David Hilbert and the foundations of the theory of plane area

Eduardo N. Giovannini[1,2] (iD)

Abstract

This paper provides a detailed study of David Hilbert's axiomatization of the theory of plane area, in the classical monograph *Foundation of Geometry* (1899). On the one hand, we offer a precise contextualization of this theory by considering it against its nineteenth-century geometrical background. Specifically, we examine some crucial steps in the emergence of the modern theory of geometrical equivalence. On the other hand, we analyze from a more conceptual perspective the significance of Hilbert's theory of area for the foundational program pursued in *Foundations*. We argue that this theory played a fundamental role in the general attempt to provide a new independent basis for Euclidean geometry. Furthermore, we contend that our examination proves relevant for understanding the requirement of "purity of the method" in the tradition of modern synthetic geometry.

1 Introduction

Chapter IV of David Hilbert's classical *Foundations of Geometry*, first published in 1899, develops the theory of plane polygonal area. This section of the influential monograph is usually praised not only for its unprecedented level of rigor, being the first modern axiomatization of this central part of elementary geometry, but also for the many innovative and original results contained therein. Among these, one can mention the systematic study of different relations of geometrical equivalence, the

---

✉ Eduardo N. Giovannini
  eduardo.giovannini@univie.ac.at

1 Department of Philosophy, University of Vienna, Universitätsstraße 7, 1010 Vienna, Austria

2 CONICET and Universidad Nacional del Litoral, Bv. Pellegrini 2750, S3000 Santa Fe, Argentina

construction of the theory of area independently of continuity assumptions (viz. the axiom of Archimedes), as well as a sophisticated but elementary proof of the central geometrical proposition known as De Zolt's postulate. Notwithstanding, despite the importance of this chapter, it has received less attention from historians and philosophers of mathematics than other sections of *Foundations*. [1]

This paper aims to fill this gap in the specialized literature by offering the first detailed historical discussion of Hilbert's axiomatic investigations into the theory of plane area. We will undertake this task by closely examining the development of this theory in *Foundations*. In addition, Hilbert's notes for lecture courses on the foundations of mathematics will also be taken into account. These important sources offer a unique landscape to elaborate a more accurate historical account of his work.[2] Hilbert's theory of plane area will be investigated with an eye to *two interpretative points*.

The first point concerns the *historical background* of Hilbert's investigations. We will argue that, to a significant extent, his axiomatization of the theory of area was the culmination of a rich and intense foundational debate, which took place during the second half of the nineteenth century. This debate was triggered by the emergence of the modern theory of geometrical equivalence, which investigates criteria for the equality of area of polygonal figures on the basis of its decomposition and composition into polygonal components, respectively congruent. The main issue in these discussions concerned the role and logical status of a geometrical proposition known as "De Zolt's postulate." This central proposition states that if a polygon is divided into polygonal parts in any given way, then the union of all but one of these parts is not equivalent (i.e., equal in area) to the given polygon. In discussing methodological and epistemological issues related to this new "geometrical axiom," geometers involved in this debate delivered novel insights for the modern synthetic reconstruction of Euclidean geometry. The contextualization of Hilbert's investigations within this specific geometrical background yields a better historical assessment of his contributions in *Foundations* and sheds new light on a central episode in the emergence of modern axiomatic geometry. In particular, a welcome offshoot of the present investigation is a better historical appraisal of the contributions of important nineteenth-century geometers, such as Friedrich Schur, to the foundations of modern geometry.

The second interpretative issue relates to the historical and conceptual *significance* of Hilbert's theory of plane area for the general axiomatic program pursued in *Foundations*. As is well known, this program aimed at providing a new *independent basis* for elementary Euclidean geometry, by removing the dependence on continuity and (implicit) numerical assumptions from the classical theories of proportion and plane area. In this regard, a key technical innovation was the construction of a purely geometrical calculus of segments, which allowed the derivation of the (abstract) algebraic structure of an ordered field from the axioms for the Euclidean plane. In this paper, we will argue that the problem of obtaining an adequate proof of the so-called De

---

[1] Hilbert's theory of plane area in *Foundations* has been analyzed recently by Baldwin (2018a, b) and Baldwin and Mueller (2019). These articles offer excellent expositions of the central ideas and results achieved by Hilbert. Nevertheless, the nineteenth-century geometrical background upon which Hilbert developed his theory is not taken into particular consideration.

[2] Hilbert's notes for lecture courses on the foundations of geometry, corresponding to the period 1891–1902, have been published in Hallett and Majer (2004).

Zolt's postulate was, for Hilbert, a central issue in the modern axiomatic development of the theory of plane area. More specifically, we will contend that a significant challenge was to deliver a rigorous proof of this proposition that was not only strictly geometrical—in the sense of avoiding numerical considerations—but also independent of the Archimedean axiom.

The paper consists of *two thematic parts*. The first part provides a historical examination of the development of the theory of plane area in the second half of the nineteenth century, which set the stage for Hilbert's axiomatic investigations. A central aspect of this geometrical background was a clear distinction between a "synthetic" and a "metrical" approach to the study of polygonal areas. While the former was identified with the theory of geometrical equivalence, the latter consisted in the (now standard) method of measuring the area of polygonal figures by means of (positive) real numbers. To put these geometrical developments into proper context, Sect. 2 presents a brief overview of Euclid's theory of area in the *Elements*. Next, in Sect. 3, we analyze several critical steps in the emergence of the modern theory of equivalence. In particular, Sect. 3.1 discusses some novel results in the study of geometrical equivalence, while Sect. 3.2 focuses on the contributions of the Italian mathematician Antonio De Zolt. Section 4 provides then a detailed analysis of the *immediate background* of Hilbert's work in *Foundations*. Section 4.1 explores the connections between the modern theory of magnitudes and the foundations of the theory of plane area in the works of Otto Stolz. In turn, in Sect. 4.2, we examine Schur's geometrical proof of the comparability of plane polygons.

The second part of the paper offers a detailed account of Hilbert's theory of plane area. In Sect. 5 we analyze Hilbert's initial reflections on the role and significance of De Zolt's postulate in the theory of plane area, as reported in his notes for lecture courses on the foundations of geometry. This axiomatic development of the theory of plane area is then examined in Sect. 6. On the one hand, in Sect. 6.1, we discuss a series of technical innovations and conceptual clarifications advanced by Hilbert concerning the theory of geometrical equivalence. On the other hand, in Sect. 6.2, we provide a thorough reconstruction of the central proof of De Zolt's postulate in *Foundations*. Finally, Sect. 7 presents some concluding remarks.[3]

2 Euclid's theory of area in the *Elements*: an overview

The modern debate on the foundations of the theory of equivalence was significantly motivated by Euclid's theory of plane area in the *Elements*. In particular, a central issue concerned the role that the common notions played in the development of this theory. Hilbert also repeatedly referred to Euclid's treatment of plane areas in the classical Greek text, especially when establishing methodological requirements for the founda-

---

[3] Terminological note. We will use the term "area" to denote the *surface content*, or simply the *content*, of a plane rectilinear figure. Hilbert generally used the expression "Flächeninhalt" to refer to this notion. In turn, we will employ the expression "measure of area" for the metrical concept of area of a polygonal figure. The corresponding German term used consistently by Hilbert is "Flächenmaß." The term "polygon" will always be used in the sense of *simple polygon*.

