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FINITE GK-DIMENSIONAL PRE-NICHOLS ALGEBRAS OF QUANTUM LINEAR SPACES AND OF CARTAN TYPE

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Abstract. We study pre-Nichols algebras of quantum linear spaces and of Cartan type with finite GK-dimension. We prove that except for a short list of exceptions involving only roots of order 2, 3, 4, 6, any such pre-Nichols algebra is a quotient of the distinguished pre-Nichols algebra introduced by Angiono generalizing the De Concini-Kac-Procesi quantum groups. There are two new examples, one of which can be thought of as G_2 at a third root of one.

Contents

1. Introduction

1.1. Overview.

1.1.1. *The problem.* Let $\Bbbk$ be a field. Let GK-dim be an abbreviation of Gelfand-Kirillov dimension, see [KL]. In this paper we contribute to the ongoing program of classifying Hopf algebras with finite GK-dim. See [B+, G, L] and references therein.

Let H be a Hopf algebra and let ${}^H_H\mathcal{YD}$ be the category of Yetter-Drinfeld modules over H. Assume that H is pointed (similar arguments apply more generally if its coradical is a Hopf subalgebra). Basic invariants of H are

(i) the group of grouplikes $\Gamma = G(H)$,
(ii) the diagram $R = \oplus_{n\in\mathbb{N}_0} R^n$, a graded connected Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$,
(iii) the infinitesimal braiding $V := R^1$, an object in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$.

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See [AS3]. Assume that Γ has finite growth. In order to classify those H with finite GK-dim, one first needs to understand all such R with finite GK-dim. As a coalgebra R is coradically graded and connected; in other words, it is strictly graded as in [Sw, Section 11.2, see p. 232 and Lemma 11.2.1]. Strictly graded Hopf algebras R in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with $R^1 \simeq V$ are called post-Nichols algebras of V; also, graded Hopf algebras R in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ generated by $R^1 \simeq V$ are called pre-Nichols algebras of V. See §2.5.

The Nichols algebra $\mathscr{B}(V)$ is isomorphic to the subalgebra of R generated by V; see [A] for an introduction to Nichols algebras. When $\operatorname{char} \Bbbk = 0$ and $\dim H < \infty$ (thus Γ is finite), it was conjectured in [AS2] that $R = \mathscr{B}(V)$; this is known to be true when Γ is abelian by [An1]. The validity of this conjecture says that the classification of the finite-dimensional Nichols algebras in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is a substantial step towards the problem of classifying finite-dimensional pointed Hopf algebras with group Γ. When $\operatorname{char} \Bbbk > 0$ or $\dim H = \infty$, the conjecture fails to be true and the knowledge of the Nichols algebras is not enough. Thus, towards classifying pointed Hopf algebras with group Γ and finite GK-dim, we do not see how to avoid the consideration of the following questions:

(A) classify all $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that $\mathscr{B}(V)$ has finite GK-dim,
(B) for such V classify all post-Nichols algebras with finite GK-dim.

Question (B) appears to be difficult to handle directly. However Lemma 2.2 below, proved in [AAH3], reduces Question (B) for V as in (A) to

(C) classify all pre-Nichols algebras of V^* with finite GK-dim.

As usual it is more flexible to deal with classes of braided vector spaces rather than classes of groups Γ and correspondingly pre-Nichols algebras as braided Hopf algebras; see §2.3 for unexplained vocabulary.

1.1.2. *Eminent pre-Nichols algebras.* For Question (C) we point out that all pre-Nichols algebras of V form a poset $\mathfrak{Pre}(V)$ with $T(V)$ minimal and $\mathscr{B}(V)$ maximal; those with finite GK-dim form a saturated subposet $\mathfrak{Pre}_{\mathrm{fGK}}(V)$, cf. §2.5. When $\operatorname{char} \Bbbk = 0$ and the braiding is the usual flip, the Nichols algebra is just the symmetric algebra and the pre-Nichols algebras with finite GK-dim are the universal enveloping algebras of the finite-dimensional $\mathbb{N}$-graded Lie algebras generated in degree one. Thus $\mathfrak{Pre}_{\mathrm{fGK}}(V)$ is hardly computable when $\dim V \geq 2$. Similar considerations are valid when the braiding is the super flip of a super vector space, see §2.9.2. But if $\dim V = 1$ and $\operatorname{char} \Bbbk = 0$, then $\mathfrak{Pre}_{\mathrm{fGK}}(V) = \mathfrak{Pre}(V)$ has obviously a minimal element. We introduce in this paper the notion of *eminent* pre-Nichols algebra as one that is a minimum in $\mathfrak{Pre}_{\mathrm{fGK}}(V)$. That is, a pre-Nichols algebra $\widehat{\mathscr{B}}$ of a braided vector V is eminent if

(a) GK-dim $\widehat{\mathscr{B}} < \infty$;
(b) if $\mathscr{B}$ is a pre-Nichols algebra of V with GK-dim $\mathscr{B} < \infty$, then there exists a morphism of pre-Nichols algebras $\widehat{\mathscr{B}} \to \mathscr{B}$, necessarily surjective.

The existence of an eminent pre-Nichols algebra $\widehat{\mathscr{B}}$ reduces Question (C) to the determination of all pre-Nichols algebra quotients of $\widehat{\mathscr{B}}$, that is its homogeneous Hopf ideals starting in degree (at least) 2. Presently there is no general recipe to decide whether a braided vector space admits an eminent pre-Nichols algebra. In this paper we shall show that many braided vector spaces of diagonal type have eminent pre-Nichols algebras.

1.1.3. *Distinguished pre-Nichols algebras.* From now on we assume that $\Bbbk$ is algebraically closed and $\operatorname{char}\Bbbk = 0$. In this paper we deal with Question (C) for braided vector spaces V of diagonal type, i.e. with braiding determined by a matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}}$ with entries in $\Bbbk^\times$ where $\theta \in \mathbb{N}$ and $\mathbb{I} = \{1, \dots, \theta\}$. See §2.8 for precise definitions.

First we need to discuss Question (A) for this class. Finite-dimensional Nichols algebras of diagonal type, i.e. those with GK-dim = 0, were classified in [H1] through the notion of (generalized) root system. More generally the list of all Nichols algebras of diagonal type with finite root system is given in *loc. cit.* It was conjectured in [AAH1], and verified in various cases [R], [AA1, AAH2], that Nichols algebras of diagonal type with finite GK-dim are precisely those with finite root system. We recall this as Conjecture 2.6. We shall assume in a few proofs that Conjecture 2.6 is valid in dimensions ≤ 5.

Let $\mathscr{B}(V) = T(V)/\mathcal{J}(V)$ be a finite-dimensional Nichols algebra of diagonal type. The *distinguished* pre-Nichols algebra of V introduced in [An3] is the quotient $\widetilde{\mathscr{B}}(V) := T(V)/\mathcal{I}(V)$, where $\mathcal{I}(V)$ is the ideal of $T(V)$ generated by the defining relations of $\mathcal{J}(V)$ given in [An1] but excluding the powers of the Cartan root vectors and including the quantum Serre relations at Cartan vertices. Detailed presentations of $\mathcal{J}(V)$ and $\mathcal{I}(V)$ are available in [AA2, §4]. The notion of Cartan root requires the theory of Weyl groupoid that would led us too far from the goal of this paper. Indeed in Cartan type all roots are so and the distinguished pre-Nichols algebras are the positive parts of the quantum groups of [DKP]. Originally $\widetilde{\mathscr{B}}(V)$ was introduced as a tool for understanding the relations of $\mathscr{B}(V)$; several results on $\widetilde{\mathscr{B}}(V)$ were established in [An3]. The graded duals of the distinguished pre-Nichols algebras have been presented by generators and relations in [AAR].

Unlike the notion of eminent pre-Nichols algebra, we lack at the moment a concise abstract definition of $\widetilde{\mathscr{B}}(V)$ that could be adapted beyond finite dimensional $\mathscr{B}(V)$ of diagonal type; but see §1.2.1 for quantum linear spaces.

In [An3] the author and one of us asked whether a distinguished pre-Nichols algebra is eminent (in the terminology just introduced). Recall that the classification in [H1] was organized in [AA2] in various types: Cartan, standard, super, modular, super modular and UFO. Here we address the question above when $\mathscr{B}(V)$ is either a quantum linear space or of Cartan type. The recent paper [ACS] treats super and standard types, the remaining ones being the subject of work in progress.

1.2. **The main results.** In the present paper we focus on braided vector spaces of diagonal type of two kinds. Fix V of diagonal type as in §2.8, with braiding given by the matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}}$.

1.2.1. *Quantum linear spaces.* Here we assume that $\mathfrak{q}$ satisfies $q_{ij}q_{ji} = 1$ for all $i \neq j \in \mathbb{I}$. We extend the notion of distinguished pre-Nichols algebra to quantum linear spaces even when they are infinite-dimensional. Namely we define the *distinguished pre-Nichols algebra* $\widetilde{\mathscr{B}}(V)$ as the one presented by generators $(x_i)_{i\in\mathbb{I}}$ and relations $x_ix_j - q_{ij}x_jx_i$, for all $i \neq j \in \mathbb{I}$. If $1 \neq q_{ii}$ is a root of unit for all i, then $\dim \mathscr{B}(V) < \infty$, V is of Cartan type $A_1 \times \cdots \times A_1$ and any root is Cartan, so this definition is consistent with the one given in [An3] and discussed above.

We need some notation to state our first Theorem. Set

$$\begin{aligned} &\mathbb{I}^{\infty} = \{i \in \mathbb{I}\colon q_{ii} \notin \mathbb{G}_\infty\}, && \mathbb{I}^{N} = \{i \in \mathbb{I}\colon \operatorname{ord} q_{ii} = N\},\ N \geq 1, \\ &\mathbb{I}^{t} = \bigcup_{N>3} \mathbb{I}^{N}, && \mathbb{I}^{\pm} = \{i \in \mathbb{I}\colon q_{ii} = \pm 1\} = \mathbb{I}^{1} \sqcup \mathbb{I}^{2}. \end{aligned} \tag{1.1}$$

Thus $\mathbb{I} = \mathbb{I}^{\pm} \sqcup \mathbb{I}^{3} \sqcup \mathbb{I}^{t} \sqcup \mathbb{I}^{\infty}$. For $\star \in \mathbb{N} \cup \{\pm, t, \infty\}$, let $V^\star$ be the subspace of V spanned by $(x_i)_{i\in\mathbb{I}^\star}$ and $\mathfrak{q}^\star$ the restriction of $\mathfrak{q}$ to $V^\star$. Then

$$V = V^{\pm} \oplus V^{3} \oplus V^{t} \oplus V^{\infty}.$$

Theorem 1.1. *Assume that Conjecture* 2.6 *is true.*

(a) *The distinguished pre-Nichols algebra $\widetilde{\mathscr{B}}(V^\star)$ is eminent, $\star \in \{3, t, \infty\}$.*

(b) *Let $\mathscr{B}$ be a finite GK-dimensional pre-Nichols algebra of V; let $\mathscr{B}^{\pm,3}$, respectively $\mathscr{B}^{t}$, $\mathscr{B}^{\infty}$ be the subalgebra of $\mathscr{B}$ generated by $V^{\pm} \oplus V^{3}$, respectively V^{t}, V^{∞}. Then there is a decomposition*

$$\mathscr{B} \simeq \mathscr{B}^{\pm,3} \underline{\otimes} \mathscr{B}^{t} \underline{\otimes} \mathscr{B}^{\infty}. \tag{1.2}$$

(c) *Assume that V has a basis $\{x_1, x_2\}$ with $x_1 \in V^3$, $x_2 \in V^1$. Then*

$$\breve{\mathscr{B}}(V) = T(V)/\langle (\operatorname{ad}_c x_1)^4(x_2),\ (\operatorname{ad}_c x_2)^2(x_1)\rangle$$

is an eminent pre-Nichols algebra of V and has GK-dim $= 6$.

See §2.1 for the meaning of $\underline{\otimes}$. Parts (a) and (b) follow from Proposition 3.2 whose proof assumes that Conjecture 2.6 is true. Part (c) is Proposition 3.3. Although $\breve{\mathscr{B}}(V)$ of part (c) is not the distinguished pre-Nichols algebra of the quantum plane V, it can be thought of as the distinguished one of the braided vector space of Cartan type G_2, but degenerated in the sense that the parameter is a primitive third root of unity. Via suitable bosonizations, $\breve{\mathscr{B}}(V)$ provides new examples of pointed Hopf algebras with finite GK-dim.

Let $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(V)$. By (a) and (1.2) we have a surjective map of pre-Nichols algebras $\mathscr{B}^{\pm,3} \underline{\otimes} \widetilde{\mathscr{B}}(V^t) \underline{\otimes} \widetilde{\mathscr{B}}(V^\infty) \to \mathscr{B}$. Therefore it remains to understand $\mathscr{B}^{\pm,3}$. Towards this, we know:

- The pre-Nichols algebras of $V^{\pm}$ with finite GK-dim are (up to a twist) the enveloping superalgebras $U(\mathfrak{n})$, where $\mathfrak{n} = \oplus_{j\in\mathbb{N}} \mathfrak{n}^j$ is a finite-dimensional Lie superalgebra generated by $\mathfrak{n}^1 \simeq V$, see §2.9.2.
- By Proposition 3.2, $\mathscr{B}^{2,3} \simeq \mathscr{B}^{2} \underline{\otimes} \mathscr{B}^{3}$.
- For instance, if $V^1 = 0$, then there is a surjective map of pre-Nichols algebras $\mathscr{B}^{2} \underline{\otimes} \widetilde{\mathscr{B}}(V^3) \underline{\otimes} \widetilde{\mathscr{B}}(V^t) \underline{\otimes} \widetilde{\mathscr{B}}(V^\infty) \to \mathscr{B}$.

Towards $\mathscr{B}^{1,3}$ we know Part (c) and §3.3. It is natural to ask:

Question 1.2. Assume that $V = V^1 \oplus V^3$, $\dim V^1 = 1$ and $\dim V^3 = 2$. Is the distinguished pre-Nichols algebra $\widetilde{\mathscr{B}}(V)$ eminent?

1.2.2. *Connected Cartan type.* Here $\mathfrak{q}$ is of finite Cartan type, i.e.

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \qquad -\operatorname{ord} q_{ii} < a_{ij} \leq 0, \qquad i \neq j \in \mathbb{I},$$

where $\mathbf{a} = (a_{ij})_{i,j\in\mathbb{I}}$ is a Cartan matrix of finite type with connected Dynkin diagram. In §4 we recall the possibilities for such $\mathfrak{q}$. They depend on a root of unity q, whose order is denoted by N. In the following statement the symbols $x_{1112}, x_{2221}, x_{2112}, x_{1221}$ are defined in (2.1).

Theorem 1.3.

(a) *The distinguished pre-Nichols algebra* $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ *is eminent except in the following cases:* A_2 *with* $N = 3$,

$$A_\theta, \quad \theta \geq 2, \quad N = 2; \qquad D_\theta, \quad \theta \geq 4, \quad N = 2; \qquad G_2, \quad N = 4, 6. \tag{1.3}$$

(b) *Suppose* $\mathfrak{q}$ *is of type* A_2 *with* $N = 3$. *Then*

$$\widehat{\mathscr{B}} = \Bbbk\langle x_1, x_2 | x_{1112}, x_{2221}, x_{2112}, x_{1221}\rangle$$

is an eminent pre-Nichols algebra of $\mathfrak{q}$, *and* $\operatorname{GK-dim} \widehat{\mathscr{B}} = 5$.

This answers (partially) a question in [An3].

The proof of (a) is given in Lemmas 4.12, 4.13, 4.15, 4.16, 4.17, 4.18. For the cases listed in (1.3) the determination of the poset $\mathfrak{Pre}_{\mathrm{fGK}}(V)$ remains an open problem except for G_2, with $N = 4, 6$ that was solved in [ACS]. See Section 5 for partial results; answers to Questions 5.2, 5.5, 5.7, 5.9 and 5.11 would shed light on the issue. The proof of (b) is given in Proposition 4.11. The eminent pre-Nichols algebra $\widehat{\mathscr{B}}$ is introduced and studied in §4.2.2. There we show that $\widehat{\mathscr{B}}$ properly covers the distinguished pre-Nichols algebra $\widetilde{\mathscr{B}}(V)$, which has $\operatorname{GK-dim} \widetilde{\mathscr{B}}(V) = 3$.

2. Preliminaries

2.1. **Conventions.** For $n \leq m \in \mathbb{N}_0$, put $\mathbb{I}_{n,m} = \{k \in \mathbb{N}_0 \colon n \leq k \leq m\}$ and $\mathbb{I}_m = \mathbb{I}_{1,m}$. Given a positive integer N, we denote by $\mathbb{G}_N$ the group of N-th roots of unity in $\Bbbk^\times$, and by $\mathbb{G}'_N \subset \mathbb{G}_N$ the subset of those of order N. The group of all roots of unity is denoted by $\mathbb{G}_\infty$ and $\mathbb{G}'_\infty := \mathbb{G}_\infty - \{1\}$.

The subalgebra generated by a subset X of an associative algebra is denoted by $\Bbbk\langle X\rangle$.

All Hopf algebras are assumed to have bijective antipode. If H is a Hopf algebra, the group of group-like elements is denoted by $G(H)$, while $\mathcal{P}(H)$ is the subspace of primitive elements. By $\operatorname{gr} H$ we mean the graded coalgebra associated to the coradical filtration.

If A and B are algebras in ${}^H_H\mathcal{YD}$, we denote by $A \underline{\otimes} B = (A \otimes B, \mu_{A\underline{\otimes}B})$ the algebra with multiplication $\mu_{A\underline{\otimes}B} = (\mu_A \otimes \mu_B)(\mathrm{id}_A \otimes c_{B,A} \otimes \mathrm{id}_B)$, where μ_A and μ_B are the multiplications of A and B, respectively.

2.2. **Gelfand-Kirillov dimension.** We refer to [KL] for general information on this topic. The following useful statement is immediate from the definition of GK-dim. Let R be a ring and let $M = \oplus_{n\in\mathbb{N}_0} M_n$ be a direct sum of R-modules M_n which are free of finite rank (we say M is a locally finite graded R-module). The Poincaré series of M is

$$P_M := \sum_{n\in\mathbb{N}_0} \operatorname{rank} M_n \, X^n \in \mathbb{Z}[[X]].$$

Lemma 2.1. *Let* $\mathbb{L}$ *and* $\mathbb{F}$ *be fields and let* $T = \oplus_{n\in\mathbb{N}_0} T_n$ *and* $U = \oplus_{n\in\mathbb{N}_0} U_n$ *be two locally finite graded algebras generated in degree one over* $\mathbb{L}$ *and* $\mathbb{F}$ *respectively. If* $P_T = P_U$, *then* $\operatorname{GK-dim} T = \operatorname{GK-dim} U$. □

Actually [KL, 12.6.2] shows that the Poincaré series of a graded finitely generated algebra provides its GK-dim.

