\gamma}$ is bounded from $L_{w^\gamma}^{\frac{p_j}{\gamma}}(\mathbb{R}^n)$ to $L_{w^\gamma}^{\frac{q}{\gamma}}(\mathbb{R}^n)$. Thus, we have that

$$\|wT_{\alpha,b}^m f\|_q \lesssim \sum_{j=0}^{m-1} \|w^\gamma M_{\theta_j\gamma}(|T_{\alpha,b}^j f|^\gamma)\|_{q/\gamma}^{1/\gamma} + \|wf\|_p$$

$$\lesssim \sum_{j=0}^{m-1} \|w^\gamma (T_{\alpha,b}^j f)^\gamma\|_{p_j/\gamma}^{1/\gamma} + \|wf\|_p = \sum_{j=0}^{m-1} \|wT_{\alpha,b}^j f\|_{p_j} + \|wf\|_p.$$

Since $1/p_j = 1/p - (j\delta + \alpha)/n$ and $w \in A_{p,p_j}$, and recalling that $\|b\|_{\Lambda(\delta)} = 1$, we apply the inductive hypothesis to get

$$\|wT_{\alpha,b}^m f\|_q \lesssim \sum_{j=0}^{m-1} \|wf\|_p + \|wf\|_p \lesssim \|wf\|_p.$$

$\square$

P r o o f   o f   T h e o r e m 2.2. Fix $f \in L_w^r(\mathbb{R}^n)$. It will be enough to show that for every ball B,

$$\frac{\|w\chi_B\|_\infty}{|B|} \int_B |T_{\alpha,b}^m f(x) - a_B| dx \lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\tilde{\delta}}{n}},$$

for some constant a_B.

For a ball $B \subset \mathbb{R}^n$, set $f_1 = f\chi_{2B}$, $f_2 = f - f_1$ and $a_B = \frac{1}{|B|} \int_B T_{\alpha,b}^m f_2$. Then,

$$\frac{\|w\chi_B\|_\infty}{|B|} \int_B |T_{\alpha,b}^m f(x) - a_B| dx$$

$$\leq \frac{\|w\chi_B\|_\infty}{|B|} \int_B |T_{\alpha,b}^m f_1(x)| dx + \frac{\|w\chi_B\|_\infty}{|B|} \int_B |T_{\alpha,b}^m f_2(x) - a_B| dx$$

$$= I_1 + I_2.$$

For I_1 we write

$$I_1 = \frac{\|w\chi_B\|_\infty}{|B|} \int_B \left| \int_{2B} (b(x) - b(y))^m K_\alpha(x - y) f(y) dy \right| dx.$$

By using Tonelli's theorem and the fact that $b \in \Lambda(\delta)$, we obtain

$$I_1 \lesssim \frac{\|w\chi_B\|_\infty}{|B|} \int_{2B} |f(y)| \left(\int_B |b(x) - b(y)|^m |K_\alpha(x - y)| dx \right) dy$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m |3B|^{\frac{\delta m}{n} - 1} \|w\chi_B\|_\infty \int_{2B} |f(y)| \left(\int_{2B} |K_\alpha(x - y)| dx \right) dy$$

We notice that for $x, y \in 2B$, if R is the radius of B, then $x \in B(y, 4R)$ so we can use Lemma 3.2 (since $S_\alpha^* \subset S_\alpha$), Hölder's inequality and the condition on w to have

$$I_1 \lesssim \|b\|_{\Lambda(\delta)}^m |3B|^{\frac{m\delta + \alpha}{n} - 1} \|w\chi_B\|_\infty \|fw\|_r \|w^{-1}\chi_{3B}\|_{r'}$$

$$\leq \|b\|_{\Lambda(\delta)}^m \|fw\|_r |3B|^{\frac{m\delta + \alpha}{n} - \frac{1}{r}} \|w\chi_{3B}\|_\infty \frac{\|w^{-1}\chi_{3B}\|_{r'}}{|3B|^{1/r'}}$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\tilde{\delta}}{n}}.$$

