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Were strong inflaton field fluctuations the cause of the big bang?*

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In this paper, I propose a model to describe the birth of the universe taking into account back reaction effects produced by the inflaton field fluctuations, with self-interactions included. These fluctuations $\delta\varphi = \varphi(x) - \langle\varphi\rangle$ would have been very important at the Planck scales and their self-interactions could have been the fuel to the primordial expansion of the universe.

Keywords: Preinflation; strong inflaton field fluctuations.

1. Introduction and Motivation

The theory of preinflation was introduced with the aim to describe the primordial evolution of the universe,[1–4] since the universe began to expand from a size of the order of the Planck length, and a maximum Hubble parameter, which has been decreasing since then throughout the history of the universe. However, one can make the question about how did the universe achieve this value for the Hubble parameter? A few years ago, a model of preinflation was introduced,[5] in which the universe evolved to an asymptotic de Sitter expansion. In that work, the scalar metric fluctuations were studied that appear as a geometric response to the inflaton field fluctuations by means of geometrical displacement from a Riemann manifold to an extended one, through the Relativistic Quantum Geometry (RQG) formalism,[6]

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but without taking into account the self-interactions of these fluctuations. Other approaches on extended manifolds have been considered in the framework of the Weyl connections.[7–11] In such formalism, back-reaction effects[12–17] are described as nonperturbative geometric fluctuations with respect to the background Riemann manifold.[18]

In this work, we introduce a new model, in which the universe began to expand with a null Hubble parameter, which increases until reach its maximum value. As is known, this is impossible if we consider the universe as an isolated system.[19] Therefore, the universe must have started its expansion by taking energy from somewhere. We shall explore the possibility that it has collected the necessary energy to begin its expansion from the fluctuations of the inflaton field, which are self-interacting. These fluctuations will be considered as the generators of a Weyl manifold that takes into account the departure from the background dynamics, which is described on a semi-Riemannian manifold (with the Levi-Civita connections).

2. Einstein–Hilbert Action and Boundary Terms

We consider a relativistic physical system, whose background dynamics is represented on a semi-Riemannian manifold, in a such manner that its Einstein–Hilbert action is

$$\mathcal{I} = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} + \mathcal{L}_m\right]. \tag{1}$$

Here, R is the background scalar curvature, $\kappa = 8\pi G$ and $\mathcal{L}_m$ is the Lagrangian density that describes the background physical fields. After varying the action, we obtain that

$$\delta\mathcal{I} = \int d^4x\sqrt{-g}\left[\delta g^{\alpha\beta}\left(G_{\alpha\beta} + \kappa T_{\alpha\beta}\right) + g^{\alpha\beta}\delta R_{\alpha\beta}\right] = 0, \tag{2}$$

where the background stress tensor is $T_{\alpha\beta} = 2\frac{\delta\mathcal{L}_m}{\delta g^{\alpha\beta}} - g_{\alpha\beta}\mathcal{L}_m$. In the case that $\delta\Theta = g^{\alpha\beta}\delta R_{\alpha\beta}$ is nonzero, it alters the dynamics of the system, because it acts as a source on the background equation of motion. We shall consider the background dynamics on the semi-Riemannian manifold described by the Levi-Civita connections, and the boundary terms will be described using an extension of the former manifold in the following manner:

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ \begin{array}{c} \alpha \\ \beta\,\gamma \end{array} \right\} + b\,\delta\varphi^{\alpha} g_{\beta\gamma}. \tag{3}$$

We shall use the connections (3), in order to define the varied Ricci tensor $\delta R_{\alpha\beta}$, as an extended version of the Palatini identity[20]

$$\delta R^{\mu}_{\alpha\beta\mu} = \delta R_{\alpha\beta} = \left(\delta\Gamma^{\mu}_{\alpha\mu}\right)_{\|\beta} - \left(\delta\Gamma^{\mu}_{\alpha\beta}\right)_{\|\mu}. \tag{4}$$

