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On Minimal Faithful Representations of a Class of Nilpotent Lie Algebras

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Abstract. In this work we consider 2-step nilradicals of parabolic subalgebras of the simple Lie algebra A_n and describe a new family of faithful nil-representations of the nilradicals $\mathfrak{n}_{a,c}$, $a, c \in \mathbb{N}$. We obtain a sharp upper bound for the minimal dimension $\mu(\mathfrak{n}_{a,c})$ and for several pairs (a, c) we obtain $\mu(\mathfrak{n}_{a,c})$.

1. Introduction

All the vector spaces and linear transformations considered in this paper are assumed to be finite dimensional over the complex numbers $\mathbb{C}$. Let $\mathfrak{n}$ be a complex finite dimensional Lie algebra and let $\mu(\mathfrak{n})$ and $\mu_{nil}(\mathfrak{n})$ denote the minimal dimension of a faithful representation and nil-representation, respectively, of $\mathfrak{n}$. By Ado-Iwasawa's Theorem, see for instance [12, page 202], these numbers are well defined. Clearly, these are invariants of $\mathfrak{n}$ and $\mu(\mathfrak{n}) \leq \mu_{nil}(\mathfrak{n})$.

In the theory of Lie algebras, it is important to study $\mu(\mathfrak{n})$ and to find good upper bounds for it. On the one hand, this is motivated by a Milnor's conjecture (see [15]), which asserts that any solvable Lie algebra $\mathfrak{n}$ satisfies $\mu(\mathfrak{n}) \leq \dim \mathfrak{n} + 1$. However, the answer to Milnor's conjecture is negative (see, for instance, [3] and [6] for counterexamples). On the other hand, for computational mathematics, it is interesting to construct faithful representations of a given Lie algebra $\mathfrak{n}$ of the smallest possible dimension; but, in general, this is not easy. In [5], [8] and [11], the authors obtain several methods for such constructions for a given nilpotent Lie algebra $\mathfrak{n}$; but the upper or lower bounds achieved for μ and μ_{nil} are far from being sharp.

In [16], Reed established that $\mu(\mathfrak{n}) \leq 1 + (\dim \mathfrak{n})^k$ for any k-step nilpotent Lie algebra $\mathfrak{n}$. After that, Burde established in [4], that $\mu(\mathfrak{n}) < \dfrac{3}{\sqrt{\dim \mathfrak{n}}} 2^{\dim \mathfrak{n}}$.

A major goal in this area is to prove or disprove that there exists a polynomial $p \in \mathbb{C}[t]$ such that $\mu(\mathfrak{n}) \leq p(\dim \mathfrak{n})$, for all Lie algebras $\mathfrak{n}$ (or at least for a wide class). In fact, the value of μ or μ_{nil} has been obtained only for few families (see [7], [9], [14], [17], [18], [19]).

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For $a, b, c \in \mathbb{N}$, the 2-step nilradical of a parabolic subalgebra of the simple Lie algebra A_n of dimension $ab + bc + ac$, $\mathfrak{n}_{a,b,c}$, is the Lie algebra with basis

$$\mathcal{B} = \{X_{i,j},\ Y_{j,k},\ Z_{i,k} : i = 1, \ldots, a;\ j = 1, \ldots, b;\ k = 1, \ldots, c\}$$

and non-zero brackets

$$[X_{i,j}, Y_{j,k}] = Z_{i,k}$$

for $i = 1, \ldots, a;\ j = 1, \ldots, b$ and $k = 1, \ldots, c$. This Lie algebra has an standard faithful nilrepresentation $(\pi_S, \mathbb{C}^{a+b+c})$ that, in terms of the canonical basis of $\mathbb{C}^{a+b+c}$, is given by

$$\pi_S\left(\sum_{i=1}^{a}\sum_{j=1}^{b} x_{ij}X_{i,j} + \sum_{j=1}^{b}\sum_{k=1}^{c} y_{jk}Y_{j,k} + \sum_{i=1}^{a}\sum_{k=1}^{c} z_{ik}Z_{i,k}\right) = \begin{array}{cl} \left[\begin{array}{c|ccc|ccc} & x_{11} & \cdots & x_{1b} & z_{11} & \cdots & z_{1c} \\ 0 & \vdots & & \vdots & \vdots & & \vdots \\ & x_{a1} & \cdots & x_{ab} & z_{a1} & \cdots & z_{ac} \\ \hline & & & & y_{11} & \cdots & y_{1c} \\ 0 & & 0 & & \vdots & & \vdots \\ & & & & y_{b1} & \cdots & y_{bc} \\ \hline 0 & & 0 & & & 0 & \end{array}\right] & \begin{array}{l} \}\,a \\ \}\,b \\ \}\,c \end{array} \\ \quad a \qquad\qquad b \qquad\qquad\qquad c & \end{array} \tag{1}$$

Thus

$$\mu(\mathfrak{n}_{a,b,c}) \leq \dim \pi_S = a + b + c \qquad \text{for all } a, b, c \in \mathbb{N}.$$

We point out that the nilradical $\mathfrak{n}_{1,b,1}$ is the Heisenberg Lie algebra of dimension $2b+1$. In [4] it is proved that

$$\mu(\mathfrak{n}_{1,b,1}) = b + 2 = \dim \pi_S.$$

In [10], 2-step nilradicals of type A are considered and, in particular, it is proved that if either $c = b - a$ or $a = c$ and $b \leq 2a$, then

$$\mu(\mathfrak{n}_{a,b,c}) = a + b + c = \dim \pi_S.$$

These previous examples show that $\mu(\mathfrak{n}_{a,b,c})$ coincides with the dimension of the standard representation π_S, for certain classes of nilradicals. A first question is whether this is a general phenomenon.

To answer negatively this question, consider the Lie algebra $\mathfrak{n}_{1,1,c}$. This is a well-known nilradical of type A, which arise as an extension of $\mathbb{C}$ by $\mathbb{C}^{2c}$. The Betti numbers of this Lie algebra have been obtained in [2] and later, its adjoint cohomology has been computed in [1].

Consider the following example:

Example 1.1. *For $\mathfrak{n}_{1,1,6}$, one has the following faithful nil-representation of dimension 7*

$$x_{11}X_{11} + \sum_{k=1}^{6} y_{1k}Y_{1k} + \sum_{k=1}^{6} z_{1k}Z_{1k} \longrightarrow \begin{bmatrix} 0 & 0 & x_1 & 0 & z_{11} & z_{12} & z_{13} \\ 0 & 0 & 0 & x_1 & z_{14} & z_{15} & z_{16} \\ 0 & 0 & 0 & 0 & y_{11} & y_{12} & y_{13} \\ 0 & 0 & 0 & 0 & y_{14} & y_{15} & y_{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

therefore

$$\mu(\mathfrak{n}_{1,1,6}) < 8 = \dim \pi_S.$$

We shall prove in this work that,

$$\mu(\mathfrak{n}_{1,1,c}) = \left\lceil 2\sqrt{2c} \right\rceil < 2 + c \qquad \text{for} \quad c \geq 6. \tag{2}$$

An interesting generalization of the Lie algebra $\mathfrak{n}_{1,1,c}$ is the class of Lie algebras given by the family $\mathfrak{n}_{a,1,c}$ for $a \leq c$. In particular, these Lie algebras have maximal dimension center among all nilradicals of type A of fixed dimension.