2.3. **Braided Hopf algebras.** A pair (V, c) where V is a vector space and $c \in GL(V^{\otimes 2})$ satisfies the braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$$

is called a braided vector space. A braided vector space with compatible algebra and coalgebra structures as in [T] is called a braided Hopf algebra. For instance the tensor algebra $T(V)$ has a canonical structure of (graded connected) braided Hopf algebra such that the elements of degree 1 are primitive. Also the tensor coalgebra $T^c(V)$ becomes a braided Hopf algebra by the twisted shuffle product; see e.g. [R, Proposition 9]. There is a homogeneous morphism of braided Hopf algebras $\Omega\colon T(V) \to T^c(V)$ determined by $\Omega(v) = v$, $v \in V$; its image is the Nichols algebra of V, denoted $\mathscr{B}(V)$. In fact Ω is the quantum symmetrizer, see e.g. [A, Section 3.3].

Another description: let $\mathcal{J}(V)$ be the largest element of the set $\mathfrak{S}$ of graded Hopf ideals of $T(V)$ trivially intersecting $\Bbbk \oplus V$. Then $\mathscr{B}(V) \simeq T(V)/\mathcal{J}(V)$.

2.4. **Principal realizations.** Theorems 1.1 and 1.3 are relevant for the classification of Hopf algebras with finite GK-dim. Indeed a braided vector space arises (up to a mild condition) as a Yetter-Drinfeld module over a Hopf algebra; this is called a realization. Realizations are not unique and we single out a class of them for braidings of diagonal type. Let H be a Hopf algebra. A YD-pair is a couple $(g, \chi) \subset G(H) \times \mathrm{Hom}_{\mathrm{Alg}}(H, \Bbbk)$ satisfying

$$\chi(h)g = \chi(h_{(2)})h_{(1)}g\mathcal{S}(h_{(3)}), \qquad h \in H.$$

Compare with [AS1, p. 671]. This compatibility guarantees that $\Bbbk_g^\chi$ (i. e. H acting and coacting on $\Bbbk$ by χ and g, respectively) is a Yetter-Drinfeld module over H. Let $(V, c^{\mathfrak{q}})$ be a braided vector space of diagonal type. Following [AS1, p. 673], a principal realization of $(V, c^{\mathfrak{q}})$ over H is a family $(g_i, \chi_i)_{i \in \mathbb{I}}$ of YD-pairs such that $q_{ij} = \chi_j(g_i)$ for all i, j. In this case $V = \bigoplus_i \Bbbk_{g_i}^{\chi_i} \in {}^H_H\mathcal{YD}$.

2.5. **Pre-Nichols and post-Nichols algebras.** We present in detail the objects of interest in this paper.

- Let $\mathscr{B} = \bigoplus_{n \in \mathbb{N}_0} \mathscr{B}^n$ be a graded connected braided Hopf algebra with $\mathscr{B}^1 \simeq V$. Then $\mathscr{B}$ is a *pre-Nichols* algebra of V if it is generated by $\mathscr{B}^1$. In this case there are epimorphisms of (graded) braided Hopf algebras

$$T(V) \twoheadrightarrow \mathscr{B} \twoheadrightarrow \mathscr{B}(V).$$

Hence the set $\mathfrak{Pre}(V)$ of isomorphism classes of pre-Nichols algebras of V is partially ordered with $T(V)$ minimal and $\mathscr{B}(V)$ maximal:

$$\begin{tikzcd}
 & & & T(V) \arrow[dlll, twoheadrightarrow] \arrow[d, twoheadrightarrow] \arrow[drrr, twoheadrightarrow] & & & \\
\mathscr{B} \arrow[drrr, twoheadrightarrow] & \dots & \dots & \mathscr{B}' \arrow[d, twoheadrightarrow] & \dots & \dots & \mathscr{B}'' \arrow[dlll, twoheadrightarrow] \\
 & & & \mathscr{B}(V) & & &
\end{tikzcd}$$

- Dually, a graded connected braided Hopf algebra $\mathcal{E} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{E}^n$ with $\mathcal{E}^1 \simeq V$ is a *post-Nichols* algebra of V if it is coradically graded. Thus we have monomorphisms of (graded) braided Hopf algebras

$$\mathscr{B}(V) \hookrightarrow \mathcal{E} \hookrightarrow T^c(V).$$

Hence the set $\mathfrak{Post}(V)$ of isomorphism classes of post-Nichols algebras of V is partially ordered with $T^c(V)$ maximal and $\mathscr{B}(V)$ minimal:

$$\begin{tikzcd}
 & & & \mathscr{B}(V) \arrow[dlll, hook'] \arrow[d, hook] \arrow[drrr, hook] & & & \\
\mathscr{B} \arrow[drrr, hook] & \dots & \dots & \mathscr{B}' \arrow[d, hook] & \dots & \dots & \mathscr{B}'' \arrow[dlll, hook'] \\
 & & & T(V) & & &
\end{tikzcd}$$

The only pre-Nichols which is also a post-Nichols algebra of V is $\mathscr{B}(V)$ itself.

2.6. **Eminent pre- and post-Nichols algebras.** For the purposes of classifying Hopf algebras with finite GK-dim, it is important to describe the (partially ordered) subset $\mathfrak{Post}_{\mathrm{fGK}}(V)$ of $\mathfrak{Post}(V)$ consisting of post-Nichols algebras with finite GK-dim. In this paper we are mainly interested in the (partially ordered) subset $\mathfrak{Pre}_{\mathrm{fGK}}(V)$ of $\mathfrak{Pre}(V)$ consisting of pre-Nichols algebras with finite GK-dim. The reason to start with this is given by the following result:

Lemma 2.2 ([AAH3])**.** *Let $\mathscr{B}$ be a pre-Nichols algebra of V and let $\mathcal{E} = \mathscr{B}^d$ be the graded dual of $\mathscr{B}$. Then* $\operatorname{GK-dim} \mathcal{E} \leq \operatorname{GK-dim} \mathscr{B}$. *If $\mathcal{E}$ is finitely generated, then the equality holds.* □

A first approximation to the determination of $\mathfrak{Post}_{\mathrm{fGK}}(V)$ and $\mathfrak{Pre}_{\mathrm{fGK}}(V)$ is through the following notion.

Definition 2.3.

(a) A pre-Nichols algebra $\widehat{\mathscr{B}}$ is *eminent* if it is the minimum of $\mathfrak{Pre}_{\mathrm{fGK}}(V)$; i. e. there is an epimorphism of braided Hopf algebras $\widehat{\mathscr{B}} \twoheadrightarrow \mathscr{B}$ that is the identity on V for any $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(V)$.

(b) A post-Nichols algebra $\widehat{\mathcal{E}}$ is *eminent* if it is the maximum of $\mathfrak{Post}_{\mathrm{fGK}}(V)$; that is for any $\mathcal{E} \in \mathfrak{Post}_{\mathrm{fGK}}(V)$, there is a monomorphism of braided Hopf algebras $\mathcal{E} \hookrightarrow \widehat{\mathcal{E}}$ that is the identity on V.

Beware that there are braided vector spaces without eminent pre-Nichols algebras; e. g., if $\dim V > 1$ and the braiding is the usual flip, then $\mathfrak{Pre}_{\mathrm{fGK}}(V)$ has infinite chains. An intermediate situation could be described as follows.

Definition 2.4. A family $(\widehat{\mathscr{B}}_i)_{i\in I} \subset \mathfrak{Pre}_{\mathrm{fGK}}(V)$ is *eminent* if

(a) for any $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(V)$, there exists $i \in I$ and an epimorphism of braided Hopf algebras $\widehat{\mathscr{B}}_i \twoheadrightarrow \mathscr{B}$ that is the identity on V, and

(b) $(\widehat{\mathscr{B}}_i)_{i\in I}$ is minimal among the families in $\mathfrak{Pre}_{\mathrm{fGK}}(V)$ satisfying (a).

Eminent families of post-Nichols algebras are defined similarly.

All the notions above about braided Hopf algebras related to braided vector spaces have a counterpart for Yetter-Drinfeld modules. Namely, suppose that (V, c) is realized in ${}^H_H\mathcal{YD}$ for some Hopf algebra H. Then $\mathfrak{Pre}^H(V)$ is the subset of $\mathfrak{Pre}(V)$ of pre-Nichols algebras that belong to ${}^H_H\mathcal{YD}$; similarly we have $\mathfrak{Pre}^H_{\mathrm{fGK}}(V)$, $\mathfrak{Post}^H(V)$, $\mathfrak{Post}^H_{\mathrm{fGK}}(V)$, and also H-eminent pre-Nichols or post-Nichols algebras.

2.7. **The adjoint representation and q-brackets.** Any Hopf algebra R in ${}^H_H\mathcal{YD}$ comes equipped with the (left) *adjoint* representation $\mathrm{ad}_c\colon R \to \operatorname{End} R$, given by

$$(\mathrm{ad}_c\, x)y = \mu(\mu \otimes \mathcal{S})(\mathrm{id} \otimes c)(\Delta \otimes \mathrm{id})(x \otimes y), \qquad x, y \in R,$$

where μ, Δ and $\mathcal{S}$ denote the multiplication, comultiplication and antipode of R, respectively. The adjoint action of a primitive element $x \in R$ is

$$(\mathrm{ad}_c\, x)y = xy - (x_{(-1)} \cdot y)x_{(0)}, \qquad y \in R.$$

Given $x_{i_1}, x_{i_2} \dots, x_{i_k} \in R$, put

$$x_{i_1 i_2 \dots i_k} = (\mathrm{ad}_c\, x_{i_1}) \dots (\mathrm{ad}_c\, x_{i_{k-1}})x_{i_k}. \tag{2.1}$$

We also set $x_{(k\,h)} = x_{k\,(k+1)\,(k+2)\dots h}$ for $k < h$.

On the other hand, the *braided commutator* is defined by

$$[x, y]_c = xy - (x_{(-1)} \cdot y)x_{(0)}, \qquad x, y \in R.$$

We refer to [AA2, Introduction] for a more detailed treatment.

2.8. **Nichols algebras of diagonal type.** Fix a natural number θ and let $\mathbb{I} = \mathbb{I}_\theta$. Any matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}}$ with coefficients in $\Bbbk^\times$ determines a braided vector space *of diagonal type* $(V, c^{\mathfrak{q}})$, where

$$V \text{ has a basis } (x_i)_{i\in\mathbb{I}}, \qquad c^{\mathfrak{q}}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \qquad i, j \in \mathbb{I}. \tag{2.2}$$

The *Dynkin diagram* associated to $\mathfrak{q}$ is a non-oriented graph with θ vertices. The vertex i is labelled by q_{ii}, and there is an edge between i and j if and only if $\widetilde{q}_{ij} := q_{ij}q_{ji} \neq 1$; in this case, the edge is labeled by $\widetilde{q}_{ij}$. Thus we may speak of the connected components of this diagram and by abuse of notation of $\mathfrak{q}$. The following useful result says that a connected component with at least 2 vertices one of them labelled by 1 gives rise to an infinite GK-dimensional Nichols algebra.

Lemma 2.5 ([AAH1, Lemma 2.8])**.** *Let U be a braided vector space of diagonal type with Dynkin diagram*

$$\overset{q}{\circ} \xrightarrow{\quad r \quad} \overset{1}{\circ}\,, \qquad r \neq 1.$$

Then $\operatorname{GK-dim} \mathscr{B}(U) = \infty$. □

Let $\alpha_1, \dots, \alpha_\theta$ be the canonical basis of $\mathbb{Z}^\theta$. From the braiding matrix $\mathfrak{q}$ we obtain a $\Bbbk^\times$-valued bilinear form on $\mathbb{Z}^\theta$, still denoted $\mathfrak{q}$ and determined by $\mathfrak{q}(\alpha_i, \alpha_j) = q_{ij}$, $i, j \in \mathbb{I}$. Put also

$$\widetilde{\mathfrak{q}}(\alpha, \beta) := \mathfrak{q}(\alpha, \beta)\mathfrak{q}(\beta, \alpha), \qquad \alpha,\, \beta \in \mathbb{Z}^\theta. \tag{2.3}$$

For sake of brevity, we use $\mathfrak{q}_{\alpha\beta} = \mathfrak{q}(\alpha, \beta)$ and $\widetilde{\mathfrak{q}}_{\alpha\beta} = \widetilde{\mathfrak{q}}(\alpha, \beta)$ as well.

The braided vector space $(V, c^{\mathfrak{q}})$ as in (2.2) is realized in ${}^{\mathbb{Z}^\theta}_{\mathbb{Z}^\theta}\mathcal{YD}$ by declaring

$$\deg(x_i) = \alpha_i, \qquad \alpha_i \cdot x_j = q_{ij} x_j, \qquad i, j \in \mathbb{I}. \tag{2.4}$$

The algebra $T(V)$ becomes $\mathbb{Z}^\theta$-graded. Thus any quotient algebra R of $T(V)$ by a graded ideal inherits the grading: $R = \bigoplus_{\alpha\in\mathbb{Z}^\theta} R^\alpha$. We keep the notation deg for this degree. Furthermore, if R is an algebra obtained as a quotient of $T(V)$ by a graded ideal I (thus a subobject in ${}^{\mathbb{Z}^\theta}_{\mathbb{Z}^\theta}\mathcal{YD}$), then the braiding on the homogeneous subspaces is given by

$$c(u\otimes v) = \mathfrak{q}_{\alpha,\beta}\, v\otimes u, \qquad u\in R^\alpha,\ v\in R^\beta. \tag{2.5}$$

We shall use (2.5) many times. The braided commutators satisfy

$$[u, vw]_c = [u, v]_c w + \mathfrak{q}_{\alpha\beta} v[u, w]_c, \tag{2.6}$$

$$[uv, w]_c = \mathfrak{q}_{\beta\gamma}[u, w]_c v + u[v, w]_c, \tag{2.7}$$

$$\big[[u, v]_c, w\big]_c = \big[u, [v, w]_c\big]_c - \mathfrak{q}_{\alpha\beta} v[u, w]_c + \mathfrak{q}_{\beta\gamma}[u, w]_c v, \tag{2.8}$$

for homogeneous elements $u\in R^\alpha$, $v\in R^\beta$, $w\in R^\gamma$.

In the diagonal setting (2.2) we set as usual $\mathcal{J}_{\mathfrak{q}} = \mathcal{J}(V)$, $\mathscr{B}_{\mathfrak{q}} = \mathscr{B}(V)$, $\widetilde{\mathscr{B}}_{\mathfrak{q}} = \widetilde{\mathscr{B}}(V)$, etc. Nichols algebras *of diagonal type* (i. e. those arising from braided vector spaces of diagonal type) have been intensively studied. The classification of all matrices $\mathfrak{q}$ such that $\mathscr{B}_{\mathfrak{q}}$ has finite root system was provided in [H1]; the defining relations of these Nichols algebras are given in [An1, An2]. Clearly, finite dimensional Nichols algebras of diagonal type have finite root system. It was conjectured that those of finite GK-dim share the same property.

Conjecture 2.6 ([AAH1, Conjecture 1.5])**.** *The root system of a Nichols algebra of diagonal type with finite GK-dimension is finite.*

The validity of Conjecture 2.6 would imply the classification of finite GK-dimensional Nichols algebras of diagonal type. There is strong evidence supporting it. The conjecture holds when $\theta = 2$ [AAH2, Thm. 4.1], when the braiding is of affine Cartan type [AAH2, Thm. 1.2], or when $\mathfrak{q}$ is generic, that is $q_{ii}\notin\mathbb{G}_\infty$, and $q_{ij}q_{ji} = 1$ or $q_{ij}q_{ji}\notin\mathbb{G}_\infty$, for all $i\neq j\in\mathbb{I}$ [R, AA1].

We include for completeness proofs of the following well-known results.

Lemma 2.7. *Let* $0\neq v, w\in T(V)$ *be homogeneous primitive elements with* $\deg v = \alpha$ *and* $\deg w = \beta$. *Then* $(\operatorname{ad}_c v)w$ *is primitive if and only if* $\widetilde{\mathfrak{q}}_{\alpha\beta} = 1$.

Proof. Using (2.5), compute $\Delta((\operatorname{ad}_c v)w) = \Delta(vw - \mathfrak{q}_{\alpha\beta}\, wv) =$

$$\begin{aligned}
&= (v\otimes 1 + 1\otimes v)(w\otimes 1 + 1\otimes w) - \mathfrak{q}_{\alpha\beta}\,(w\otimes 1 + 1\otimes w)(v\otimes 1 + 1\otimes v)\\
&= vw\otimes 1 + v\otimes w + \mathfrak{q}_{\alpha\beta}\, w\otimes v + 1\otimes vw\\
&\quad - \mathfrak{q}_{\alpha\beta}\,(wv\otimes 1 + w\otimes v + \mathfrak{q}_{\beta\alpha}\, v\otimes w + 1\otimes wv)\\
&= (\operatorname{ad}_c v)w\otimes 1 + 1\otimes(\operatorname{ad}_c v)w + \big(1 - \widetilde{\mathfrak{q}}_{\alpha\beta}\big)v\otimes w.
\end{aligned}$$

□

Lemma 2.8. *Let R be a graded braided Hopf algebra. If W is any braided subspace of R contained in $\mathcal{P}(R)$ then* GK-dim $\mathscr{B}(W)\leq$ GK-dim R.

Proof. We follow [AS4, Lemma 5.4]. Since the elements of W are primitive, the subalgebra $\Bbbk\langle W\rangle$ is a braided Hopf subalgebra of R; by definition of the Nichols algebra it follows that $\operatorname{gr}\Bbbk\langle W\rangle$ projects onto $\mathscr{B}(W)$, so GK-dim $\mathscr{B}(W)\leq$ GK-dim $\operatorname{gr}\Bbbk\langle W\rangle$. But GK-dim $\operatorname{gr}\Bbbk\langle W\rangle\leq$ GK-dim $\Bbbk\langle W\rangle$ by [KL, Lemma 6.5], and this proves the desired inequality. □

2.9. **Pre-Nichols algebras of diagonal type.** Let $(V, c^{\mathfrak{q}})$ be a braided vector space of diagonal type associated to the matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}}$. Recall that $\widetilde{q}_{ij} = q_{ij}q_{ji}$, $i \neq j$. We write $\mathfrak{Pre}^{\mathbb{Z}^\theta}_{\mathrm{fGK}}(V)$ for $\mathfrak{Pre}^{\Bbbk\mathbb{Z}^\theta}_{\mathrm{fGK}}(V)$, cf. (2.4).