Unless otherwise noted, all translations in what follows are by the author.

tions of this central part of Euclidean geometry. It will be beneficial for our subsequent discussion to present a brief *overview* of the theory of plane area in the *Elements*.[4]

As is well known, Euclid developed a theory of the *comparison* of polygonal areas, not a theory of measure of areas in the modern sense, i.e., as numerical functions that assign (positive real) numbers to every rectilinear figure. In general terms, Euclid's method consisted in studying the equality of area or content of polygonal figures based on the possibility of decomposing and composing them into polygonal parts, congruent in pairs, respectively. This strictly geometrical approach to the study of areas, known as the "theory of equivalence," was then fundamentally grounded on the relation of geometrical congruence. Moreover, the common notions played a central function in this method, since the derivation of the equality of area of two polygons by the procedure of adding and removing congruent figures was essentially based on the properties of equality, addition, and subtraction formulated in these general principles. Recall that, according to Heiberg's critical edition, the *Elements* contains the following five common notions:

CN1: *Things which are equal to the same thing are also equal to one another.*
CN2: *If equals be added to equals, the wholes are equal.*
CN3: *If equals be subtracted from equals, the remainders are equal.*
CN4: *Things which coincide with one another are equal to one another.*
CN5: *The whole is greater than the part.*

The main results about the equality of area of polygonal figures are presented in the propositions I.35-I.45 of the *Elements*. This set of propositions provides, as it were, the *foundation* of Euclid's theory of equivalence. The fourteen propositions of Book II are also immediately related to the notion of polygonal area; particularly, in Proposition II.14 Euclid shows how to construct a square equal in area to a given polygonal figure. Finally, Book VI also contains important propositions about plane areas (especially, VI.1, VI.25, VI.28, and VI.29), obtained through the application of the theory of proportion, previously developed in Book V.[5]

Proposition I.35 marks then the beginning of Euclid's studies of polygonal areas[6]:

I.35. *Parallelograms which are on the same base and in the same parallels are equal to one another.*

The general idea of the proof is as follows: by applying I.29 and I.34, Euclid shows first that the triangle ABE is equal (i.e., congruent) to the triangle DCF. But if the

---

[4] For detailed studies of Euclid's theory of area, see Mueller (1981) and De Risi (2020).

[5] A detail examination of Euclid's use of the theory of area in the study of similar figures can be found in Błaszczyk and Petiurenko (2020).

[6] Proposition I.34 states that in every parallelogram the opposite sides and angles are congruent, and the diagonal divides it into two congruent parts. It should be noted that the word employed is not "parallel-logram," but "parallelogrammic region" (παραλληλόγραμμονχώριον). Some scholars have argued that, with this expression, Euclid is not alluding to the geometrical figure, but to its *content* or area. In other words, by "parallelogrammic region" he might be referring to the more abstract *representation of the content* of a figure. On this reading, see De Risi (2020). However, the precise explanation of this notion is a complex matter, for it requires an account of Euclid's conception of magnitudes as "abstract objects" distinguishable from geometrical figures, that is, of the Greek understanding of the so-called method of "method of definitions by abstraction."

Fig. 1 *Elements*, I. 35
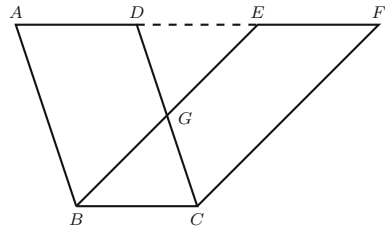
same triangle *DGE* is *subtracted* from each one of those triangles, one obtains the trapeziums *ABGD* and *EGCF*, which must be "equal" (in area) by CN3. Then, if the triangle *BCG* is *added* to those trapeziums, one obtains the parallelograms *ABCD* and *EBCF*, which also must be equal (in area) by CN2.[7]

As can be noticed in the proof of I.35, Euclid's strategy to establish the equivalence of a pair of plane polygons consisted in the addition and subtraction of other polygonal figures congruent in pairs. This procedure was essentially grounded on the properties of equality, addition, and subtraction laid down in the common notions, particularly in CN1-CN3. In other words, the systematic use of the common notions in the study of plane polygons was grounded on the assumption that polygonal areas were a class of geometrical magnitudes. Moreover, Euclid did not introduce any specific term to distinguish the equality of area or content from the more basic relation of congruence. Nevertheless, the application of the common notions in this context suggests the distinction between two different notions of equality of area or equivalence of plane polygons.

The first notion of "equality" is grounded on CN2, that is, on the "equality by the *addition* of equals." Accordingly, two polygons are equal in area if they result from adding figures, respectively congruent or, in other words, if they are composed by the same (i.e., congruent) polygonal parts. In turn, the second notion is based on CN3, namely on the criterion of "equality by *difference* of equals." Two polygons are equal in area if they can be obtained by subtracting 'equal' figures from "equal figures." More precisely, according to this second notion, two polygons are said to be equal in area if it is possible to add to them "equal" figures, and obtain a pair of polygons equivalent (by addition). Euclid did not distinguish in any part of the *Elements* between these two criteria of equality of area for plane rectilinear figures. However, the employment of *both* CN2 and CN3 in most of the fundamental propositions about polygonal areas, as illustrated in the proof of I.35, reveals that the second notion is the one operating in his theory. The precise description and investigation of these two criteria of equality of area became a central issue in the modern development of the theory of geometrical equivalence.

Propositions I.36 states that parallelograms with equal bases and in the same parallels are also equal (in area). In turn, in I.37 and I.38, Euclid proves that the same conditions apply for the case of triangles, that is, that triangles with the same (or equal) bases and in the same parallels are equal (in area). Furthermore, the *partial* converses

---

[7] A complete proof of this proposition requires the consideration of different cases, depending on whether the sides have points in common. This was already noticed by Proclus in his commentary to the first book of the *Elements* (Fig. 1).
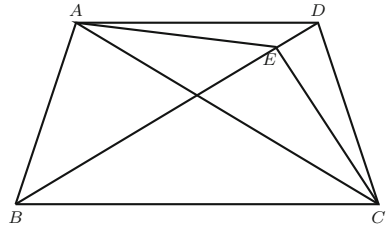
of the last two propositions are formulated in I.39 and I.40, respectively. In particular, I.39 asserts that:

I.39. *Equal triangles which are on the same base and on the same side are also in the same parallels.*

Euclid demonstrates this proposition by means of a *reductio* or an indirect argument. Here ABC and DBC are the two triangles equal in area, which are on the same base BC and on the same side (Fig. 2). Let AD be joined. We must prove that BC is parallel to AD. Let us assume that AD is not parallel to BC. Then, it is possible to draw from A a parallel to BC, which might be called AE (I.31). Let EC be joined. Since the triangles ABC and EBC are on the same base and on the same parallels, they must be equal in area (I.37). But ABC is equal in area to DBC, so DBC must also be equal in area to EBC (CN1). Euclid then claims that this implies that "the greater would be equal to the less: which is impossible." Therefore, AE is not parallel to BC. In the same manner, one can prove that any other straight line drawn from A, and different from AD, cannot be parallel to BC; therefore, AD is parallel to BC.

Euclid did not assign any particular foundational role to this proposition; it has been pointed out by historians of Greek mathematics that he never used I.39 again, or similarly I.40, in the proofs of any other proposition throughout the *Elements*. However, the modern discussions of the theory of equivalence will bestow primary importance upon this proposition. This might be explained by the fact that this is the first proposition about polygonal areas where CN5 comes into play. To be more precise, in the proof of I.39 Euclid did not reach the contradiction by applying the general principle "the whole is greater than the part," but instead by resorting to the sentence "the greater would be equal to the lesser: which is impossible." It has been argued by authoritative scholars that this is also the case in other similar *reductio* arguments in which Euclid aims to compare figures.[8] In any event, it is clear that I.39 is the first proposition in the *Elements* where one encounters a relation of *order* for polygonal areas, and that this relation bears an essential connection to the criterion formulated in CN5.