If we consider a relativistic flow[a]

$$\delta R_{\alpha\beta} = \lambda(x)\, \delta g_{\alpha\beta}. \tag{8}$$

By using the fact that $\delta(g^{\alpha\beta} g_{\alpha\beta}) = 0$ on the varied action (2), we obtain $\delta\mathcal{I} = \int d^4x\sqrt{-g}[G_{\alpha\beta} - \lambda(x)g_{\alpha\beta} + \kappa\, T_{\alpha\beta}] = 0$, so that the background Einstein equations result

$$G_{\alpha\beta} - \lambda(x)g_{\alpha\beta} + \kappa\, T_{\alpha\beta} = 0. \tag{9}$$

The dynamical background equations will be $\nabla_\beta\, G^{\alpha\beta} = 0$, with the field equations

$$\nabla_\beta\, T^{\alpha\beta} = \frac{1}{\kappa}\, g^{\alpha\beta}\, \frac{\partial\lambda(x)}{\partial x^\beta}. \tag{10}$$

From the expression (8), we obtain the flux equation $\delta\Phi = g^{\alpha\beta}\delta R_{\alpha\beta} = \lambda(x)\, g^{\alpha\beta}\delta g_{\alpha\beta}$, which takes the explicit form

$$\Box\delta\varphi + [2b + (1-\xi^2)]\delta\varphi_\alpha\delta\varphi^\alpha = -\frac{2}{3b}\, \lambda(x)[4(1-\xi^2) - b]\delta(\delta\varphi). \tag{11}$$

Using the gauge $b = -(1/2)(1-\xi^2)$, the nonlinear terms in the flux $\delta\Phi$ are removed, and therefore Eq. (11) holds $\Box\delta\varphi = 6\,\lambda(x)\delta(\delta\varphi)$, that on hypersurfaces $dt = 0$, takes the form

$$\Box\delta\varphi = 6\,\lambda(x)\delta\varphi. \tag{12}$$

In this paper, we shall study the important case where the self-interactions produce the mass of the inflaton field fluctuations $\delta\varphi$:[19]

$$\lambda(t) = -\frac{1}{6}\, \frac{\delta^2\bar{V}(\phi)}{\delta\phi^2}, \tag{13}$$

which could be determinant for the birth of the universe's expansion. Note that for this choice we obtain the equation for a massive scalar field on a curved background

$$\Box\delta\varphi + \frac{\delta^2\bar{V}(\phi)}{\delta\phi^2}\, \delta\varphi = 0, \tag{14}$$

with a squared mass: $m^2 = \frac{\delta^2\bar{V}(\phi)}{\delta\phi^2} > 0$.

[a]The covariant derivative is null: $\nabla_\nu\, g_{\alpha\beta} = 0$, but on the extended manifold it is nonzero

$$g_{\alpha\beta\|\nu} = -b\big(\delta\varphi_\beta\, g_{\alpha\nu} + \delta\varphi_\alpha\, g_{\nu\beta}\big) + 2(1-\xi^2)g_{\alpha\beta}\,\delta\varphi_\nu, \tag{5}$$

where b, and ξ^2 are some dimensionless couplings to be determined. Therefore, we can define the variation of the metric tensor on the extended manifold

$$\delta g_{\alpha\beta} = g_{\alpha\beta\|\nu}\, dx^\nu = -b\left(\delta\varphi_\alpha\, dx_\beta + \delta\varphi_\beta\, dx_\alpha\right) + 2(1-\xi^2)g_{\alpha\beta}\,\delta(\delta\varphi), \tag{6}$$

$$\delta g^{\alpha\beta} = g^{\alpha\beta}_{\|\nu}\, dx^\nu = b\big(\delta\varphi^\alpha\, dx^\beta + \delta\varphi^\beta\, dx^\alpha\big) - 2(1-\xi^2)g^{\alpha\beta}\,\delta(\delta\varphi), \tag{7}$$

where $\delta(\delta\varphi) = \frac{\partial\delta\varphi}{\partial x^\mu}\, dx^\mu \equiv \delta\varphi_\mu dx^\mu$ (we denote $\delta\varphi_{,\mu} \equiv \delta\varphi_\mu$), and the background line element is $dl^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$.