In this paper we will consider the particular case $b = 1$, and in order to simplify the notation we denote $\mathfrak{n}_{a,c} := \mathfrak{n}_{a,1,c}$.

The article is organized as follows. In §2 we review some of the standard facts on the representations of $\mathfrak{n}_{a,c}$ for every $a, c \in \mathbb{N}$. In §3, we present the construction of two types of representations and give the value of $\mu(\mathfrak{n}_{a,c})$ for a large class of pairs (a, c). Finally, in §4 we generalize the previous faithful nil-representations and obtain a bound for $\mu(\mathfrak{n}_{a,c})$ (see Theorem 4.5).

2. Faithful nil-representations for $\mathfrak{n}_{a,c}$

Let V be a finite dimensional vector space and let $\mathfrak{n}$ be a nilpotent Lie algebra. A representation (π, V) of $\mathfrak{n}$ on V is a Lie algebra homomorphism $\pi : \mathfrak{n} \to \mathfrak{gl}(V)$ and a *nil-representation* is, by definition, a representation whose $\pi(X)$ is a nilpotent endomorphism for every $X \in \mathfrak{n}$.

Let $a, c \in \mathbb{N}$ and let $\mathfrak{n}_{a,c}$ be the Lie algebra introduced in §1. In order to simplify the notation, let

$$\mathcal{B} = \{X_i, Y_j, Z_{i,j} : i = 1, \ldots, a \text{ and } j = 1, \ldots, c\}$$

be the basis of $\mathfrak{n}_{a,c}$ (see §1). We will denote by

$$\mathfrak{z}(\mathfrak{n}_{a,c}) = \operatorname{span}\{Z_{i,j} : i = 1, \ldots, a \text{ and } j = 1, \ldots, c\}$$

the center of $\mathfrak{n}_{a,c}$. From (1), we obtain the standard faithful nil-representation $(\pi_S, \mathbb{C}^{a+1+c})$ that, in terms of the canonical basis of $\mathbb{C}^{a+1+c}$, is given by

$$\pi_S\left(\sum_{i=1}^{a} x_iX_i + \sum_{i=1}^{c} y_jY_j + \sum_{i,j=1}^{a,c} z_{i,j}Z_{i,j}\right) = \begin{array}{c} \left[\begin{array}{c|c|cccc} & x_1 & z_{1,1} & \cdots & z_{1,c} \\ & x_2 & z_{2,1} & \cdots & z_{2,c} \\ 0 & \vdots & \vdots & & \\ & x_a & z_{a,1} & \cdots & z_{a,c} \\ \hline & 0 & y_1 & & y_c \\ \hline 0 & & & 0 & \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}x\\x\\x\\x\end{array}}\right\} a \\ \}\, 1 \\ \}\, c \end{array} \\ \underbrace{\qquad\qquad}_{a}\underbrace{\quad}_{1}\underbrace{\qquad\qquad}_{c} \end{array} \tag{3}$$

Then

$$\mu(\mathfrak{n}_{a,c}) \leq a + 1 + c.$$

From now on, we will assume that $a \leq c$, since the Lie algebra $\mathfrak{n}_{a,c}$ is isomorphic to $\mathfrak{n}_{c,a}$. It is clear that $\mathfrak{a} = \operatorname{span}\{Y_j, Z_{i,j} : i = 1, \ldots, a \text{ and } j = 1, \ldots, c\}$ is an abelian Lie subalgebra of $\mathfrak{n}_{a,c}$. From [13], we obtain $\mu_{nil}(\mathfrak{a}) = \left\lceil 2\sqrt{(a+1)c} \right\rceil$. Therefore

$$\left\lceil 2\sqrt{(a+1)c} \right\rceil \quad \leq \quad \mu_{nil}(\mathfrak{n}_{a,c}) \qquad \text{for every } a, c \in \mathbb{N}. \tag{4}$$

Remark 2.1. *Let $\mathfrak{n}$ be a nilpotent Lie algebra and let $\mathfrak{z}(\mathfrak{n})$ be the center of $\mathfrak{n}$. We know that a representation (π, V) is faithful if and only if $\pi|_{\mathfrak{z}(\mathfrak{n})}$ is injective.*

3. The representations π_0 and π_1

We now fix the following notation. Let $a, c \in \mathbb{N}$ such that $a \leq c$ and let $q = \left\lfloor \sqrt{\frac{c}{a+1}} \right\rfloor$. Let $M_{m,n}$ be the set of the complex matrices of size $m \times n$ and let $\{E_{i,j}\} \subseteq M_{m,n}$ be the canonical basis of $M_{m,n}$. Let $r = 0$ or $r = 1$ and let $z_r = \left\lceil \frac{c}{q+r} \right\rceil$.

Definition 3.1. *Consider the usual basis for $\mathfrak{n}_{a,c}$ and let $\pi_r : \mathfrak{n}_{a,c} \to \mathfrak{gl}((q+r)(a+1) + z_r)$ be the linear map defined by:*

1. *For $i = 1, \ldots, a$:*
$$\pi_r(X_i) = \sum_{p=1}^{q+r} E_{i+(p-1)a,\,p+(q+r)a}.$$

2. *For $j = 1, \ldots, c$:*
$$\pi_r(Y_j) = E_{a(q+r)+t,\,(a+1)(q+r)+u},$$
where $t = \left\lceil \frac{j}{z_r} \right\rceil$ and $u = j - \left(\left\lceil \frac{j}{z_r} \right\rceil - 1\right) z_r$.

3. *For $i = 1, \dots, a$ and $j = 1, \dots, c$:*

$$\pi_r(Z_{i,j}) = \pi_r(X_i)\pi_r(Y_j) = [\pi_r(X_i), \pi_r(Y_j)].$$

Remark 3.2. *It is easy to see that, by construction, π_r is a representation of $\mathfrak{n}_{a,c}$ for $r = 0, 1$.*

Example 3.3. *Let $a = 3$ and $c = 9$ then $q = 1$.*

(i) If $r = 0$ we obtain $z_0 = 9$. Then the representation $(\pi_0, \mathbb{C}^{13})$ is

$$\pi_0\left(\sum_{i=1}^{3} x_i X_i + \sum_{j=1}^{9} y_j Y_j + \sum_{i=1}^{3}\sum_{j=1}^{9} z_{i,j} Z_{i,j}\right) = \begin{bmatrix} 0&0&0&x_1&z_{1,1}&z_{1,2}&z_{1,3}&z_{1,4}&z_{1,5}&z_{1,6}&z_{1,7}&z_{1,8}&z_{1,9} \\ 0&0&0&x_2&z_{2,1}&z_{2,2}&z_{2,3}&z_{24}&z_{2,5}&z_{2,6}&z_{2,7}&z_{2,8}&z_{2,9} \\ 0&0&0&x_3&z_{3,1}&z_{3,2}&z_{3,3}&z_{3,4}&z_{3,5}&z_{3,6}&z_{3,7}&z_{3,8}&z_{3,9} \\ 0&0&0&0&y_1&y_2&y_3&y_4&y_5&y_6&y_7&y_8&y_9 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Notice that in this case, π_0 coincides with the standard representation (3) *of* $\mathfrak{n}_{3,9}$.

