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2016 EPL 113 50005

(http://iopscience.iop.org/0295-5075/113/5/50005)

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

March 2016

EPL, **113** (2016) 50005
www.epljournal.org
doi: 10.1209/0295-5075/113/50005

Dissipative effects in nonlinear Klein-Gordon dynamics

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received 10 January 2016; accepted in final form 22 March 2016
published online 1 April 2016

PACS 05.90.+m – Other topics in statistical physics, thermodynamics, and nonlinear dynamical systems
PACS 02.30.Jr – Partial differential equations
PACS 03.50.-z – Classical field theories

Abstract – We consider dissipation in a recently proposed nonlinear Klein-Gordon dynamics that admits exact time-dependent solutions of the power-law form $e_q^{i(kx-wt)}$, involving the q-exponential function naturally arising within the nonextensive thermostatistics ($e_q^z \equiv [1+(1-q)z]^{1/(1-q)}$, with $e_1^z = e^z$). These basic solutions behave like free particles, complying, for all values of q, with the de Broglie-Einstein relations $p = \hbar k$, $E = \hbar\omega$ and satisfying a dispersion law corresponding to the relativistic energy-momentum relation $E^2 = c^2p^2 + m^2c^4$. The dissipative effects explored here are described by an evolution equation that can be regarded as a nonlinear generalization of the celebrated telegraph equation, unifying within one single theoretical framework the nonlinear Klein-Gordon equation, a nonlinear Schrödinger equation, and the power-law diffusion (porous-media) equation. The associated dynamics exhibits physically appealing traveling solutions of the q-plane wave form with a complex frequency ω and a q-Gaussian square modulus profile.

Introduction. – The spatio-temporal behavior of a wide family of physical systems and processes is described by nonlinear partial differential equations [1–3]. This has stimulated an increasing research activity on the dynamics associated with a class of evolution equations that includes nonlinear versions of the Schrödinger [3–5] and the Fokker-Planck [6–12] ones. Our main focus here will be on a family of telegraph-like equations describing dissipative effects in the context of a recently advanced nonlinear Klein-Gordon dynamics (NLKGD) [4] related to nonextensive statistical mechanics and the associated nonadditive entropies [13–15]. The free-particle nonlinear Klein-Gordon equation proposed in [4] has a nonlinearity in the mass term which, in contrast to what happens in the standard linear case, is proportional to a power of the wave function $\Phi(x,t)$. The salient feature of the NLKGD introduced in [4] is that it exhibits localized solutions where the space-time dependence of the wave function $\Phi(x,t)$ occurs solely through the combination $x - vt$. Consequently, one has a space translation at a constant velocity v without change in the wave function's shape. These solutions are known as q-plane waves and are compatible, for all values of q, with the Planck and de Broglie relations, satisfying $E = \hbar\omega$ and $p = \hbar k$, with $E^2 = c^2p^2 + m^2c^4$. It was shown in [4] that there is also a nonlinear Schrödinger equation (often referred to as the NRT Schrödinger equation) with a nonlinearity in the Laplacian term, that also admits q-plane wave solutions, which are compatible with the nonrelativistic relation $E = p^2/2m$. Under Galilean transformations, the q-plane wave solutions of the NRT Schrödinger equation recover the transformations rules of the linear Schrödinger equation [16]. The NRT equation satisfied by the q-plane waves can be obtained from a field theory based upon an action variational principle [17]. These properties suggest that the q-plane wave solutions of both the nonlinear NRT Schrödinger and the Klein-Gordon equations studied in [4] can be regarded as a new field-theoretical description of particle dynamics that may be relevant in diverse areas of physics, including nonlinear optics, superconductivity, and plasma physics [17,18].

As already mentioned, the dissipative nonlinear Klein-Gordon dynamics that we are going to explore here is described by a family of telegraph-like equations. The standard telegraph equation constitutes a cornerstone of mathematical physics, with deep theoretical significance and many applications [19–28]. Historically the telegraph equation was first formulated to describe leaky electrical transmission lines. In one dimension it has the form

$$\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}-\frac{\partial^2\Psi}{\partial x^2}+\delta\frac{\partial\Psi}{\partial t}=0. \quad (1)$$

This equation corresponds to phenomena intermediate between wave propagation and diffusion. It can also be regarded as a wave equation with a damping effect described by the term having the first time derivative. It admits a statistical interpretation in terms of a Poisson process (dichotomous diffusion) associated with particles that move with constant speed and change the direction of motion at random times [20–22]. It has profound (and surprising) connections with quantum mechanics, being intimately related to the Dirac equation [21]. The applications of the telegraph equation are diverse. We can mention correlated random walks [19], tunneling processes [23], diffusion phenomena in optics [24,25], and cosmic-ray transport [26,27].

