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2013 J. Phys. B: At. Mol. Opt. Phys. 46 065202

(http://iopscience.iop.org/0953-4075/46/6/065202)

J. Phys. B: At. Mol. Opt. Phys. **46** (2013) 065202 (6pp) doi:10.1088/0953-4075/46/6/065202

A Jost function description of zero-energy resonance and transparency effects in elastic collisions

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Received 18 December 2012, in final form 7 February 2013
Published 1 March 2013
Online at stacks.iop.org/JPhysB/46/065202

Abstract
Under certain circumstances, an elastic cross section at very low energies could differ by orders of magnitude above or below any reasonable estimate. The first case occurs near a zero-energy resonance, while the second one is akin to the well-known Ramsauer–Townsend effect. In spite of their intrinsic similarity, both effects are usually explained in very different ways, either in terms of poles of the scattering matrix or by means of a partial-wave analysis, respectively. In this paper we show that a unified description is actually possible. We demonstrate that in a zero-energy resonance or transparency, the standard $\epsilon^{2\ell}$ threshold law of the ℓ-wave cross section is changed into $\epsilon^{2(\ell-1)}$ (ϵ^{-1} for $\ell = 0$) or $\epsilon^{2(\ell+1)}$, respectively. Finally, we show that while the zero-energy resonance occurs whenever the scattering length associated to an individual zero of the Jost function diverges, the transparent scattering is a collective effect, where the sum of all the individual scattering lengths plays a relevant role.

(Some figures may appear in colour only in the online journal)

1. Introduction

In the early 1920s, the low-energy electron collision experiments developed by Ramsauer (1921a, 1921b) and the electron swarm experiments pioneered by John Sealy Townsend and his DPhil student Victor Bailey (Townsend and Bailey 1921, 1922a, 1922b, 1922c) led to the independent discovery of a pronounced minimum exhibited by the total cross section for the scattering of electrons by argon at energies of the order of 1 eV. As Ramsauer stated in his talk at the *Naturforscherversammlung* at Bad Nauheim in September 1920, 'while (the cross section of) other gases—as far as they have been investigated to date—approach a constant value with decreasing velocity of the beam, the cross section of argon becomes extraordinarily small' (Mehra and Rechenberg 2000). Subsequent measurements by Ramsauer himself (Ramsauer 1923, Ramsauer and Kollath 1929) extended this discovery to other noble gases, showing that the cross sections observed at the minimum were two orders of magnitude smaller than estimates obtained from measurements in gas kinetic experiments. It was not until the end of the 1920s, when Schrödinger's formalism of quantum mechanics had been established (Schrödinger 1926), that Holtsmark (1929, 1930) managed to explain this marked transparency of rare gases over a small range of low energies by means of a partial-wave description of scattering processes, as developed by Faxén and himself (Faxén and Holtsmark 1927) a few years before.

On the following decade, the first measurements of the scattering of slow neutrons by liquid hydrogen (Halpern *et al* 1937, Brickwedde *et al* 1938) showed that the cross section of *ortho*-hydrogen was several times larger than that of *para*-hydrogen. Subsequent measurements with gaseous targets (Alvarez and Pitzer 1940, Sutton *et al* 1947) finally led to a full confirmation of these early findings (see, e.g. Squires and Stewart (1953)). These results did not only confirm the feasibility of a method proposed by Schwinger and Teller (1937) for determining the triplet and singlet scattering amplitudes, but also that some kind of zero-energy resonance was occurring in the latter due to the presence of a virtual state.

These two experimental findings showed that under certain circumstances, an elastic cross section at very low energies could differ by orders of magnitude from any reasonable estimate of its geometrical size. In recent years, this fact has recovered its original relevance due to its importance in the study of ultracold atomic collisions, and the thrust experienced in this area by the development of laser cooling and trapping techniques.