(ii) If $r = 1$ we obtain $z_1 = 5$. Then the representation $(\pi_1, \mathbb{C}^{13})$ is

$$\pi_1\left(\sum_{i=1}^{3} x_i X_i + \sum_{j=1}^{9} y_j Y_j + \sum_{i=1}^{3}\sum_{j=1}^{9} z_{i,j} Z_{i,j}\right) = \begin{bmatrix} 0&0&0&0&0&0&x_1&0&z_{1,1}&z_{1,2}&z_{1,3}&z_{1,4}&z_{1,5} \\ 0&0&0&0&0&0&x_2&0&z_{2,1}&z_{2,2}&z_{2,3}&z_{2,4}&z_{2,5} \\ 0&0&0&0&0&0&x_3&0&z_{3,1}&z_{3,2}&z_{3,3}&z_{3,4}&z_{3,5} \\ 0&0&0&0&0&0&0&x_1&z_{1,6}&z_{1,7}&z_{1,8}&z_{1,9}&0 \\ 0&0&0&0&0&0&0&x_2&z_{2,6}&z_{2,7}&z_{2,8}&z_{2,9}&0 \\ 0&0&0&0&0&0&0&x_3&z_{3,6}&z_{3,7}&z_{3,8}&z_{3,9}&0 \\ 0&0&0&0&0&0&0&0&y_1&y_2&y_3&y_4&y_5 \\ 0&0&0&0&0&0&0&0&y_6&y_7&y_8&y_9&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Theorem 3.4. *Let $a, c \in \mathbb{N}$ and let $\left(\pi_r, \mathbb{C}^{(a+1)(q+r)+z_r}\right)$ be the representations defined above. Then π_r is faithful for $r = 0, 1$.*

Proof. From Remark 2.1 we obtain that π_r is faithful if and only if $\pi_r|_{\mathfrak{z}(\mathfrak{n}_{a,c})}$ is injective. Then

$$\begin{aligned} 0 = \pi_r\left(\sum_{i=1}^{a}\sum_{j=1}^{c} z_{i,j} Z_{i,j}\right) &= \sum_{i=1}^{a}\sum_{j=1}^{c} z_{i,j}\pi_r(Z_{i,j}) = \sum_{i=1}^{a}\sum_{j=1}^{c} z_{i,j}\pi_r(X_i)\pi_r(Y_j) \\ &= \sum_{i=1}^{a}\sum_{j=1}^{c} z_{i,j}\left(\sum_{p=1}^{q+r} E_{i+(p-1)a,p+(q+r)a}\right) E_{a(q+r)+t,(a+1)(q+r)+u}, \end{aligned}$$

where $t = \left\lceil \frac{j}{z_r} \right\rceil$ and $u = j - \left(\left\lceil \frac{j}{z_r} \right\rceil - 1\right) z_r$. Thus

$$\sum_{i=1}^{a}\sum_{j=1}^{c} z_{i,j} E_{i+\left(\left\lceil \frac{j}{z_r} \right\rceil - 1\right)a,(a+1)(q+r)+j-\left(\left\lceil \frac{j}{z_r} \right\rceil - 1\right)z_r} = 0.$$

It is easy to see that we must have $z_{i,j} = 0$ for every $i = 1, \dots, a$, $j = 1, \dots, c$, and the proof is complete. $\square$

Remark 3.5. *If* $\sqrt{\frac{c}{a+1}} = q + \alpha$ *with* $0 \leq \alpha < 1$*, we obtain*

$$2\sqrt{(a+1)c} = 2(a+1)\sqrt{\frac{c}{a+1}} = 2(a+1)q + 2(a+1)\alpha.$$

It follows that

$$\left\lceil 2\sqrt{(a+1)c}\right\rceil = 2(a+1)q + \lceil 2(a+1)\alpha\rceil, \tag{5}$$

and

$$2\sqrt{(a+1)c} \leq 2(a+1)q + \lceil 2(a+1)\alpha\rceil.$$

Hence

$$4(a+1)c \leq 4(a+1)^2q^2 + 4\lceil 2(a+1)\alpha\rceil(a+1)q + \lceil 2(a+1)\alpha\rceil^2,$$

thus

$$c \leq (a+1)q^2 + \lceil 2(a+1)\alpha\rceil q + \left\lfloor \frac{\lceil 2(a+1)\alpha\rceil^2}{4(a+1)}\right\rfloor. \tag{6}$$

We can now formulate the main result of this section.

Theorem 3.6. *Let* $a, c \in \mathbb{N}$ *and let* $\sqrt{\frac{c}{a+1}} = q + \alpha$ *with* $0 \leq \alpha < 1$*. If*

$$\lceil 2(a+1)\alpha\rceil < 2\sqrt{a+1} \qquad \text{or} \qquad 2(a+1) - 2\sqrt{a+1} < \lceil 2(a+1)\alpha\rceil$$

then

$$\mu(\mathfrak{n}_{a,c}) = \left\lceil 2\sqrt{(a+1)c}\right\rceil.$$

Proof. Assume that $\lceil 2(a+1)\alpha\rceil < 2\sqrt{a+1}$, then $\dfrac{\lceil 2(a+1)\alpha\rceil^2}{4(a+1)} < 1$. Since $c \in \mathbb{N}$, we obtain from (6) that

$$c \leq (a+1)q^2 + \lceil 2(a+1)\alpha\rceil q.$$

It follows that

$$\left\lceil \frac{c}{q}\right\rceil \leq (a+1)q + \lceil 2(a+1)\alpha\rceil,$$

and therefore

$$(a+1)q + \left\lceil \frac{c}{q}\right\rceil \leq 2(a+1)q + \lceil 2(a+1)\alpha\rceil \overset{(5)}{=} \left\lceil 2\sqrt{(a+1)c}\right\rceil. \tag{7}$$

Consider the faithful nil-representation π_0. Then equations (7) and (4) imply that

$$\mu(\mathfrak{n}_{a,c}) = \left\lceil 2\sqrt{(a+1)c}\right\rceil.$$

Assume now that $2(a+1) - 2\sqrt{a+1} < \lceil 2(a+1)\alpha \rceil$. Then

$$\begin{aligned} 2(a+1) - \lceil 2(a+1)\alpha \rceil &< 2\sqrt{a+1} \\ 4(a+1)^2 - 4(a+1)\lceil 2(a+1)\alpha \rceil + \lceil 2(a+1)\alpha \rceil^2 &< 4(a+1) \\ (a+1) - \lceil 2(a+1)\alpha \rceil + \frac{\lceil 2(a+1)\alpha \rceil^2}{4(a+1)} &< 1 \\ \frac{\lceil 2(a+1)\alpha \rceil^2}{4(a+1)} &< 1 + \lceil 2(a+1)\alpha \rceil - (a+1). \end{aligned}$$

Therefore

$$\left\lfloor \frac{\lceil 2(a+1)\alpha \rceil^2}{4(a+1)} \right\rfloor \leq \lceil 2(a+1)\alpha \rceil - (a+1). \tag{8}$$