Euclid did not offer an explicit definition of the relations of "greater" and "lesser" in area. Nevertheless, his geometrical practice indicates that he conceived the relation of

---

[8] De Risi (2020) lists several propositions in which Euclid uses the latter sentence, in various similar forms, in the course of indirect proofs. A clear example is the proof of the proposition I.6 of the *Elements*, in which Euclid compares triangles with respect to the relation of *congruence*. In this recent work, De Risi also puts into question the authenticity of CN4 and CN5 on the basis of a conceptual analysis of their role in Euclid's theory of plane area.

order for polygonal areas as grounded on the relation of *inclusion*, or more precisely, on the mereological relation of parthood. According to this conception, a polygon **P** is said to be greater in area than another polygon **Q**, if there is a polygon **P′** *properly contained* in **P**, such that **P′** is equal in area to **Q**. The precise understanding of how Euclid conceived this relation of order is a major and disputed issue among scholars. Euclid's demonstrative practice suggests that the fact that a figure is a proper part of another did not need to be derived propositionally, but was usually *inferred diagrammatically*. Thus, Euclid's understanding of the relation of order is connected to the central interpretative issue of the role of diagrams in his geometrical practice. For our interest, focused on the modern theory of equivalence, it might be sufficient to point out that Euclid established a connection between the relation of order for polygonal figures and CN5.

Proposition I.41 deserves also here a particular mention:

I.41. *If a parallelogram has the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.*

To prove this proposition, Euclid relies basically on the equality of area of triangles with equal bases and altitudes, which was established in I.37. An immediate corollary of this proposition is that "every triangle is equal in area to a parallelogram with equal base and half altitude." Although Euclid did not draw this conclusion from Proposition I.41, this corollary will play an important role in the modern theory of equivalence.

The cluster of propositions I.42-I.45 features a crucial moment in the systematic study of plane areas carried out in Book I of the *Elements*. Euclid achieves there a series of results, usually known as the "application of areas," which in general terms show how any rectilinear figure can be transformed into a parallelogram or a rectangle *equal in area*, with a given side. This technique provides a procedure to add, subtract, and compare any pair of plane polygons in relation to their areas. Moreover, the method of "application of areas" has paramount importance in Book II, where Euclid proves important results about the relations between straight line segments and polygonal areas. This Book also contains the crucial proposition II.14, where Euclid "completes" his method of transformations of areas by showing how to construct a *square* equal in area to any given polygonal figure.

Let us discuss in some detail the content of this cluster of propositions of Book I. We will direct our attention to the *meaning* of the method advanced by Euclid, rather than to the proofs of these propositions. In I.42, Euclid shows how to construct a parallelogram, in a given angle, equal to a given triangle. In turn, I.43 is not a construction problem but a *theoretical* proposition, famously known as the Gnomon theorem. This theorem, which plays a crucial role in the proofs of the following two propositions, states that:

I.43. *In any parallelogram the complements of the parallelograms about the diameter are equal to one another.*

Here one must prove that the parallelograms $EBGK$ and $HKFD$, the "complements" about the diameter, are equal in area. Euclid commences by using I.34 to show

Fig. 3 *Elements*, I. 43
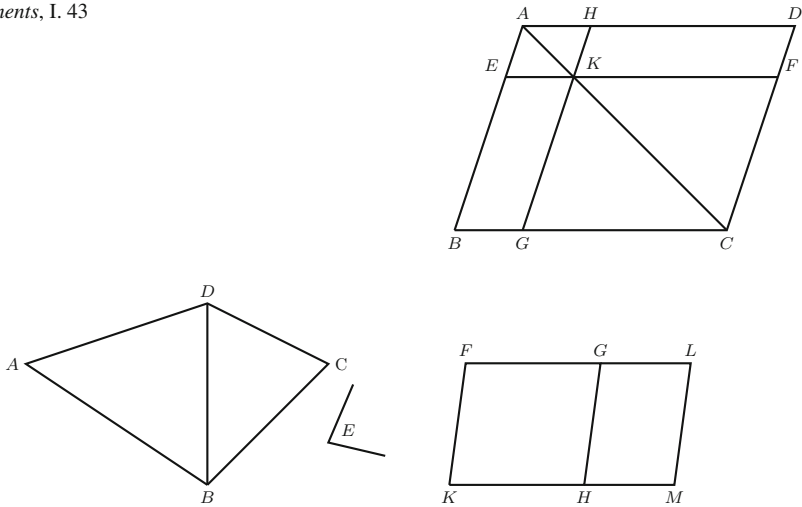
Fig. 4 *Elements*, I. 45

that the pairs of triangles ABC and ACD, AEK and AKH, KGC, and KCF are, respectively, equal (i.e., congruent). But the triangle AEK "together with" the triangle KGC is equal (in area) to the triangle AHK "together with" the triangle KCF (CN2). If these pairs of triangles are *subtracted*, respectively, from the whole triangles ABC and ACD, the remainders will be equal in area by CN3. Hence, the parallelograms $EBGK$ and $HKFD$ are equal in area (Fig. 3).

Now, the Propositions I.44 and I.45 provide the very core of Euclid's method of transformations of areas. Let us analyze them in turn.

I.44. *To a given straight line to apply, in a given rectilinear angle, a parallelogram equal to a given triangle.*

The details of the proof are not relevant here; for our purposes, it will be sufficient to stress that Euclid shows how to transform any triangle, but also *any parallelogram*, into another parallelogram with a given angle and *with a given side*. In his influential editorial notes to the *Elements*, Heath points out that this proposition is one of the "most impressive results" of Greek geometry. The last step in the method of transformation of areas is to show how to construct a parallelogram (or a rectangle), equal in area to *any rectilinear figure*. This is precisely the construction problem tackled in the next Proposition I.45 (Fig. 4).

I.45. *To construct, in a given rectilinear angle, a parallelogram equal to a given rectilinear figure.*

The proof involves the following ideas. First, the given polygonal figure is decomposed into triangles; this "triangulation" is obtained by drawing all diagonals from *one vertex* of the figure, chosen arbitrarily.[9] Next, utilizing I.42, one constructs the

---

[9] Euclid proves I.45 for the case that the polygonal figure an arbitrary quadrilateral, which can be thus decomposed into two triangles. However, the same argument can be repeated and applied to a polygon on *n-sides*, which proves the theorem in general.

parallelogram $FKHG$ equal (in area) to the triangle ABD, in the angle HKF equal to the given angle E. Then, one "applies" to the side GH a parallelogram $GHML$ equal (in area) to the other triangle DBC and with the angle GHM equal to the angle E. In other words, one places the new parallelogram $GHML$ *adjacent* to the first constructed parallelogram $FKHG$, at the *common side GH*. The remainder of the proof consists in showing, by means of a relatively involved argument, that the figure thus obtained is indeed a *parallelogram.*

By relying on I.44 and I.45, one can easily prove that any polygonal figure can be transformed into a parallelogram with a given angle and *with a given side* (viz. with a given *height*). Nevertheless, the theoretical significance of this corollary—let us call it I.45B—cannot be underestimated, for it makes truly operational the addition and subtraction of two-dimensional figures: *any pair of polygonal figures* can always be added (or subtracted) by transforming them into two rectangles with a common height. Euclid did not draw this immediate consequence from the latter couple of propositions, although influential historians of Greek mathematics have pointed out that he *implicitly* used I.45B in the proofs of other important propositions throughout the *Elements*, such as VI.25.[10] Euclid's reluctance to *explicitly* formulate this corollary is tightly bound to deep and difficult interpretative issues regarding the meaning of the method of transformation of areas in Greek geometry. We briefly address two main problems, for they will prove to be highly relevant for our subsequent discussion of the modern geometrical theory of equivalence.