It might be reasonable to assume that both the zero-energy resonance and the low-energy transparency effects are prone for a unified description. However, in most of the quantum mechanics textbooks, as for instance in those by Schiff (1949), Merzbacher (1961), Messiah (1964) or Gasiorowicz (1974), just to mention some of the most traditional and well-known books, the theoretical explanations of these two effects seem to run over separate paths. On one side, even though it might be rather cumbersome to exactly determine the Ramsauer–Townsend minimum, the low-energy transparency effect can be easily explained by means of a partial wave analysis (Bohm 1951). On the other hand, the zero-energy resonance effect is generally explained in terms of poles of the scattering matrix (Newton 1966).

The aim of this paper is to explore the viability of a unified description of the zero-energy resonance and the low-energy transparency effects in terms of the Jost function. Actually, it is well-known that the function introduced by Jost in 1947 (Jost 1947, Jost and Pais 1951) provides a natural and very convenient framework for dealing with the zero-energy resonance effect. Being analytically continued to complex values of the impulse, its zeros are shown to be univocally related to bound, virtual and resonant states of the corresponding Hamiltonian. In particular, this formalism helps to explain the appearance of a zero-energy resonance as an emerging effect produced by a zero of the s-wave Jost function near the origin of the impulse plane, i.e. a bound or virtual state with a very small energy. In contrast, since the transparency effect is clearly unrelated to any single zero of the Jost function, a similar explanation in terms of Jost functions might seem impossible or at least impractical. Here we will show that this is not the case, and that an explanation of the low-energy transparency in terms of the Jost function is possible and that it can provide some new insights on two effects that rank among the most interesting manifestations of the quantum world.

2. Definition of the Jost function

Let us start by summarizing some relevant and basic concepts of the Jost function description of non-relativistic scattering processes as given, for example, by Taylor (1972). Consider a two-particle system of reduced mass m moving with relative energy $E = \hbar^2 p^2/2m$ and angular momentum ℓ. The interaction between the particles is provided by a real spherically symmetric potential $V(\mathrm{r})$, where r is the distance among them. The corresponding radial Schrödinger equation reads

$$\left(\frac{\mathrm{d}^2}{\mathrm{dr}^2} + p^2 - \frac{\ell(\ell+1)}{\mathrm{r}^2} - \frac{2m}{\hbar^2}V(\mathrm{r})\right)\psi_\ell(p,\mathrm{r}) = 0.$$

Now we employ a length R, characteristic of the potential $V(\mathrm{r})$, to scale this equation as follows

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - U(r)\right)\psi_\ell(k,r) = 0,$$

where we have defined the dimensionless distance $r = \mathrm{r}/R$, wave number $k = pR$ and potential

$$U(r) = 2mR^2\gamma\, V(\mathrm{r})/\hbar^2.$$

Here we have arbitrarily introduced a coupling parameter γ, which for the time being we set equal to 1. Let us assume that the potential $U(r)$ is continuous for $r > 0$, except perhaps at a finite and discrete set of finite discontinuities, and satisfies the conditions $r^3 U(r) \to 0$ when $r \to \infty$ and $r^{3/2} U(r) \to 0$ when $r \to 0$, which guarantee that the bound spectrum of $U(r)$ is non-degenerate and finite in number and that the standard assumptions of the scattering theory apply, as described for instance by Taylor (1972).

We introduce regular solutions $\psi_\ell(k,r)$ with the condition that for $r \to 0$ they have to behave as the free regular solution, namely,

$$\psi_\ell(k,r) \approx \frac{(kr)^{\ell+1}}{(2\ell+1)!!}.$$

It can be shown that for $r \to \infty$, the regular solution $\psi_\ell(k,r)$ approaches a linear combination of free solutions

$$\psi_\ell(k,r) \approx \frac{\mathrm{i}}{2}\left[f_\ell(k)\,\mathrm{e}^{\mathrm{i}(kr-\ell\pi/2)} - f_\ell(-k)\,\mathrm{e}^{-\mathrm{i}(kr-\ell\pi/2)}\right], \quad (1)$$