For I_2, we first estimate the difference $|T_{\alpha,b}^m f_2(x) - (T_{\alpha,b}^m f_2)_B|$ for every $x \in B$. Since

$$|T_{\alpha,b}^m f_2(x) - (T_{\alpha,b}^m f_2)_B| \leq \frac{1}{|B|} \int_B |T_{\alpha,b}^m f_2(x) - T_{\alpha,b}^m f_2(y)| dy,$$

we analyze $A = |T_{\alpha,b}^m f_2(x) - T_{\alpha,b}^m f_2(y)|$. If $x, y \in B$

$$A \leq \int_{(2B)^c} |(b(x) - b(z))^m K_\alpha(x - z) - (b(y) - b(z))^m K_\alpha(y - z)| |f(z)| dz$$

$$\leq \int_{(2B)^c} |b(x) - b(z)|^m |K_\alpha(x-z) - K_\alpha(y-z)||f(z)|dz$$

$$+ \int_{(2B)^c} |(b(x) - b(z))^m - (b(y) - b(z))^m||K_\alpha(y-z)||f(z)|dz$$

$$\leq \int_{(2B)^c} |b(x) - b(z)|^m |K_\alpha(x-z) - K_\alpha(y-z)||f(z)|dz$$

$$+ |b(x) - b(y)|$$

$$\times \sum_{k=0}^{m-1} \int_{(2B)^c} |b(x) - b(z)|^{m-1-k}|b(y) - b(z)|^k|K_\alpha(y-z)||f(z)|dz$$

$$= I_3 + I_4.$$

By the definition of $\Lambda(\delta)$, we get that

$$I_3 \lesssim \|b\|_{\Lambda(\delta)}^m \int_{(2B)^c} |x - z|^{\delta m}|K_\alpha(x-z) - K_\alpha(y-z)||f(z)|dz$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} |x - z|^{\delta m} \frac{|x-y|^\eta}{|x-z|^{n-\alpha+\eta}}|f(z)|dz$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty} \frac{2^{j\delta m}|B|^{\frac{\delta m}{n}}}{2^{j(n-\alpha+\eta)}|B|^{1-\frac{\alpha}{n}}} \int_{2^{j+1}B \setminus 2^j B} |f(z)|dz.$$

Then, by Hölder's inequality, the definition of $\tilde{\delta}$ and the fact that $w \in A_{r,\infty}$, we deduce that

$$I_3 \lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\tilde{\delta}}{n}} \sum_{j=1}^{\infty} 2^{j(\tilde{\delta}-\eta)} \frac{\|w^{-1}\chi_{2^{j+1}B}\|_{r'}}{|2^{j+1}B|^{1/r'}}$$

$$\lesssim \frac{\|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\tilde{\delta}}{n}}}{\|w\chi_B\|_\infty}.$$

In order to estimate I_4, we use that $b \in \Lambda(\delta)$ and $K_\alpha \in S_\alpha^*$ to get that

$$I_4 \lesssim \|b\|_{\Lambda(\delta)}|x-y|^\delta \|b\|_{\Lambda(\delta)}^{m-1}$$

$$\times \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|x-z|^{\delta(m-1-k)}|y-z|^{\delta k}}{|y-z|^{n-\alpha}}|f(z)|dz$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m |B|^{\frac{\delta}{n}} \sum_{j=1}^{\infty} |2^{j+1}B|^{\frac{\delta(m-1)}{n}-1+\frac{\alpha}{n}} \int_{2^{j+1}B \setminus 2^j B} |f(z)|dz.$$

Then, by Hölder's inequality and condition $w \in A_{r,\infty}$ we obtain

$$I_4 \lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\delta}{n}} \sum_{j=1}^{\infty} |2^{j+1}B|^{\frac{\delta(m-1)+\alpha}{n} - \frac{1}{r}} \frac{\|w^{-1}\chi_{2^{j+1}B}\|_{r'}}{|2^{j+1}B|^{1/r'}}$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{m\delta+\alpha}{n} - \frac{1}{r}} \|w\chi_B\|_\infty^{-1} \sum_{j=1}^{\infty} 2^{j\left(\frac{\delta(m-1)+\alpha}{n} - \frac{1}{r}\right)}$$

$$= \frac{\|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\widetilde{\delta}}{n}}}{\|w\chi_B\|_\infty} \sum_{j=1}^{\infty} 2^{j(\widetilde{\delta}-\delta)} \lesssim \frac{\|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\widetilde{\delta}}{n}}}{\|w\chi_B\|_\infty}$$

since $\widetilde{\delta} < \delta$.

Therefore,

$$A \lesssim \frac{\|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\widetilde{\delta}}{n}}}{\|w\chi_B\|_\infty}$$

which yields, after taking averages,

$$I_2 \lesssim \frac{\|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\widetilde{\delta}}{n}}}{\|w\chi_B\|_\infty},$$

and, thus,

$$\frac{\|w\chi_B\|_\infty}{|B|} \int_B |T_{\alpha,b}^m f(x) - a_B| dx \lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_r |B|^{\frac{\widetilde{\delta}}{n}}.$$

It remains to take supremum over all balls B to get the desired result. $\square$