q-plane waves and nonlinear evolution equations. – The q-plane waves arise naturally within a theoretical framework where the Boltzmann-Gibbs (BG) entropy and statistical mechanics are generalized through the introduction of a power-law entropic functional S_q characterized by an index q (the BG entropy being recovered in the limit $q \to 1$). Recent progress along these lines of enquiry includes, for instance, nonlinear extensions of various important equations of physics and new forms of the Central Limit Theorem [29]. The q-Gaussian distributions, which generalize the standard Gaussian distribution and arise from the optimization of the q-entropy [13], or as solutions of the corresponding nonlinear Fokker-Planck equation [10], play a central role within these developments. They have found several interesting applications to the analysis of recent experimental findings [14]. These applications concern diverse physical systems including, among others, i) cold atoms in dissipative optical lattices [30]; ii) quasi–two-dimensional dusty plasmas [31]; iii) ions in radio frequency traps interacting with a buffer gas [32]; iv) RKKY spin glasses, like CuMn and AuFe [33]; v) overdamped motion of vortices in type-II superconductors [7,9,34]. More generally, q-exponential distributions have also been applied to several physical scenarios. As recent examples we can mention the description of the transverse momentum spectra in high-energy proton-proton and proton-antiproton collisions [35], heat baths with a finite number of effective degrees of freedom [36], universal financial [37] and biological [38] laws, among others.

Of the three nonlinear dynamical equations admitting q-plane wave solutions advanced in [4] (the NRT Schrödinger, and the q-nonlinear Klein-Gordon and Dirac equations) the NRT Schrödinger equation is the one that has been more intensively studied so far. Recent advances along these lines are the investigation of an associated field theory [17], of the effects of Galilean transformations [16], of quasi-stationary, wave packet, and uniformly accelerated solutions [5,39], and of its relation with the Bohmian formulation of quantum mechanics [40].

The nonlinear Klein-Gordon dynamics introduced in [4] is governed by the field equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right]-\nabla^2\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] + q\frac{m^2c^2}{\hbar^2}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{2q-1}=0, \quad (2)$$

where $\vec{x}\in\mathbb{R}^d$, $\nabla=\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_d}\right)$ is the d-dimensional ∇-operator, $q\geq 1$ and the real, positive constant Φ_0 leads to correct physical dimensionalities for all terms (this scaling becomes irrelevant only in the limit case of the linear Klein-Gordon equation, that is, for $q=1$). The constant Φ_0 constitutes a parameter characterizing the evolution equation (2) itself (that is, it should not be regarded as part of the initial conditions). The dynamical equation (2) can be obtained within a classical field theory derived from an appropriate Lagrangian variational principle [41].

The q-plane wave solutions of the field equation (2) are given by a q-exponential function evaluated on a pure imaginary argument, which corresponds to the principal value of

$$\exp_q(iu)=[1+(1-q)iu]^{\frac{1}{1-q}}\,;\ \exp_1(iu)\equiv\exp(iu), \quad (3)$$

where $u\in\mathbb{R}$. The basic relations satisfied by the above function are [42]

$$\begin{aligned}\exp_q(\pm iu) &= \cos_q(u)\pm i\sin_q(u),\\ \cos_q(u) &= r_q(u)\cos\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\},\\ \sin_q(u) &= r_q(u)\sin\left\{\frac{1}{q-1}\arctan[(q-1)u]\right\},\\ r_q(u) &= \left[1+(1-q)^2u^2\right]^{1/[2(1-q)]}. \end{aligned} \quad (4)$$

It is clear from eqs. (4) that a q-exponential with a pure imaginary argument, $\exp_q(iu)$, exhibits an oscillatory behavior with a u-dependent amplitude $r_q(u)$. It can immediately be verified that the function $\exp_q(iu)$ is of square integrable type for $1<q<3$, the concomitant integral being divergent both for $q\leq 1$ and $q\geq 3$.