where $f_\ell(k)$ is the so-called ℓ-wave Jost function of impulse k. Since the Schrödinger equation and the boundary conditions are real, so is the regular wave function $\psi_\ell(k,r)$, and therefore the Jost function verifies $f_\ell(-k) = f_\ell(k)^*$ for real values of k. Comparing with the asymptotic form of any solution of the radial Schrödinger equation

$$\psi_\ell(k,r) \propto \mathrm{e}^{\mathrm{i}(kr-\ell\pi/2)} + s_\ell(k)\,\mathrm{e}^{-\mathrm{i}(kr-\ell\pi/2)},$$

we obtain the following relation with the S matrix elements,

$$s_\ell(k) = \frac{f_\ell(-k)}{f_\ell(k)} = \mathrm{e}^{-2\mathrm{i}\delta_\ell(k)},$$

where we have defined the phase shifts

$$\delta_\ell(k) = -\arg f_\ell(k).$$

It is clear that through these phase shifts, the Jost functions contain all of the information about the asymptotic behaviour of the radial wave function. For instance, the elastic cross section is given by

$$\sigma(k) = \sum_{\ell=0}^{\infty}\sigma_\ell(k),$$

where the partial-wave cross sections $\sigma_\ell(k)$ are usually written as

$$\sigma_\ell(k) = \frac{4\pi R^2(2\ell+1)}{k^2}\sin^2\delta_\ell(k), \quad (2)$$

or, in terms of the Jost functions,

$$\sigma_\ell(k) = \frac{4\pi R^2(2\ell+1)}{k^2}\left(\frac{\mathrm{Im} f_\ell(k)}{|f_\ell(k)|}\right)^2, \qquad (3)$$

which implies the so-called *unitarity limit* (Newton 1966)

$$\sigma_\ell(k) \leqslant \frac{4\pi R^2(2\ell+1)}{k^2}. \qquad (4)$$

In particular, we see that $\sigma_\ell(k)$ reaches its maximum value when $\delta_\ell(k) = \pi/2$ (modulo π), i.e. Re $(f_\ell(k)) = 0$. Comparing the two expressions (2) and (3) for $\sigma_\ell(k)$ in terms of the phase shift $\delta_\ell(k)$ and the Jost function $f_\ell(k)$ respectively, it might be thought that the former is all that is needed to fully develop a non-relativistic scattering theory and that the last expression—which also involves the modulus of the Jost function—is somehow unnecessary. We can say that as long as we are interested in elastic processes, this observation seems to be fair and sound. However, as it will be shown in the following sections, equation (3) will provide us with a much better starting point for the description of the zero-energy resonance and transparency effects.

To date, we have considered the impulse k as a real physical observable. However, one of the most powerful techniques in scattering theory comes from extending physical (real) parameters to the complex plane. Two important properties of the extended Jost function are relevant for the following sections (Taylor 1972). Firstly, the Jost function has a kind of reflection symmetry about the imaginary axis,

$$f_\ell(-k^*) = f_\ell(k)^*, \qquad (5)$$

which is an extension to the whole k-plane of the relation $f_\ell(-k) = f_\ell(k)^*$ for real k mentioned before. The above equation implies that the Jost function is real onto the imaginary axis. Secondly, if k_o is a zero of $f_\ell(k)$, then $-k_o$ cannot be a zero at the same time, namely $f_\ell(k_o) = 0 \Rightarrow f_\ell(-k_o) \neq 0$; otherwise the regular solution will be identically zero for such a value of k.