It follows that

$$\begin{aligned} c &\overset{(6)}{\leq} (a+1)q^2 + \lceil 2(a+1)\alpha \rceil q + \left\lfloor \frac{\lceil 2(a+1)\alpha \rceil^2}{4(a+1)} \right\rfloor r \\ &\overset{(8)}{\leq} (a+1)q^2 + \lceil 2(a+1)\alpha \rceil q + \lceil 2(a+1)\alpha \rceil - (a+1) \\ &= (a+1)q(q+1) + \lceil 2(a+1)\alpha \rceil (q+1) - (a+1)(q+1). \end{aligned}$$

Thus

$$\left\lceil \frac{c}{q+1} \right\rceil \leq (a+1)q + \lceil 2(a+1)\alpha \rceil - (a+1).$$

and therefore

$$\begin{aligned} (a+1)(q+1) + \left\lceil \frac{c}{q+1} \right\rceil &\leq 2(a+1)q + \lceil 2(a+1)\alpha \rceil \\ &\overset{(5)}{=} \left\lceil 2\sqrt{(a+1)c} \right\rceil. \end{aligned} \tag{9}$$

Consider the faithful nil-representation π_1. By equations (9) and (4), we obtain that

$$\mu(\mathfrak{n}_{a,c}) = \left\lceil 2\sqrt{(a+1)c} \right\rceil,$$

which completes the proof. □

Corollary 3.7. *If $a = 1$ or $a = 2$ then*

$$\mu(\mathfrak{n}_{a,c}) = \left\lceil 2\sqrt{(a+1)c} \right\rceil \qquad \textit{for every } c \geq a.$$

Proof. If $a = 1$ or $a = 2$ we obtain $\sqrt{a+1} < 2$. Thus

$$2(a+1) - 2\sqrt{a+1} < 2\sqrt{a+1},$$

by Theorem 3.6 we conclude that $\mu(\mathfrak{n}_{a,c}) = \left\lceil 2\sqrt{(a+1)c} \right\rceil$. □

Theorem 3.8. *Let $a, c \in \mathbb{N}$ and let $\sqrt{\dfrac{c}{a+1}} = q + \alpha$ with $0 \leq \alpha < 1$. If* *(i)* $2\sqrt{a+1} \leq \lceil 2(a+1)\alpha \rceil \leq 2(a+1) - 2\sqrt{a+1}$ *and*

(ii) $c \geq (a - \sqrt{a+1})^2(a+1)$

then

$$\mu(\mathfrak{n}_{a,c}) = \left\lceil 2\sqrt{(a+1)c} \right\rceil \quad or \quad \mu(\mathfrak{n}_{a,c}) = \left\lceil 2\sqrt{(a+1)c} \right\rceil + 1.$$

Proof. From *(i), we obtain*

$$\frac{1}{2\sqrt{a+1}} \leq \frac{\lceil 2(a+1)\alpha \rceil}{4(a+1)} \leq \frac{1}{2}\left(1 - \frac{1}{\sqrt{a+1}}\right). \tag{10}$$

Therefore

$$\begin{aligned} c &\overset{(6)}{\leq} (a+1)q^2 + \lceil 2(a+1)\alpha \rceil q + \frac{\lceil 2(a+1)\alpha \rceil^2}{4(a+1)} \\ &\overset{(10)}{\leq} (a+1)q^2 + \lceil 2(a+1)\alpha \rceil q + \frac{\lceil 2(a+1)\alpha \rceil}{2} - \frac{\lceil 2(a+1)\alpha \rceil}{2\sqrt{a+1}} \\ &= ((a+1)q + \lceil 2(a+1)\alpha \rceil - (a+1))(q+1) + \\ &\quad -\frac{\lceil 2(a+1)\alpha \rceil}{2} + (a+1) - \frac{\lceil 2(a+1)\alpha \rceil}{2\sqrt{a+1}}. \end{aligned} \tag{11}$$

Also, since $2\sqrt{(a+1)} \leq \lceil 2(a+1)\alpha \rceil$, *we obtain*

$$-\frac{\lceil 2(a+1)\alpha \rceil}{2} + (a+1) - \frac{\lceil 2(a+1)\alpha \rceil}{2\sqrt{a+1}} \leq a - \sqrt{a+1}. \tag{12}$$

Combining (11) *and* (12), *we have*

$$\frac{c}{q+1} \leq (a+1)q + \lceil 2(a+1)\alpha \rceil - (a+1) + \frac{a - \sqrt{a+1}}{q+1}.$$

Thus

$$\left\lceil \frac{c}{q+1} \right\rceil \leq (a+1)q + \lceil 2(a+1)\alpha \rceil - (a+1) + \left\lceil \frac{a - \sqrt{a+1}}{q+1} \right\rceil. \tag{13}$$

Consider π_1. *From* (13) *we obtain*

$$\mu(\mathfrak{n}_{a,c}) \leq 2(a+1)q + \lceil 2(a+1)\alpha \rceil + \left\lceil \frac{a - \sqrt{a+1}}{q+1} \right\rceil.$$

Under the hypothesis $c \geq (a - \sqrt{a+1})^2(a+1)$, *we obtain*

$$q + 1 > \sqrt{\frac{c}{a+1}} \geq a - \sqrt{a+1}.$$

Then it follows that

$$\begin{aligned} \mu(\mathfrak{n}_{a,c}) &\leq 2(a+1)q + \lceil 2(a+1)\alpha \rceil + 1 \\ &\overset{(5)}{=} \left\lceil 2\sqrt{(a+1)c} \right\rceil + 1. \end{aligned}$$

The proof is completed. □

Corollary 3.9. *Let $a = 3$.*

(1) If $c \neq (2k+1)^2$ for $k \in \mathbb{N}$ then
$$\mu(\mathfrak{n}_{3,c}) = \left\lceil 4\sqrt{c} \right\rceil.$$

(2) If $c = (2k+1)^2$ for some $k \in \mathbb{N}$ then
$$\mu(\mathfrak{n}_{3,c}) \leq \left\lceil 4\sqrt{c} \right\rceil + 1.$$

Proof. If $a = 3$, then $2\sqrt{a+1} = 2(a+1) - 2\sqrt{a+1} = 4$. Therefore:

- If $\lceil 8\alpha \rceil \neq 4$, we obtain by Theorem 3.6 that $\mu(\mathfrak{n}_{3,c}) = \lceil 4\sqrt{c} \rceil$.
- If $\lceil 8\alpha \rceil = 4$, Theorem 3.8 implies that
$$\lceil 4\sqrt{c} \rceil \leq \mu(\mathfrak{n}_{3,c}) \leq \lceil 4\sqrt{c} \rceil + 1 \qquad \text{for } c \geq 4.$$

Consider the case $\lceil 8\alpha \rceil = 4$, which implies $\frac{3}{8} < \alpha \leq \frac{1}{2}$. Assume that $c \neq (2k+1)^2$ for every $k \in \mathbb{N}$. On the one hand we obtain, by equation (5), that
$$\left\lceil 2\sqrt{4c} \right\rceil = 8q + \lceil 8\alpha \rceil = 8q + 4.$$

On the other hand,
$$c = 4(q+\alpha)^2 \leq 4\left(q + \frac{1}{2}\right)^2 = 4q^2 + 4q + 1.$$
Since $c \neq (2q+1)^2$ we obtain that $c \leq 4q(q+1)$ and then $\frac{c}{q+1} \leq 4q$.