P r o o f   o f   T h e o r e m 2.3. Let $B \subset \mathbb{R}^n$ be a ball and $x \in B$. Let $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$. Then,

$$T_{\alpha,b}^m f(x) - (T_{\alpha,b}^m f)_B = T_{\alpha,b}^m f_1(x) - (T_{\alpha,b}^m f_1)_B$$

$$+ \sum_{k=0}^m c_k \bigg[(b(x) - b_B)^{m-k} T_\alpha((b - b_B)^k f_2)(x)$$

$$- \frac{1}{|B|} \int_B (b(z) - b_B)^{m-k} T_\alpha((b - b_B)^k f_2)(z) dz \bigg].$$

We can rewrite the above identity in the following form

$$T_{\alpha,b}^m f(x) - (T_{\alpha,b}^m f)_B = \sigma_1(x) - (\sigma_1)_B$$

$$+ \sum_{k=0}^m c_k \left[\sigma_2(x,u,k) - (\sigma_2(\cdot,u,k))_B + \sigma_3(x,u,k) - (\sigma_3(\cdot,u,k))_B \right],$$

where

$$\sigma_1(x) = T_{\alpha,b}^m f_1(x),$$

$$\sigma_2(x, u, k) = (b(x) - b_B)^{m-k} \left(T_\alpha((b - b_B)^k f_2)(x) - T_\alpha((b - b_B)^k f_2)(u) \right),$$

$$\sigma_3(x, u, k) = (b(x) - b_B)^{m-k} T_\alpha((b - b_B)^k f_2)(u).$$

For σ_1, since $w \in A_{\frac{n}{m\delta + \alpha}, \infty}$, there exists $1 < p < \frac{n}{m\delta + \alpha}$ such that $w \in A_{p,\infty}$. We take $\frac{1}{q} = \frac{1}{p} - \frac{m\delta + \alpha}{n}$, so $q > p$ and $w \in A_{q,\infty}$ and, moreover, $w \in A_{p,q}$. By applying Hölder's inequality with q and q' and the boundedness of $T_{\alpha,b}^m$ from $L_w^p(\mathbb{R}^n)$ to $L_w^q(\mathbb{R}^n)$ (Theorem 2.1) we obtain that

$$\fint_B |\sigma_1(x)| dx \leq \frac{C}{|B|} \left(\int_B |T_{\alpha,b}^m f_1(x) w(x)|^q \right)^{1/q} \|w^{-1}\chi_B\|_{q'}$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \frac{\|f w \chi_B\|_p}{|B|} \|w^{-1}\chi_B\|_{q'}.$$

Since $\frac{1}{p} = \frac{\alpha + (m-1)\delta}{n} + \frac{1}{q} + \frac{\delta}{n}$, we can apply again Hölder's inequality and the fact that $w \in A_{q,\infty}$ to get

$$\fint_B |\sigma_1(x)| dx \lesssim \|b\|_{\Lambda(\delta)}^m \|f w\|_{\frac{n}{\alpha + (m-1)\delta}} |B|^{\delta/n} \frac{\|w^{-1}\chi_B\|_{q'}}{|B|^{1/q'}}$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \|f w\|_{\frac{n}{\alpha + (m-1)\delta}} |B|^{\delta/n} \|w \chi_B\|_\infty^{-1}.$$

In order to estimate σ_2 we use the inequality

$$\fint_B |b(x) - b_B|^{m-k} dx \leq \|b\|_{\Lambda(\delta)}^{m-k} |B|^{\frac{\delta(m-k)}{n}}$$

and Lemma 3.5 to obtain

$$\fint_B |\sigma_2(x, u, k)| dx \lesssim \|b\|_{\Lambda(\delta)}^k \frac{\|f w\|_{\frac{n}{\alpha + (m-1)\delta}}}{\|w \chi_B\|_\infty} |B|^{\frac{\delta(k-m+1)}{n}} \fint_B |b(x) - b_B|^{m-k} dx$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \|f w\|_{\frac{n}{\alpha + (m-1)\delta}} \|w \chi_B\|_\infty^{-1} |B|^{\delta/n}.$$

Consequently, since

$$\sum_{k=0}^m c_k [\sigma_3(x, u, k) - (\sigma_3(\cdot, u, k))_B] = \left(T_{\alpha,b}^m f(x) - (T_{\alpha,b}^m f)_B \right)$$

$$- (\sigma_1(x) - (\sigma_1)_B) - \sum_{k=0}^m c_k \left[\sigma_2(x, u, k) - (\sigma_2(\cdot, u, k))_B \right]$$

by first assuming that $T_{\alpha,b}^m f : L_w^{\frac{n}{\alpha + (m-1)\delta}} \hookrightarrow \mathbb{L}_w(\delta)$, then

$$\frac{1}{|B|} \int_B \left| \sum_{k=0}^m c_k \left[\sigma_3(x, u, k) - (\sigma_3(\cdot, u, k))_B \right] dx \right|$$

$$\leq \frac{1}{|B|} \int_B |T_{\alpha,b}^m f(x) - (T_{\alpha,b}^m f)_B| dx + \frac{2}{|B|} \int_B |\sigma_1(x)| dx$$

$$+ \sum_{k=0}^m c_k \frac{2}{|B|} \int_B |\sigma_2(x, u, k)| dx$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \|fw\|_{\frac{n}{\alpha+(m-1)\delta}} \|w\chi_B\|_\infty^{-1} |B|^{\delta/n}.$$