3. Zeros of the Jost function

If the Jost function has a zero in the upper half of the k plane, we see in equation (1) that the regular solution decreases exponentially for $r \to \infty$. This means that $\psi_\ell(k_o, r)$ is a bound solution of the radial Schrödinger equation with energy $E_o = \hbar^2 k_o^2/2mR^2$. Since this energy has to be real and negative, k_o has to be located onto the imaginary axis. Similarly, if $\psi_\ell(k_o, r)$ represents a bound state of energy $E_o = -\hbar^2|k_o|^2/2mR^2$, then it has to be exponentially bounded, and equation (1) implies that $f_\ell(\mathrm{i}|k_o|) = 0$. In other words, there is a one-to-one relation between zeros of the Jost function $f_\ell(k)$ in the positive imaginary axis and ℓ-wave bound states.

If we reduce the coupling parameter γ, the potential becomes less attractive, and each zero would move downwards along the imaginary axis and eventually reach the origin. From then on, it can be shown (Taylor 1972) that the zero will continue along the negative imaginary axis if $\ell = 0$ or move into the fourth quadrant tangentially to the real axis if $\ell > 0$. These zeros no longer represent bound states, and are referred to as virtual and resonant states, respectively. For certain potentials and for a decreasing coupling parameter, even a virtual state might also reach a point below the origin where it would also move into the fourth quadrant.

Due to equation (5), for each zero on the fourth quadrant, there has to be another one on the third quadrant symmetrically located about the imaginary axis. Note that when the coupling parameter γ is decreased so that a zero associated to a bound or a virtual state reaches a point where it is about to abandon the imaginary axis, that same point has to be reached by another zero moving upwards. Both zeros have to meet in order to move apart from the imaginary axis 'symmetrically'.

4. Effective range expansion

To understand the behaviour of the cross section at low energies, in particular the zero-energy resonance and transparency effects we are interested in, we need to expand the Jost function in powers of k. Under certain restrictive conditions on the potential $U(r)$, the Jost function can be written as

$$f_\ell(k) = g_\ell(k) + \mathrm{i}k^{2\ell+1}h_\ell(k),$$

where $g(k)$ and $h(k)$ are even functions in k. Expanding them in powers of k, we obtain

$$f_\ell(k) = [g_{0\ell} + g_{2\ell}k^2 + O(k^4)] + \mathrm{i}k^{2\ell+1}[h_{0\ell} + h_{2\ell}k^2 + O(k^4)]\,. \qquad (6)$$

This equation resembles the well-known effective range expansion for $\cot\delta_\ell(k)$ as first introduced in nuclear scattering by Bethe (1949) on the basis of an idea by Schwinger (1947)

$$(a_\ell\, k)^{2\ell+1}\cot\delta_\ell(k) = -1 + \tfrac{1}{2} r_\ell\, a_\ell\, k^2 + O(k^4). \qquad (7)$$

Here we have modified the standard definition (Taylor 1972) of the scattering length a_ℓ for $\ell \neq 0$ so as to make it actually a length (Macri and Barrachina 2002). Except for that, equation (7) coincides with the standard effective range expansion, where the scattering length a_ℓ and the effective range r_ℓ can be written in terms of the real coefficients in equation (6) as follows,

$$a_\ell = \left(\frac{h_{0\ell}}{g_{0\ell}}\right)^{1/(2\ell+1)} \quad \text{and} \quad r_\ell = \frac{2}{a_\ell}\left(\frac{h_{2\ell}}{h_{0\ell}} - \frac{g_{2\ell}}{g_{0\ell}}\right).$$

Note that only two parameters, a_ℓ and r_ℓ, are relevant for the calculation of the phase shift up to the second order in k, while we have included four parameters in the expansion of the Jost function itself. This might seem as an unjustified redundancy since both (6) and (7) are second order expansions in k. Thus, the same order of approximation would be reached when they are inserted in (2) or (3) and, in principle, the two expansions would provide similar results. However this is not always the case. The reason is that the scattering length a_ℓ and the effective range r_ℓ might present singularities, while the coefficients in the effective range expansion (6) of the Jost function do not. As we shall see in a moment, this simple difference implies that the extra parameters are not only relevant but absolutely essential to describe the effects we are interested in.