2.9.1. *Pre-Nichols algebras under twist-equivalence.* Let $\mathfrak{p} = (p_{ij})_{i,j\in\mathbb{I}}$ be another braiding matrix such that

$$q_{ii} = p_{ii}, \qquad \widetilde{q}_{ij} = \widetilde{p}_{ij}, \qquad i, j \in \mathbb{I}.$$

In this case, $(V, c^{\mathfrak{q}})$ and the braided vector space $(W, c^{\mathfrak{p}})$ with basis $(y_i)_{i\in\mathbb{I}}$ are said to be *twist-equivalent.*

Lemma 2.9. *There is an isomorphism of posets* $\mathfrak{Pre}^{\mathbb{Z}^\theta}_{\mathrm{fGK}}(W) \simeq \mathfrak{Pre}^{\mathbb{Z}^\theta}_{\mathrm{fGK}}(V)$.

Proof. Let $\sigma : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk^\times$ be the bilinear form, hence a 2-cocycle, given by $\sigma(\alpha_i, \alpha_j) = \begin{cases} p_{ij}q_{ij}^{-1}, & i \leq j, \\ 1, & i > j \end{cases}$. Let $T(V)_\sigma$ be the corresponding cocycle deformation of $T(V)$, i. e. with multiplication

$$u \cdot_\sigma v = \sigma(\alpha, \beta)uv, \qquad u \in T(V)^\alpha,\ v \in T(V)^\beta,\ \alpha, \beta \in \mathbb{Z}^\theta. \tag{2.9}$$

By the proof of [AS3, Prop. 3.9] the linear map $\varphi : W \to V$, $\varphi(y_i) = x_i$, $i \in \mathbb{I}$, induces an isomorphism $\varphi : T(W) \to T(V)_\sigma$ of Hopf algebras in ${}^{\mathbb{Z}^\theta}_{\mathbb{Z}^\theta}\mathcal{YD}$. Let I be a Hopf ideal of $T(V)$ that belongs to ${}^{\mathbb{Z}^\theta}_{\mathbb{Z}^\theta}\mathcal{YD}$; then it is also a Hopf ideal of $T(V)_\sigma$ and $\operatorname{GK-dim} T(V)/I = \operatorname{GK-dim} T(V)_\sigma/I$ by Lemma 2.1. $\square$

2.9.2. *Pre-Nichols algebras of super symmetric algebras.* Assume that $\widetilde{q}_{ij} = 1 = q_{ii}^2$, for all $j \neq i \in \mathbb{I}$. Then $V = V_0 \oplus V_1$ is a super vector space where V_j is spanned by those x_i's such that $q_{ii} = (-1)^j$, $j = 0, 1$. Let $\mathfrak{p} = (p_{ij})_{i,j\in\mathbb{I}}$ be the matrix corresponding to the associated super symmetry. Then

- The pre-Nichols algebras of $(V, c^{\mathfrak{p}})$ are the enveloping superalgebras $U(\mathfrak{n})$, where $\mathfrak{n} = \oplus_{j\in\mathbb{N}}\mathfrak{n}^j$ is a graded Lie superalgebra generated by $\mathfrak{n}^1 \simeq V$.
- $\mathfrak{Pre}_{\mathrm{fGK}}(V, c^{\mathfrak{p}})$ consists of the enveloping superalgebras $U(\mathfrak{n})$, where $\mathfrak{n} = \oplus_{j\in\mathbb{N}}\mathfrak{n}^j$ is a graded Lie superalgebra generated by $\mathfrak{n}^1 \simeq V$ with $\dim \mathfrak{n} < \infty$.
- Hence $\mathfrak{Pre}^{\mathbb{Z}^\theta}_{\mathrm{fGK}}(V, c^{\mathfrak{p}})$ consists of the enveloping superalgebras $U(\mathfrak{n})$, where $\mathfrak{n} = \oplus_{\beta\in\mathbb{Z}^\theta}\mathfrak{n}^\beta$ is a finite-dimensional $\mathbb{Z}^\theta$-graded Lie superalgebra generated by $\mathfrak{n}^1 = \oplus_{i\in\mathbb{I}}\mathfrak{n}^{\alpha_i} \simeq V$. In particular $\mathfrak{Pre}^{\mathbb{Z}^\theta}_{\mathrm{fGK}}(V, c^{\mathfrak{p}}) \subsetneq \mathfrak{Pre}_{\mathrm{fGK}}(V, c^{\mathfrak{p}})$.
- By Lemma 2.9, $\mathfrak{Pre}^{\mathbb{Z}^\theta}_{\mathrm{fGK}}(V, c^{\mathfrak{q}})$ is isomorphic as a poset to the set of isomorphism classes of finite-dimensional $\mathbb{Z}^\theta$-graded Lie superalgebras as in the previous point.

3. Quantum linear spaces

In this section we investigate finite GK-dimensional pre-Nichols algebras of quantum linear spaces. These are Nichols algebras of braided vector spaces of diagonal type with totally disconnected Dynkin diagram. More precisely, fix a matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}}$ and a vector space V with basis $(x_i)_{i\in\mathbb{I}}$ and braiding given by $c^{\mathfrak{q}}(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$, $i, j \in \mathbb{I}$. In this section we assume that

$$q_{ij}q_{ji} = 1, \qquad i \neq j \in \mathbb{I}. \tag{3.1}$$

Then $\mathscr{B}_{\mathfrak{q}}$ is presented by generators $(x_i)_{i\in\mathbb{I}}$ and relations

$$x_{ij} = 0, \quad \text{if } i < j, \tag{3.2}$$

$$x_i^{N_i} = 0, \quad \text{if } q_{ii} \in \mathbb{G}'_\infty, \qquad \text{where } N_i := \operatorname{ord} q_{ii} \in \mathbb{N} \cup \infty; \tag{3.3}$$

here we are using the notation (2.1). It has a PBW-basis:

$$\{x_1^{a_1} x_2^{a_2} \cdots x_\theta^{a_\theta} : 0 \le a_i < N_i \text{ if } q_{ii} \in \mathbb{G}'_\infty;\ 0 \le a_i \text{ otherwise}\}. \tag{3.4}$$

As defined in the Introduction, the *distinguished pre-Nichols algebra* $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ *of* V is presented by generators $(x_i)_{i\in\mathbb{I}}$ and relations (3.2); it is a domain of GK-dim θ. Recall the partition $\mathbb{I} = \mathbb{I}^{\pm} \sqcup \mathbb{I}^{3} \sqcup \mathbb{I}^{t} \sqcup \mathbb{I}^{\infty}$ where as in (1.1) we set

$$\mathbb{I}^{\infty} = \{i \in \mathbb{I}\colon q_{ii} \notin \mathbb{G}_\infty\}, \qquad \mathbb{I}^{\pm} = \{i \in \mathbb{I}\colon q_{ii} = \pm 1\},$$

$$\mathbb{I}^{N} = \{i \in \mathbb{I}\colon q_{ii} \in \mathbb{G}'_N\},\ N \ge 3, \qquad \mathbb{I}^{t} = \bigcup_{N>3} \mathbb{I}^{N}.$$

For $\star \in \{\pm, 3, t, \infty\}$, let $V^\star$ be the subspace of V spanned by $(x_i)_{i\in\mathbb{I}^\star}$ and $\mathfrak{q}^\star$ the restriction of $\mathfrak{q}$ to $V^\star$. Then $V = V^{\pm} \oplus V^{3} \oplus V^{t} \oplus V^{\infty}$. As we have seen in §2.9.2 the $\mathbb{Z}^\theta$-graded pre-Nichols algebras of $V^{\pm}$ are twistings of enveloping algebras of nilpotent Lie superalgebras with suitable properties; in particular, there is no eminent pre-Nichols algebra of $V^{\pm}$.

3.1. **Reduction to order** ≤ 3**.** Below we consider various braided vector spaces of diagonal type, see §2.8 for the recipe of the Dynkin diagram that encodes the matrix $\mathfrak{q}$ that determines the braiding.

Remark 3.1. Let $i \ne j \in \mathbb{I}$. Recall that $x_{ij} := (\operatorname{ad}_c x_i) x_j = x_i x_j - q_{ij} x_j x_i$. The braiding of the 3-dimensional subspace $\Bbbk x_i + \Bbbk x_{ij} + \Bbbk x_j \subset T(V)$ is easily computed, and the corresponding Dynkin diagram is either

$$\underset{i}{\overset{q_{ii}}{\circ}} \xrightarrow{\quad q_{ii}^2 \quad} \underset{ij}{\overset{q_{ii}q_{jj}}{\circ}} \xrightarrow{\quad q_{jj}^2 \quad} \underset{j}{\overset{q_{jj}}{\circ}}\ , \tag{3.5}$$

or it is disconnected if the label of some edge is 1. Indeed,

$$q_{i,ij} = q_{ii}q_{ij}, \quad q_{ij,i} = q_{ii}q_{ji}, \quad q_{ij,ij} = q_{ii}\widetilde{q}_{ij}q_{jj}, \quad q_{ij,j} = q_{ij}q_{jj}, \quad q_{j,ij} = q_{ji}q_{jj}$$

so $\widetilde{q}_{i,ij} = q_{ii}^2 \widetilde{q}_{ij}$ and $\widetilde{q}_{ij,j} = q_{jj}^2 \widetilde{q}_{ij}$. Since $\widetilde{q}_{ij} = 1$ because we are in the quantum linear space situation, (3.5) is the Dynkin diagram of $\Bbbk x_i + \Bbbk x_{ij} + \Bbbk x_j$.

Proposition 3.2. *Let* $i, j \in \mathbb{I}$ *such that* $4 < \operatorname{ord} q_{ii} + \operatorname{ord} q_{jj}$. *Assume that Conjecture* 2.6 *is true. Then* $x_{ij} = 0$ *holds in any finite GK-dimensional pre-Nichols algebra of* $\mathfrak{q}$.

We point out that Conjecture 2.6 is needed only to discard cases (1), (5) and (8) below, that require the conjecture only for dimension 3.

Proof. Let $\mathscr{B}$ be a pre-Nichols algebra of $\mathfrak{q}$, so there is a braided Hopf algebra map $T(V) \to \mathscr{B}$. Let y_1, y_2, y_3 denote the image of x_i, x_j, x_{ij}, respectively, and consider $W := \Bbbk y_1 + \Bbbk y_2 + \Bbbk y_3$. By Lemma 2.7 we have $W \subset \mathcal{P}(\mathscr{B})$, hence Lemma 2.8 warranties GK-dim $\mathscr{B}(W) \le$ GK-dim $\mathscr{B}$.

Assume $y_3 \ne 0$, so W is 3-dimensional by a degree argument and its Dynkin diagram $\mathfrak{D}$ is (3.5). We show that GK-dim $\mathscr{B}(W) = \infty$.

Consider the subspaces $V_1 = \Bbbk y_1 \oplus \Bbbk y_3$, $V_2 = \Bbbk y_3 \oplus \Bbbk y_2 \subset W$; denote their corresponding Dynkin diagrams by $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively. From $q_{ij}q_{ji} = 1$ it follows $x_{ij} = -q_{ij}x_{ji}$, so the image of x_{ji} in $\mathscr{B}$ is not zero.

We split the proof in several cases according to the possibilities for $\operatorname{ord} q_{ii}$ and $\operatorname{ord} q_{jj}$.

Case 1 ($q_{ii} \notin \mathbb{G}_\infty$ or $q_{jj} \notin \mathbb{G}_\infty$). This essentially goes back to [R]. Assume first $q_{ii} \notin \mathbb{G}_\infty$. If $\operatorname{GK-dim} \mathscr{B} < \infty$, it follows from [AAH1, Lemmas 2.6 and 2.7] that there exists a natural number k such that $(k)^!_{q_{ii}} \prod_{h=0}^{k-1}(1-q_{ii}^h) = 0$, which contradicts $q_{ii} \notin \mathbb{G}_\infty$. The case $q_{jj} \notin \mathbb{G}_\infty$ is similar: since the image of x_{ji} is not zero, we may apply the same argument as with q_{ii}.

Case 2 ($q_{ii} = 1$ or $q_{jj} = 1$). We may suppose $q_{jj} = 1$; if $q_{ii} = 1$, the same argument applies. By the previous case, we may assume q_{ii} is a root of unity, and by hypothesis its order must be $N_i > 3$. The diagram $\mathfrak{D}_1$ is

$$\overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{60pt}{0.5pt}} \overset{q_{ii}}{\underset{ij}{\circ}}\,, \qquad \text{Cartan type} \quad \begin{pmatrix} 2 & 2-N_i \\ 2-N_i & 2 \end{pmatrix}.$$

If $\operatorname{GK-dim} \mathscr{B}(V_1) < \infty$ then [AAH2] implies that the Cartan matrix is of finite type. Thus we conclude $N_i = 3$, a contradiction.

Case 3 ($q_{ii} = 1$ or $q_{jj} = -1$). Assume that $q_{jj} = 1$. By Case 1, we may assume that q_{ii} is a root of unity; by hypothesis, its order is ≥ 3. By [AAH2], $\operatorname{GK-dim} \mathscr{B}(V_1) = \infty$ since the Dynkin diagram of V_1 is

$$\overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{60pt}{0.5pt}} \overset{-q_{ii}}{\underset{ij}{\circ}}$$

and this does not appear in [H1, Table 1]; indeed it is of Cartan, but not finite, type. The case $q_{ii} = -1$ is treated similarly.

Case 4 ($q_{ii}, q_{jj} \in \mathbb{G}_\infty - \mathbb{G}_2$). Now W has connected Dynkin diagram

$$\mathfrak{D} = \overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{60pt}{0.5pt}} \overset{q_{ii}q_{jj}}{\underset{ij}{\circ}} \overset{q_{jj}^2}{\rule[3pt]{60pt}{0.5pt}} \overset{q_{jj}}{\underset{j}{\circ}}\,.$$

If the Nichols algebra of V_1 is finite GK-dimensional, by exhaustion of [H1, Table 1] we conclude that q_{ii}, q_{jj} and $\mathfrak{D}_1$ satisfy one of the following:

(1) $q_{ii} \in \mathbb{G}'_3, q_{ii}q_{jj} = -1,\quad \overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{30pt}{0.5pt}} \overset{-1}{\underset{ij}{\circ}}$

(2) $q_{ii} \in \mathbb{G}'_4,\ q_{jj} = q_{ii},\quad \overset{q_{ii}}{\underset{i}{\circ}} \overset{-1}{\rule[3pt]{30pt}{0.5pt}} \overset{-1}{\underset{ij}{\circ}}$

(3) $q_{ii} \in \mathbb{G}'_3, q_{jj} = \pm q_{ii},\quad \overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{30pt}{0.5pt}} \overset{\pm q_{ii}^2}{\underset{ij}{\circ}}$

(4) $q_{ii} \in \mathbb{G}'_3, q_{jj} = q_{ii}^2,\quad \overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{30pt}{0.5pt}} \overset{1}{\underset{ij}{\circ}}$

(5) $q_{ii} \in \mathbb{G}'_6, q_{ii}q_{jj} = -1,\quad \overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{30pt}{0.5pt}} \overset{-1}{\underset{ij}{\circ}}$

(6) $q_{ii} \in \mathbb{G}'_5, q_{jj} = q_{ii}^2,\quad \overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{30pt}{0.5pt}} \overset{q_{ii}^3}{\underset{ij}{\circ}}$

(7) $q_{jj} \in \mathbb{G}'_9, q_{ii} = q_{jj}^3,\quad \overset{q_{jj}^3}{\underset{i}{\circ}} \overset{q_{jj}^6}{\rule[3pt]{30pt}{0.5pt}} \overset{q_{jj}^4}{\underset{ij}{\circ}}$

(8) $q_{ii} \in \mathbb{G}'_5, q_{ii}q_{jj} = -1,\quad \overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{30pt}{0.5pt}} \overset{-1}{\underset{ij}{\circ}}$

In the rest of the proof, we discard one by one all these possibilities.

(2) Now W is of Cartan type with Dynkin diagram and Cartan matrix:

$$\mathfrak{D} = \overset{q_{ii}}{\underset{i}{\circ}} \overset{-1}{\rule[3pt]{2cm}{0.4pt}} \overset{-1}{\underset{ij}{\circ}} \overset{-1}{\rule[3pt]{2cm}{0.4pt}} \overset{q_{ii}}{\underset{j}{\circ}}, \ q_{ii} \in \mathbb{G}'_4; \qquad \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

Since this matrix is of affine type, $\operatorname{GK-dim} \mathscr{B}(W) = \infty$ by [AAH2].

(3) Assume first $q_{jj} = -q_{ii}$. Then

$$\mathfrak{D}_2 = \overset{-q_{ii}^2}{\underset{ij}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{2cm}{0.4pt}} \overset{-q_{ii}}{\underset{j}{\circ}}, \quad q_{ii} \in \mathbb{G}'_3,$$

which is not arithmetic. By [AAH2] we see that $\operatorname{GK-dim} \mathscr{B}(V_2) = \infty$. Next, when $q_{jj} = q_{ii}$, W is of Cartan type with Dynkin diagram and Cartan matrix:

$$\mathfrak{D} = \overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{2cm}{0.4pt}} \overset{q_{ii}^2}{\underset{ij}{\circ}} \overset{q_{ii}^2}{\rule[3pt]{2cm}{0.4pt}} \overset{q_{ii}}{\underset{j}{\circ}}, \ q_{ii} \in \mathbb{G}'_3; \qquad \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix},$$

which is affine, so $\operatorname{GK-dim} \mathscr{B}(W) = \infty$ by [AAH2]. (4) Since $q_{ii}^2 \neq 1$, we have $\operatorname{GK-dim} \mathscr{B}(V_1) = \infty$ by [AAH1, Lemma 2.8]. (6) In this case

$$\mathfrak{D}_2 = \overset{q_{ii}^3}{\underset{ij}{\circ}} \overset{q_{ii}^4}{\rule[3pt]{2cm}{0.4pt}} \overset{q_{ii}^2}{\underset{j}{\circ}}, \quad q_{ii} \in \mathbb{G}'_5,$$

is of indefinite Cartan type, so $\operatorname{GK-dim} \mathscr{B}(V_2) = \infty$ by [AAH2]. (7) Similarly,

$$\mathfrak{D}_2 = \overset{q_{jj}^4}{\underset{ij}{\circ}} \overset{q_{jj}^2}{\rule[3pt]{2cm}{0.4pt}} \overset{q_{jj}}{\underset{j}{\circ}}, \quad q_{jj} \in \mathbb{G}'_9,$$

is indefinite Cartan, so $\operatorname{GK-dim} \mathscr{B}(V_2) = \infty$. In the remaining cases, $\mathfrak{D}$ is

(1) $\mathfrak{D} = \overset{\omega}{\underset{i}{\circ}} \overset{\omega^2}{\rule[3pt]{2cm}{0.4pt}} \overset{-1}{\underset{ij}{\circ}} \overset{\omega}{\rule[3pt]{2cm}{0.4pt}} \overset{-\omega^2}{\underset{j}{\circ}}, \quad \omega \in \mathbb{G}'_3.$

(5) $\mathfrak{D} = \overset{-\omega^2}{\underset{i}{\circ}} \overset{\omega}{\rule[3pt]{2cm}{0.4pt}} \overset{-1}{\underset{ij}{\circ}} \overset{\omega^2}{\rule[3pt]{2cm}{0.4pt}} \overset{\omega}{\underset{j}{\circ}}, \quad \omega \in \mathbb{G}'_3.$

(8) $\mathfrak{D} = \overset{\zeta}{\underset{i}{\circ}} \overset{\zeta^2}{\rule[3pt]{2cm}{0.4pt}} \overset{-1}{\underset{ij}{\circ}} \overset{\zeta^3}{\rule[3pt]{2cm}{0.4pt}} \overset{-\zeta^4}{\underset{j}{\circ}}, \quad \zeta \in \mathbb{G}'_5.$

Now (1) and (5) are equal up to permutation of the indexes. Only here we need to assume the validity of Conjecture 2.6. Indeed, these diagrams do not appear in [H1, Table 2], so $\operatorname{GK-dim} \mathscr{B}(W) = \infty$ in all cases. □

3.2. **A pre-Nichols algebra of type G_2.** We assume $(V, c^{\mathfrak{q}})$ has the following Dynkin diagram

$$\overset{\omega}{\underset{1}{\circ}} \qquad\qquad \overset{1}{\underset{2}{\circ}}, \qquad \omega \in \mathbb{G}'_3.$$

Proposition 3.3. *The algebra* $\breve{\mathscr{B}}_{\mathfrak{q}} := T(V)/\langle x_{11112}, x_{221}\rangle$ *is an eminent pre-Nichols algebra of* $(V, c^{\mathfrak{q}})$ *and* $\operatorname{GK-dim} \breve{\mathscr{B}}_{\mathfrak{q}} = 6$.