One central aspect is the view that, with the *implicit* derivation of I.45B, Euclid provides an elementary method to "measure" the area of any polygonal figure. In fact, if the height of the constructed rectangle is conceived as the "unit length," then this proposition shows how to "calculate" its (measure of) area, i.e., by measuring the length of the corresponding base. In other words, the (measure of) area of a constructed rectangle with unit height would be equal to (the length of) its base. Although *mathematically plausible*, this interpretation of Euclid's method faces very fundamental problems.[11] First, the idea that the (measure of) area of a polygon is equal to (the length of) a segment fragrantly violates the fundamental tenet of *homogeneity* in Greek mathematics, according to which one can only compare, and operate with, magnitudes *of the same kind*. In short, equating plane areas to lengths of segments, and thus comparing different kinds of geometrical magnitudes, is incompatible with the Greek notion of *geometrical magnitude*, for which a geometrical quantity can never be considered independently from the corresponding geometrical figure. Second, the

---

[10] Proposition VI.25 reads: "To construct one and the same figure similar to a given rectilinear figure and equal to another given rectilinear figure."

[11] In an often-quoted passage of his editorial notes, Heath proposed to identify the given height of equivalent parallelogram, constructed in I.44, with a "unit length":

This proposition [i.e., I.44] will always remain one of the most impressive in all geometry when account is taken (I) of the great importance of the result obtained, the transformation of a parallelogram of any shape into another with the same angle and of equal area but with one side of any given length, e.g., a *unit* length. (Heath 1956, pp. 342–343)

As is well known, this reading has been fiercely defended by the advocates of the so-called geometric algebra interpretation of Book II of the *Elements*. For a detail discussion, see Unguru and Rowe (1981, 1982) and Corry (2013).

introduction of a unit length presupposes a general and abstract concept of number, which can be applied to measure any kind of geometrical magnitude; this conception was absent in the Greek mathematical tradition.

Another crucial matter consists in explaining how exactly the results on the "applications of areas" ground a procedure to compare any pair of polygonal figures with respect to their areas. Again, we can address this issue only schematically. Using I.45B, one can transform any pair of polygons into equivalent rectangles with a given *height*. To compare both figures, one only needs to determine whether their bases coincide or not. If the bases coincide, then by CN4 the two rectangles are not only congruent but also "equal in area." In turn, if one rectangle is a proper part of the other, then by CN5 the former would be lesser in area than the latter. Now, Euclid's complex theory of congruence prescribes that to establish that the two rectangles are congruent, one has to rely ultimately on some kind of *superposition argument*. In addition, if this is not the case, we have seen that under certain circumstances, Euclid allows himself to conclude that one figure is a proper part of another by means of *diagrammatic inferences*. Therefore, for brevity's sake, Euclid's procedure to compare polygonal areas depends heavily on intuitive or empirical arguments related to the "movement of figures," as well as on diagrammatic inferences concerning the mereological relation of parthood.[12] Naturally, this will be contested vigorously during the emergence of the modern theory of equivalence.

This concludes our overview of Euclid's theory of area in the *Elements*. Let us focus now on the emergence of the modern theory of geometrical equivalence.

3 The emergence of the modern theory of equivalence

We still lack a detailed and comprehensive historical study of the emergence of the modern geometrical theory of equivalence in the second half of the nineteenth century.[13] This historical development is notably interesting and complex, for it raised many methodological, foundational, and epistemological issues for the modern synthetic reconstruction of Euclidean geometry. Pedagogical concerns also had considerable relevance.[14] In this section, we will remain content with presenting some key ideas which directly or indirectly provided the background and motivation for Hilbert's investigations. Section 3.1 presents some initial results of the modern theory of equivalence. Section 3.2 briefly analyzes the key contributions of the Italian mathematician Antonio De Zolt to the emergence of this geometrical theory.

---

[12] For an insightful analysis of the role played by CN4 and CN5 in Euclid's method of application of areas, as well as the use of "diagrammatic inferences," see De Risi (2020).

[13] The most important studies are still the classical paper of Amaldi (1900) and the more recent work by Volkert (1999). In these works, one can find excellent accounts of the main steps in the development of the modern theory of equivalence. This section is greatly indebted to these accounts.

[14] The pedagogical concerns were connected to the remarkable array of geometry textbooks published in Italy in the second half of the nineteenth century. They aimed at replacing Euclid's *Elements* as the teaching source in secondary schools. For details, see Vecchi (1915) and Giacardi and Scoth (2014).

3.1 The Wallace–Bolyai–Gerwien theorem

The emergence of the modern theory of equivalence is usually traced back to the discovery of an important theorem which connects the notions of measure of area and geometrical "equivalence."[15] The theorem asserts that two polygons with equal measure of area can always be decomposed into the same number of polygonal parts (particularly, of triangles), respectively congruent. This theorem was first posed as a question by the English mathematician William Wallace in 1814, and proved in the affirmative by John Lowry in the same year.[16] However, the true impact of this result took place almost two decades later, when it was *independently* rediscovered and proved by two different mathematicians. In 1832, Farkas Bolyai, the father of János Bolyai, one of the creators of hyperbolic geometry, provided a new but *sketchy* proof of the theorem in question. Then, one year later, in 1833, the German mathematician and lieutenant Paul Gerwien made a notable contribution by offering a very detailed and rigorous proof of the theorem, which also included a generalization to spherical polygons.[17] Thus, this theorem is now known as the Wallace–Bolyai–Gerwien theorem.

Regarding the latter proof, Gerwien's strategy was to prove the theorem first for the case of *triangles* and then arrive at the general result by showing how any polygon can be decomposed into a finite number of triangles. The details are not important for our present discussion[18], although we should mention that Gerwien's close examination of the validity of this theorem for the case of triangles contributed to a more rigorous explanation of the concept of decomposition of a polygon. Moreover, Gerwien concluded his essay with the following important conclusion:

> The present essay reveals that the equality of rectilinear figures can be defined as follows: Equal figures are those which are composed of the same pieces. (Gerwien 1833a, p. 234)

This final remark is perhaps the first modern attempt to provide an *explicit* characterization of the relation of "equality of area." In fact, this definition is pretty close to what was later called "equivalence by decomposition" or "equidecomposition," namely: two polygons are equidecomposable if it is possible to decompose them into the same number of polygonal components congruent in pairs. As is well known, Hilbert coined the term "equidecomposition" [*zerlegungsgleichheit*] in the second German edition of *Foundations* Hilbert (1903).[19] It is worth mentioning that Gerwien could not offer an explicit definition of the concept "area" or "content," but only characterized the relation "to have the same surface." A precise definition of area as a *class*

---

[15] Legendre (1806) introduced the term "equivalence," to distinguish terminologically the notion of equality of area from the idea of equality as congruence.

[16] See Wallace (1814) and Jackson (1912).

[17] Cf. Gerwien (1833a, b).

[18] For a detailed analysis of Gerwien's contributions, and its impact on subsequent discussions, see Volkert (1999). For the significance of this theorem in the development of the theory of equivalence in the second half of the nineteenth century, see Amaldi (1900) and Simon (1906).

[19] This relation is sometimes also called "equivalence by dissection" or "scissors congruence."

of equivalence of equidecomposable polygons was first achieved by members of the Peano School in the last decade of the nineteenth century.[20]

The notion of equivalence by decomposition was adopted by the French mathematician Jean-Marie Duhamel, who carried out the first critical discussion of the *foundations* of the theory of equivalence in the nineteenth century. Duhamel presented this examination in the second part of his mathematical-philosophical treatise *Des méthodes dans les sciences de raisonnement* (1866), which exerted a significant influence in subsequent investigations. These reflections were mainly elaborated in an appendix entitled "Note sur l'équivalence"[21], which advanced novel ideas not only from a technical but also from a *methodological* point of view. One important methodological insight concerned the adoption of a *unique* criterion to establish the equality of area of two plane figures, namely the relation of equidecomposition. Since this relation was based on the criterion of "equality by *addition* of equals" (Euclid's CN2), Duhamel restricted the use of the criterion of "equality by *difference* of equals" (CN3) from the development of the theory of equivalence. To cope with this methodological requirement, he offered new proofs of some relevant theorems about the equivalence of polygons, where the application of the principle "if equivalent figures are subtracted from equivalent figures the remaining figures are equivalent" was essential. An interesting example is the proof of the theorem "two parallelograms on the same base and with the same altitude are equivalent", which corresponds to proposition I.35 of the *Elements*.