By considering the representation π_1, we obtain
$$\left\lceil 2\sqrt{4c} \right\rceil \leq \mu(\mathfrak{n}_{3,c}) \leq \dim \pi_1 \leq 4(q+1) + 4q = \left\lceil 2\sqrt{4c} \right\rceil$$
and the result follows. □

Remark 3.10. *The previous Theorems have shown that there is a large class of pairs (a, c) for which an excellent bound for $\mu(\mathfrak{n}_{a,c})$ is achieved. However, there is still a large class for which these representations are not optimal. Assume that $2\sqrt{a+1} \leq \lceil 2(a+1)\alpha \rceil \leq 2(a+1) - 2\sqrt{a+1}$ and $c < (a - \sqrt{a+1})^2(a+1)$. Set $q = \left\lfloor \sqrt{\frac{c}{a+1}} \right\rfloor$ and $\alpha = \sqrt{\frac{c}{a+1}} - q$ as before. Recall that $\left\lceil 2\sqrt{c(a+1)} \right\rceil = 2(a+1)q + \lceil 2(a+1)\alpha \rceil$ and $c = (a+1)(q+\alpha)^2$. Then*
$$\frac{1}{\sqrt{a+1}} - \frac{1}{2(a+1)} < \alpha \leq 1 - \frac{1}{\sqrt{a+1}}. \tag{14}$$
On the one hand we obtain, by equation (14):
$$c \leq (a+1)(q+1)^2 - 2\sqrt{a+1}(q+1) + 1.$$
Therefore
$$\frac{c}{q+1} \leq (a+1)(q+1) - 2\sqrt{a+1} + \frac{1}{q+1} \qquad \textit{and} \qquad \left\lceil \frac{c}{q+1} \right\rceil < (a+1)(q+1) - 2\sqrt{a+1} + \frac{3}{2}.$$

By considering the representation π_1 of dimension $(a+1)(q+1)+\left\lceil \frac{c}{q+1}\right\rceil$, we obtain

$$\begin{aligned}\dim\pi_1 &< 2(a+1)(q+1)+\frac{3}{2}-2\sqrt{a+1}\\ &= 2(a+1)q+2(a+1)+\frac{3}{2}-2\sqrt{a+1}-\lceil 2(a+1)\alpha\rceil+\lceil 2(a+1)\alpha\rceil\\ &= 2(a+1)q+\lceil 2(a+1)\alpha\rceil+\frac{3}{2}+\Big(2(a+1)-2\sqrt{a+1}-\lceil 2(a+1)\alpha\rceil\Big)\\ &\le 2(a+1)q+\lceil 2(a+1)\alpha\rceil+\frac{3}{2}+2(a+1)-4\sqrt{a+1}.\end{aligned}$$

Therefore

$$\dim\pi_1 \le \left\lceil 2\sqrt{c(a+1)}\right\rceil+\left\lfloor 2\left(\sqrt{a+1}-1\right)^2-\frac{1}{2}\right\rfloor \quad \text{for } a\ge 3.$$

Notice that for $a\ge 12$ we already have $\left\lfloor 2\left(\sqrt{a+1}-1\right)^2-\frac{1}{2}\right\rfloor > a$, so we must consider new representations in order to improve the bound for $\mu(\mathfrak{n}_{a,c})$.

4. New faithful representations for $\mathfrak{n}_{a,c}$. Extending π_0 and π_1.

Let q and α be as before. In the previous section we have obtained minimal representations π_r for a large class of pairs (a,c). In particular, if $\alpha=0$, we have obtained the value of $\mu(\mathfrak{n}_{a,c})$. We will consider now $0<\alpha<1$.

Let $r,s\in\mathbb{N}$ such that $2\le s$, $(s-1)\left\lceil\frac{a}{s}\right\rceil<a$ and $r<s$. Set

$$d_1=qa+r\left\lceil\frac{a}{s}\right\rceil \quad\text{and}\quad d_2=(s-1)\left\lceil\frac{c}{qs+1}\right\rceil+\left\lceil\frac{c-q(s-1)\left\lceil\frac{c}{qs+1}\right\rceil}{q+1}\right\rceil.$$

Definition 4.1. *Consider the usual basis for $\mathfrak{n}_{a,c}$ and define the linear map $\pi_{r/s}:\mathfrak{n}_{a,c}\to\mathfrak{gl}(d_1+q+1+d_2)$ in the following way:*

1. *For $1\le i\le a$:*
$$\pi_{r/s}(X_i)=\sum_{k=1}^{q}E_{(k-1)a+i,d_1+k}+E_{(q-1)a+r\left\lceil\frac{a}{s}\right\rceil+i,d_1+q+1}.$$
2. *For $1\le j\le (q-1)d_2+\left\lceil\frac{c}{qs+1}\right\rceil$:*
$$\pi_{r/s}(Y_j)=E_{d_1+1+\left\lceil\frac{j}{d_2}\right\rceil,d_1+q+1+j-\left\lfloor\frac{j-1}{d_2}\right\rfloor d_2}.$$
3. *For $(q-1)d_2+\left\lceil\frac{c}{qs+1}\right\rceil+1\le j\le qd_2$:*
$$\pi_{r/s}(Y_j)=E_{d_1+q,j-(q-1)d_2+d_1+q+1}+E_{d_1+q+1,d_1+q+1+j-(q-1)d_2-\left\lceil\frac{c}{qs+1}\right\rceil}.$$
4. *For $qd_2+1\le j\le c$:*
$$\pi_{r/s}(Y_j)=E_{d_1+q+1,d_1+q+1+(s-1)\left\lceil\frac{c}{qs+1}\right\rceil+j-qd_2}.$$
5. *For $1\le i\le a$ and $1\le j\le c$:*
$$\pi_{r/s}(Z_{i,j})=\pi_{r/s}(X_i)\pi_{r/s}(Y_j)=[\pi_{r/s}(X_i),\pi_{r/s}(Y_j)].$$

Example 4.2. *Consider $a = 5$ and $c = 25$. Then $q = 2$ and we can choose $s = 2$ or $s = 3$.*

(i) If $s = 2$, then $r = 1$ and we obtain

$$\pi_{1/2}\left(\sum_{i=1}^{5} x_i X_i + \sum_{j=1}^{25} y_j Y_j + \sum_{i=1}^{5}\sum_{j=1}^{25} z_{i,j} Z_{i,j}\right) =$$