Proof. We first claim that the elements x_{11112} and x_{221} are primitive in $T(V)$. This is verified by a direct computation, see [S].

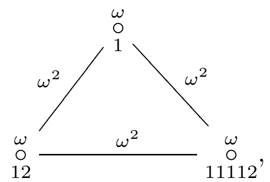
Second, we claim that the relations $x_{11112} = 0$ and $x_{221} = 0$ hold in any finite GK-dimensional pre-Nichols algebra $\mathscr{B}$ of $(V, c^{\mathfrak{q}})$.

Assume first $x_{11112} \neq 0$ in $\mathscr{B}$. Then also $x_{12} \neq 0$. From Lemma 2.7 and the previous claim, we have a braided subspace

$$W = \Bbbk x_1 + \Bbbk x_{12} + \Bbbk x_{11112} \subset \mathcal{P}(\mathscr{B}),$$

so Lemma 2.8 gives $\text{GK-dim}\, \mathscr{B}(W) \leq \text{GK-dim}\, \mathscr{B}$. By a degree argument, W has dimension three; from direct computation its Dynkin diagram is

(3.6)



Cartan type $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$.

Since the Cartan matrix is of affine type A_2^1, we have $\text{GK-dim}\, \mathscr{B}(W) = \infty$ by [AAH2, Theorem 1.2(a)]. Thus $\text{GK-dim}\, \mathscr{B} = \infty$.

Assume now $x_{221} \neq 0$ in $\mathscr{B}$. Then $x_{21} \neq 0$, and since $q_{12}q_{21} = 1$, we have $x_{12} = -q_{12}x_{21} \neq 0$. Consider $W' = \Bbbk x_1 + \Bbbk x_{12} + \Bbbk x_{221}$. We may now use the same argument as above. Indeed, $W' \subset \mathcal{P}(\mathscr{B})$ has Dynkin diagram (3.6) replacing x_{11112} by x_{221}, so $\text{GK-dim}\, \mathscr{B}(W') = \infty$ by the same reason as $\text{GK-dim}\, \mathscr{B}(W) = \infty$. Hence $\text{GK-dim}\, \mathscr{B} = \infty$. Thus $\breve{\mathscr{B}}_{\mathfrak{q}} \twoheadrightarrow \mathscr{B}$.

The verification of $\text{GK-dim}\, \breve{\mathscr{B}}_{\mathfrak{q}} = 6$ is postponed to Proposition 4.5. □

3.3. **A further reduction.** Let $\mathscr{B}$ be a finite GK-dimensional pre-Nichols algebra of $\mathscr{B}_{\mathfrak{q}}$. We are naturally led to consider

$$E := \{(i,j)\colon\ i \in \mathbb{I}^3, j \in \mathbb{I}^1,\ x_{ij} \neq 0 \text{ in } \mathscr{B}\}. \tag{3.7}$$

That is, $(i,j) \in E$ means that $\operatorname{ord} q_{ii} = 3$, $q_{jj} = 1$ and $x_{ij} \neq 0$ in $\mathscr{B}$.

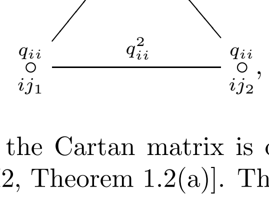
Remark 3.4. If $(i,j) \in E$, the braided vector space $\Bbbk x_i \oplus \Bbbk x_{ij} \subset \mathscr{B}$ is of Cartan type A_2 by Remark 3.1.

Lemma 3.5. *If $(i,j_1), (i,j_2) \in E$ then $j_1 = j_2$.*

Proof. Since x_{ij_1} and x_{ij_2} are $\mathbb{Z}^\theta$-homogeneous,

$$c(x_{ij_1} \otimes x_{ij_2}) = \mathfrak{q}(\alpha_i + \alpha_{j_1}, \alpha_i + \alpha_{j_2})\, x_{ij_2} \otimes x_{ij_1} = q_{ii}q_{ij_2}q_{j_1 i}q_{j_1 j_2}\, x_{ij_2} \otimes x_{ij_1}.$$

Assume $j_1 \neq j_2$. Then x_i, x_{ij_1} and x_{ij_2} have pairwise different $\mathbb{Z}^\theta$-degrees, so they span a 3-dimensional braided subspace $W = \Bbbk x_i + \Bbbk x_{ij_1} + \Bbbk x_{ij_2} \subset \mathcal{P}(\mathscr{B})$. Now the Dynkin diagram of W is



$q_{ii} \in \mathbb{G}_3'$, Cartan type $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$.

Since the Cartan matrix is of affine type $A_2^{(1)}$, we have $\text{GK-dim}\, \mathscr{B}(W) = \infty$ by [AAH2, Theorem 1.2(a)]. Thus $\text{GK-dim}\, \mathscr{B} = \infty$, a contradiction. □

4. Cartan type

In this section we determine the finite GK-dimensional pre-Nichols algebras of braided vector spaces of finite Cartan type under some restrictions.

We fix a matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}}$ of non-zero scalars such that $q_{ii} \neq 1$ for all $i \in \mathbb{I}$ and a braided vector space $(V, c^{\mathfrak{q}})$ with braiding given by $c^{\mathfrak{q}}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, $i, j \in \mathbb{I}_\theta$, in a basis $\{x_1, \dots, x_\theta\}$. Let $N_i = \operatorname{ord} q_{ii} \in \mathbb{N} \cup \infty$.

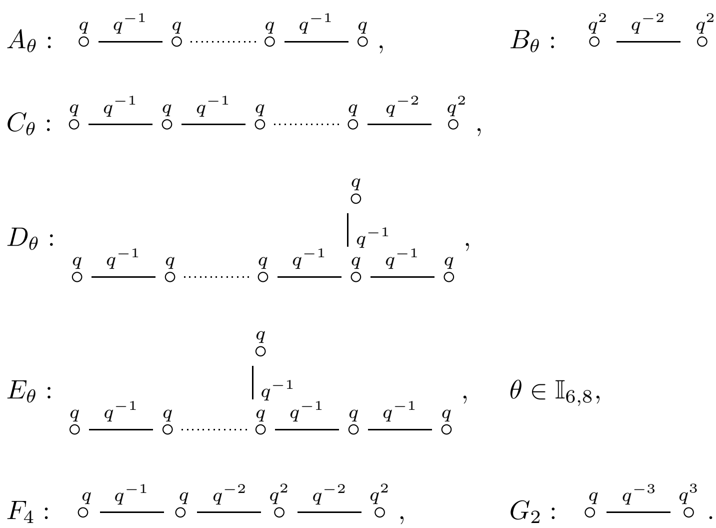
Recall that $\mathfrak{q}$, or $(V, c^{\mathfrak{q}})$, is of *Cartan type* if there exists a Cartan matrix $\mathbf{a} = (a_{ij})_{i,j\in\mathbb{I}}$ such that $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$ for all i, j. Let $i \in \mathbb{I}$. If $N_i = \infty$, then a_{ij} are uniquely determined. Otherwise, we impose

$$-N_i < a_{ij} \leq 0, \qquad j \in \mathbb{I}. \tag{4.1}$$

In this way we say that $(V, c^{\mathfrak{q}})$, is of Cartan type $\mathbf{a}$.

We follow the terminology of [K]. Cartan matrices are arranged in three families, namely: finite, affine and indefinite. We say that $\mathfrak{q}$, or $(V, c^{\mathfrak{q}})$, belongs to one of these families if the corresponding $\mathbf{a}$ does.

In this section we assume that $\mathfrak{q}$ is of *connected finite Cartan type* and that $\dim \mathscr{B}_{\mathfrak{q}} < \infty$. Thus the possible Dynkin diagrams of $\mathfrak{q}$ have the following form, where q is a root of unity in $\Bbbk$ of order $N > 1$:



We refer to the survey[AA2] for restrictions on N and other features of $\mathscr{B}_{\mathfrak{q}}$ in each case. The quantum Serre relations are the following elements of $T(V)$:

$$(\operatorname{ad}_c x_i)^{1-a_{ij}} x_j, \qquad i,\, j \in \mathbb{I},\, i \neq j. \tag{4.2}$$

By [AS2, Lemma A.1] these are primitive in any pre-Nichols algebra. Let $\widetilde{\mathscr{B}}_{\mathfrak{q}} = T(V)/\mathcal{I}_{\mathfrak{q}}$ be the distinguished pre-Nichols algebra of $(V, c^{\mathfrak{q}})$, see §1.1.3.

Remark 4.1. From the detailed presentation in [AA2, §4] we see that the quantum Serre relations (4.2) generate $\mathcal{I}_{\mathfrak{q}}$ in the following cases:

- when $\mathbf{a}$ is of type A_2 or B_2 [AA2, pp. 397, 399, 400],
- when $\mathbf{a}$ is of type G_2 and $N \neq 4, 6$ [AA2, pp. 410, 411],
- when $\mathbf{a}$ is simply-laced and $N > 2$ [AA2, pp. 397, 404, 407],
- when $\mathbf{a}$ is of type B, C, or F and $N > 4$ [AA2, pp. 399, 402, 409].

4.1. **Quantum Serre relations.** Let $\mathbf{a} = (a_{ij})_{i,j\in\mathbb{I}}$ be a symmetrizable indecomposable generalized Cartan matrix and $\mathbf{d} \in GL_\theta(\mathbb{Z})$ diagonal such that $\mathbf{da}$ is symmetric. The datum $(\mathbf{a}, \mathbf{d})$ is equivalent to an irreducible Cartan datum as in [Lu, 1.1.1] by setting

$$\cdot : \mathbb{I}\times\mathbb{I} \to \mathbb{Z}, \qquad i\cdot j = d_i a_{ij}, \qquad i, j \in \mathbb{I}.$$

Let $\mathfrak{g} = \mathfrak{g}(\mathbf{a})$ be the associated Kac-Moody algebra which has a triangular decomposition $\mathfrak{g}(\mathbf{a}) = \mathfrak{g}^+ \oplus \mathfrak{h} \oplus \mathfrak{g}^-$.

Let $q \in \Bbbk^\times$ and consider the Dynkin diagram

$$\cdots\cdots \overset{q^{d_i}}{\underset{i}{\circ}} \xrightarrow{\quad q^{d_i a_{ij}} \quad} \overset{q^{d_j}}{\underset{j}{\circ}} \cdots\cdots \tag{4.3}$$

Let $\mathfrak{q}$ be any matrix with Dynkin diagram (4.3) and $(V, c^{\mathfrak{q}})$ be the corresponding braided vector space with basis $(x_i)_{i\in\mathbb{I}}$. Notice that $\mathfrak{q}$ is of Cartan type but it is not necessarily of type $\mathbf{a}$ as (4.1) may not hold.

Let $\breve{\mathscr{B}}_{\mathfrak{q}} = T(V)$ modulo the ideal $\mathfrak{K}_{\mathfrak{q}}$ generated by the quantum Serre relations $(\operatorname{ad}_c x_i)^{1-a_{ij}}(x_j)$, $i \neq j \in \mathbb{I}$, which is a pre-Nichols algebra of V.

Proposition 4.2. GK-dim $\breve{\mathscr{B}}_{\mathfrak{q}} \geq \dim \mathfrak{g}^+$.

Proof. If $\xi \in \Bbbk$, $\xi^2 = q$, then $\mathfrak{p} = (\xi^{d_i a_{ij}})$ has Dynkin diagram (4.3). Let $(W, c^{\mathfrak{q}})$ be the corresponding braided vector space with basis $(\widehat{x}_i)_{i\in\mathbb{I}}$.

Claim 1. GK-dim $\breve{\mathscr{B}}_{\mathfrak{q}}$ = GK-dim $\breve{\mathscr{B}}_{\mathfrak{p}}$.

Proof. By the proof of [AS3, Proposition 3.9] (or the proof of Lemma 2.9) there is a homogeneous linear isomorphism $\psi : T(V) \to T(W)$ determined by $\psi(x_i) = \widehat{x}_i$ for all $i \in \mathbb{I}$ and satisfying [AS3, Remarks 3.10]. Hence $\psi(\mathfrak{K}_{\mathfrak{q}}) = \mathfrak{K}_{\mathfrak{p}}$ and ψ induces a homogeneous linear isomorphism $\psi : \breve{\mathscr{B}}_{\mathfrak{q}} \to \breve{\mathscr{B}}_{\mathfrak{p}}$. Then apply Lemma 2.1. □

Let now $\mathbf{f}$ be the $\mathbb{Q}(v)$-algebra defined in [Lu, 1.2.5], where v is an indeterminate and let ${}_{\mathcal{A}}\mathbf{f}$ be the $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$-subalgebra spanned by the quantum divided powers of the generators of $\mathbf{f}$ [Lu, 1.4.7]. By [Lu, 14.4.3], ${}_{\mathcal{A}}\mathbf{f}$ is a free $\mathcal{A}$-module and

$$P_{{}_{\mathcal{A}}\mathbf{f}} = P_{\mathbf{f}}. \tag{4.4}$$

Consider $\Bbbk$ as $\mathcal{A}$-module via $v \mapsto \xi$. Then we have the algebras ${}_{\Bbbk}\mathbf{f} = \Bbbk \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f}$ and ${}_{\Bbbk}\widetilde{\mathbf{f}}$ defined in [Lu, 33.1.1] (which is nothing else than $\breve{\mathscr{B}}_{\mathfrak{p}}$). By [Lu, 1.4.3], the quantum Serre relations hold in ${}_{\Bbbk}\mathbf{f}$, hence we have a surjective algebra map $\breve{\mathscr{B}}_{\mathfrak{p}} = {}_{\Bbbk}\widetilde{\mathbf{f}} \twoheadrightarrow {}_{\Bbbk}\mathbf{f}$. Thus

$$\text{GK-dim}\, \breve{\mathscr{B}}_{\mathfrak{p}} \geq \text{GK-dim}\, \mathbf{f}. \tag{4.5}$$

On the other hand, let $\Bbbk_0$ be $\Bbbk$ as $\mathcal{A}$-module via $v \mapsto 1$. Then ${}_{\Bbbk_0}\widetilde{\mathbf{f}} \simeq U(\mathfrak{g}^+)$ by [Lu, 33.1.1] and $\dim_{\mathbb{Q}(v)} \mathbf{f}_\nu = \dim_{\Bbbk_0}({}_{\Bbbk_0}\widetilde{\mathbf{f}}_\nu)$ by [Lu, 33.1.3]; that is

$$\text{GK-dim}\, \mathbf{f} = \text{GK-dim}({}_{\Bbbk_0}\widetilde{\mathbf{f}}) = \dim \mathfrak{g}^+, \tag{4.6}$$

where the first equality holds by Lemma 2.1. The Proposition follows. □

Example 4.3. Let $\mathbf{a} = \begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$. Then (4.3) takes the form $\overset{q}{\underset{1}{\circ}} \xrightarrow{\ q^{-5}\ } \overset{q^5}{\underset{2}{\circ}}$ with $q \in \Bbbk^\times$. If $q_{12} \in \Bbbk^\times$ and $q_{21} := q_{12}^{-1} q^{-5}$, then $\mathfrak{q} = \begin{pmatrix} q & q_{12} \\ q_{21} & q^5 \end{pmatrix}$ has the Dynkin diagram above. Here $\breve{\mathscr{B}}_{\mathfrak{q}} = \Bbbk\langle x_1, x_2\rangle$ modulo the relations

$$x_2^2x_1 - q_{21}(2)_q\, x_2x_1x_2 + qq_{21}^2\ x_1x_2^2,$$
$$x_1^6x_2 - 3q_{12}^2\ x_1^4x_2x_1^2 + 3q_{12}^4\ x_1^2x_2x_1^4 - q_{12}^6\ x_2x_1^6.$$

In this setting Proposition 4.2 gives $\operatorname{GK-dim} \breve{\mathscr{B}}_{\mathfrak{q}} = \infty$.

Example 4.4. Let $\mathbf{a} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. Then (4.3) takes the form $\overset{q}{\underset{1}{\circ}} \overset{q^{-3}}{\text{———}} \overset{q^3}{\underset{2}{\circ}}$ with $q \in \Bbbk^\times$. If $q_{12} \in \Bbbk^\times$ and $q_{21} := q_{12}^{-1}q^{-3}$, then $\mathfrak{q} = \begin{pmatrix} q & q_{12} \\ q_{21} & q^3 \end{pmatrix}$ has the Dynkin diagram above. Here $\breve{\mathscr{B}}_{\mathfrak{q}} = \Bbbk\langle x_1, x_2\rangle$ modulo the relations

$$x_2^2x_1 - q_{21}(2)_{q^3}\, x_2x_1x_2 + q_{21}^2q^3\ x_1x_2^2,$$
$$x_1^4x_2 - q_{12}(4)_q\, x_1^3x_2x_1 + q_{12}^2q\binom{4}{2}_q\ x_1^2x_2x_1^2 - q_{12}^3(4)_q\, x_1^3x_2x_1 + q_{12}^4q^6\, x_2x_1^4.$$

In this situation Proposition 4.2 establishes $\operatorname{GK-dim} \breve{\mathscr{B}}_{\mathfrak{q}} \geq 6$.