3. Preinflation and the Birth of the Universe

We consider a single scalar field ϕ in the Lagrangian of the action (1), which is minimally coupled to gravity and drives the expansion of the universe $\mathcal{L}_m = -\left[\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi)\right]$, which is related to the stress tensor: $T_{\mu\nu} = 2\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} - g_{\mu\nu}\,\mathcal{L}_m$. In this paper, we shall consider natural units, so that $c = \hbar = 1$. In order to describe the background dynamics with a variable time scale, we shall consider the line element[21]

$$dl^2 = e^{-2\int \gamma(t)dt}dt^2 - a_0^2\, e^{2\int H(t)dt}\,\delta_{ij}\,dx^i dx^j, \tag{15}$$

such that $H(t)$ is the Hubble parameter on the background metric and $\gamma(t)$ describes the time scale of the background metric. This should be the case in an emergent accelerated universe in which the time scale can be considered variable with the expansion. From the point of view of a co-moving relativistic observer, its "clock" is accelerated with respect to an observer which is in a noninertial frame and a physical time that evolves as $d\tau = \sqrt{g_{00}}dx^0 = e^{-\int \gamma(t)\,dt}\,dt$. This has sense because the observer in the co-moving frame is accelerated with respect to whole in the noninertial frame. The action explicitly written is

$$\mathcal{I} = \int d^4x\,\sqrt{-g}\left\{\frac{\mathcal{R}}{16\pi G} - \left[\frac{\dot{\phi}^2}{2}\,e^{2\int \gamma(t)\,dt} - V(\phi)\right]\right\}, \tag{16}$$

which can be rewritten as

$$\begin{aligned}\mathcal{I} &= \int d^4x\,\sqrt{-g}\,e^{2\int \gamma(t)dt}\left\{\frac{\bar{\mathcal{R}}}{16\pi G} - \left[\frac{\dot{\phi}^2}{2} - \bar{V}(\phi)\right]\right\}\\ &\equiv \int d^4x\sqrt{-g}\,e^{2\int \gamma(t)dt}\left\{\frac{\bar{\mathcal{R}}}{16\pi G} + \bar{\mathcal{L}}_m\right\},\end{aligned} \tag{17}$$

where $\bar{\mathcal{L}}_m = \bar{\mathcal{L}}_m\,e^{-2\int \gamma(t)dt}$. The scalar field ϕ in (17) will be solution of a new equation because it is embedded in an effective background volume $\bar{v} = \sqrt{-g}\,e^{2\int \gamma(t)dt} = a_0^3\,e^{\int \gamma(t)dt}\,e^{3\int H(t)dt}$. The redefined potential is $\bar{V}(\phi) = V(\phi)\,e^{-2\int \gamma(t)dt}$, and there is an effective scalar curvature $\bar{\mathcal{R}} = \mathcal{R}\,e^{-2\int \gamma(t)\,dt}$, which is related with the new line element

$$d\bar{l}^2 = \bar{g}_{\alpha\beta}\,dx^\alpha dx^\beta = dt^2 - a_0^2\,e^{2\int [H(t)+\gamma(t)/3]dt}\,\delta_{ij}\,dx^i dx^j. \tag{18}$$