$$\left[\begin{array}{cccccccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} & z_{1,5} & z_{1,6} & z_{1,7} & z_{1,8} & z_{1,9} & z_{1,10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 & z_{2,1} & z_{2,2} & z_{2,3} & z_{2,4} & z_{2,5} & z_{2,6} & z_{2,7} & z_{2,8} & z_{2,9} & z_{2,10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 & z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} & z_{3,5} & z_{3,6} & z_{3,7} & z_{3,8} & z_{3,9} & z_{3,10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & z_{4,1} & z_{4,2} & z_{4,3} & z_{4,4} & z_{4,5} & z_{4,6} & z_{4,7} & z_{4,8} & z_{4,9} & z_{4,10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & z_{5,1} & z_{5,2} & z_{5,3} & z_{5,4} & z_{5,5} & z_{5,6} & z_{5,7} & z_{5,8} & z_{5,9} & z_{5,10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & z_{1,11} & z_{1,12} & z_{1,13} & z_{1,14} & z_{1,15} & z_{1,16} & z_{1,17} & z_{1,18} & z_{1,19} & z_{1,20} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & z_{2,11} & z_{2,12} & z_{2,13} & z_{2,14} & z_{2,15} & z_{2,16} & z_{2,17} & z_{2,18} & z_{2,19} & z_{2,20} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & z_{3,11} & z_{3,12} & z_{3,13} & z_{3,14} & z_{3,15} & z_{3,16} & z_{3,17} & z_{3,18} & z_{3,19} & z_{3,20} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & x_1 & z_{4,11}+z_{1,16} & z_{4,12}+z_{1,17} & z_{4,13}+z_{1,18} & z_{4,14}+z_{1,19} & z_{4,15}+z_{1,20} & z_{4,16}+z_{1,21} & z_{4,17}+z_{1,22} & z_{4,18}+z_{1,23} & z_{4,19}+z_{1,24} & z_{4,20}+z_{1,25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & x_2 & z_{5,11}+z_{2,16} & z_{5,12}+z_{2,17} & z_{5,13}+z_{2,18} & z_{5,14}+z_{2,19} & z_{5,15}+z_{2,20} & z_{5,16}+z_{2,21} & z_{5,17}+z_{2,22} & z_{5,18}+z_{2,23} & z_{5,19}+z_{2,24} & z_{5,20}+z_{2,25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & z_{3,16} & z_{3,17} & z_{3,18} & z_{3,19} & z_{3,20} & z_{3,21} & z_{3,22} & z_{3,23} & z_{3,24} & z_{3,25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & z_{4,16} & z_{4,17} & z_{4,18} & z_{4,19} & z_{4,20} & z_{4,21} & z_{4,22} & z_{4,23} & z_{4,24} & z_{4,25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & z_{5,16} & z_{5,17} & z_{5,18} & z_{5,19} & z_{5,20} & z_{5,21} & z_{5,22} & z_{5,23} & z_{5,24} & z_{5,25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{11} & y_{12} & y_{13} & y_{14} & y_{15} & y_{16} & y_{17} & y_{18} & y_{19} & y_{20} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{16} & y_{17} & y_{18} & y_{19} & y_{20} & y_{21} & y_{22} & y_{23} & y_{24} & y_{25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

(ii) If $s = 3$, consider $r = 2$. Then we obtain

$$\pi_{2/3}\left(\sum_{i=1}^{5} x_i X_i + \sum_{j=1}^{25} y_j Y_j + \sum_{i=1}^{5}\sum_{j=1}^{25} z_{i,j} Z_{i,j}\right) =$$

$$\left[\begin{array}{ccccccccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} & z_{1,5} & z_{1,6} & z_{1,7} & z_{1,8} & z_{1,9} & z_{1,10} & z_{1,11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 & z_{2,1} & z_{2,2} & z_{2,3} & z_{2,4} & z_{2,5} & z_{2,6} & z_{2,7} & z_{2,8} & z_{2,9} & z_{2,10} & z_{2,11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 & z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} & z_{3,5} & z_{3,6} & z_{3,7} & z_{3,8} & z_{3,9} & z_{3,10} & z_{3,11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & z_{4,1} & z_{4,2} & z_{4,3} & z_{4,4} & z_{4,5} & z_{4,6} & z_{4,7} & z_{4,8} & z_{4,9} & z_{4,10} & z_{4,11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & z_{5,1} & z_{5,2} & z_{5,3} & z_{5,4} & z_{5,5} & z_{5,6} & z_{5,7} & z_{5,8} & z_{5,9} & z_{5,10} & z_{5,11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & z_{1,12} & z_{1,13} & z_{1,14} & z_{1,15} & z_{1,16} & z_{1,17} & z_{1,18} & z_{1,19} & z_{1,20} & z_{1,21} & z_{1,22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & z_{2,12} & z_{2,13} & z_{2,14} & z_{2,15} & z_{2,16} & z_{2,17} & z_{2,18} & z_{2,19} & z_{2,20} & z_{2,21} & z_{2,22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & z_{3,12} & z_{3,13} & z_{3,14} & z_{3,15} & z_{3,16} & z_{3,17} & z_{3,18} & z_{3,19} & z_{3,20} & z_{3,21} & z_{3,22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & z_{4,12} & z_{4,13} & z_{4,14} & z_{4,15} & z_{4,16} & z_{4,17} & z_{4,18} & z_{4,19} & z_{4,20} & z_{4,21} & z_{4,22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & x_1 & z_{5,12}+z_{1,16} & z_{5,13}+z_{1,17} & z_{5,14}+z_{1,18} & z_{5,15}+z_{1,19} & z_{5,16}+z_{1,20} & z_{5,17}+z_{1,21} & z_{5,18}+z_{1,22} & z_{5,19} & z_{5,20}+z_{1,23} & z_{5,21}+z_{1,24} & z_{5,22}+z_{1,25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & z_{2,16} & z_{2,17} & z_{2,18} & z_{2,19} & z_{2,20} & z_{2,21} & z_{2,22} & 0 & z_{2,23} & z_{2,24} & z_{2,25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & z_{3,16} & z_{3,17} & z_{3,18} & z_{3,19} & z_{3,20} & z_{3,21} & z_{3,22} & 0 & z_{3,23} & z_{3,24} & z_{3,25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & z_{4,16} & z_{4,17} & z_{4,18} & z_{4,19} & z_{4,20} & z_{4,21} & z_{4,22} & 0 & z_{4,23} & z_{4,24} & z_{4,25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & z_{5,16} & z_{5,17} & z_{5,18} & z_{5,19} & z_{5,20} & z_{5,21} & z_{5,22} & 0 & z_{5,23} & z_{5,24} & z_{5,25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{12} & y_{13} & y_{14} & y_{15} & y_{16} & y_{17} & y_{18} & y_{19} & y_{20} & y_{21} & y_{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{16} & y_{17} & y_{18} & y_{19} & y_{20} & y_{21} & y_{22} & 0 & y_{23} & y_{24} & y_{25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

Notice that for $\mathfrak{n}_{5,25}$, $\dim \pi_{1/2} = 26$ *and* $\dim \pi_{2/3} = 28$, *while* $\dim \pi_S = 31$.