The d-dimensional q-plane wave solution of eqs. (2) is given by

$$\Phi(\vec{x},t)=\Phi_0\,\exp_q\left[i(\vec{k}\cdot\vec{x}-\omega t)\right], \quad (5)$$

If we take into account that $\mathrm{d}\exp_q(z)/\mathrm{d}z=[\exp_q(z)]^q$ and $\mathrm{d}^2\exp_q(z)/\mathrm{d}z^2=q[\exp_q(z)]^{2q-1}$ we obtain, for the $(d+1)$-dimensional d'Alembertian operator,

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right]\left(\frac{\Phi}{\Phi_0}\right)= -q\left[\left(\frac{\omega}{c}\right)^2-\left(\sum_{n=1}^{d}k_n^2\right)\right]\left(\frac{\Phi}{\Phi_0}\right)^{2q-1}. \quad (6)$$

Using the above relation one verifies that the q-plane wave ansatz (5) satisfies the field equation (2) if the frequency ω and the momentum k satisfy $\omega^2 = c^2k^2 + m^2c^4/\hbar^2$. Making now, through the celebrated de Broglie and Planck relations, the identifications $\vec{k} \to \vec{p}/\hbar$ and $\omega \to E/\hbar$, it is clear that the q-plane waves are solutions of eq. (2) satisfying $E^2 = c^2p^2 + m^2c^4$. That is, they comply with the energy spectrum of a relativistic free particle for all values of q. Therefore, eq. (2), together with its solution eq. (5), constitute promising candidates for describing interesting types of physical phenomena.

Generalized telegraph equation and nonlinear Klein-Gordon dynamics with dissipation. – The structure of the nonlinear Klein-Gordon equation in d-dimensional space [4] comprises two parts: a term corresponding to the linear $(d+1)$-dimensional wave equation plus a nonlinear mass term proportional to a power of the wave function. Here we are going to introduce a family of evolution equations endowed with a more general power-law nonlinear term (that incorporates the one appearing in the nonlinear Klein-Gordon as a particular instance) that preserve the q-plane wave solutions of the NLKGD.

Let us consider the equation of motion,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] - \nabla^2\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] + q\sum_{i=1}^{L}\delta_i\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{\alpha_i^{(1)}}\left(\frac{\partial}{\partial t}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right]\right)^{\alpha_i^{(2)}} = 0, \quad (7)$$

characterized by the $(3L+1)$ parameters q, δ_i, $\alpha_i^{(1)}$, and $\alpha_i^{(2)}$, with $i = 1,\ldots,L$. As in the case of the nonlinear Klein-Gordon equation, the constant Φ_0 guarantees the correct dimensionalities of the different terms appearing in (7). The parameters q, $\alpha_i^{(1)}$, and $\alpha_i^{(2)}$ are dimensionless, while the dimensions of the δ_i's depend on the values of the exponents $\alpha_i^{(1)}$, and $\alpha_i^{(2)}$. It can be verified after some algebra that the evolution equation (7) admits solutions of the q-plane wave form (from now on we adopt the notation $\Psi = \Phi/\Phi_0$),

$$\Psi = [1 + (1-q)i(\vec{k}\cdot\vec{x} - \omega t)]^{\frac{1}{1-q}}, \quad (8)$$

with $q > 1$, provided that the exponents $\alpha_i^{(1)}$, and $\alpha_i^{(2)}$ comply with the consistency relation,

$$\alpha_i^{(1)} + q\alpha_i^{(2)} = 2q - 1, \qquad i = 1,\ldots,L. \quad (9)$$

and the wave number vector $\vec{k}$ is related to the frequency ω through the dispersion relation,

$$-\frac{\omega^2}{c^2} + k^2 + \sum_{i=1}^{L}\delta_i(-i\omega)^{\alpha_i^{(2)}} = 0, \quad (10)$$

where $k^2 = \vec{k}\cdot\vec{k}$. Note that the parameters $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ have to comply with (9) in order to have exact q-plane wave solutions. Otherwise eq. (7) does not admit this kind of dynamics and it is outside the scope of the present work. Note, however, that even with these constraints, the problem still has $(2L+1)$ parameters, and comprises, consequently, a rich variety of dynamical possibilities. The q-plane wave (8) constitutes a solution of (7) for any q. However, we shall consider only $q > 1$, yielding $\lim_{|\vec{k}\cdot\vec{x}|\to\infty}|\Psi|^2 = 0$, while for $q < 1$ one has $\lim_{|\vec{k}\cdot\vec{x}|\to\infty}|\Psi|^2 = \infty$. In the case of $L = 1$, $\delta_1 = m^2c^2/\hbar^2$, and $\alpha_1^{(2)} = 0$ eq. (7) coincides with the nonlinear Klein-Gordon equation proposed in [4] which, in turn, reduces to the standard Klein-Gordon equation in the limit $q \to 1$. On the other hand, for $L = 1$, $\delta_1 > 0$, $\alpha_1^{(2)} = 1$, and $q \to 1$ the standard linear telegraph equation is recovered. Other relevant equations are obtained as particular limit cases of (7). For instance, the limit $c \to \infty$ corresponds to equations respectively equivalent to the NRT nonlinear Schrödinger equation (for $L = 1$, δ_1 pure imaginary, and $\alpha_1^{(2)} = 1$) and to the porous-media equation (for $L = 1$, δ_1 real, $q \neq 1$, and $\alpha_1^{(2)} = 1$). In summary, it is clear that several of the nonlinear differential equations of current interest associated with the q-thermostatistical formalism, as well as their linear counterparts, are particular instances of (7). This equation also comprises new dynamical scenarios which, as we shall presently see, exhibit dissipation effects. In particular, $L = 2$ with $\delta_1 = m^2c^2/\hbar^2$, and $\alpha_1^{(2)} = 0$ corresponds to the nonlinear Klein-Gordon equation (2) with an extra term involving the first time derivative of the field. Generically (that is, except for a discrete set of particular values of δ_2 and $\alpha_2^{(1)}$) the associated dynamics shows dissipation. In what follows we shall discuss general features of this dissipative dynamics, which characterizes the last mentioned $L = 2$ case, as well as more general instances of the generalized telegraph equation (7).