The stress tensor in (17) is $\bar{T}_{\alpha\beta} = 2\frac{\delta\bar{\mathcal{L}}_m}{\delta\bar{g}^{\mu\nu}} - \bar{g}_{\mu\nu}\,\bar{\mathcal{L}}_m$. The redefined potential with back-reaction contributions included is

$$\begin{aligned}\bar{\Upsilon}(\phi) &= \left[V(\phi) + \frac{1}{(48\pi\,G)}\,\frac{\delta^2 V(\phi)}{\delta\phi^2}\right]e^{-2\int \gamma(t)dt}\\ &= \left[V(\phi) - \frac{\lambda(t)}{(8\pi\,G)}\right]e^{-2\int \gamma(t)\,dt},\end{aligned} \tag{19}$$

where we have made use of the Einstein equations with back-reaction effects included (9), and the fact that the cosmological parameter $\lambda(x^\mu) \equiv \lambda(t)$ is

$\lambda(t) = -\frac{1}{6}\frac{\delta^2 \bar{V}(\phi)}{\delta\phi^2}$. In that case, the dynamics of the scalar field ϕ is given by

$$\ddot{\phi} + [3\,H + \gamma]\dot{\phi} + \frac{\delta\bar{\Upsilon}}{\delta\phi} = 0 \tag{20}$$

that describes the dynamics of the background scalar field evolving on the background metric (18). The term $3H\,\dot{\phi}$ is due to the expansion of the universe, but the term $\gamma\,\dot{\phi}$, with $\gamma(t) < 0$ describes a nontrivial time scale. When $\gamma > 0$, it describes a friction parameter that produces (only when $\gamma > 3H$), the reheating of the universe. However, in our case, parameter $\gamma(t)$ is negative, and cannot be physically interpreted as a friction one, but as one that describes the energy that the quantum fluctuations $\delta\varphi$ delivers to the universe, to fuel its initial expansion.

The background Einstein equations are

$$3\,H^2 = 8\pi\,G\left[\frac{\dot{\phi}^2}{2} + \bar{\Upsilon}(\phi)\right] = 8\pi\,G\,\rho, \tag{21}$$

$$-(3\,H^2 + 2\dot{H} + 2\gamma\,H) = 8\pi\,G\left[\frac{\dot{\phi}^2}{2} - \bar{\Upsilon}(\phi)\right] = 8\pi\,G\,P, \tag{22}$$

such that P and ρ are, respectively, the pressure and the energy density on the action (16)

$$P\,e^{2\int\gamma(t)dt}\,\delta^i{}_j = -T^i{}_j = -\bar{T}^i{}_j, \quad \rho\,e^{2\int\gamma(t)dt} = T^0{}_0 = \bar{T}^0{}_0, \tag{23}$$

and the diagonal components of the stress tensor for a perfect fluid are $T^\mu{}_\nu = \mathrm{diag}(\rho, -P, -P, -P)$. Therefore, the effective equation of state $\omega = P/\rho$, during preinflation will be $\omega = \frac{\frac{\dot{\phi}^2}{2} - \bar{\Upsilon}(\phi)}{\frac{\dot{\phi}^2}{2} + \bar{\Upsilon}(\phi)}$.

In order to describe a scenario that describes a universe which starts with a null Hubble parameter to reach its maximum value at the end of preinflation, we shall propose ***a toy model***, in which H is related with the parameter γ by the expression

$$3\,H[\phi(\tau)] + \gamma[\phi(\tau)] = \epsilon\,H_0, \tag{24}$$

where H_0 is constant. Using the Einstein equations (21) and (22), we obtain that

$$6\,H^2 + 2[\dot{\phi}\,H' + \epsilon\,H_0 - 3\,H^2] - 16\pi\,G\,\bar{\Upsilon}[\phi(t)] = 0. \tag{25}$$