5. Zero-energy resonances

It is clear that the Jost function vanishes at $k = 0$ each time a bound state is at the verge of becoming a virtual state (for $\ell = 0$) or a resonance (for $\ell \neq 0$). In terms of the effective range expansion, this means that the coefficient $g_{0\ell}$ vanishes, and therefore the scattering length a_ℓ diverges. As can be seen from equation (6) in the previous section, this zero is simple for s-waves and double for $\ell > 0$. So we have to analyse these two cases separately. For s-waves, the Jost function reads $f_0(k) = g_{00} + ikh_{00} + g_{20}k^2 + o(k^3)$ and replacing in the partial cross section (3) we get

$$\sigma_0(k) \approx 4\pi R^2 \frac{{h_{00}}^2 + 2h_{00}h_{20}k^2}{{g_{00}}^2 + ({h_{00}}^2 + 2g_{00}g_{20})k^2}, \tag{8}$$

which, in the low-energy limit reads

$$\sigma_0(k) \approx 4\pi R^2 \frac{h_{00}^2}{g_{00}^2} = 4\pi R^2 a_0^2.$$

Now, for $g_{00} = 0$ (i.e. $a_0 = h_{00}/g_{00} \to \infty$), equation (8) diverges like

$$\sigma_0(k) \approx \frac{4\pi R^2}{k^2},$$

i.e. the unitarity limit equation (4) is reached. Note that equation (8) cannot be written in terms of the coefficients a_ℓ and r_ℓ alone. The same occurs for $\ell > 0$, namely

$$\sigma_\ell(k) \approx 4\pi R^2 (2\ell + 1) \frac{{h_{0\ell}}^2 k^{4\ell}}{(g_{0\ell} + g_{2\ell}k^2)^2}.$$

For $g_{0\ell} \neq 0$ the partial cross section behaves like

$$\sigma_\ell(k) \approx 4\pi R^2 (2\ell + 1) \left(\frac{h_{0\ell}}{g_{0\ell}}\right)^2 k^{4\ell} = 4\pi (a_\ell R)^2 (2\ell + 1)\, (a_\ell k)^{4\ell},$$

which is nothing else but the elastic scattering version of the celebrated threshold law founded by Wigner (1948) for the energy dependence of the cross sections close to the opening of an inelastic channel. However, whenever $g_{0\ell} = 0$, the ℓ-wave cross section reads

$$\sigma_\ell(k) \approx 4\pi R^2 (2\ell + 1) \left(\frac{h_{0\ell}}{g_{2\ell}}\right)^2 k^{4(\ell-1)},$$

which goes to zero slower than in the absence of a resonance. Actually, for the $\ell > 0$ cases, the threshold behaviour of the ℓ-wave cross section changes to that of an '$\ell - 1$' wave. Note, however, that this modification of the regular $k^{4\ell}$ behaviour of the ℓ partial cross section does not strictly represents a resonance since the unitarity limit (4) is not reached unless the whole real part of the Jost function vanishes, i.e. $g_\ell = 0$. In particular, the previous equation violates equation (4) for $k > (g_{2\ell}/h_{0\ell})^{2/(4\ell-2)}$, thus establishing a limit for its validity. Even then, the validity of the unitarity limit can be extended by the simple prospect of including one extra term in the previous equation, namely

$$\sigma_\ell(k) \approx 4\pi R^2 (2\ell + 1) \frac{{h_{0\ell}}^2 k^{4\ell}}{(g_{0\ell} + g_{2\ell}k^2)^2 + (h_{0\ell}k^{2\ell+1})^2},$$

so that, whenever $g_{0\ell} = 0$, the ℓ-wave cross section reads

$$\sigma_\ell(k) \approx 4\pi R^2 (2\ell + 1) \frac{k^{4(\ell-1)}}{(g_{2\ell}/h_{0\ell})^2 + k^{4\ell-2}}.$$