This last example gains more relevance when the parameter $q \in \Bbbk^\times$ specializes to a root of unity with small order.

Proposition 4.5. *Let* $\mathbf{a}$ *and* $q \in \Bbbk^\times$ *as in Example* 4.4*.*

(a) *If* $q \in \mathbb{G}_3'$ *then* $\operatorname{GK-dim} \breve{\mathscr{B}}_{\mathfrak{q}} = 6$*.*

(b) *If* $q \in \mathbb{G}_2'$ *then* $x_{112}^2 = 0$ *in* $\breve{\mathscr{B}}_{\mathfrak{q}}$*.*

Proof. Let $q \in \Bbbk^\times$. Put $x_{1^32^2} = [x_{112}, x_{12}]$. By direct computation, in $\breve{\mathscr{B}}_{\mathfrak{q}}$ the following relations hold:

$$\begin{aligned}
x_{12}x_2 &= q_{12}q^3x_2x_{12},\\
x_{112}x_2 &= q_{12}^2q^3x_2x_{112} + q_{12}q^2(q-1)(2)_qx_{12}^2,\\
x_{1112}x_2 &= q_{12}^3q^3x_2x_{1112} + q_{12}q(q^2-q-1)x_{1^32^2} + q_{12}^2q^2(q-1)(3)_qx_{12}x_{112},\\
x_{1^32^2}x_2 &= q_{12}^3q^6x_2x_{1^32^2} + q_{12}^2q^3(q-1)^2(2)_qx_{12}^3,\\
x_1x_{1^32^2} &= q_{12}^2q^3x_{1^32^2}x_1 + x_{1112}x_{12} - q_{12}^2q^3x_{12}x_{1112},\\
(2)_qx_{1112}x_{12} &= q_{12}^2q^3(2)_qx_{12}x_{1112} + q_{12}q(q-1)(3)_qx_{112}^2.
\end{aligned}$$

(a) Here $q \in \mathbb{G}_3'$, so the last relation above becomes

$$x_{1112}x_{12} = q_{12}^2x_{12}x_{1112}.$$

Substituting this in the penultimate equation we get

$$x_1x_{1^32^2} = q_{12}^2x_{1^32^2}x_1.$$

These equalities imply more commutations:

$$\begin{aligned}
x_{1^32^2}x_{12} &= q_{12}x_{12}x_{1^32^2}, & x_{112}x_{1^32^2} &= q_{12}x_{1^32^2}x_{112},\\
x_{1112}x_{112} &= q_{12}x_{112}x_{1112}, & x_{1112}x_{1^32^2} &= q_{12}^3x_{1^32^2}x_{1112}.
\end{aligned}$$

Now we claim that $\breve{\mathscr{B}}_{\mathfrak{q}}$ is linearly spanned by

$$\mathbb{B} = \{x_2^{n_1}x_{12}^{n_2}x_{1^32^2}^{n_3}x_{112}^{n_4}x_{1112}^{n_5}x_1^{n_6} : 0 \leq n_1, \dots, n_6\}.$$

Denote by $\mathcal{I}$ the linear span of $\mathbb{B}$. Since $1 \in \mathcal{I}$, it is enough to show that $\mathcal{I}$ is left ideal of $\breve{\mathscr{B}}_{\mathfrak{q}}$. If we multiply $x_2^{n_1}x_{12}^{n_2}x_{1^32^2}^{n_3}x_{112}^{n_4}x_{1112}^{n_5}x_1^{n_6}$ by x_1 on the left, we can use the previously deduced commutations between the (powers of the) x_α's to successively rearrange the terms until we get a linear combination of elements in $\mathbb{B}$. The claim follows. Let $K = \{2, 12, 1^3 2^2, 112, 1112, 1\}$ be ordered by

$$2 > 12 > 1^3 2^2 > 112 > 1112 > 1.$$

This order is convex, that is for $\alpha, \beta \in K$ with $\alpha > \beta$, the braided commutator $[x_\alpha, x_\beta]_c$ is a sum of monomials in the letters γ such that $\beta < \gamma < \alpha$. Indeed this follows from the equalities above.

Consider next the lexicographical order on $\mathbb{B}$ induced by the order of K and the corresponding $\mathbb{N}_0$-filtration $\mathcal{F}$ on $\breve{\mathscr{B}}_{\mathfrak{q}}$. Let $\operatorname{gr}_{\mathcal{F}}(\breve{\mathscr{B}}_{\mathfrak{q}})$ be the associated graded algebra. By the convexity of the order, there is a natural projection from a quantum polynomial algebra $\Bbbk_{\mathfrak{q}}[y_1, \dots, y_6] \twoheadrightarrow \operatorname{gr}_{\mathcal{F}}(\breve{\mathscr{B}}_{\mathfrak{q}})$, hence $\operatorname{GK-dim} \operatorname{gr}_{\mathcal{F}}(\breve{\mathscr{B}}_{\mathfrak{q}}) \leq 6$. By [KL, Proposition 6.6] and Example 4.4 we also have $\operatorname{GK-dim} \operatorname{gr}_{\mathcal{F}}(\breve{\mathscr{B}}_{\mathfrak{q}}) = \operatorname{GK-dim} \breve{\mathscr{B}}_{\mathfrak{q}} \geq 6$, so the equality holds.

(b) This follows by specialization at $q = -1$ in the relation

$$(2)_q x_{1112} x_{12} = q_{12}^2 q^3 (2)_q x_{12} x_{1112} + q_{12} q (q-1)(3)_q x_{112}^2.$$

$\square$

Remark 4.6. Let us point out the relevance of (b). By Kharchenko's theory [Kh], $\breve{\mathscr{B}}_{\mathfrak{q}}$ has a PBW-basis. By Proposition 4.2 we know $\operatorname{GK-dim} \breve{\mathscr{B}}_{\mathfrak{q}} \geq 6$ but, when $q \in \mathbb{G}'_2$, the root $2\alpha_1 + \alpha_2$ will not contribute to $\operatorname{GK-dim} \breve{\mathscr{B}}_{\mathfrak{q}}$ by (b) above. So even if $\mathbf{a}$ is of type G_2, one should not expect that the PBW generators are just those related to the six positive roots of G_2, as was the case in the proof of (a).

4.2. **Type A_2.** In this and the next subsections we seek eminent (families of) pre-Nichols algebras in order to determine finite GK-dim pre-Nichols algebras of braidings of finite Cartan type. The distinguished pre-Nichols algebra will serve as the principal guide in our exploration.

4.2.1. *Type A_2 with $N > 3$.*

Lemma 4.7. *Assume* $\mathbf{a}$ *is of Cartan type* A_2 *with* $N > 3$. *If* $\mathscr{B}$ *is a finite GK-dimensional pre-Nichols algebra of* $\mathfrak{q}$, *then* $x_{112} = 0$ *and* $x_{221} = 0$ *in* $\mathscr{B}$, *i. e. the distinguished pre-Nichols algebra* $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ *is eminent, cf. Definition* 2.3.

Proof. Assume $x_{iij} \neq 0$ for some $i \neq j \in \mathbb{I}_2$; the 3-dimensional braided subspace $W := \Bbbk x_j \oplus \Bbbk x_i \oplus \Bbbk x_{iij} \subset \mathcal{P}(\mathscr{B})$ has $\operatorname{GK-dim} \mathscr{B}(W) < \infty$.

Consider the braided subspace $W_1 = \Bbbk x_i \oplus \Bbbk x_{iij} \subset W$. By direct computation, the braiding on W_1 is of Cartan type with the following Dynkin diagram and Cartan matrix:

$$\underset{i}{\overset{q_{ii}}{\circ}} \xrightarrow{\quad q_{ii}^3 \quad} \underset{iij}{\overset{q_{ii}^3}{\circ}}, \quad A_1 = \begin{pmatrix} 2 & 3-N \\ 1-M & 2 \end{pmatrix}, \quad M = \begin{cases} N/3, & \text{if } 3|N, \\ N, & \text{otherwise.} \end{cases}$$

If either $N = 5$ or $N > 6$, it is evident that the Cartan matrix A_1 is not finite, so $\operatorname{GK-dim} \mathscr{B}(W_1) = \infty$ by [AAH2, Theorem 1.2 (b)]. This contradicts $\operatorname{GK-dim} \mathscr{B} < \infty$.

For the remaining cases (i. e. $N = 4$ and $N = 6$), we consider the whole W. Since $\widetilde{\mathfrak{q}}(\alpha_j, 2\alpha_i + \alpha_j) = (q_{ji} q_{ij})^2 q_{jj}^2 = 1$, the braiding on W is of Cartan type with the following Dynkin diagram and Cartan matrix

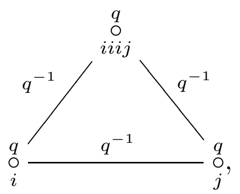
$$\overset{q_{ii}}{\underset{j}{\circ}} \overset{q_{ii}^{-1}}{\rule[3pt]{5em}{0.4pt}} \overset{q_{ii}}{\underset{i}{\circ}} \overset{q_{ii}^{3}}{\rule[3pt]{5em}{0.4pt}} \overset{q_{ii}^{3}}{\underset{iij}{\circ}}, \qquad A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 3-N \\ 0 & 1-M & 2 \end{pmatrix}.$$

Now it is straightforward to verify that if $N = 4$ or 6, then A is of affine type, which contradicts $\operatorname{GK-dim} \mathscr{B}(W) < \infty$ by [AAH2, Theorem 1.2 (b)]. $\square$

4.2.2. *Type A_2 with $N = 3$.* Here is the first restriction.

Lemma 4.8. *Assume* **a** *is of Cartan type A_2 with $N = 3$. Let $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}$. Then $x_{iiij} = 0$ and $x_{jiij} = 0$ in $\mathscr{B}$ for all $i \neq j \in \mathbb{I}_2$.*

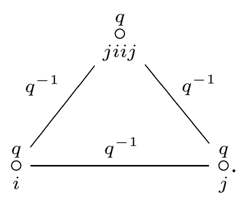
Proof. Since x_{iij} is primitive, using that $\widetilde{\mathfrak{q}}(\alpha_i, 2\alpha_i+\alpha_j) = q^4q^{-1} = 1$ and $\widetilde{\mathfrak{q}}(\alpha_j, 2\alpha_i+\alpha_j) = q^2q^{-2} = 1$, we get $x_{iiij}, x_{jiij} \in \mathcal{P}(\mathscr{B})$ by Lemma 2.7. Assume first $x_{iiij} \neq 0$ in $\mathscr{B}$. The braided subspace $\Bbbk x_i \oplus \Bbbk x_j \oplus \Bbbk x_{iiij} \subset \mathcal{P}(\mathscr{B})$ has finite GK-dim Nichols algebra. The Dynkin diagram is



Cartan type $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$

The Cartan matrix is of affine type $A_2^{(1)}$, and by [AAH2, Theorem 1.2(a)] this contradicts $\operatorname{GK-dim} \mathscr{B} < \infty$.

If $x_{jiij} \neq 0$, the same argument leads to a contradiction. Indeed, by direct computations, the Dynkin diagram of $U = \Bbbk x_i \oplus \Bbbk x_j \oplus \Bbbk x_{jiij}$ is



so $\operatorname{GK-dim} \mathscr{B}(U) = \infty$ by [AAH2, Theorem 1.2(a)]. $\square$

Remark 4.9. Denote $\widehat{\mathscr{B}} = T(V)/\langle x_{1112}, x_{2221}, x_{2112}, x_{1221}\rangle$. The defining ideal of $\widehat{\mathscr{B}}$ is a Hopf ideal by the proof of Lemma 4.8. Let $\pi\colon \widehat{\mathscr{B}} \to \mathscr{B}(V)$ denote the natural projection. Let $\widehat{\mathscr{Z}}$ be the subalgebra of $\widehat{\mathscr{B}}$ generated by

$$z_1 := x_2^3, \qquad z_2 := x_{221}, \qquad z_3 := x_{112}, \qquad z_4 := x_1^3, \qquad z_5 := x_{12}^3.$$

The next results are devoted to prove that $\widehat{\mathscr{B}}$ is eminent.

Lemma 4.10.

(a) *Given $i \neq j \in \mathbb{I}_2$, the following relations hold in $\widehat{\mathscr{B}}$:*

$$[x_{ij}, x_{iij}]_c = 0 = [x_{ij}, x_{jji}]_c; \qquad [x_{iij}, x_{jji}]_c = 0.$$

(b) *$\widehat{\mathscr{Z}}$ is a normal braided Hopf subalgebra of $\widehat{\mathscr{B}}$.*

(c) *The z_i's q-commute; $B = \{z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} z_5^{n_5} : n_i \in \mathbb{N}_0\}$ is a basis of $\widehat{\mathscr{Z}}$.*

(d) *$\widehat{\mathscr{Z}} = {}^{\operatorname{co}\pi}\widehat{\mathscr{B}}$.*

Proof.

(a) Just compute using (2.8):

$$\begin{aligned}
\big[x_{ij}, x_{iij}\big]_c &= \big[x_i, [x_j, x_{iij}]_c\big]_c - q_{ij}x_j\big[x_i, x_{iij}\big]_c + q_{jj}q_{ji}^2\big[x_i, x_{iij}\big]_c x_j = 0;\\
\big[x_{ij}, x_{jji}\big]_c &= \big[x_i, [x_j, x_{jji}]_c\big]_c - q_{ij}x_j\big[x_i, x_{jji}\big]_c + q_{ii}q_{ij}^2\big[x_i, x_{jji}\big]_c x_j = 0;\\
\big[x_{iij}, x_{jji}\big]_c &= \big[[x_i, x_{ij}]_c, x_{jji}\big]_c\\
&= \big[x_i, [x_{ij}, x_{jji}]_c\big]_c - q_{ii}q_{ij}x_{ij}\big[x_i, x_{jji}\big]_c + q_{ii}^2q_{ij}\big[x_i, x_{jji}\big]_c x_{ij}\\
&= \big[x_i, [x_{ij}, x_{jji}]_c\big]_c = 0.
\end{aligned}$$

(b) We claim that the generators of $\widehat{\mathscr{Z}}$ are annihilated by the braided adjoint action of $\widehat{\mathscr{B}}$. Fix $i \in \mathbb{I}_2$. By definition $(\operatorname{ad}_c x_i)z_2 = 0 = (\operatorname{ad}_c x_i)z_3$. In $T(V)$ we have $(\operatorname{ad}_c x_i)x_i^3 = x_i^4 - q_{ii}^3x_i^4 = 0$, and if $j \neq i$ then

$$\begin{aligned}
x_{jjji} = (\operatorname{ad}_c x_j)^3x_i &= \sum_{k=0}^{3}(-1)^k q_{ji}^k q_{jj}^{k(k-1)/2}\binom{3}{k}_{q_{jj}} x_j^{3-k}x_ix_j^k\\
&= x_j^3x_i - q_{ji}^3x_ix_j^3 = -q_{ji}^3(\operatorname{ad}_c x_i)x_j^3.
\end{aligned}\tag{4.7}$$

Thus $\operatorname{ad}_c x_i$ annihilates z_1 and z_4. Finally, we proceed with z_5. From (a) we get the commutation $x_{112}x_{12} = q_{11}^2q_{12}x_{12}x_{112}$ in $\widehat{\mathscr{B}}$. Then using (2.6)

$$\begin{aligned}
(\operatorname{ad}_c x_1)z_5 &= x_{112}x_{12}^2 + q_{11}q_{12}x_{12}x_{112}x_{12} + q_{11}^2q_{12}^2x_{12}^2x_{112}\\
&= q_{12}^2(q_{11}^4 + q_{11}^3 + q_{11}^2)x_{12}^2x_{112} = 0.
\end{aligned}\tag{4.8}$$

For $(\operatorname{ad}_c x_2)z_5$, notice that on the one hand

$$\begin{aligned}
\big[x_{12}, -\big]_c^3 x_2 &= \sum_{k=0}^{3}(-1)^k(q_{12}q_{22})^k q_{22}^{k(k-1)/2}\binom{3}{k}_{q_{22}} x_{12}^{3-k}x_2x_{12}^k\\
&= x_{12}^3x_i - q_{12}^3x_ix_{12}^3 = -q_{12}^3(\operatorname{ad}_c x_2)x_{12}^3.
\end{aligned}\tag{4.9}$$

On the other hand, using $[x_{12}, x_2]_c = q_{12}^2q_{22}x_{221}$ and (a) we get

$$\begin{aligned}
\big[x_{12}, -\big]_c^3 x_2 &= \Big[x_{12}, \big[x_{12}, [x_{12}, x_2]_c\big]_c\Big]_c\\
&= q_{12}^2q_{22}\Big[x_{12}, [x_{12}, x_{221}]_c\Big]_c = \Big[x_{12}, 0\Big]_c = 0,
\end{aligned}\tag{4.10}$$

so $(\operatorname{ad}_c x_2)z_5 = 0$. This shows that $\widehat{\mathscr{Z}}$ is a normal subalgebra.

Next we verify that $\Delta(z_i) \in \widehat{\mathscr{Z}} \otimes \widehat{\mathscr{Z}}$ for $i \in \mathbb{I}_5$. This is clear for $i \in \mathbb{I}_4$, because those elements are primitive in $T(V)$; for $i = 5$ we compute in $T(V)$:

$$\begin{aligned}
\Delta(x_{12}^3) =& x_{12}^3 \otimes 1 + 1 \otimes x_{12}^3\\
&+ (q^{-1} - q^{-2})x_{112} \otimes x_{221} + (1 - q^{-1})^3q_{21}^3x_1^3 \otimes x_2^3\\
&+ (1 - q)^2q_{21}^3x_{1112} \otimes x_2^2 - (1 - q^{-1})^2q^{-1}x_1^2 \otimes x_{2221}\\
&+ (q - 1)x_1 \otimes [x_{12}, x_{221}]_c - (1 - q^{-1})q_{21}[x_{12}, x_{112}]_c \otimes x_2.
\end{aligned}\tag{4.11}$$

Using (a) and the defining relations of $\widehat{\mathscr{B}}$ we see that $\widehat{\mathscr{Z}}$ is a Hopf subalgebra.