On the other hand, if we suppose that (2.2) holds, it is easy to see that $T_{\alpha,b}^m f : L_w^{\frac{n}{\alpha+(m-1)\delta}}(\mathbb{R}^n) \hookrightarrow \mathbb{L}_w(\delta)$. $\qquad \square$

P r o o f   o f   T h e o r e m 2.4. We proceed by induction. We must point out that the case $m = 0$ was already proved in [1]. As in the proof of Theorem 2.1 we have that

$$\|wT_{\alpha,b}^m f\|_q \lesssim \|wM_{0,\gamma}^\sharp(T_{\alpha,b}^m f)\|_q.$$

We shall now use the second part of Lemma 3.3, since we have that $K_\alpha \in S_\alpha \cap H_{\alpha,\Phi}$. Thus, we obtain that

$$\|wT_{\alpha,b}^m f\|_q \lesssim \sum_{j=0}^{m-1} \|wM_{\theta_j,\gamma}(|T_{\alpha,b}^j f|)\|_q + \|wM_{m\delta+\alpha,\widetilde{\Phi}} f\|_q,$$

where we have assumed, without loss of generality, that $\|b\|_{\Lambda(\delta)} = 1$.

From the hypotheses on the weight w and the Young function $\widetilde{\Phi}$, by Theorem 3.1 we know that $\|wM_{m\delta+\alpha,\widetilde{\Phi}} f\|_q \lesssim \|wf\|_p$. The proof now follows in the same way as the proof of Theorem 2.1. $\qquad \square$

P r o o f   o f   T h e o r e m 2.5. Take f, f_1, f_2 and a_B as in the proof of Theorem 2.2, and define I_1 and I_2 likewise.

Since in I_1 we have only used the size condition S_α^*, the estimation is the same, by taking into account that $w^\beta \in A_{r/\beta,\infty}$ yields $w \in A_{r,\infty}$ for any $\beta \geq 1$.

For I_2 we proceed similarly but we have to use now that $K_\alpha \in H_{\alpha,\Phi,m}(\delta)$ with $\Phi^{-1}(t) \lesssim t^{\frac{\beta-1}{r}}$ for some $1 < \beta < r$ and all $t > 0$. We split the average into I_3 and I_4 as in the proof of Theorem 2.2. The last one can be controlled in the same form. The difference will be in I_3. Recall that

$$I_3 = \int_{(2B)^c} |b(x) - b(z)|^m |K_\alpha(x - z) - K_\alpha(y - z)||f(z)| dz,$$

for $x \in B$.

By the definition of $\Lambda(\delta)$, we get that

$$I_3 \lesssim \|b\|_{\Lambda(\delta)}^m \int_{(2B)^c} |x - z|^{\delta m} |K_\alpha(x - z) - K_\alpha(y - z)||f(z)| dz$$

$$\lesssim \|b\|_{\Lambda(\delta)}^m \sum_{j=1}^{\infty} |2^{j+1}B|^{\frac{\delta m}{n}} \int_{2^{j+1}B \setminus 2^j B} |K_\alpha(x-z) - K_\alpha(y-z)||f(z)|dz.$$

Now, since $K_\alpha \in H_{\alpha,\Phi,m}(\delta)$, $w^\beta \in A_{r/\beta,\infty}$ and $\Phi^{-1}(t)t^{1/r}t^{1/(r/\beta)'} \lesssim t$, we can prodeed as in (3.5) with $k = m$ to obtain

$$I_3 \lesssim \frac{\|b\|_{\Lambda(\delta)}^m |B|^{\frac{\tilde{\delta}}{n}} \|fw\|_r}{\|w\chi_B\|_\infty}.$$

$\square$

P r o o f   o f   T h e o r e m 2.6. We proceed as in the proof of Theorem 2.3. We must only use the corresponding hypothesis on the kernel, that guarantees the validity of Theorem 2.4 and Lemma 3.6, which are immediate from the fact that $S_\alpha^* \cap H_{\alpha,\Phi,m}(\delta) \subset S_\alpha^* \cap H_{\alpha,\Phi}$ (see Remark 2.6). $\square$

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