Theorem 4.3. *Let $a, c, r, s \in \mathbb{N}$ such that $a \leq c$, $s \geq 2$, $(s-1)\left\lceil \frac{a}{s} \right\rceil < a$ and $r < s$. Then $\pi_{r/s}$ is a faithful nil-representation of $\mathfrak{n}_{a,c}$.*

Proof. From Remark 2.1 we obtain that $\pi_{r/s}$ is faithful if and only if $(\pi_{r/s})|_{\mathfrak{z}(\mathfrak{n}_{a,c})}$ is injective. Moreover, we obtain the following:

$$\pi_{r/s}\left(\sum_{i=1}^{a}\sum_{j=1}^{c} z_{i,j}Z_{i,j}\right) = \sum_{u=1}^{(q-1)a+r\lceil\frac{a}{s}\rceil}\sum_{v=d_1+q+2}^{d_1+q+1+d_2} z_{u-a\lfloor\frac{u}{a}\rfloor,v-d_1-q-1+d_2\lfloor\frac{u}{a}\rfloor}E_{u,v} \qquad (A)$$

$$+\sum_{u=d_1-a+1}^{qa}\sum_{v=d_1+q+2}^{d_1+q+1+d_2-\lceil\frac{c}{qs+1}\rceil}\Big(z_{u-(q-1)a,v-d_1-q-1+(q-1)d_2}$$

$$+z_{u-(q-1)a-r\lceil\frac{a}{s}\rceil,v-d_1-q-1+d_2(q-1)+\lceil\frac{c}{qs+1}\rceil}\Big)E_{u,v} \qquad (B)$$

$$+\sum_{u=d_1-a+1}^{qa}\sum_{v=d_1+q+2+d_2-\lceil\frac{c}{qs+1}\rceil}^{d_1+q+1+(s-1)\lceil\frac{c}{qs+1}\rceil} z_{u-(q-1)a,v-d_1-q-1+(q-1)d_2}E_{u,v} \qquad (C)$$

$$+\sum_{u=d_1-a+1}^{qa}\sum_{v=d_1+q+2+(s-1)\lceil\frac{c}{qs+1}\rceil}^{d_1+q+1+(s-1)\lceil\frac{c}{qs+1}\rceil+c-qd_2}\Big(z_{u-(q-1)a,v-d_1-q-1+(q-1)d_2}$$

$$+z_{u-(q-1)a-r\lceil\frac{a}{s}\rceil,v-d_1-q-1+d_2q-(s-1)\lceil\frac{c}{qs+1}\rceil}\Big)E_{u,v} \qquad (D)$$

$$+\sum_{u=d_1-a+1}^{qa}\sum_{v=d_1+q+2+(s-1)\lceil\frac{c}{qs+1}\rceil+c-qd_2}^{d_1+q+1+d_2} z_{u-(q-1)a,v-d_1-q-1+(q-1)d_2}E_{u,v} \qquad (E)$$

$$+\sum_{u=qa+1}^{d_1}\sum_{v=d_1+q+2}^{d_1+q+1+d_2-\lceil\frac{c}{qs+1}\rceil} z_{u-qa+a-r\lceil\frac{a}{s}\rceil,v-d_1-q-1+d_2(q-1)+\lceil\frac{c}{qs+1}\rceil}E_{u,v} \qquad (F)$$

$$+\sum_{u=qa+1}^{d_1}\sum_{v=d_1+q+2+(s-1)\lceil\frac{c}{qs+1}\rceil}^{d_1+q+1+(s-1)\lceil\frac{c}{qs+1}\rceil+c-qd_2} z_{u-qa+a-r\lceil\frac{a}{s}\rceil,v-d_1-q-1+d_2q-(s-1)\lceil\frac{c}{qs+1}\rceil}E_{u,v}. \qquad (G)$$

Since the sums (A), (B), (C), $(D$, (E), (F) and (G) are mutually disjoint, we obtain that if $\pi_{r/s}\left(\sum_{i=1}^{a}\sum_{j=1}^{c} z_{i,j}Z_{i,j}\right) = 0$ then:

1. From (A): $z_{i,j} = 0$ for $1 \le i \le a$ and $1 \le j \le (q-1)d_2$;
2. From (A): $z_{i,j} = 0$ for $1 \le i \le r\lceil\frac{a}{s}\rceil$ and $(q-1)d_2+1 \le j \le qd_2$;
3. From (B): $z_{i,j} + z_{i-r\lceil\frac{a}{s}\rceil,j+\lceil\frac{c}{qs+1}\rceil} = 0$ for $r\lceil\frac{a}{s}\rceil+1 \le i \le a$ and $(q-1)d_2+1 \le j \le qd_2-\lceil\frac{c}{qs+1}\rceil$;
4. From (C): $z_{i,j} = 0$ for $r\lceil\frac{a}{s}\rceil+1 \le i \le a$ and $qd_2-\lceil\frac{c}{qs+1}\rceil+1 \le j \le (q-1)d_2+(s-1)\lceil\frac{c}{qs+1}\rceil$;
5. From (D): $z_{i,j} + z_{i-r\lceil\frac{a}{s}\rceil,j+d_2-(s-1)\lceil\frac{c}{qs+1}\rceil} = 0$ for $r\lceil\frac{a}{s}\rceil+1 \le i \le a$ and $(q-1)d_2+(s-1)\lceil\frac{c}{qs+1}\rceil+1 \le j \le (s-1)\lceil\frac{c}{qs+1}\rceil+c-d_2$;
6. From (E): $z_{i,j} = 0$ for $r\lceil\frac{a}{s}\rceil+1 \le i \le a$ and $(s-1)\lceil\frac{c}{qs+1}\rceil+c-d_2+1 \le j \le qd_2$;

7. From (F): $z_{i,j} = 0$ for $a - r\left\lceil \frac{a}{s} \right\rceil + 1 \leq i \leq a$ and $(q-1)d_2 + \left\lceil \frac{c}{qs+1} \right\rceil + 1 \leq j \leq qd_2$;
8. From (G): $z_{i,j} = 0$ for $a - r\left\lceil \frac{a}{s} \right\rceil + 1 \leq i \leq a$ and $qd_2 + 1 \leq j \leq c$.

Here we have the following possibilities:

- If $r\left\lceil \frac{a}{s} \right\rceil \leq a - r\left\lceil \frac{a}{s} \right\rceil$, then it remains to prove

$$z_{ij} = 0, \quad \text{for} \quad 1 \leq i \leq r\left\lceil \frac{a}{s} \right\rceil,\ qd_2 + 1 \leq j \leq c, \tag{15}$$

$$z_{ij} = 0, \quad \text{for} \quad r\left\lceil \frac{a}{s} \right\rceil + 1 \leq i \leq a - r\left\lceil \frac{a}{s} \right\rceil,\ (q-1)d_2 + 1 \leq j \leq (q-1)d_2 + \left\lceil \frac{c}{qs+1} \right\rceil, \tag{16}$$

$$z_{ij} = 0, \quad \text{for} \quad r\left\lceil \frac{a}{s} \right\rceil + 1 \leq i \leq a - r\left\lceil \frac{a}{s} \right\rceil,\ (q-1)d_2 + \left\lceil \frac{c}{qs+1} \right\rceil + 1 \leq j \leq qd_2 - \left\lceil \frac{c}{qs+1} \right\rceil, \tag{17}$$

$$z_{ij} = 0, \quad \text{for} \quad \begin{array}{l} r\left\lceil \frac{a}{s} \right\rceil + 1 \leq i \leq a - r\left\lceil \frac{a}{s} \right\rceil, \\ (q-1)d_2 + (s-1)\left\lceil \frac{c}{qs+1} \right\rceil + 1 \leq j \leq (s-1)\left\lceil \frac{c}{qs+1} \right\rceil + c - d_2, \end{array} \tag{18}$$

$$z_{ij} = 0, \quad \text{for} \quad r\left\lceil \frac{a}{s} \right\rceil + 1 \leq i \leq a - r\left\lceil \frac{a}{s} \right\rceil,\ qd_2 + 1 \leq j \leq c, \tag{19}$$

$$z_{ij} = 0, \quad \text{for} \quad a - r\left\lceil \frac{a}{s} \right\rceil + 1 \leq i \leq a,\ (q-1)d_2 + 1 \leq j \leq (q-1)d_2 + \left\lceil \frac{c}{qs+1} \right\rceil. \tag{20}$$

  In this case we obtain:

  (i) (16) and (17) follow by combining (3) and (2),
  (ii) (18) and (19) follow by combining (5) and (7),
  (iii) (20) follows by combining (3) and (i),
  (iv) (15) follows by combining (5) and (ii).