Duhamel distinguishes between two cases, depending on whether the sides opposite to the base have points in common or not. The interesting case is when neither *E* nor *F* lie between *C* and *D* (Fig. 5b). The idea of the proof is as follows: Let *G* be the intersection point between *AE* and *BD*. Subdivide *BD* in *n equal* segments, with length less than *BG*. Next, from each one of the points $G_1, G_2, \ldots, G_n$, draw parallel lines to the base *AB*. The resulting partial parallelograms in *ABDC* will be all congruent as well as the resulting partial parallelograms in *ABFE*. Now, the two lowest partial parallelograms have the common base *AB* and stand in the situation described in the first case (Fig. 5a); then, they are *equidecomposable*. And the same applies to each one of the partial parallelograms that compose *ABDC* and *ABFE*, for they are, respectively, congruent to the lowest ones with the common base *AB*. Hence, *ABDC* and *ABFE* can be decomposed in the same number of partial parallelograms, respectively equidecomposable, and consequently they can be decomposed in the same number of polygonal parts, respectively congruent.[22]

A central aspect of the proof of this second case, explicitly acknowledged by Duhamel, is that it presupposes not only the possibility of subdividing a given segment in any number of parts of the same length but also the Archimedean property of line segments. More precisely, if the distance between points D and E were an infinitesi-

---

[20] For the development of the method of "definition by abstraction" in nineteenth-century geometry, see Mancosu (2016).

[21] See Duhamel (1866, pp. 445–450).

[22] This proof presupposes then that congruent parallelograms are equidecomposable. Duhamel provided first a proof of this proposition in Duhamel (1866, pp. 351–352). Moreover, to obtain the desired decomposition of *ABDC* and *ABFE*, one only needs to copy the dividing lines of the lowest parallelograms in the other partial parallelograms.
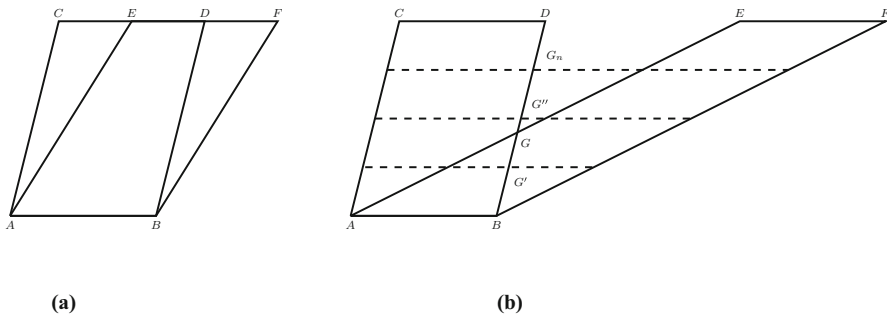
Fig. 5 Duhamel ([1866](#)) proof of *Elements* I. 35

mal (non-Archimedean) quantity, then the segment BG would be incommensurable to the side BD and no finite number of parallelograms contained in $ABCD$ would ever complete the parallelogram $ABFE$.[23] This revealed that Archimedes' axiom was a necessary condition to build the theory of equivalence upon the relation of equidecomposition. As is well known, this metageometrical result was first rigorously proved by Hilbert in the first edition of *Foundations* (1899). Up to the publication of this work, all modern presentations of the geometrical theory of equivalence were based on the relation of *equidecomposition*.

The critical considerations advanced by Duhamel were taken up by the Italian mathematician Aureliano Faifofer, in the highly influential textbook *Elementi di geometria*, first published in 1878.[24] In particular, he followed to a great extent the methodological guidelines laid down, but not thoroughly executed, by Duhamel. First, Faifofer provided explicit definitions of the relation of geometrical equivalence—in terms of equidecomposition—and addition of polygons. Second, he formulated the properties corresponding to CN1 and CN2—i.e., transitivity and additivity—as specific *geometrical* propositions about polygonal areas and provided the corresponding proofs. Following Duhamel's "purity of the method" requirement of avoiding the use of CN3, Faifofer provided new proofs of other propositions where this principle had been used. In this regard, his proof of the Gnonom theorem (*Elements, I.43*) was particularly innovative by circumventing the critical use of this Euclidean principle. In sum, Faifofer's presentation of the theory of equivalence was a notable improvement in the systematic development of this geometrical theory.

In the first edition of the textbook, a striking aspect of Faifofer's development of the theory of equivalence was the lack of an explicit definition of the relation of *lesser* and *greater* in area. In other words, there was no discussion of the criteria of non-equivalence of polygonal figures. Naturally, this had an immediate impact on the proofs of those propositions that appealed to the general principle "the whole is greater than the part" (CN5). A crucial example is the proposition I.39 of the *Elements*, which Faifofer tried to prove without using the latter common notion. This proof

---

[23] Cf. Duhamel ([1866](#), p. 448).

[24] Faifofer's *Elementi* became rapidly a main textbook for teaching elementary geometry in Italian secondary schools. This textbook went under 22 editions, the last published in 1925. For more details, see Vecchi ([1915](#)) and Giacardi and Scoth ([2014](#)).
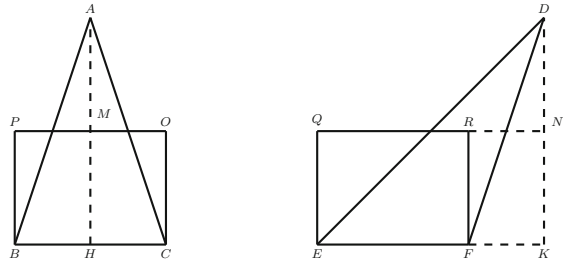
faced important shortcomings, which were identified by the Italian mathematician Antonio de Zolt shortly afterward, in a monograph that prompted an intense and notable debate on the foundations of the theory of equivalence. The publication of De Zolt's monograph sparked a decisive momentum in the modern development of the geometrical theory of equivalence.

3.2 De Zolt's postulate in the theory of equivalence

In 1881, De Zolt published a short monograph titled *Principii della eguaglianza di poligoni*. In the Preface, the author declared that the work aimed to offer a systematic examination of this central part of elementary geometry. After a brief assessment of some recent presentations of the theory of equivalence, De Zolt's focused his attention on Faifofer's *Elementi*. His main criticism of this work concerned the proof of the theorem corresponding to the proposition I.39 of the *Elements*. Let us briefly examine this proof (Fig. 6).

Theorem 275. *If two triangles are equivalent and have equal bases, then their altitudes are also equal.* (Faifofer 1878, p. 167)

Similar to Euclid's proof, the geometrical argument delivered by Faifofer was also a *reductio*. Consider the rectangles $PBCO$ and $QEFR$ equivalent to the given triangles ABC and DEF, with equal bases and half altitudes. By transitivity, $PBCO$ is equivalent to $QEFR$. Now, assume that the corresponding altitudes MH and NK of these rectangles are unequal. In that case, Faifofer observed, "one of the rectangles, that which has the lesser altitude, would be equal to a part of the other; and this excludes the possibility that the two rectangles are equivalent" (p. 167). Therefore, the altitudes of the rectangles, and respectively of the two triangles, must be equal.