We shall assume a wave vector $\vec{k}$ with real components (this choice can be regarded as a choice determining the form of the initial form of the wave function at $t = 0$). The frequency of the q-plane wave solutions is then determined by solving the dispersion relation (10) for ω. In general we are going to have a complex frequency,

$$\omega = \omega_a + i\omega_b, \quad (11)$$

the imaginary part ω_b corresponding to the dissipation effects implied by the evolution equation (7). Indeed, the presence of a complex frequency arising from a dispersion relation is the standard signature of dissipative behaviour in processes described by partial differential equations (see, for instance, [43]). It is worth stressing, however, that the analysis of dissipation relations is usually done in terms of standard, exponential wave solutions. In contrast, we are here dealing with dissipation within a dynamics based on power-law q-plane waves, a subject that still remains largely unexplored. An interesting example of this kind of process has been discussed in connection with Landau damping in plasma oscillations [44].

The qualitative features of the dynamics associated with the q-plane wave solutions can be clarified by considering the behavior of its squared wave function profile. We have,

$$|\Psi|^2 = A\left[1-(1-q)(\vec{x}-\vec{x}_0)^T\mathcal{B}(\vec{x}-\vec{x}_0)\right]^{\frac{1}{1-q}}, \qquad (12)$$

where

$$A = [1+(1-q)\omega_b t]^{\frac{2}{1-q}} \equiv (e_q^{\omega_b t})^2, \qquad (13)$$

$$\vec{x}_0 = \frac{\omega_a t}{k^2}\vec{k}, \qquad (14)$$

and $\mathcal{B}$ is an $L\times L$ matrix with elements $\beta_{ij} = (q-1)k_ik_j/[1+(1-q)\omega_b t]^2$. In (12), following a standard notational convention, $(\vec{x}-\vec{x}_0)$ is to be understood as a column vector, while $(\vec{x}-\vec{x}_0)^T$ stands for the concomitant row vector. We see that the squared modulus profile $|\Psi|^2$ has the shape of a multi-valuated q-Gaussian with both its center $\vec{x}_0$ and amplitude A being time dependent. The center $\vec{x}_0$ (where $|\Psi|^2$ adopts its maximum value A) moves uniformly with a constant velocity $\mathrm{d}\vec{x}_0/\mathrm{d}t = \vec{k}\omega_a/k^2$. In the dissipative case, corresponding to $\omega_b < 0$ (remember that we are considering $q > 1$), the amplitude A decreases according to a q-exponential law. That is, we have, $A = \exp_q^2(\omega_b t)$. Equations (8) and (12) are written with respect to an arbitrary Cartesian spatial reference frame. If we choose a reference frame oriented in such a way that the x_1-axis points in the direction of the wave vector $\vec{k}$, then Ψ becomes independent of the remaining $(d-1)$ spatial coordinates, and $|\Psi|^2$ can be expressed in terms of one single coordinate $x = x_1$,

$$|\Psi|^2 = A\left[1-(1-q)\beta(x-x_0)^2\right]^{\frac{1}{1-q}} \equiv Ae_q^{-\beta(x-x_0)^2}, \qquad (15)$$

with A given by (13) and

$$\beta = \frac{(q-1)k^2}{[1+(1-q)\omega_b t]^2}. \qquad (16)$$

We see that in the dissipative case we have $\beta \to 0$ when $t\to\infty$. That is, the q-plane wave solution becomes less localized as it evolves. From now one, we are going to work with a reference frame oriented as explained above, so that we are going to consider an effective one-dimensional problem.