It is important to point out that this kind of effect can also be observed in inelastic collisions. As we already mentioned, in 1948 Wigner demonstrated that the energy dependence of cross sections near the threshold of an opening reaction channel with ℓ symmetry is governed by the celebrated threshold law $\sigma_\ell \propto k^{2\ell+1}$. However, in order to deal with some deviations which were observed in recent experiments, the following generalization was proposed (Macri and Barrachina 2003, see also Barrachina and Macri (2004))

$$\sigma_\ell \propto \frac{k^{2\ell+1}}{|f_\ell(k)|^2},$$

where $f_\ell(k)$ is the elastic ℓ-wave Jost function corresponding to the final state interaction. If the system is far from a resonance, the Jost function is nearly constant and the numerator gives the usual Wigner law. Else, the Jost function depends strongly on k and the denominator describes the observed deviations. In particular, at a zero-energy resonance, the Wigner threshold law drastically changes from a $k^{2\ell+1}$ to a $k^{2\ell-3}$ dependence for $\ell > 0$ and a $k^{2\ell-1}$ dependence for $\ell = 0$.

6. Zero-energy transparency

The ℓ-wave scattering length a_ℓ changes its sign regularly when the control parameter γ varies. Actually it is positive for a bound state (i.e. when the zero is in the positive imaginary axis) and negative otherwise. Therefore there have to be values of γ for which it vanishes. In this work, we will show that when this occurs for $\ell = 0$, the s-wave partial cross section goes to zero instead of reaching a constant value at $k = 0$; while $\sigma_\ell(k)$ vanishes faster than $k^{4\ell}$ whenever $a_\ell = 0$. This effect was partially discussed by Greenhow (1993) but only for the particular case of the square-well potential and with no relation to the scattering length neither to the Jost function description employed here. Actually, we show that this effect occurs whenever the coefficient of lowest order in the imaginary part of the Jost function, $h_{0\ell}$, vanishes, and so does a_ℓ. For a general potential, the partial cross section at the energy threshold reads

$$\sigma_\ell(k) = 4\pi R^2\ (2\ell + 1) \left(\frac{h_{2\ell}}{g_{0\ell}}\right)^2 k^{4(\ell+1)}.$$

Let us point out that, again, it is not possible to express this result in terms of the effective range r_ℓ since it diverge like ${a_\ell}^{-2(\ell+1)}$ for $a_\ell \to 0$. Actually it can be easily shown that $r_\ell\, {a_\ell}^{2(\ell+1)}$ converges to $2(g_{0\ell}h_{2\ell} - g_{2\ell}h_{0\ell})/g_{0\ell}^2$ in this limit. Thus, when $h_{0\ell}$ vanishes the ℓ-wave cross section has an $\ell + 1$ wave behaviour and, in particular, the partial cross section σ_0 vanishes along with the other ℓ-wave components. This implies that the scattering might be negligible at very low energies, i.e. the particles become 'transparent' to each other.

Let us investigate how we can understand this effect by means of the zeros k_n of the Jost function on the complex k plane. We consider a potential that vanishes beyond some finite range. Let us point out that the same conclusions could be reached for more general interactions, but with more involved math. For s-waves we can write