(c) We show that any pair of generators of $\widehat{\mathscr{Z}}$ q-commute. By definition of $\widehat{\mathscr{B}}$, both x_1 and x_2 q-commute with z_2 and z_3, so z_4 and z_1 q-commute with z_2 and z_3. Secondly, (4.7) implies that z_1 and z_4 q-commute. Thirdly, (a) shows that z_5 q-commutes with z_3 and z_2, and also that z_2 and z_3 q-commute. Lastly, z_5

q-commutes with z_4 by (4.8), and with z_1 by (4.9) and (4.10). Hence B linearly generates $\widehat{\mathscr{Z}}$.

The linear independence is proven by steps.

Step 1. The set $\{z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} \colon n_i \in \mathbb{N}_0\}$ is linearly independent.

Proof. Consider the Hopf algebra $\widehat{\mathscr{B}}\#\Bbbk\mathbb{Z}^2$; let A denote the subalgebra generated by $z_1, \dots, z_4$ and $\mathbb{Z}^2$. Since all the generators of A are either skew-primitives or group-likes, it follows that A itself is a pointed Hopf algebra. Notice that $z_1, \dots, z_4 \in \mathcal{P}(\widehat{\mathscr{Z}})$ are linearly independent. Indeed, they are non-zero because their $\mathbb{Z}$-degree is 3, so they are linearly independent since their $\mathbb{Z}^2$-degrees are pairwise different (here we are using that the defining ideal of $\widehat{\mathscr{B}}$ is a Hopf ideal generated by $\mathbb{Z}^2$-homogeneous elements of $\mathbb{Z}$-degree 4). Hence the infinitesimal braiding of A contains the braided vector space $\Bbbk z_1 \oplus \cdots \oplus \Bbbk z_4$, which is quantum linear space with all points labeled by 1. Thus $\left\{z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} g \colon n_i \in \mathbb{N}_0, g \in \mathbb{Z}^2\right\} \subset A$ is linearly independent. □

Step 2. The element z_5 does not belong to the left ideal $\widehat{\mathscr{B}}\langle z_1, z_2, z_3, z_4\rangle$.

Proof. We verify this using [GAP]. □

The ideal $\widehat{\mathscr{B}}\langle z_1, z_2, z_3, z_4\rangle$ is a Hopf ideal because the generators are primitive. Denote the quotient by R and consider the projection $\pi_R \colon \widehat{\mathscr{B}} \twoheadrightarrow R$.

Step 3. The set $\{\pi_R(z_5)^n \colon n \in \mathbb{N}_0\}$ is linearly independent.

Proof. Consider the Hopf algebra $R\#\mathbb{Z}^2$. The subalgebra generated by $\pi_R(z_5)$ and $\mathbb{Z}^2$ is a pointed Hopf algebra. Moreover, its infinitesimal braiding contains $\pi_R(z_5)$, which is a non-zero point by Step 2 and is labeled by 1. Now proceed as in the proof of Step 1. □

Step 4. We have $(\mathrm{id} \otimes \pi_R)\Delta(z_5^n) = \sum_{k=0}^{n} \binom{n}{k} z_5^k \otimes \pi_R(z_5)^{n-k}$ for all $n \in \mathbb{N}_0$.

Proof. The case $n = 0$ is obvious, and $n = 1$ follows from (4.11). An standard inductive argument for braided comultiplication yields the desired result. □

Step 5. The set B is linearly independent.

Proof. Let $\sum_{n_1,\dots,n_5 \in \mathbb{N}_0} \lambda_{n_1,\dots,n_5} z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} z_5^{n_5} = 0$. Assume there exists n_5 such that $\lambda_{n_1,\dots,n_5} \neq 0$ for some $n_1, \dots, n_4 \in \mathbb{N}_0$; take N as the maximal one. By Step 3 there is a linear map $f \colon R \to \Bbbk$ such that $f(\pi_R(z_5)^n) = \delta_{n,N}$ for all $n \in \mathbb{N}_0$. Now using Step 4 we compute

$$
\begin{aligned}
0 &= (\mathrm{id} \otimes f)(\mathrm{id} \otimes \pi_R)\Delta \left(\sum_{n_1,\dots,n_5 \in \mathbb{N}_0} \lambda_{n_1,\dots,n_5} z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} z_5^{n_5} \right) \\
&= \sum_{n_1,\dots,n_5 \in \mathbb{N}_0} \lambda_{n_1,\dots,n_5} (\mathrm{id} \otimes f)(\mathrm{id} \otimes \pi_R) \left(\left(\prod_{i=1}^{4} \sum_{j=0}^{n_i} \binom{n_i}{j} z_i^j \otimes z_i^{n_i - j} \right) \Delta z_5^{n_5} \right) \\
&= \sum_{n_1,\dots,n_4 \in \mathbb{N}_0,\ n_5 \leq N} \lambda_{n_1,\dots,n_5} (\mathrm{id} \otimes f) \left(\sum_{j=0}^{n_5} z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} z_5^j \otimes \pi_R(z_5^{n_5 - j}) \right) \\
&= \sum_{n_1,\dots,n_4 \in \mathbb{N}_0} \lambda_{n_1,\dots,n_4,N} z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} \otimes 1.
\end{aligned}
$$

This contradicts Step 1. □

(d) Since $\Delta(z_i) \in \widehat{\mathscr{Z}} \otimes \widehat{\mathscr{Z}}$ and $\widehat{\mathscr{Z}}$ is normal, the right ideal $\widehat{\mathscr{B}}\widehat{\mathscr{Z}}^{+}$ is a Hopf ideal. By [A+, Proposition 3.6 (c)] we get that the equality $\widehat{\mathscr{Z}} = {}^{\operatorname{co}\pi}\widehat{\mathscr{B}}$ is equivalent to $\mathscr{B}_{\mathfrak{q}} \simeq \widehat{\mathscr{B}}/\widehat{\mathscr{B}}\widehat{\mathscr{Z}}^{+}$. This last isomorphism holds because the diagram

$$\begin{tikzcd} \mathcal{J}_{\mathfrak{q}} \arrow[r, hook] \arrow[d] & T(V) \arrow[d] \arrow[dr] & \\ \widehat{\mathscr{B}}\widehat{\mathscr{Z}}^{+} \arrow[r, hook] & \widehat{\mathscr{B}} \arrow[r, two heads, "\pi"] & \mathscr{B}_{\mathfrak{q}} \end{tikzcd}$$

commutes. □

Proposition 4.11.

(a) *There is an extension of braided Hopf algebras*

$$\Bbbk \to \widehat{\mathscr{Z}} \hookrightarrow \widehat{\mathscr{B}} \twoheadrightarrow \mathscr{B}_{\mathfrak{q}} \to \Bbbk.$$

(b) *The pre-Nichols algebra $\widehat{\mathscr{B}}$ is eminent and* $\operatorname{GK-dim} \widehat{\mathscr{B}} = 5$.

Proof.

(a) Follows from Lemma 4.10 (d).

(b) We know that $\widehat{\mathscr{B}}$ covers all elements of $\mathfrak{Pre}_{\mathrm{fGK}}$ by Lemma 4.8; it remains to show that $\widehat{\mathscr{B}}$ itself belongs to $\mathfrak{Pre}_{\mathrm{fGK}}$. By [A+, Proposition 3.6 (d)] there is a right $\widehat{\mathscr{Z}}$-linear isomorphism $\mathscr{B}_{\mathfrak{q}} \otimes \widehat{\mathscr{Z}} \simeq \widehat{\mathscr{B}}$. Since $\mathscr{B}_{\mathfrak{q}}$ is finite dimensional, this implies that $\widehat{\mathscr{B}}$ is finitely generated as a $\widehat{\mathscr{Z}}$-module. Now [KL, Proposition 5.5] provides $\operatorname{GK-dim} \widehat{\mathscr{B}} = \operatorname{GK-dim} \widehat{\mathscr{Z}} = 5$. □

4.3. **Type B_2.**

Lemma 4.12. *Assume that* **a** *is of Cartan type B_2. Then the distinguished pre-Nichols algebra $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is eminent.*

Proof. Here $N > 2$. We may fix a braiding matrix $\mathfrak{q}$ such that $q_{11} = q_{22}^2$, so $q = q_{22}$ and $\widetilde{q_{12}} = q^{-2}$. Let $\mathscr{B}$ be a finite GK-dimensional pre-Nichols algebra of V. It is enough to prove that $x_{112} = 0 = x_{2221}$ in $\mathscr{B}$.

Assume first $x_{112} \neq 0$, and consider the 3-dimensional braided subspace $W := \Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk x_{112} \subset \mathcal{P}(\mathscr{B})$. Then $\operatorname{GK-dim} \mathscr{B}(W) < \infty$ from Lemma 2.8. We split the proof according to the several possibilities for N.

♡ $N = 3$. Now the braiding on W is of Cartan type

$$\overset{q^2}{\underset{1}{\circ}} \overset{q^{-2}}{\rule[0.5ex]{3em}{0.4pt}} \overset{q}{\underset{2}{\circ}} \overset{q^{-2}}{\rule[0.5ex]{3em}{0.4pt}} \overset{q^2}{\underset{112}{\circ}}, \qquad A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

Since A is of affine type $C_2^{(1)}$, this contradicts [AAH2, Theorem 1.2(a)].

♡ $N = 6$. In this case the braiding on $W_2 := \Bbbk x_2 \oplus \Bbbk x_{112} \subset W$ is

$$\overset{q}{\underset{2}{\circ}} \overset{q^{-2}}{\rule[0.5ex]{3em}{0.4pt}} \overset{q^5}{\underset{112}{\circ}}, \qquad \text{Cartan type } \begin{pmatrix} 2 & -2 \\ -4 & 2 \end{pmatrix}.$$

The Cartan matrix is of indefinite type, and by [AAH2, Theorem 1.2(b)] this contradicts $\operatorname{GK-dim} \mathscr{B}(W) < \infty$.

$\heartsuit$ $N \neq 3, 6$. The Dynkin diagram of $W_1 := \Bbbk x_1 \oplus \Bbbk x_{112} \subset W$ is

$$\mathfrak{D}_1 = \underset{1}{\overset{q^2}{\circ}} \overset{q^6}{\rule[3pt]{60pt}{0.4pt}} \underset{112}{\overset{q^5}{\circ}} .$$

Since $\operatorname{GK-dim} \mathscr{B}(W_1) < \infty$, it follows from [AAH2, Theorem 1.2(b)] that the associated root system is finite. Now $\mathfrak{D}_1$ is connected; by exhaustion on [H1, Table 1], we deduce that we must have $N = 4$ or $N = 8$. We turn again to W_2, whose Dynkin diagram is easily computed in each case:

$$\heartsuit\heartsuit N = 4: \qquad \underset{2}{\overset{q}{\circ}} \overset{q^{-2}}{\rule[3pt]{60pt}{0.4pt}} \underset{112}{\overset{q}{\circ}}, \qquad \text{Cartan type } \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

$$\heartsuit\heartsuit N = 8: \qquad \underset{2}{\overset{q}{\circ}} \overset{q^{-2}}{\rule[3pt]{60pt}{0.4pt}} \underset{112}{\overset{q^5}{\circ}}, \qquad \text{Cartan type } \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

In any case the Cartan matrix is of affine type $A_1^{(1)}$, so $\operatorname{GK-dim} \mathscr{B}(W_2) = \infty$ by [AAH2, Theorem 1.2(b)].

Assume $x_{2221} \neq 0$ in $\mathscr{B}$. The subspace $U := \Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk x_{2221} \subset \mathcal{P}(\mathscr{B})$ has dimension 3 and $\operatorname{GK-dim} \mathscr{B}(U) < \infty$. Now $U_1 := \Bbbk x_1 \oplus \Bbbk x_{2221} \subset U$ has connected Dynkin diagram

$$\underset{1}{\overset{q^2}{\circ}} \overset{q^2}{\rule[3pt]{60pt}{0.4pt}} \underset{2221}{\overset{q^5}{\circ}},$$

and it is finite by [AAH2, Theorem 1.2(b)]. By exhaustion on [H1, Table 1] we deduce that $N = 4$. Then the Dynkin diagram of U is of Cartan type

$$\underset{2}{\overset{q}{\circ}} \overset{-1}{\rule[3pt]{60pt}{0.4pt}} \underset{1}{\overset{-1}{\circ}} \overset{-1}{\rule[3pt]{60pt}{0.4pt}} \underset{2221}{\overset{q}{\circ}}, \qquad A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

Since A is of affine type $C_2^{(1)}$, this contradicts [AAH2, Theorem 1.2(a)]. $\square$

4.4. **Type G_2.**

Lemma 4.13. *Assume that* $\mathbf{a}$ *is of Cartan type* G_2*. Then the quantum Serre relations hold in any* $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}$*. In particular, the distinguished pre-Nichols algebra* $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ *is eminent if* $N \neq 4, 6$*.*

Proof. Here $N > 3$. Let $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(V)$; we show first that the quantum Serre relations $x_{11112} = 0 = x_{221}$ hold in $\mathscr{B}$.

Start assuming $x_{11112} \neq 0$. Then the 3-dimensional subspace $W := \Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk x_{11112} \subset \mathcal{P}(\mathscr{B})$ satisfies $\operatorname{GK-dim} \mathscr{B}(W) \leq \operatorname{GK-dim} \mathscr{B}$ by Lemma 2.8. The Dynkin diagram of $W_1 := \Bbbk x_1 \oplus \Bbbk x_{11112} \subset W$ is

$$\mathfrak{D}_1 = \underset{1}{\overset{q}{\circ}} \overset{q^5}{\rule[3pt]{60pt}{0.4pt}} \underset{11112}{\overset{q^7}{\circ}} .$$

Since $\operatorname{GK-dim} \mathscr{B}(W_1) < \infty$, it follows from [AAH2, Theorem 1.2(b)] that the root system of $\mathfrak{D}_1$ is finite. We split the proof according to the several possibilities for N.

$\heartsuit$ $N = 5$. The diagram $\mathfrak{D}_1$ is disconnected, but we might consider instead $W_2 := \Bbbk x_{11112} \oplus \Bbbk x_2 \subset W$, that satisfies $\operatorname{GK-dim} \mathscr{B}(W_2) < \infty$ as well. By direct computation W_2 is of indefinite Cartan type:

$$\overset{q^2}{\underset{11112}{\circ}} \xrightarrow{\quad q^{-6} \quad} \overset{q^3}{\underset{2}{\circ}}, \qquad \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix},$$

which is in contradiction with [AAH2, Theorem 1.2(a)].

♡ $N \neq 5$. Now $\mathfrak{D}_1$ is connected and finite; by inspection on [H1, Table 1], we must have $N = 4$ or $N = 6$.

♡♡$N = 4$. In this case W_2 is of Cartan type

$$\overset{q^3}{\underset{11112}{\circ}} \xrightarrow{\quad q^{-6} \quad} \overset{q^3}{\underset{2}{\circ}}, \qquad \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

which is of affine type $A_1^{(1)}$, now contradicting [AAH2, Theorem 1.2(b)].

♡♡$N = 6$. In this case the Dynkin diagram of W is of Cartan type

$$\overset{q}{\underset{11112}{\circ}} \xrightarrow{\quad q^{-1} \quad} \overset{q}{\underset{1}{\circ}} \xrightarrow{\quad q^{-3} \quad} \overset{q^3}{\underset{2}{\circ}}, \qquad A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}.$$

By [AAH2, Theorem 1.2(b)] this contradicts $\operatorname{GK-dim} \mathscr{B}(W) < \infty$, since A is of affine type $G_2^{(1)}$.

Assume now $x_{221} \neq 0$ in $\mathscr{B}$. The subspace $U := \Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk x_{221} \subset \mathcal{P}(\mathscr{B})$ has dimension 3 and $\operatorname{GK-dim} \mathscr{B}(U) < \infty$. Consider two possibilities for N.

♡$N \neq 4$. Now $U_1 := \Bbbk x_1 \oplus \Bbbk x_{221} \subset U$ has connected Dynkin diagram

$$\overset{q}{\underset{1}{\circ}} \xrightarrow{\quad q^{-4} \quad} \overset{q^7}{\underset{221}{\circ}}.$$

By exhaustion on [H1, Table 1] we conclude that this diagram is never finite, which contradicts [AAH2, Theorem 1.2(b)], as $\operatorname{GK-dim} \mathscr{B}(U_1) < \infty$.

♡$N = 4$. In this case the braiding on U is of Cartan type

$$\overset{q}{\underset{1}{\circ}} \xrightarrow{\quad q^{-3} \quad} \overset{q^3}{\underset{2}{\circ}} \xrightarrow{\quad q^{-3} \quad} \overset{q^3}{\underset{221}{\circ}}, \qquad A = \begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Since A is of affine type $G_2^{(1)}$, this contradicts [AAH2, Theorem 1.2(a)].

Thus the quantum Serre relations hold in $\mathscr{B}$. By Remark 4.1 this proves the assertion regarding $N \neq 4, 6$. □

4.5. **Type A_3.**

Lemma 4.14. *If* $\mathbf{a}$ *is of Cartan type A_3 with $N > 2$, then $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is eminent.*

Proof. As $N > 2$, the ideal $\mathcal{I}_{\mathfrak{q}}$ is generated by the quantum Serre relations $x_{13} = 0$ and $x_{iij} = 0$ for $|j - i| = 1$, cf. [AA2, p. 397]. Let $\mathscr{B} \in \mathfrak{Pre}_{\text{fGK}}(\mathfrak{q})$. Then $x_{13} = 0$ holds in $\mathscr{B}$ since the braided vector space $\Bbbk x_1 \oplus \Bbbk x_3$ satisfies the hypothesis in Proposition 3.2.

Turn to x_{iij} for some fix $i, j \in \mathbb{I}_3$ with $|j - i| = 1$; in this case $\Bbbk x_i \oplus \Bbbk x_j$ is of Cartan type A_2. If $N > 3$, then $x_{iij} = 0$ in $\mathscr{B}$ by Lemma 4.7. Only the case $N = 3$ remains. Now we have $\mathfrak{q}(2\alpha_i + \alpha_j, 2\alpha_i + \alpha_j) = q_{ii}^4 \widetilde{q_{ij}}^2 q_{jj} = q^5 q^{-2} = 1$. Using [AAH1, Lemma 2.8], in order to guarantee $x_{iij} = 0$ in $\mathscr{B}$ it is enough to find $k \in \mathbb{I}_3$ such that $\widetilde{\mathfrak{q}}(\alpha_k, 2\alpha_i + \alpha_j) \neq 1$. It is straightforward to verify that the unique $k \in \mathbb{I}_3$ different from i and j does the trick. □

4.6. **Types B_3 and C_3.**

Lemma 4.15. *The distinguished pre-Nichols algebra $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is eminent if either*

(i) $\mathbf{a}$ *is of type B_3, or* (ii) $\mathbf{a}$ *is of type C_3.*

Proof. Let $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{q})$. Then $x_{13} = 0$ holds in $\mathscr{B}$. Indeed, the braided vector space $\Bbbk x_1 \oplus \Bbbk x_3$ satisfies the hypothesis in Proposition 3.2. Similarly, since $\Bbbk x_2 \oplus \Bbbk x_3$ is of type B_2, it follows from Lemma 4.12 that the quantum Serre relations involving x_2 and x_3 hold in $\mathscr{B}$.