Evidently, a contradiction only arises if one assumes that two rectangles with equal bases and unequal altitudes *cannot* be equivalent; or, alternatively, that if one rectangle is a proper part of another, they cannot be equivalent. De Zolt noticed that, although this fact was intuitively evident, a rigorous exposition of the theory of equivalence demanded an explicit justification for it. More precisely, he pointed out that this notable gap in Faifofer's proof was a consequence of the fact that the relation of *non-equivalence* did not receive adequate treatment in his exposition:

First of all, we notice that the non-equivalence of two figures is much more complex than it may seem at first. And in fact, to say that two figures are not

equivalent is to affirm that: if one of them is divided in any given way and in as many parts as one wants, it is not possible; however, you arrange these parts, to compose with them the other figure. [25] (De Zolt 1881, p. 12)

Thus, De Zolt claimed that, to obtain a rigorous introduction of the relation of ordering for polygonal figures, the following proposition must be included either as an axiom or as a theorem of the theory of equivalence:

If a polygon is divided into parts in a given way, it is not possible, when one of these parts is omitted, to recompose the remaining parts in such way that they cover entirely the polygon. (De Zolt 1881, p. 12)

This proposition is now known as "De Zolt's postulate." In his monograph, De Zolt attempted to prove this "fundamental proposition" in the theory of equivalence, but only managed to sketch a somewhat confusing and clearly flawed argument, which was unanimously criticized by his contemporaries.[26] The details of the argument are not important here; on the contrary, let us briefly comment on two relevant conceptual issues. The very formulation of De Zolt's postulate was anchored on a novel conception of the relation of ordering for polygonal figures. Unlike the criterion suggested by CN5, this conception was not grounded on the (mereological) relation of parthood, but on the operations of *decomposition* and *addition*. De Zolt explained this new understanding as follows:

When two polygons are not equal, they can be divided, as it was proved, such that all the parts of one of them appear in the other, and in the latter there are parts which are not in the former. In this way, divisibility constitutes the positive character of the *non-equality* of the two polygons; of which it will be said *lesser* the one which is divisible so that all its parts can figure in the other; and this one, *greater*.[27] (De Zolt 1881, p. 36)

This description suggests the following alternative definition of the relation of (strict) order for polygons: "a polygon **P** is *greater* (in area) than another polygon **Q** (in symbols, $\mathbf{Q} < \mathbf{P}$), if and only if there exists another polygon **R** such that $\mathbf{Q} + \mathbf{R} = \mathbf{P}$." As a matter of fact, this definition is built upon a "strong trichotomy" principle (using modern terminology) that states that for any polygons **P**, **Q**, there exists a polygon **R** such that *exactly one* of the following conditions holds: $\mathbf{P} = \mathbf{Q}$, $\mathbf{P} = \mathbf{Q} + \mathbf{R}$, or $\mathbf{Q} = \mathbf{P} + \mathbf{R}$. In his monograph, De Zolt formulated (a version of) this trichotomy law and attempted a proof by appealing to his new geometrical postulate[28]; this shows that he

---

[25] "Notiamo anzitutto come la non-equivalenza di due figure sia fatto assai piú complesso di quanto, a tutta prima, possa sembrare. E infatti, dire che due figure non sono equivalenti é affermare che: divisa una di esse figure in un modo equalsivoglia e in quante si vogliano parti, non é possibile, comunque si dispongano tal parti, comporre con esse l'altra figura."

[26] Some initial critical reactions to De Zolt's alleged "proof" of his novel postulate can be found in De Paolis (1886) and Faifofer (1886).

[27] "Allorché due poligoni non son eguali, si possono dividere, come fu dimostrate, in modo che tutte le parti di uno di essi figurino nell'altro, ed in questo sianvi parti che non sono in quello. Cosí fatta divisibilitá forma il carattere positivo di *diseguaglianza* dei due poligoni; dei quali, si dira *minore* quello che é divisibile in modo che sue parti tutte possano figure nell'altro; e questo, *maggiore*."

[28] See De Zolt (1881, §5).

understood that the key role of this fundamental proposition was to guarantee that plane polygons can be (linearly) ordered with respect to their areas. In sum, the formulation of De Zolt's postulate was intimately connected to a new conception of *non-equivalence*. Furthermore, this can also be appreciated in the fact that De Zolt *never* equated his postulate to the Euclidean principle "the whole is greater than the part."

The second issue refers to De Zolt's "quasi-axiomatic" development of the theory of equivalence. Another salient aspect of De Zolt's monograph was the explicit formulation and proof of several fundamental properties of geometrical equivalence (viz. equidecomposition) and non-equivalence, such as transitivity, additivity, subtraction, and trichotomy, among others. This standpoint will become a central methodological requirement in the modern theory of equivalence; namely, the explicit derivation as *geometric theorems* of the fundamental properties of equivalence, non-equivalence, and addition, previously stated as general principles of magnitudes in Euclid's common notions.

The publication of De Zolt's monograph marked a new era in the modern investigations into the theory of equivalence. The initial reaction was to include De Zolt's postulate as a *new axiom* of geometry, as can be noticed in the expositions of this theory presented in Faifofer ([1882]) and De Paolis ([1884]). In this regard, the widespread view was that a detailed proof of this geometrical proposition seemed too complicated and involved for a rigorous but still elementary exposition of the theory of equivalence, intended to teaching geometry in secondary schools. This standpoint was also adopted in other notable geometry textbooks published in Italy some years later, such as Lazzeri and Bassani ([1891]), Veronese and Gazzinaga ([1900]), and the influential Enriques and Amaldi ([1903]).

However, the search for a proof of De Zolt's postulate prompted an intense and fruitful foundational debate, which took place during the 1890s, primarily at the *Periodico di Mathematica* and the *Bollettino dell'Associazione "Mathesis"*.[29] These highly influential journals also had a strong interest in mathematical education. These critical discussions aimed not only to yield a rigorous proof of the "fundamental proposition" in the theory of equivalence but also to avoid as much as possible the deployment of non-elementary means. A close examination of this fascinating debate is beyond the scope of the present article; nevertheless, we should point out that, as a result of these discussions, two important proofs of De Zolt's postulate were obtained, namely Veronese ([1894/1895]) and Lazzeri ([1895]). A salient trait of these proofs was the appeal to a geometrical notion of measure of area, introduced using the classical theory of proportion. Finally, this debate also had significant ramifications in Germany and France, as we will analyze in the next section.

4 The immediate background of Hilbert's theory of plane area

Although Hilbert added a short reference to De Zolt's monograph in the second German edition of *Foundation* (1903), there is no clear indication that he had any *direct* informa-

---

[29] Some important moments in the notable Italian debate on the foundations of the theory of equivalence have been surveyed by Amaldi ([1900]) and more recently by (Bartocci [2012], pp. 248–258). I would like to thank an anonymous referee for calling my attention to the latter book.

tion about the "Italian" debate on the foundations of the theory of equivalence. These discussions had had, however, some repercussions in the German-speaking world, mainly through the works of Réthy (1891) and Rausenberger (1893). On the contrary, Hilbert's early interest in the theory of equivalence came from different sources. As can be noticed in his lecture courses, Hilbert's was deeply influenced by Otto Stolz, Friedrich Schur, and Wilhelm Killing. This section aims to reconstruct this immediate background of Hilbert's axiomatic investigations. Section 4.1 examines the contributions of Stolz to these problems; in turn, Sect. 4.2 focuses on Schur's important, but often neglected, work.

4.1 Stolz and the modern theory of magnitudes

Otto Stolz's engagement with the geometrical theory of equivalence was triggered by a more general concern on the fundamental notion of *extensive magnitude*. As is well known, in the first volume of the influential treatise *Vorlesungen über allgemeine Arithmetik* (1885), Stolz laid the groundwork for the modern theory of magnitudes by providing the first "axiomatic" characterization of this central mathematical concept. His axiomatic system consisted of fourteen "conditions" [*Bedingungen*], which every set of (geometrical) elements must satisfy in order to constitute a "system of absolute magnitudes." Stolz's conditions read as follows[30]:

1)   If $A = B$, then $B = A$;
2)   If $A > B$, then $B < A$ (and conversely);
3)   For every pair of magnitudes A, B, *exactly one* of the following conditions holds: $A = B$ or $A > B$ or $A < B$;
4)   If $A = B$ and $B = C$, then $A = C$;
5)   If $A = B$ and $B > C$, then $A > C$.
6)   If $A > B$ and $B > C$, then $A > C$.
7) $(A + B) + C = A + (B + C)$;
8) $A + B = B + A$;
9)   If $A = A'$ and $B = B'$, then $A + B = A' + B'$;
10) If $A > A'$ and $B = B'$, then $A + B > A' + B'$;
11) $A + B > A$;
12) If $A > B$, then there is in the system one and only one magnitude X such as $B + X = A$;
13) For each member A of the system and each positive integer n, there is an X in the system such that $nX = A$.
14) If $A > B$, there is a multiple of B which is greater than A: $pB > A$.