An interesting special case is given by $L = 2$, $\alpha_1^{(1)} = 2q-1$, $\alpha_1^{(2)} = 0$, $\delta_1 = m^2c^2/\hbar^2$, $\alpha_2^{(1)} = -1$, $\alpha_2^{(2)} = 2$, and $\delta_2 = \delta > 0$. This case yields a nondissipative, time-reversible dynamics. The dispersion relation is

$$-\left(\frac{1}{c^2}+\delta\right)\omega^2 + k^2 + \frac{m^2c^2}{\hbar^2} = 0. \qquad (17)$$

This dispersion relation is consistent with the relativistic energy-momentum relation with an effective velocity of light $c^* = \frac{c}{\sqrt{1+\delta c^2}} < c$ and an effective mass $m^* = m\sqrt{1+\delta c^2} \geq m$. For zero rest mass ($m = 0$) we obtain the evolution equation,

$$\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2\Psi}{\partial x^2} + \bar{\delta}\frac{1}{\Psi}\left(\frac{\partial\Psi}{\partial t}\right)^2 = 0, \qquad (18)$$

where $\bar{\delta} \equiv q\delta$. This notation stresses the fact that the structure of the above equation is q-independent. This equation has the remarkable property of admitting q-plane wave soliton-like solutions that propagate without changing shape and with constant velocity, *for all values* $q > 1$. In contrast to what happens with the standard linear wave equation, these q-plane waves are the only traveling solutions of the form $f(kx-\omega t)$ admitted by (18). The effective velocity of these solutions is q-dependent and given by $c_q = c/\sqrt{1+\frac{\bar{\delta}}{q}c^2}$.

Conclusions. – We have explored dissipation effects in the nonlinear Klein-Gordon field theory recently introduced in [4]. These effects are described by a parameterized evolution equation constituting a nonlinear generalization of the celebrated telegraph equation. This equation incorporates as particular instances various nonlinear evolution equations that have recently received increasing attention, such as the power-law diffusion equation (porous-media equation) and the NRT nonlinear Schrödinger and Klein-Gordon equations (the last one corresponding to the NLKGD). The linear Klein-Gordon and telegraph equations are also recovered as particular limit cases. The nonlinear telegraph equation admits q-plane wave solutions which are generically characterized by a dispersion relation leading to complex frequencies. As these solutions evolve, their square modulus profile preserves a q-Gaussian shape with its center moving at a constant velocity. However, when the frequency has a negative imaginary part, these solutions exhibit clear signs of dissipative behaviour: the maximum value of the profile decays in time according to a power law, and the solution becomes less localized. The decay in time is a basic property also shared by the standard exponential plane wave solutions associated with linear equations in dissipative regimes. However, the behaviour of the q-plane waves is richer because they exhibit some degree of localization. This allows us to determine how the maximum (center) of the solution moves, and how its shape and degree of localization change in time.

The q-plane wave solutions of the NRT nonlinear Schrödinger equation, and of the nonlinear Klein-Gordon equation without dissipation behave like solitons, in the sense that they travel rigidly, at a uniform velocity, without changing shape. In the case of a complex frequency with negative imaginary part, the solutions of our nonlinear telegraph equation lose the soliton-like character due to the dissipative effects. They still move at a constant speed, but the shape of the solution changes with time. For special values of the parameters, however, dissipation disappears and the soliton-like behavior is recovered. Of special interest is a particular case, given by eq. (18). This equation turns out to be q-independent. However, it admits q-plane waves solutions for all values of q. It would be interesting to explore in more detail to what extent the q-plane wave solutions of (18) are solitons. To this aim, it would be necessary to investigate analytically or numerically the stability of these solutions, and whether two of them traveling in opposite directions can survive after a collision, retaining their original forms. It is worth mentioning that these questions are still open also for the q-plane wave solutions of both the NRT nonlinear Schrödinger equation and for the nonlinear Klein-Gordon equation without dissipation.

The nonlinear telegraph equation advanced here may be useful for describing a variety of physical systems or processes such as wave guides and electrical transmission lines with nonlinear amplitude-dependent dissipation, and nonlinear non-Poissonian dichotomous diffusion processes. Any further developments concerning the aforementioned theoretical issues, or dealing with possible applications of the nonlinear telegraph equation, will be very welcome.

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Partial financial support from Grant 401512/2014-2 of CNPq (Brazilian agency), from FAPERJ (Brazilian agency), and from the John Templeton Foundation, is acknowledged.

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