Step 1. If $\mathbf{a}$ is of Cartan type B_3, then the quantum Serre relations hold in any finite GK-dim pre-Nichols algebra.

Proof. Here $\Bbbk x_1 \oplus \Bbbk x_2$ has Dynkin diagram $\overset{q^2}{\circ}\overset{q^{-2}}{\rule[3pt]{40pt}{0.4pt}}\overset{q^2}{\circ}$, type A_2. Hence, if $\operatorname{ord} q^2 > 3$, we know from Lemma 4.7 that the quantum Serre relations between x_1 and x_2 hold in $\mathscr{B}$. Let us show that in the cases $\operatorname{ord} q^2 = 2, 3$ the same happens.

♡ $\operatorname{ord} q^2 = 2$. If $x_{112} \neq 0$ in $\mathscr{B}$, we get a subspace $\Bbbk x_2 \oplus \Bbbk x_3 \oplus \Bbbk x_{112} \subset \mathcal{P}(\mathscr{B})$ of dimension 3 with the following Dynkin diagram

$$\overset{-1}{\underset{2}{\circ}}\overset{-1}{\rule[3pt]{80pt}{0.4pt}}\overset{q}{\underset{3}{\circ}}\overset{-1}{\rule[3pt]{80pt}{0.4pt}}\overset{-1}{\underset{112}{\circ}}, \qquad \text{Cartan type} \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

This matrix is of affine type $C_2^{(1)}$, hence $\operatorname{GK-dim} \mathscr{B} = \infty$, a contradiction.

Similarly, the assumption $x_{221} \neq 0$ yields a subspace of $\mathcal{P}(\mathscr{B})$ with braiding

$$\begin{array}{ccccc} & & \overset{-1}{\underset{221}{\circ}} & & \\ & & -1\Big| & & \\ \overset{-1}{\underset{1}{\circ}} & \overset{-1}{\rule[3pt]{70pt}{0.4pt}} & \overset{-1}{\underset{2}{\circ}} & \overset{-1}{\rule[3pt]{70pt}{0.4pt}} & \overset{q}{\underset{3}{\circ}}, \end{array} \qquad \text{Cartan type} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}.$$

The Cartan matrix is of affine type $B_3^{(1)}$, and again $\operatorname{GK-dim}(\mathscr{B}) = \infty$.

♡ $\operatorname{ord} q^2 = 3$. Notice that

$$\mathfrak{q}(2\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2) = q^6 = 1, \qquad \widetilde{\mathfrak{q}}(2\alpha_1 + \alpha_2, \alpha_3) = q^{-2} \neq 1;$$
$$\mathfrak{q}(\alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_2) = q^6 = 1, \qquad \widetilde{\mathfrak{q}}(\alpha_1 + 2\alpha_2, \alpha_3) = q^{-4} \neq 1.$$

Assuming $x_{112} \neq 0$ in $\mathscr{B}$ we get $\Bbbk x_3 \oplus \Bbbk x_{112} \subset \mathcal{P}(\mathscr{B})$ with Dynkin diagram $\overset{q}{\circ}\overset{q^{-2}}{\rule[3pt]{40pt}{0.4pt}}\overset{1}{\circ}$. Then by [AAH1, Lemma 2.8] it follows that $\operatorname{GK-dim} \mathscr{B} = \infty$, a contradiction. By the same argument, we can not have $x_{221} \neq 0$ in $\mathscr{B}$. □

The assertion (i) for $N > 4$ follows since, in that case, $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations, cf. Remark 4.1.

Step 2. If $\mathbf{a}$ is of Cartan type B_3 with $N = 3$, then $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is eminent.

Proof. By [AA2, pp. 399, 400], $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations and $[x_{3321}, x_{32}]_c = 0$. Given $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}$, let us show that $[x_{3321}, x_{32}]_c \in \mathcal{P}(\mathscr{B})$. Using

$x_{13} = 0$ a straightforward computation gives

$$\Delta(x_{3321}) = x_{3321} \otimes 1 + 1 \otimes x_{3321} + (1 - q_{33})x_{332} \otimes x_1 + (1 - q_{33})q_{33}x_3 \otimes x_{321} + (1 - q_{33})(1 - q_{22})x_3^2 \otimes x_{21}.$$

With this we compute

$$\begin{aligned}\Delta([x_{3321}, x_{32}]_c) =& [x_{3321}, x_{32}]_c \otimes 1 + 1 \otimes [x_{3321}, x_{32}]_c \\ &- (1 - q_{33})^2(1 - q_{22})q_{12}\, x_3^2 \otimes x_{221} \\ &- (1 - q_{33})q_{12}q_{13}q_{23}q_{33}\, x_{3332} \otimes x_{21} \\ &+ (1 - q_{33})q_{33}^2 q_{12}q_{13}\, x_{332} \otimes (x_{321} - [x_{32}, x_1]_c) \\ &- (1 - q_{33})q_{23}q_{13}\, x_{33321} \otimes x_2 \\ &+ (1 - q_{33})q_{13}q_{12}\, [x_{332}, x_{32}]_c \otimes x_1 \\ &+ (1 - q_{33})q_{33}\, x_3 \otimes (q_{13}q_{23}q_{33}[x_{3321}, x_2]_c + [x_{321}, x_{32}]_c)\end{aligned}$$

The third and fourth terms vanish in $\mathscr{B}$ by Step 1. For the fifth term, a straightforward computation involving $x_{13} = 0$ shows that $x_{321} = [x_{32}, x_1]_c$. The last three terms also vanish, but they require a more detailed analysis.

♡ $x_{33321} = 0$ in $\mathscr{B}$. Notice that

$$\Delta(x_{33321}) = x_{33321} \otimes 1 + 1 \otimes x_{33321} + (1 - q_{33})x_{3332} \otimes x_1 - (1 - q_{33}^2)q_{32}x_{332} \otimes x_{31},$$

so this element is primitive in $\mathscr{B}$ by Step 1. Assuming $x_{33321} \neq 0$ we get a subspace $\Bbbk x_1 \oplus \Bbbk x_{33321} \subset \mathcal{P}(\mathscr{B})$ where the braiding is given by

$$\underset{1}{\overset{q^2}{\circ}} \xrightarrow{\ q^{-1}\ } \underset{33321}{\overset{q^2}{\circ}}, \qquad \text{Cartan type } \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \qquad \text{affine type } A_1^{(1)}.$$

this contradicts GK-dim $\mathscr{B} < \infty$ by [AAH2, Theorem 1.2].

♡ $[x_{332}, x_{32}]_c = 0$ in $\mathscr{B}$. Now we have

$$\Delta([x_{332}, x_{32}]_c) = [x_{332}, x_{32}]_c \otimes 1 + 1 \otimes [x_{332}, x_{32}]_c - (1 - q_{33})^2 q_{23}\, x_3^2 \otimes [x_{32}, x_2]_c - (1 - q_{33})q_{33}q_{23}\, x_{3332} \otimes x_2.$$

The element $[x_{32}, x_2]_c$ is primitive in $\mathscr{B}$, so it vanishes by the same reason that x_{223} does (cf. proof of Lemma 4.12). So $[x_{332}, x_{32}]_c \in \mathcal{P}(\mathscr{B})$ by Step 1. If it is non-zero, consider $\Bbbk x_1 \oplus \Bbbk[x_{332}, x_{32}]_c \subset \mathcal{P}(\mathscr{B})$ where the braiding is

$$\underset{x_1}{\overset{q^2}{\circ}} \xrightarrow{\ q^{-1}\ } \underset{[x_{332}, x_{32}]}{\overset{q^2}{\circ}}, \qquad \text{Cartan type } \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \qquad \text{affine type } A_1^{(1)},$$

thus we get the same contradiction as with x_{33321}.

♡ $q_{13}q_{23}q_{33}[x_{3321}, x_2]_c + [x_{321}, x_{32}]_c = 0$. Denote this element by r. Then

$$\begin{aligned}\Delta(r) =& r \otimes 1 + 1 \otimes r + (1 - q_{33})q_{22}q_{12}q_{13}\, x_{32} \otimes (x_{321} - [x_{32}, x_1]) \\ &+ (1 - q_{33})q_{33}q_{12}q_{13}q_{23}\, \big[x_3, [x_{32}, x_2]\big] \otimes x_1 \\ &- (1 - q_{33})q_{22}q_{12}q_{13}q_{23}\, x_3 \otimes \big[[x_{32}, x_2], x_1\big].\end{aligned}$$

Since $[x_{32}, x_2] = 0$ and $x_{321} - [x_{32}, x_1] = 0$ in $\mathscr{B}$, it follows that r is primitive. If $r \neq 0$ we consider $\Bbbk x_2 \oplus \Bbbk r \subset \mathcal{P}(\mathscr{B})$. The Dynkin diagram is computed:

$$\underset{2}{\overset{q^2}{\circ}} \xrightarrow{\quad q^{-1} \quad} \underset{r}{\overset{q^2}{\circ}}, \qquad \text{Cartan type } \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \qquad \text{affine type } A_1^{(1)},$$

thus we get the same contradiction as before.

Using these three $\heartsuit$ we get $[x_{3321}, x_{32}]_c \in \mathcal{P}(\mathscr{B})$. If this element is non-zero, consider $U = \Bbbk x_3 \oplus \Bbbk[x_{3321}, x_{32}]_c \subset \mathcal{P}(\mathscr{B})$. We compute the braiding:

$$\mathfrak{q}(\alpha_1 + 2\alpha_2 + 3\alpha_3, \alpha_1 + 2\alpha_2 + 3\alpha_3) = 1, \quad \widetilde{\mathfrak{q}}(\alpha_1 + 2\alpha_2 + 3\alpha_3, \alpha_3) = q^{-1} \neq 1.$$

From [AAH1, Lemma 2.8] it follows that $\operatorname{GK-dim} \mathscr{B}(U) = \infty$, but this contradicts $\operatorname{GK-dim} \mathscr{B} < \infty$. Then $[x_{3321}, x_{32}]_c = 0$ in $\mathscr{B}$ and Step 2 holds. $\square$

Step 3. If $\mathbf{a}$ is of Cartan type B_3 with $N = 4$, then $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is eminent.

Proof. By [AA2, pp. 399, 400], $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations and $[x_{123}, x_2]_c = 0$. We claim that this element is primitive in any $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}$. Indeed, using that $x_{13} = 0$ in $\mathscr{B}$, we get

$$\begin{aligned}\Delta([x_{123}, x_2]_c) =& [x_{123}, x_2]_c \otimes 1 + 1 \otimes [x_{123}, x_2]_c \\ & - (1 - \widetilde{q_{12}}) q_{32} x_1 \otimes x_{223} + (1 - \widetilde{q_{23}}) q_{32} [x_{12}, x_2]_c \otimes x_3.\end{aligned}$$

By straightforward computations, $[x_{12}, x_2]_c \in \mathcal{P}(\mathscr{B})$ and it vanishes by the same reason that x_{221} does (cf. proof of Lemma 4.7). Since $x_{223} = 0$, the claim follows.

Assume $[x_{123}, x_2]_c \neq 0$. Inside $\mathcal{P}(\mathscr{B})$ we have the 2-dimensional subspace $U = \Bbbk x_3 \oplus \Bbbk[x_{123}, x_2]_c$ where the braiding is given by

$$\underset{x_3}{\overset{q}{\circ}} \xrightarrow{\quad -1 \quad} \underset{[x_{123}, x_2]}{\overset{-q}{\circ}}, \qquad \text{Cartan type } \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Since this matrix is of affine type $A_1^{(1)}$, from [AAH2, Theorem 1.2(b)] it follows $\operatorname{GK-dim} \mathscr{B}(U) = \infty$, contradicting $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}$. $\square$

Step 4. If $\mathbf{a}$ is of Cartan type C_3, then the quantum Serre relations hold in any finite GK-dim pre-Nichols algebra.

Proof. Now $\Bbbk x_1 \oplus \Bbbk x_2$ has Dynkin diagram $\overset{q}{\circ} \xrightarrow{q^{-1}} \overset{q}{\circ}$, type A_2. If $N > 3$, then the quantum Serre relations in x_1 and x_2 hold by Lemma 4.7. For the case $N = 3$, let i, j such that $\{i, j\} = \{1, 2\}$ and suppose $x_{iij} \neq 0$ in $\mathscr{B}$. Since $\mathfrak{q}(2\alpha_i + \alpha_j, 2\alpha_i + \alpha_j) = q^5 q^{-2} = 1$ and $\widetilde{\mathfrak{q}}(2\alpha_i + \alpha_j, \alpha_3) = \widetilde{q_{i3}}^2 \widetilde{q_{j3}} \neq 1$, we get $\operatorname{GK-dim} \mathscr{B} = \infty$ by [AAH1, Lemma 2.8]. $\square$

The assertion (ii) for $N > 4$ follows since, in that case, $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations, see Remark 4.1.

Step 5. If $\mathbf{a}$ is of Cartan type C_3 with $N = 3$, then $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is eminent.

Proof. Following [AA2, pp. 401, 402]) we see that $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations and $\big[[x_{123}, x_2]_c, x_2\big]_c = 0$. Given $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}$, let us show that this element is primitive in $\mathscr{B}$. Using $x_{13}=0$ it follows that

$$\begin{aligned}\Delta([x_{123},x_2]_c) =&[x_{123},x_2]_c\otimes 1+1\otimes [x_{123},x_2]_c+(1-\widetilde{q_{12}})x_{123}\otimes x_2\\ &+(1-q^2)x_{12}\otimes x_{32}+(1-\widetilde{q_{12}})x_1\otimes (x_{23}x_2-q_{32}x_2x_{23})\\ &+(1-\widetilde{q_{23}})q_{32}[x_{12},x_2]_c\otimes x_3.\end{aligned}$$

By straightforward computations, $[x_{12},x_2]_c=q_{12}^2qx_{221}$ in $T(V)$, and so $[x_{12},x_2]_c$ vanishes in $\mathscr{B}$ by Step 4. Then we obtain

$$\begin{aligned}\Delta\big(\big[[x_{123},x_2]_c,x_2\big]_c\big) &= \big[[x_{123},x_2]_c,x_2\big]_c\otimes 1+1\otimes \big[[x_{123},x_2]_c,x_2\big]_c\\ &+(1-q^2)q_{32}q_{22}[x_{12},x_2]_c\otimes x_{32}+(1-\widetilde{q_{12}})q_{22}q_{32}^2x_1\otimes x_{2223},\end{aligned}$$

and now the claim follows from Step 4.

If $\big[[x_{123},x_2]_c,x_2\big]_c\neq 0$, consider $U=\Bbbk x_1\oplus \Bbbk\big[[x_{123},x_2]_c,x_2\big]_c\subset \mathcal{P}(\mathscr{B})$. By [AAH1, Lemma 2.8], since $\mathfrak{q}(\alpha_1+3\alpha_2+\alpha_3,\alpha_1+3\alpha_2+\alpha_3)=q^{12}q^{-9}=1$ and $\widetilde{\mathfrak{q}}(\alpha_1+3\alpha_2+\alpha_3,\alpha_1)=q^2q^{-3}\neq 1$, we have GK-dim $\mathscr{B}(U)=\infty$. This contradicts $\mathscr{B}\in\mathfrak{Pre}_{\text{fGK}}$. ☐

Step 6. If $\mathbf{a}$ is of Cartan type C_3 with $N=4$, then $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is eminent.

Proof. By [An1, Theorem 3.1], $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations and $[x_{123},x_{23}]_c=0$. Let us show that this element is primitive in any pre-Nichols algebra $\mathscr{B}$ of finite GK-dim.

First we claim that $[x_{123},x_3]_c=0$ in $\mathscr{B}$: using that $x_{13}=0$ we compute

$$\begin{aligned}\Delta\big([x_{123},x_3]_c\big) =&[x_{123},x_3]_c\otimes 1+1\otimes [x_{123},x_3]_c\\ &+(1-\widetilde{q_{12}}q_{23}q_{33})x_{13}\otimes x_{23}+(1-\widetilde{q_{12}})x_1\otimes [x_{23},x_3]_c.\end{aligned}$$

Since $[x_{23},x_3]_c\in\mathcal{P}(\mathscr{B})$, it vanishes in $\mathscr{B}$ by the same reason that x_{332} does (cf. proof of Lemma 4.12). So $[x_{123},x_3]_c\in\mathcal{P}(\mathscr{B})$. Hence, if it is non-zero we get a subspace $U=\Bbbk x_1\oplus[x_{123},x_3]_c\subset\mathcal{P}(\mathscr{B})$ where the braiding is given by

$$\underset{x_3}{\overset{q}{\circ}}\xrightarrow{\quad q^{-3}\quad}\underset{[x_{123},x_3]}{\overset{q}{\circ}}, \qquad \text{indefinite Cartan type } \begin{pmatrix}2 & -3\\ -3 & 2\end{pmatrix}.$$

But then GK-dim $\mathscr{B}(U)=\infty$ by [AAH2, Theorem 1.2(b)], a contradiction.

Next we compute

$$\begin{aligned}\Delta\big([x_{123},x_{23}]_c\big) &= [x_{123},x_{23}]_c\otimes 1+1\otimes [x_{123},x_{23}]_c\\ &+(1-\widetilde{q_{23}})q_{12}q_{22}q_{32}x_2\otimes [x_{123},x_3]+(1-\widetilde{q_{23}})^2q_{32}x_1\otimes [x_{12},x_2]_c.\end{aligned}$$

Using the previous claim and the fact $[x_{12},x_2]_c=q_{12}^2qx_{221}=0$ (by Step 4), we get $[x_{123},x_{23}]_c\in\mathcal{P}(\mathscr{B})$. If $[x_{123},x_{23}]_c\neq 0$, consider the subspace $W=\Bbbk x_1\oplus\Bbbk x_2\oplus\Bbbk[x_{123},x_{23}]_c\subset\mathcal{P}(\mathscr{B})$, where the braiding is

$$\underset{1}{\overset{q}{\circ}}\xrightarrow{\quad q^{-1}\quad}\underset{2}{\overset{q}{\circ}}\xrightarrow{\quad q^{-1}\quad}\underset{[x_{123},x_{23}]}{\overset{q^3}{\circ}}, \qquad \text{Cartan type } \begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -3 & 2\end{pmatrix}.$$

Since the Cartan matrix is of affine type $G_2^{(1)}$, it follows GK-dim $\mathscr{B}(W)=\infty$ by [AAH2, Theorem 1.2(a)]. This contradicts $\mathscr{B}\in\mathfrak{Pre}_{\text{fGK}}$. ☐

The result follows. ☐

4.7. **Some cases in rank** > 3. Here we assume that $\theta \geq 4$.