One of the problems of inflationary models is that initially, the scalar field can take trans Planckian values. In order to avoid this kind of problems, we shall consider a scalar field which is zero when the universe is created, and increases with time. To obtain the dynamics of the system, we shall work in the opposite way to what is usually done, i.e. for a given dynamics of the scalar field $\phi(t)$, we shall look for the solution of the potential $\bar{\Upsilon}(\phi)$ by using the dynamical equations (20)–(22) with the constriction (24). Because we are aimed to describe the birth of the universe, we must consider that the Hubble parameter is initially null: $H(t = 0) = 0$. In other words, the theory must be able to explain how the universe began to expand from an initial state of stillness. For the choice $\phi(t) = N\,\phi_0[1 - e^{-H_0\,t}]$, we obtain that $\dot\phi = H_0(N\,\phi_0 - \phi)$ and $\ddot\phi = -H_0^2(N\,\phi_0 - \phi)$, and from Eq. (25) we obtain that the effective potential is

$$\bar\Upsilon[\phi(t)] = \frac{H_0^2}{8\pi\,G}\left[\frac{1}{2(N+1)}\left(\frac{\phi(t)}{\phi_0}\right)^2 - \frac{N}{N+1}\left(\frac{\phi(t)}{\phi_0}\right) + N\right]. \tag{26}$$

Here, N is a dimensionless natural number that give us the scale of the energy for different epochs in the evolution of the universe. Furthermore, from the equation of motion (20), we obtain that

$$-\,H_0^2(N\,\phi_0 - \phi) + 3\,\epsilon\,H_0^2(N\,\phi_0 - \phi) + \frac{\delta\bar\Upsilon}{\delta\,\phi} = 0, \tag{27}$$

where $\phi_0^2 = \frac{1}{8\pi\,G}$ and $\phi(t)$ is an increasing function of t and always takes sub-Planck values: $0 \leq \phi(\tau) < N\,\phi_0 < M_p$. When back-reaction effects of $\delta\varphi$ are included, after using the expression: $\bar\Upsilon(\phi) = \bar V(\phi) + \frac{1}{48\,\pi\,G}\frac{\delta^2\bar V}{\delta\,\phi^2}$, we obtain the potential $\bar V$

$$\bar V[\phi(\tau)] = \frac{H_0^2}{8\pi\,G}\left[\frac{1}{2(N+1)}\left(\frac{\phi(t)}{\phi_0}\right)^2 - \frac{N}{N+1}\left(\frac{\phi(t)}{\phi_0}\right) + \frac{[6N(N+1)-1]}{6(N+1)}\right]. \tag{28}$$

Finally, the particular solutions for the Hubble parameter and the function $\gamma[\phi(t)]$ that comply with the dynamic equations (25) and (27) are

$$H[\phi(t)] = H_0\left[\left(\frac{\phi(t)}{\phi_0}\right) - \frac{1}{2N}\left(\frac{\phi(t)}{\phi_0}\right)^2\right], \tag{29}$$

$$\gamma[\phi(t)] = H_0\left[\frac{2+N}{1+N} - 3\left[\left(\frac{\phi(t)}{\phi_0}\right) - \frac{1}{2\,N}\left(\frac{\phi(t)}{\phi_0}\right)^2\right]\right]. \tag{30}$$

Note that $H(t=0) = 0$, so that the model describes in an effective manner the beginning of the expansion of the universe, with the posterior evolution.

Equation (12) on the metric (18) takes the form

$$\ddot{\delta\varphi} + (3H+\gamma)\,\dot{\delta\varphi} - a_0^{-2}e^{-2\int[H+\gamma/3]dt}\,\nabla^2\delta\varphi + \frac{\delta^2\bar V(\phi)}{\delta\phi^2}\,\delta\varphi = 0. \tag{31}$$

The relevant slow-roll parameters are[22]

$$\varepsilon(\phi) = \frac{1}{16\,\pi\,G}\left(\frac{\bar\Upsilon'}{\bar\Upsilon}\right)^2, \quad \eta(\phi) = -\frac{1}{8\,\pi\,G}\left(\frac{\bar\Upsilon''}{\bar\Upsilon}\right), \tag{32}$$