Without going into details, with these conditions Stolz established that any systems of "absolute magnitudes" can be conceived as an *ordered commutative (or Abelian) semigroup*, using modern algebraic terminology.[31] Moreover, the ordered Abelian

---

[30] Cf. Stolz (1885, p. 70). For better readability, we have simplified the formulations of conditions 3 and 13.

[31] A structure $\langle S, + \rangle$ is a *semigroup* if S is a set and $+$ is an associative binary operation on S. If the binary operation $+$ also satisfies the commutative property, then $\langle S, + \rangle$ is a *commutative (or Abelian) semigroup*. A structure $\langle S, +, < \rangle$ is an *ordered semigroup* if $<$ is a *total ordering* of S, such as the following *compatibility condition* holds: for all $a, b, c \in S$: if $a \leq b$, then $a + c \leq b + c$ and $c + a \leq c + b$. Needless to say, Stolz's conditions 1-12 do not form a system of *independent* axioms, for several conditions can be obtained from the others.

semigroup must also be divisible, according to the *divisibility* property stated in condition 13. Finally, Stolz claimed that if a system of (geometrical) elements also satisfies condition 14, that is, the so-called axiom of Archimedes, then it constitutes a system of absolute magnitudes *in the strict sense*.[32]

After specifying the fundamental properties which constitute the concept of (extensive) magnitude, Stolz attempted to *prove* that the set of plane polygons forms a "system of absolute magnitudes in the strict sense"; in fact, these geometrical elements represent a more interesting case than the set of straight line segments and the set of plane angles. The general idea of the proof, which is merely sketched, is to provide an explicit geometrical "interpretation" of the relations of equality ("=") and ordering ("<") and the operation of addition ("+"), and then to show that each one of the "axioms" of absolute magnitudes (in a strict sense) is satisfied under this given interpretation. Interestingly, Stolz noted that the most problematic aspect of this proof concerned the relation of ordering. More precisely, he pointed out that a significant shortcoming in classical Greek geometry was that "the comparability of any two geometrical magnitudes of the same kind is assumed from the outset, that is, without providing a proof of the possibility of the comparison by geometrical means" (Stolz 1885, p. 74).

Stolz's demand for a proof of the *possibility* of comparing any two geometrical magnitudes, particularly plane polygons, raised a pivotal conceptual and technical issue for the first time. More precisely, one should distinguish between two different senses in which two plane polygons are said to be *comparable*. The first relates to the validity of the (standard) trichotomy law: the validity of one of the three relations "$\mathbf{P} < \mathbf{Q}, \mathbf{P} = \mathbf{Q}, \mathbf{P} > \mathbf{Q}$" implies the non-validity of the other two. Stolz explicitly states this in his third axiom. In turn, the second sense refers to the fact that, for any two plane polygons $\mathbf{P}$ and $\mathbf{Q}$, *always* at least one of these three relations is valid. Thus, this second meaning is concerned with the *necessary* and *sufficient conditions* for the comparability of geometrical magnitudes, such as, for example, the Archimedean axiom. As was later *proved* by Hilbert (1899), the latter axiom is a necessary condition for the comparability of plane polygons, if one adopts the relation of equidecomposition as the criterion for the equality of area. In a subsequent work, Stolz (1894) explicitly distinguished between these two different issues involved in the possibility of comparing plane polygons.

As customary during this period, Stolz used the notion of *equidecomposition* as the criterion of equality of area of plane polygons. Moreover, his definition of the relation greater-than was grounded on the conception of ordering in terms of the operations of *decomposition* and *addition*, succinctly described as follows: "A polygon is greater than a second, if next to the pieces of the second it still contains others" (Stolz 1885, p. 75). It is plain that this definition of ordering was immediately suggested by his (axiomatic) conception of absolute magnitudes, particularly by the "axioms" 13 (i.e., the divisibility property) and 12. As for the operation of addition, he succinctly claimed that a polygon is called the sum of two other polygons, if it is composed [*zusammengesetzt*] by them.

---

[32] For a detailed study of Stolz's contributions to the modern theory of magnitude, see the excellent work by Ehrlich (2006).

Stolz focused then his attention on the problem of proving that any two plane polygons are comparable by means of a purely geometrical procedure. Surprisingly, his strategy consisted in comparing parallelograms with equal angles and altitudes (or bases) by *superimposing* them. Thus, the main idea was simply to use Euclid's well-known technique of "application of areas," in order to transform any polygonal figure into an equidecomposable parallelogram (or rectangle) with a given altitude; the transformed figures could be easily compared by placing one on top of the other. According to Stolz, the desired transformation could be immediately obtained by applying the theorem that "a triangle is equivalent (by decomposition) to a parallelogram with equal base and half altitude" and the Gnomon theorem.[33] Without further ado, Stolz concluded that any two polygons could be compared to one another, since every polygon can be decomposed into triangles through diagonals.

In a strict sense, Stolz did not provide any argument to show how, by applying this method to compare polygonal figures, one could prove that if a polygon **P** is equivalent to another **Q**, then **P** cannot be at the same time greater or lesser than **Q** (and conversely). In other words, he did not attempt to prove that, given his definition of equality of area and ordering, the trichotomy law expressed in the above condition 3 holds. Naturally, such a proof would involve a kind of indirect argument or *reductio*; however, Stolz did not even hint at how a contradiction could be obtained in this context. This critical observation was made shortly after by the German mathematician Wilhelm Killing, in a short but insightful review of Stolz's *Allgemeine Arithmetik*. In his critical recension, Killing sharply noted that, in order to obtain a contradiction, the definition of equidecomposition must be complemented by the postulation of a *new geometrical axiom*:

> The definition [of equidecomposition] uses an entirely determined decomposition and an entirely determined arrangement of the parts; so in order for the definition to be admissible, the following proposition must be assumed: If there is a decomposition of a polygon *A*, for which a certain arrangement of the parts yields a polygon *B*, then no decomposition of *A* is possible, for which a new arrangement of the parts yields a polygon *C*, in which the polygon *B* is contained as a part. [34] (Killing 1886, p. 186)

Killing recognized here the necessity of postulating a *version* of the so-called De Zolt's postulate to ground a relation of ordering for polygonal areas. Moreover, as far as we know, he discovered and formulated this new geometrical postulate without any direct knowledge of De Zolt's seminal work on the theory of equivalence.

In spite of the important gaps in Stolz's "proof" of the comparability of plane polygons, his attempt to *prove* that different kinds of geometrical objects constitute

---

[33] Stolz provided proofs for both theorems without resorting to the geometrical proposition corresponding to Euclid's CN3. Additionally, he proved in the same way that "parallelograms with equal bases and altitudes are equivalent (by decomposition), i.e., the proposition I.35.