Lemma 4.16. *In any of the following cases,* $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ *is eminent.*

(a) $\mathbf{a}$ *is of Cartan type with simply laced Dynkin diagram and* $N > 2$*.*
(b) $\mathbf{a}$ *is of type* B_θ*,* C_θ *(*$\theta \geq 4$*) or* F_4*, and* $N > 4$*.*

Proof. By Remark 4.1 and the restrictions on N, $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations. Let $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{q})$. If $a_{ij} = 0$, then $x_{ij} = 0$ holds in $\mathscr{B}$ by Proposition 3.2. If $a_{ij} \neq 0$, then there is $k \in \mathbb{I}$ such that $\{i, j, k\}$ span a subdiagram of type A_3, B_3 or C_3. Then $(\operatorname{ad} x_i)^{1-a_{ij}}(x_j) = 0$ by Lemmas 4.14 or 4.15. Thus $\widetilde{\mathscr{B}}_{\mathfrak{q}} \twoheadrightarrow \mathscr{B}$. □

In the next subsections we treat some remaining cases with small N.

4.8. **Types** B_θ**,** C_θ**,** F_4**,** $\theta > 3$**,** $N = 3, 4$**.**

Lemma 4.17. *If* $\mathbf{a}$ *is of types* B_θ*,* C_θ*, with* $\theta > 3$*, or* F_4*, and* $N = 3$ *or* 4*, then* $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ *is eminent.*

Proof. We split the proof according to the type. Let $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}$.

♡ Type F_4. Here $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations and

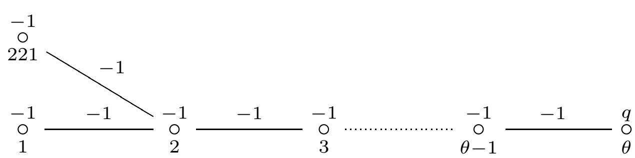
$$[x_{123}, x_{23}]_c = [x_{432}, x_3]_c = 0 \text{ if } N = 4; \qquad [x_{2234}, x_{23}]_c = 0 \text{ if } N = 3. \tag{4.12}$$

Since $N > 2$ we get $x_{14} = 0$ in $\mathscr{B}$ from Proposition 3.2. The subdiagram spanned by $\{1, 2, 3\}$ is of type C_3 thus the quantum Serre relations involving these indices hold in $\mathscr{B}$ by Lemma 4.15 (ii). Finally, $\{4, 3, 2\}$ span a diagram of type B_3 so the quantum Serre relations involving these indices hold in $\mathscr{B}$ by Lemma 4.15 (i). Moreover (4.12) are defining relations of the distinguished pre-Nichols algebra of type B_3 or C_3 for the corresponding N, hence Lemma 4.15 implies that these also vanish in $\mathscr{B}$.

♡ Type B_θ. Here $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations and

$$[x_{(i\,i+2)}, x_{i+1}]_c,\ i < \theta - 1, \text{ if } N = 4; \quad [x_{\theta\theta(\theta-1)(\theta-2)}, x_{\theta\theta-1}]_c, \text{ if } N = 3. \tag{4.13}$$

The relations involving the indices $\{\theta - 2, \theta - 1, \theta\}$ hold in $\mathscr{B}$ by Lemma 4.15 (i); also $x_{i\theta} = 0$ for any $i < \theta - 1$ by Proposition 3.2. We are left to treat the relations involving $\{1, \dots, \theta - 1\}$. If $N = 3$ we only have the quantum Serre relations, which hold by Lemma 4.14. Turn to $N = 4$. Now $\{1, \dots, \theta - 1\}$ form a subdiagram of type $A_{\theta-1}$ at a root of order 2. If $\theta - 1 \geq 4$ we apply Lemma 5.6 to get all the Serre relations except for x_{221} and $x_{\theta-2\,\theta-2\,\theta-1}$. The last one holds by Lemma 4.15 (i). For the first one, we apply [AAH2, Theorem 1.2] since the diagram



is of affine Cartan type. Now $[x_{(i\,i+2)}, x_{i+1}]_c = 0$ for $i < \theta - 2$ hold by Lemma 5.6 (e). We treat separately the last case standing.

♡♡ Type B_4 with $N = 4$. The relations x_{221} and $x_{\theta-2\,\theta-2\,\theta-1}$ hold by the same reason as above. Moreover, we also have $x_{13} = 0$. This follows from [AAH1, Lemma 2.8] since $\mathfrak{q}(\alpha_1 + \alpha_3, \alpha_1 + \alpha_3) = 1$ and $\widetilde{\mathfrak{q}}(\alpha_1 + \alpha_3, \alpha_4) \neq 1$. Finally, using (5.1) and the relations deduced so far, we get that $[x_{(13)}, x_2]_c$ is primitive in $\mathscr{B}$. Notice that

$\mathfrak{q}(\alpha_1+2\alpha_2+\alpha_3, \alpha_1+2\alpha_2+\alpha_3) = 1$ and $\widetilde{\mathfrak{q}}(\alpha_1+2\alpha_2+\alpha_3, \alpha_4) \neq 1$, so [AAH1, Lemma 2.8] applies again.

♡ Type C_θ. Here $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations and

$$[x_{(\theta-2\,\theta)}, x_{\theta-1\,\theta}]_c, \text{ if } N=4; \qquad \big[[x_{(\theta-2\,\theta)}, x_{\theta-1}]_c, x_{\theta-1}\big], \text{ if } N=3. \tag{4.14}$$

As before, Proposition 3.2 gives $x_{i\theta} = 0$ for any $i < \theta-1$; all the relations involving the indices $\{\theta-2, \theta-1, \theta\}$ hold in $\mathscr{B}$ by Lemma 4.15 (ii). It remains to verify the relations involving $\{1, \dots, \theta-1\}$. Here we only have the Serre relations. But these indices span a subdiagram of type $A_{\theta-1}$, $\theta-1 \geq 3$, at a root of unity of order 3 or 4, so they hold by Lemma 4.14. □

4.9. **Types E_6, E_7 and E_8 with $N = 2$.** By [AA2, p. 407] the distinguished pre-Nichols algebra is presented by the quantum Serre relations and

$$[x_{ijk}, x_j]_c = 0 \text{ if } i,j,k \text{ are all different and } \widetilde{q_{ij}}, \widetilde{q_{jk}} \neq 1.$$

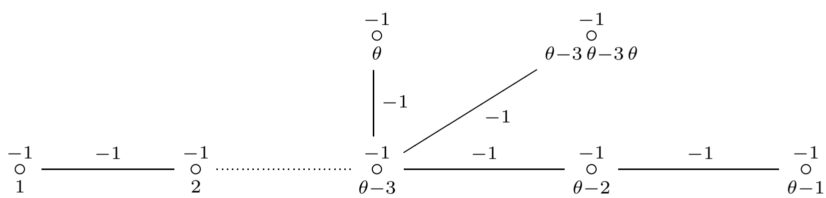
Lemma 4.18. *Assume that Conjecture* 2.6 *is true. If* $\mathbf{a}$ *is of type* E_6, E_7 *or* E_8 *with* $N=2$*, then* $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ *is eminent.*

We point out that Conjecture 2.6 is needed only for a 5-dimensional braided vector space of indefinite Cartan type.

Proof. Let $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{q})$. First we deal with the quantum Serre relations, which are always primitive. Fix $i \neq j \in \mathbb{I}_\theta$. Consider two possibilities.

♡ $\widetilde{q_{ij}} = 1$. In this case choose $k \in \mathbb{I}_\theta$ different from i and j such that $\widetilde{q_i} = 1$ but $\widetilde{q_{ik}} \neq 1$. We get $\mathfrak{q}(\alpha_i+\alpha_j, \alpha_i+\alpha_j) = 1$ and $\widetilde{\mathfrak{q}}(\alpha_i+\alpha_j, \alpha_k) \neq 1$. By [AAH1, Lemma 2.8], this warranties $x_{ij} = 0$ in $\mathscr{B}$.

♡ $\widetilde{q_{ij}} \neq 1$. In this case i and j are consecutive vertices in a subdiagram of type A_4 with $N=2$. By Lemma 5.6 (b) below, it follows that $x_{iij} = 0$ except in the following cases: $(i,j) \in \{(2,1), (\theta-3,\theta), (\theta-2,\theta-1)\}$. Fix such (i,j), assume $x_{iij} \neq 0$ and consider $\Bbbk x_1 \oplus \dots \oplus \Bbbk x_\theta \oplus \Bbbk x_{iij} \subset \mathcal{P}(\mathscr{B})$. Then the Dynkin diagram of this braided vector space is of indefinite Cartan type. We illustrate the case $(i,j) = (\theta-3,\theta)$, the other cases being similar.



Thus Conjecture 2.6 and Lemma 2.8 imply GK-dim $\mathscr{B} = \infty$.

Finally, fix i,j,k different such that $\widetilde{q_{ij}}, \widetilde{q_{jk}} \neq 1$. These are consecutive vertices in a suitable chosen subdiagram of type A_4. The Serre relations hold in $\mathscr{B}$, so by Lemma 5.6 (c) below we get that also $[x_{ijk}, x_j]_c = 0$ in $\mathscr{B}$. □

5. On the open cases

This section contain partial results towards those braidings of finite Cartan type which are still open. The detailed proofs can be found in [S].

5.1. **Type A_2 with $N = 2$.**

Lemma 5.1. *Assume* $\mathbf{a}$ *is of Cartan type* A_2 *with* $N = 2$. *Let* $\mathscr{B}$ *be a finite GK-dimensional pre-Nichols algebra of* $\mathfrak{q}$. *The following hold:*

(a) *if* $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}^{\mathbb{Z}^2}$, *then either* $x_{112} = 0$ *or* $x_{221} = 0$ *in* $\mathscr{B}$;
(b) *for different* $i, j \in \mathbb{I}_2$, $(\operatorname{ad}_c x_i)^4 x_j = 0$ *in* $\mathscr{B}$. $\square$

Question 5.2. Let $\widehat{\mathscr{B}}_1 = \Bbbk\langle x_1, x_2 | x_{221}, x_{11112}\rangle$. By Lemma 5.1 any $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}^{\mathbb{Z}^2}(V)$ is a quotient of either $\widehat{\mathscr{B}}_1$ or $\widehat{\mathscr{B}}_2 := \Bbbk\langle x_1, x_2 | x_{112}, x_{22221}\rangle$. Clearly $\widehat{\mathscr{B}}_1 \simeq \widehat{\mathscr{B}}_2$ as algebras. Is $\operatorname{GK-dim} \widehat{\mathscr{B}}_1 < \infty$?

5.2. **Type A_3 with $N = 2$.**

Lemma 5.3. *Assume* $\mathbf{a}$ *is of Cartan type* A_3 *with* $N = 2$. *Let* $\mathscr{B}$ *be a finite GK-dimensional pre-Nichols algebra of* $\mathfrak{q}$. *Then the following hold in* $\mathscr{B}$:

(a) $x_{112} = 0 = x_{332}$, $x_{213} \overset{\star}{=} 0$,
(b) $x_{22221} = 0 = x_{22223}$, $x_{11113} = 0 = x_{33331}$,
(c) *if* $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}^{\mathbb{Z}^3}$, *then at most one of* $x_{113}, x_{331}, x_{221}, x_{223}$ *is non-zero.* $\square$

Remark 5.4. The relation $\star$ is relevant because in the tensor algebra

$$\begin{aligned}\Delta([x_{(13)}, x_2]_c) = {}& [x_{(13)}, x_2] \otimes 1 + 1 \otimes [x_{(13)}, x_2] - 2q_{32} x_1 \otimes x_{223} \\ & -2q_{12}^2 q_{32} x_{221} \otimes x_3 - 2q_{12}^2 q_{32} x_2 \otimes x_{213} + 4q_{12}^2 q_{32} x_2^2 \otimes x_{13}.\end{aligned} \tag{5.1}$$

Question 5.5. By Lemma 5.3 every $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}^{\mathbb{Z}^3}$ is covered by one of

$$\widehat{\mathscr{B}} = \Bbbk\langle x_1, x_2, x_3 | x_{112}, x_{332}, x_{22221}, x_{22223}, x_{11113}, x_{33331}, x_{213}\rangle,$$
$$\widehat{\mathscr{B}}_1 = \widehat{\mathscr{B}}/\langle x_{113}\rangle, \quad \widehat{\mathscr{B}}_2 = \widehat{\mathscr{B}}/\langle x_{331}\rangle, \quad \widehat{\mathscr{B}}_3 = \widehat{\mathscr{B}}/\langle x_{221}\rangle, \quad \widehat{\mathscr{B}}_4 = \widehat{\mathscr{B}}/\langle x_{223}\rangle.$$

Are $\operatorname{GK-dim} \widehat{\mathscr{B}}_1$ or $\operatorname{GK-dim} \widehat{\mathscr{B}}_3 < \infty$? ($\widehat{\mathscr{B}}_1 \simeq \widehat{\mathscr{B}}_2$ and $\widehat{\mathscr{B}}_3 \simeq \widehat{\mathscr{B}}_4$ as algebras).

5.3. **Type A_θ, $\theta \geq 4$ with $N = 2$.** In this setting $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by

$$x_{ij} = 0, \quad |i-j| > 1; \quad x_{iij} = 0, \quad |i-j| = 1; \quad [x_{(ii+2)}, x_{i+1}]_c = 0, \quad i \in \mathbb{I}_{\theta-2}.$$

Lemma 5.6. *Assume* $\mathbf{a}$ *is of Cartan type* A_θ, $\theta \geq 4$, *with* $N = 2$. *The following hold in any finite GK-dimensional pre-Nichols algebra* $\mathscr{B}$ *of* $\mathfrak{q}$:

(a) $x_{ij} = 0$ *for any* $|i-j| > 1$;
(b) $x_{iij} = 0$ *for* $|i-j| = 1$ *and* $(i,j) \neq (2,1)$, $(\theta-1, \theta)$;
(c) $x_{iiiij} = 0$ *for* $(i,j) = (2,1)$, $(\theta-1,\theta)$;
(d) *if* $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}^{\mathbb{Z}^\theta}$, *then either* $x_{221} = 0$ *or* $x_{\theta-1\theta-1\theta} = 0$;
(e) *if* $(i,j) \in \{(2,1), (\theta-1,\theta)\}$ *and* $x_{iij} = 0$, *then* $[x_{(i-1\ i+1)}, x_i]_c = 0$. $\square$

Question 5.7. Let $\widehat{\mathscr{B}}_1$ denote the quotient of $T(V)$ by the relations

$$\begin{aligned} x_{ij} &= 0, \quad |i-j| > 1; && (\operatorname{ad}_c x_{\theta-1})^4 x_\theta = 0; \\ x_{iij} &= 0, \quad |i-j| = 1, (i,j) \neq (\theta-1,\theta); && [x_{(13)}, x_2]_c = 0. \end{aligned}$$

Similarly, define $\widehat{\mathscr{B}}_2$ by the relations

$$\begin{aligned} x_{ij} &= 0, \quad |i-j| > 1; && (\operatorname{ad}_c x_2)^4 x_1 = 0; \\ x_{iij} &= 0, \quad |i-j| = 1, (i,j) \neq (2,1); && [x_{(\theta-2\ \theta)}, x_{\theta-1}]_c = 0. \end{aligned}$$

(Clearly $\widehat{\mathscr{B}}_2 \simeq \widehat{\mathscr{B}}_1$ as algebras). Is $\operatorname{GK-dim} \widehat{\mathscr{B}}_1 < \infty$?

5.4. **Type D_θ with $N = 2$.** Here (cf. [AA2, p. 404]) the distinguished pre-Nichols algebra $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ is presented by the quantum Serre relations and a bunch of q-brackets coming from the several subdiagrams of type A_3, namely:

$$[x_{(i\,i+2)}, x_{i+1}]_c,\ i \leq \theta - 3; \qquad [x_{\theta-3\,\theta-2\,\theta}, x_{\theta-2}]_c; \qquad [x_{\theta\,\theta-2\,\theta-1}, x_{\theta-2}]_c. \tag{5.2}$$

Lemma 5.8. *Assume* **a** *is of Cartan type D_4 with $N = 2$. The following relations hold in any $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}$:*

(a) *if $i \neq j$ and $\widetilde{\mathfrak{q}}_{ij} = -1$, then $x_{iij} = 0$;*
(b) *if $i \neq j$ and $\widetilde{\mathfrak{q}}_{ij} = 1$, then $x_{kij} = 0$ for all $k \in \mathbb{I}_4$;*
(c) *if r is one of the elements in* (5.2)*, then $(\mathrm{ad}_c\, x_k)r = 0$ for all $k \in \mathbb{I}_4$.* □

Question 5.9. Let $\widehat{\mathscr{B}}$ denote the quotient of $T(V)$ by the relations (a), (b) and (c) . Is GK-dim $\widehat{\mathscr{B}} < \infty$?

Lemma 5.10. *Assume* **a** *is of Cartan type D_θ with $\theta > 4$ and $N = 2$. The following relations hold in any $\mathscr{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(V)$:*

(a) *all the defining relations of $\widetilde{\mathscr{B}}_{\mathfrak{q}}$ except $x_{\theta\,\theta-1}$ and $[x_{\theta\,\theta-2\,\theta-1}, x_{\theta-2}]_c$;*
(b) *the relations $x_{k\,\theta\,\theta-1}$ and $(\mathrm{ad}_c\, x_k)[x_{\theta\,\theta-2\,\theta-1}, x_{\theta-2}]_c$ for all $k \in \mathbb{I}_\theta$.* □

Question 5.11. Let $\widehat{\mathscr{B}}$ denote the quotient of $T(V)$ by the relations

$$\begin{aligned} x_{ij} &= 0,\ \widetilde{\mathfrak{q}}_{ij} = 1,\ (i,j) \neq (\theta, \theta-1); & [x_{\theta-3\,\theta-2\,\theta}, x_{\theta-2}]_c &= 0; \\ x_{iij} &= 0,\ \widetilde{\mathfrak{q}}_{ij} = -1; & [x_{(i\,i+2)}, x_{i+1}]_c &= 0,\ i \leq \theta-3; \\ x_{k\,\theta\,\theta-1} &= 0,\ k \in \mathbb{I}_\theta; & [x_k, [x_{\theta\,\theta-2\,\theta-1}, x_{\theta-2}]] &= 0,\ k \in \mathbb{I}_\theta. \end{aligned}$$

Is GK-dim $\widehat{\mathscr{B}} < \infty$? We conjecture that GK-dim $\widehat{\mathscr{B}} =$ GK-dim $\widetilde{\mathscr{B}}_{\mathfrak{q}} + 2$. This will be treated in a subsequent paper.

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