and the scalar spectral index that characterizes the spectrum of $\delta\varphi$, is $n_s(\phi) = 1 - 6\varepsilon(\phi) + 2\eta(\phi)$. Furthermore, the tensor index is given by $n_t(\phi) = -2\varepsilon(\phi)$ and can be defined the tensor to scalar index: $r(\phi) = -8\,n_t(\phi) = 16\,\varepsilon(\phi)$. In Fig. 3, the indices $1 - n_s(\phi)$, $n_t(\phi)$ and $r(\phi)$ are plotted for $G = 1$, $N = 10$ and $H_0 = 0.0005\,G^{-1/2}$ and $\phi_* \simeq 2\,G^{-1/2}$. Note that at the end of preinflation, when $\phi \equiv \phi_*$, it is obtained that $1 - n_s(\phi_*) \simeq 0.035$, and $|r(\phi_*)| \simeq |n_t(\phi_*)| \ll 0.1$. All the values agree with observation.[23]

4. Final Comments

In this paper, I proposed a model to describe the birth of the universe taking into account back reaction effects produced by the inflaton field fluctuations, with self-interactions included. These fluctuations, would have been the fuel to the primordial expansion of the universe. In the model, H, which is initially null, increases with time, but after some Planck times, when $\phi(t_*) = N\,\phi_0$, H reaches its maximum. On the other hand, $\gamma(t)$ becomes negative until reach its minimum value at $\phi(t_*) = N\,\phi_0$ (see Fig. 1). This epoch is known as preinflation and explains how the Hubble parameter, being initially null, can reach the maximum value from which it later decreases along all the history of the universe. The negative parameter $\gamma(t)$ cannot be physically interpreted as a friction one, but as one that describes the energy that the quantum fluctuations $\delta\varphi$ on the curved background delivers to the universe, to fuel its expansion. However, it would be interesting to study a model in which γ takes initially negative values to later be positive. A model like this could be appropriate to describe the transition between the preinflation and fresh inflationary[24] stages. Another important fact is that the equation of state during preinflation approaches asymptotically to $\omega \to -1$ (see Fig. 2), because along the primordial expansion the kinetic contribution in the energy density becomes less significative. Note the difference between the model here proposed and whole of Ref. 19, in which the universe suffers a transition from an initial stable state [such that the universe is initially at the minimum of a ϕ-quartic potential: $V(\phi) \sim \phi^4$], to an unstable evolving state that drives the initial accelerated expansion. In that case, the spacetime is initially Euclidean to change dynamically to one globally hyperbolic during the expansion. Another difference is that in the scenario described in

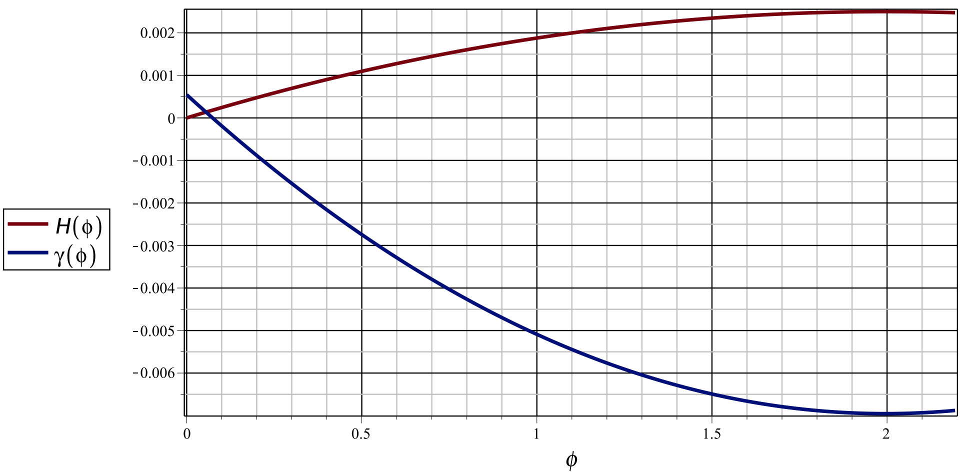

Fig. 1. Plot of $H(\phi)$, for $G = 1$, $N = 10$ and $H_0 = 0.0005\,G^{-1/2}$. During preinflation H increases with time, but reaches its maximum value at the end of preinflation, when $\phi(t_*) = N\,\phi_0 \simeq 2\,G^{-1/2}$, rather than $\gamma(t)$ becomes negative until reach its minimum value at $\phi(t_*)$.