$$f_0(k) = \tilde{f}_0(k) \prod_{n=1}^{N} (k_n - k),$$

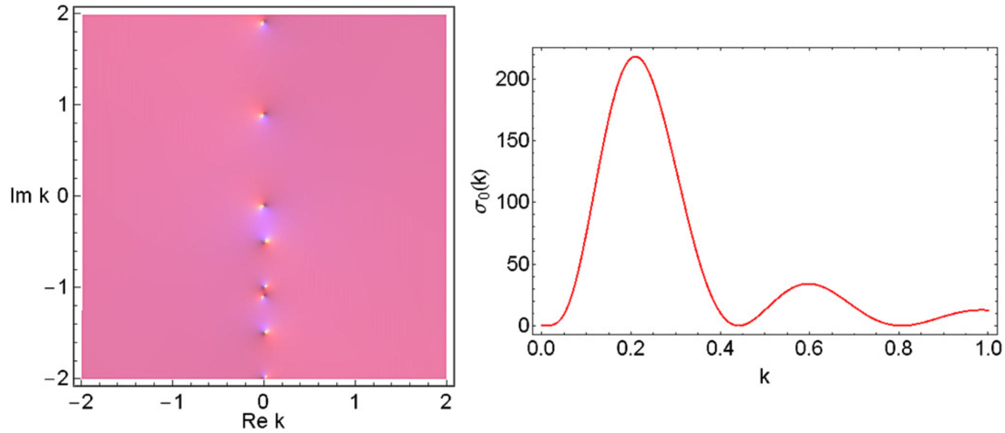

Figure 1. Left panel: modulus of the s-wave Jost function for the Hulthén potential with a coupling parameter $\gamma = \sqrt{2mZR} \sim 100.91$ in logarithmic scale showing its zeros and poles on the complex k-plane. Right panel: corresponding s-wave partial cross section.

where we have singled out all of the N (possibly infinite) zeros of the Jost function. An expansion up to the first order in k reads

$$f_0(k) \propto 1 + \mathrm{i}\tilde{a}_0 k - \left(\sum_{n=1}^{N} \frac{1}{k_n}\right) k.$$

Here we have defined the reduced scattering length $\tilde{a}_0 = -\mathrm{i}\,\mathrm{d}\ln\tilde{f}_0/\mathrm{d}k|_{k=0}$. By applying equation (5) it can be shown that this quantity is real. Similarly, since all these zeros are always located on the imaginary axis or symmetrically on the third and fourth quadrant, it is easily shown that their sum is imaginary. Thus, we characterize the contribution of each zero k_n by means of an 'individual' scattering length $\tilde{a}_n = -\mathrm{Im}\,(1/k_n)$, so that

$$f_0(k) \propto 1 + \mathrm{i}\sum_{n=0}^{N} \tilde{a}_n\, k,$$

which, upon comparing with equation (6) yields that the scattering length a_0 in the effective range expansion is related to the sum of the individual scattering lengths associated to the zeros of the Jost function in the complex k plane, namely,

$$a_0 = \sum_{n=0}^{N} \tilde{a}_n.$$

We see that while the zero-energy resonance is a kind of single-zero effect, which occurs whenever any of the individual scattering length diverges, this 'transparent' scattering is a collective effect, where all the individual scattering lengths add to zero. Let us succinctly investigate how this mechanism works. We start by assuming that all the zeros of the Jost function are located below the real axis, a situation that occurs for a coupling parameter γ that is small enough to prevent the potential from sustaining bound states. The collective contribution $\sum_{n>1} \tilde{a}_{n\neq 0}$ of all these zeros to the scattering length is negative. In return, the non-resonant part $\tilde{a}_0$ is positive, but its contribution is not enough for a_0 to vanish (except, of course, at $\gamma \to 0$). Thus for the zero-energy transparency to occur, the potential has to sustain one (or more) bound states so that the corresponding zeros on the positive imaginary axis equilibrate those in the lower half of the k plane. In other words, this effect would never occur if the potential cannot sustain bound states. Note that a similar provision is necessary for the Ramsauer–Townsend effect. In the presence of a given number n_0 of s-wave bound states, $\delta_0(0) - \delta_0(\infty) = n_0\pi$ in accordance with the theorem demonstrated by Levinson in 1949 (Levinson 1949). Thus, for $n_0 > 1$, this s-wave phase shift would equal a multiple of π at some given value of k. Then, if this impulse is close enough to threshold, where all the other partial cross sections are negligible, the Ramsauer–Townsend effect will occur.