- If $a - r\left\lceil \frac{a}{s} \right\rceil < r\left\lceil \frac{a}{s} \right\rceil$, then it remains to prove

$$z_{ij} = 0, \quad \text{for} \quad a - r\left\lceil \frac{a}{s} \right\rceil + 1 \leq i \leq a,\ (q-1)d_2 + 1 \leq j \leq (q-1)d_2 + \left\lceil \frac{c}{qs+1} \right\rceil,$$

  which follows by combining (3) and (2).

□

Corollary 4.4. *The dimension of the representation $\pi_{r/s}$ for $\mathfrak{n}_{a,c}$ is*

$$\dim \pi_{r/s} = q(a+1) + 1 + r\left\lceil \frac{a}{s} \right\rceil + \left\lceil \frac{c + (s-1)\left\lceil \frac{c}{qs+1} \right\rceil}{q+1} \right\rceil.$$

Proof. By construction we have

$$\dim \pi_{r/s} = qa + r\left\lceil \frac{a}{s} \right\rceil + q + 1 + (s-1)\left\lceil \frac{c}{qs+1} \right\rceil + \left\lceil \frac{c - (s-1)\left\lceil \frac{c}{qs+1} \right\rceil}{q+1} \right\rceil.$$

Since

$$\frac{c - q(s-1)\left\lceil \frac{c}{s+1} \right\rceil}{q+1} \leq \left\lceil \frac{c - q(s-1)\left\lceil \frac{c}{s+1} \right\rceil}{q+1} \right\rceil < \frac{c - q(s-1)\left\lceil \frac{c}{s+1} \right\rceil}{q+1} + 1,$$

then

$$\frac{c + (s-1)\left\lceil \frac{c}{s+1} \right\rceil}{q+1} \leq (s-1)\left\lceil \frac{c}{s+1} \right\rceil + \left\lceil \frac{c - q(s-1)\left\lceil \frac{c}{s+1} \right\rceil}{q+1} \right\rceil < \frac{c + (s-1)\left\lceil \frac{c}{s+1} \right\rceil}{q+1} + 1,$$

and

$$(s-1)\left\lceil \frac{c}{s+1} \right\rceil + \left\lceil \frac{c - q(s-1)\left\lceil \frac{c}{s+1} \right\rceil}{q+1} \right\rceil = \left\lceil \frac{c + (s-1)\left\lceil \frac{c}{s+1} \right\rceil}{q+1} \right\rceil,$$

and the result follows. □

Then we obtain:

Theorem 4.5. *Let $a, c, s \in \mathbb{N}$ such that $a \leq c$, $s \geq 2$ and $(s-1)\left\lceil \frac{a}{s} \right\rceil < a$. Let $q = \left\lfloor \sqrt{\frac{c}{a+1}} \right\rfloor$. Then*

$$\left\lceil 2\sqrt{(a+1)c} \right\rceil \leq \mu(\mathfrak{n}_{a,c}) \leq q(a+1) + 1 + \left\lceil \frac{a}{s} \right\rceil + \left\lceil \frac{c + (s-1)\left\lceil \frac{c}{qs+1} \right\rceil}{q+1} \right\rceil.$$

Proof. It is clear that the dimensions of the representations $\pi_{r/s}$ provide upper bounds for $\mu(\mathfrak{n}_{a,c})$. In particular, by taking $r = 1$ we obtain the minimal dimension among all $\pi_{r/s}$ with fixed s. □

Corollary 4.6. *Under the hypothesis of Theorem 4.5, if $\sqrt{\frac{c}{a+1}} = q + \frac{1}{s}$, then*

$$\mu(\mathfrak{n}_{a,c}) = \left\lceil 2\sqrt{c(a+1)} \right\rceil \qquad \text{or} \qquad \mu(\mathfrak{n}_{a,c}) = \left\lceil 2\sqrt{c(a+1)} \right\rceil + 1.$$

Proof. If $\sqrt{\frac{c}{a+1}} = q + \frac{1}{s}$ we obtain $c = \frac{(a+1)(qs+1)^2}{s^2}$.

Since $c \in \mathbb{N}$, we must have that $s^2 \mid (a+1)$. This implies that

$$\frac{c}{qs+1} = \frac{(a+1)(qs+1)}{s^2} \in \mathbb{N} \quad \text{and} \quad \left\lceil \frac{a}{s} \right\rceil = \left\lceil \frac{a+1}{s} - \frac{1}{s} \right\rceil = \frac{a+1}{s} \in \mathbb{N}.$$

Therefore

$$\begin{aligned}
\dim \pi_{1/s}(\mathfrak{n}_{a,c}) &= (a+1)q + 1 + \left\lceil \frac{a}{s} \right\rceil + \left\lceil \frac{c + (s-1)\left\lceil \frac{c}{qs+1} \right\rceil}{q+1} \right\rceil \\
&= (a+1)q + 1 + \frac{a+1}{s} + \left\lceil \frac{c + (s-1)\frac{(a+1)(qs+1)}{s^2}}{q+1} \right\rceil \\
&= (a+1)\left(q + \frac{1}{s}\right) + 1 + \left\lceil \frac{cs^2 + (s-1)(a+1)(qs+1)}{s^2(q+1)} \right\rceil \\
&= (a+1)\left(q + \frac{1}{s}\right) + 1 + \left\lceil \frac{(a+1)(qs+1)^2 + (s-1)(a+1)(qs+1)}{s^2(q+1)} \right\rceil \\
&= (a+1)\left(q + \frac{1}{s}\right) + 1 + \left\lceil \frac{(a+1)(qs+1)\left((qs+1) + (s-1)\right)}{s^2(q+1)} \right\rceil \\
&= (a+1)\left(q + \frac{1}{s}\right) + 1 + \left\lceil \frac{(a+1)(qs+1)}{s} \right\rceil \\
&= 2(a+1)\left(q + \frac{1}{s}\right) + 1 \\
&= 2\sqrt{c(a+1)} + 1 \\
&= \left\lceil 2\sqrt{c(a+1)} \right\rceil + 1.
\end{aligned}$$

And the result follows. □

Example 4.7. *Consider $\mathfrak{n}_{a,c}$ with $a = 48$ and $c = 100$. Notice that this case falls under Remark 3.10. Then we obtain $q = 1$, $\alpha = \frac{3}{7}$, and the following results:*

Representation	π_S	π_0	π_1	$\pi_{1/6}$	$\pi_{1/5}$	$\pi_{1/4}$	$\pi_{1/3}$	$\left\lceil 2\sqrt{49 \cdot 100} \right\rceil$
Dimension	*149*	*149*	*148*	*146*	*144*	*142*	*141*	*140*

Clearly, the obtained result is very sharp:

$$140 \leq \mu(\mathfrak{n}_{48,100}) \leq 141.$$

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