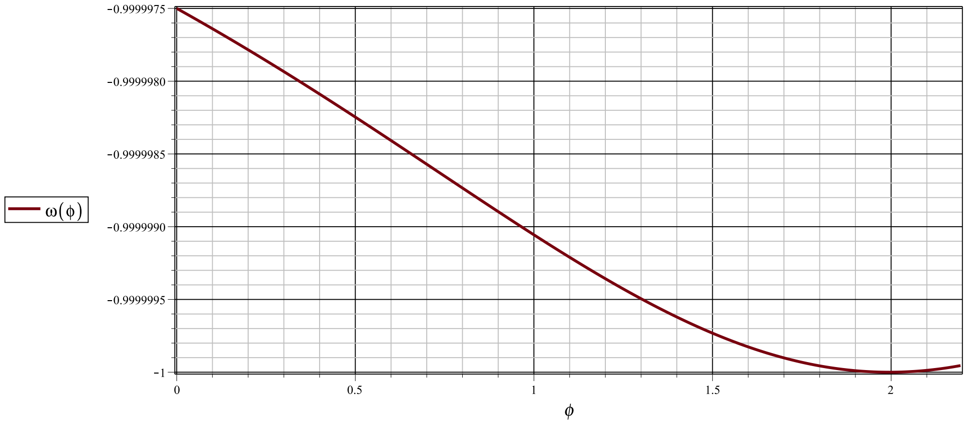

Fig. 2. Plot of $\omega = P/\rho$ during preinflation, for $G = 1$, $N = 10$ and $H_0 = 0.0005\, G^{-1/2}$ and $\phi_* \simeq 2\, G^{-1/2}$. The universe approaches to a vacuum expansion at the end of preinflation: $\omega(\phi_*) \simeq -1$.

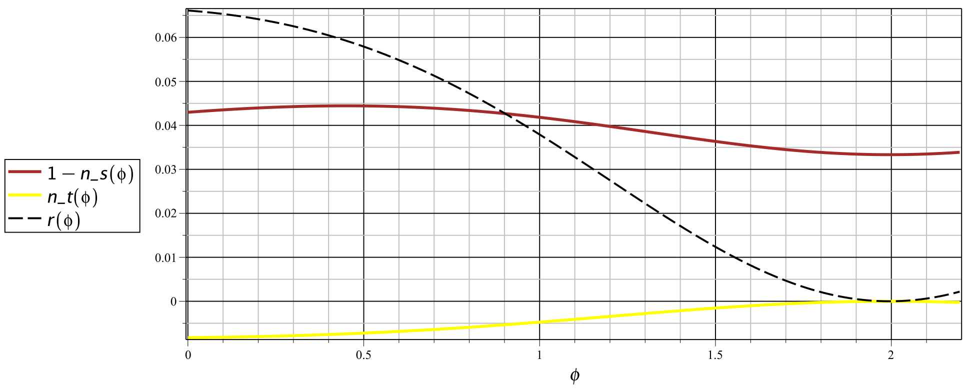

Fig. 3. Plot of indices $1 - n_s(\phi)$, $n_t(\phi)$ and $r(\phi)$, for $G = 1$, $N = 10$ and $H_0 = 0.0005\, G^{-1/2}$ and $\phi_* \simeq 2\, G^{-1/2}$. Notice that $1 - n_s(\phi_*) \simeq 0.035$ and $|r(\phi_*)| \simeq |n_t(\phi_*)| \ll 0.1$.

Ref. 19, the Hubble parameter is nonzero at the beginning. However, in the model here worked, the spacetime is always considered as hyperbolic along the expansion.

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