7. Does a Jost function's zero close to the origin guarantee a zero-energy resonance?

As we mentioned before, in scattering theory low-lying resonances are explained as poles of the S-matrix or, alternatively, as zeros of the Jost function close to the origin of the complex k-plane (Newton 1966, Taylor 1972). However, as we will show here, this can be a highly misleading conception. Let us consider, for instance, an attractive ($Z > 0$) Hulthén potential

$$V(\mathrm{r}) = -\frac{Z}{R}\frac{1}{\exp(\mathrm{r}/R) - 1}.$$

On the left panel of figure 1 we show the modulus of the s-wave Jost function $f_0(k)$ in logarithmic scale, over the complex k-plane for $\gamma = \sqrt{2mZR} \sim 100.91$. The zeros and poles of the Jost function are located at

$$k_n = -\frac{\mathrm{i}}{2}\left(n - \frac{\gamma^2}{n}\right) \qquad \text{and} \qquad k_n = -\frac{\mathrm{i}}{2}n,$$

respectively, with n a natural number. The presence of a zero at the position $k_{101} \sim -\mathrm{i}\,0.09$ with a corresponding single scattering length $\tilde{a}_{101} = -\mathrm{Im}(1/k_{101}) \sim -10.7$ might be anticipating an important enhancement of $\sigma_0(k)$ close to the energy threshold. However, on the right panel of figure 1 we show that this prediction is completely wrong. In fact, the cross section displays a 'transparency' effect, with a s-wave scattering length equal to zero which is clearly far from the previous estimate. Of course, whenever a zero of the s-wave Jost function is 'exactly' located at the origin, $\sigma_0(k)$ diverges at $k = 0$. However, we see that the presence of a zero in the vicinity of the origin does not guarantee a resonance at zero energy. However, as can be seen in the figure, the destructive interference provided by neighbours zeros cannot be sustained for every k and the cross section peaks at $k \sim 0.2$. Therefore, we can see that the proper description of both the (almost) zero-energy resonance and transparency effects require the analysis of the 'collective' contributions of all of the zeros of the Jost function.

8. Conclusions

The early explanation by Schwinger and Teller (1937) of the n–p singlet scattering in terms of a virtual state demonstrated the valuable insight that can be achieved by studying the relation between zeros of the Jost function and the parameters of an effective range expansion. In particular, the knowledge of these low-energy coefficients is relevant for the analysis of ultracold atomic and molecular collisions where, for instance, the sign of the s-wave scattering length controls the stability of an entire Bose–Einstein condensate. In this context, the study of those situations where this scattering length diverges or vanishes is relevant and of actual importance.

In this paper we have presented a unified description of the zero-energy resonance and transparency effects in an elastic scattering process in terms of the Jost function. As it is well-known, the presence of a single zero of the Jost function at the origin is responsible for the resonance. Here we demonstrated that the transparent scattering can also be explained in a similar way, but more as a collective effect, where all the individual bound, virtual and resonant states contribute so that the sum of the individual scattering lengths cancels. This theoretical framework lets us show that for the transparency effect to occur, the potential has to sustain both bound 'and' virtual or resonant states. Furthermore, we proved that in a zero-energy resonance or transparency, the standard threshold dependence of the ℓ-wave cross section is changed into that of an $\ell - 1$ or $\ell + 1$ partial wave, respectively. We showed that although a zero-energy resonance is always associated to a zero of the Jost function close to the origin, the inverse proposition is not generally true. The presence of a zero near the origin does not guarantee a resonance effect. In this situation it is even possible to find any other behaviour such as the contrasting transparency effect we presented. Let us finally point out that all these results are based on very general grounds and are basically independent of the particularities of the collision itself.

Acknowledgments

This work was supported by the Consejo Nacional de Investigaciones Científicas y Técnicas (grant no PIP 112-200801011269) and Universidad Nacional de Cuyo (grant 06/C348). The authors are also members of the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina.

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