[34] "Die Definition benutzt eine ganz bestimmte Zerlegung und eine ganz bestimmte Anordnung der Theile; damit die Definition also erlaubt ist, muss folgender Satz vorausgesetzt werden: Wenn es eine Zerlegung eines Polygons *A* giebt, für welche eine bestimmte Anordnung der Theile ein Polygon liefert, so ist keine Zerlegung von *A* möglich, für welche eine neue Anordnung der Theile ein Polygon *C* liefert, in welchem das Polygon *B* als Theil enthalten ist."

a class of "absolute" magnitudes had profound implications for the modern synthetic reconstruction of elementary geometry. Briefly, this requirement amounted to the *elimination of the concept of pure magnitude* from the foundations of geometry. More specifically, this requirement involved two main methodological and epistemological constraints: first, general principles or axioms of magnitudes must not be *directly* used in geometrical proofs; second, the geometrical propositions corresponding to those axioms must be proved as theorems; otherwise, one would commit a *petitio principii*. The systematic application of this requirement constitutes a central tenet of modern axiomatic geometry.

4.2 Friedrich Schur's proof of the comparability of plane polygons

Friedrich Schur was another important participant of the foundational debate on the geometrical theory of equivalence, whose contributions had a considerable influence on Hilbert's axiomatic views. In 1892, Schur published a short note discussing Stolz's alleged "proof" of the comparability of plane polygons.[35] This condensed paper presented some novel technical insights and introduced instructive considerations from a methodological and epistemological standpoint.

Schur focused his attention on the role played by the "general principles of magnitudes" in the problem of comparing plane polygonal figures. Interestingly, for the first time, we find an explicit requirement of "purity of the method" in connection to this general problem:

A problem as simple as the measurement of plane figures bounded by straight lines has not yet been rendered with the necessary rigor and purity of the method, as it seems to me from the available literature. Not even speaking about the [illegitimate] use of infinite processes, general axioms of magnitude are used without justification, for these are only immediately evident, when the magnitudes are straight line segments, whose comparison can be carried out by placing them on top of each other. One of such general principles of magnitudes […] is, for example, that the subtraction of equal magnitudes from equal magnitudes yields again equal magnitudes. […] But before one has not managed to measure plane figures by segments, which is only possible by the theorem to be proved, the application of this principle of magnitudes is not by any means justified. [36] (Schur 1892, pp. 2–3)

---

[35] Cf. Schur (1892).

[36] "Ein so einfaches Problem wie die Ausmessung ebener geradlinig begrenzter Figuren ist, wie es nach der mir zugänglichen Literatur den Anschein hat, noch nicht mit der hierbei möglichen Strenge und Reinheit der Methode dargestellt worden. Um garnicht zu reden von der Herbeiziehung endloser Processe, so werden mit Unrecht allgemeine Grössenaxiome benutzt, die nur dann unmittelbar klar sind, wenn diese Grössen geradlinige Strecken sind, ihre Vergleichung also durch Aufeinanderlegen bewirkt werden kann. Ein solcher allgemeiner Grössensatz […] ist z. B. der, dass die Subtraction gleicher Grössen von gleichen Grössen wieder gleiche Grössen giebt. […] Bevor es aber nicht gelungen ist die ebenen durch Strecken zu messen, was eben erst durch den zu beweisenden Satz wird, ist die Anwendung obigen Grössensatzes durch nichts gerechtfertigt."

Schur subscribed to the widespread view that the validity of the "general principles" of magnitudes was only immediately evident in the case of straight line segments, for the relation of congruence (and betweenness) turned out to be adequate for their equality, comparison, and addition. But a natural consequence of this view was that to compare polygonal areas, one only needed to establish a *correspondence* between the set of plane polygons and the linearly ordered set of straight line segments, that is, to "measure" polygons by means of segments. Thus, Schur tacitly shifted the question of the possibility of comparing plane areas to the introduction of a *measure of area*. More importantly, Schur also explicitly stressed that the notion of measure of area of a plane polygon must be introduced in an elementary and purely geometrical way. The appeal to infinite processes such as the passage to limits, which constituted the kernel of the well-known method of exhaustion, was not in accordance with the requirement of the "purity of the method."

Now, as we have seen in the previous Sect. 4.1, Stolz's purported method to compare any pair of polygonal figures consisted in transforming them into equivalent (viz. equidecomposable) rectangles with a given altitude, and then to perform the comparison by "superposing" the latter figures. More precisely, these rectangles were to be obtained by the following procedure: first, one decomposed the given polygon into triangles; second, these triangles were transformed, one by one, into equivalent and *adjacent* rectangles with the same given altitude. Nevertheless, Schur noted that the whole method of transformation of areas pended on a *crucial geometrical fact*. Consider two different triangulations of a polygon **P**. By applying the above method, one obtains two rectangles **R₁** and **R₂** equivalent to **P** and with the same given altitude. Clearly, **R₁** and **R₂** must be equivalent to one another. But is it also immediately evident that these rectangles must be *congruent* as well, that is, that their bases must coincide? According to Schur, this conclusion could only be reached by the tacit assumption of the general principle of magnitudes "the whole is greater than the part":

> However, here one passed over silently the question whether this rectangle is also uniquely determined, whether another rectangle could be obtained by another decomposition of the figure into triangles –which is the starting point. This silence can only be explained so far as the supposition, that a rectangle could be equal in area to one of its parts, is considered to be impossible readily by the general principle of magnitudes that the part cannot be equal to the whole. [37] (Schur 1892, pp. 4–5)

Strikingly, Schur did not make here any allusion to De Zolt's postulate in connection to this implicit and problematic use of Euclid's CN5 in the theory of plane area. This might suggest that he was not yet completely aware of the intense debate on this topic, which was taking place at the Italian mathematical community. Moreover, Schur demanded a proof of the central principle "the whole is greater than the part," by arguing that its application to polygonal areas was not entirely self-evident and

---

[37] "Doch ist man hierbei über die Frage mit Stillschweigen hinweggegangen, ob *dies Rechteck auch eindeutig bestimmt sei, ob nicht bei einer andern Eintheilung der Figur in Dreiecke — das ist ja der Ausgangspunkt — ein anderes Rechteck erhalten wird*. Es kann dies Stillschweigen nur so erklärt werden, dass die Annahme, ein Rechteck könne einem seiner Theile flächengleich sein, ohne Weiteres als durch den allgemeinen Grössensatz, ausgeschlossen betrachtet wird, der Theil könne dem ganzen nicht gleich sein."

beyond all doubt. Additionally, he explicitly raised some methodological constraints in relation to this proof:

> But it is clear that in the case of the above precise definition of equality this principle is by no means totally self-evident, and a first attempt to prove it leads to a procedure of exhaustion which, besides the question of the application of infinite processes, does not even seem to deliver the desired result. And yet here too, with very simple means and without a postulate, we can achieve full rigor within the scope of the given definition of equality of area. [38] (Schur 1892, pp. 4–5)

There is a relevant conceptual point to make here. By identifying this fundamental "gap" in the method of transformation of areas, Schur raised two different issues, without distinguishing them explicitly. The first concerned the already discussed problem that, if a polygon could be equivalent (viz. equidecomposable) to a proper part, then polygons would not be comparable with respect to their areas; in other words, it related to the fact that a *geometrical version* of the general principle "the whole is greater than the part" was a necessary condition for the validity of the standard trichotomy law. The second problem alluded to the (schematic) introduction of a notion of measure of area. Schur's novel insight was to conceive the constructed equivalent rectangle with a given altitude as the *measure of area* of the given polygon. However, showing that this rectangle must be *uniquely determined* by the polygon amounted to proving that this notion of measure of area was *well defined*, namely that it was independent of the triangulation of the polygon used to calculate its measure of area. Evidently, these were two different problems that should be distinguished.

In the remainder of the note, Schur attempted to prove the latter fact. His geometrical argument was very sketchy and could hardly be considered a rigorous proof. Nevertheless, his general proof strategy had a significant impact on future endeavors to prove De Zolt's postulate by *purely geometrical means*. As mentioned, Schur put forward the following notion of measure of area of a plane polygon:

> We can now consider this rectangle, which is uniquely assigned to a polygon, as the representative of its area, whereby one side of all these rectangles is given once and for all. (Schur 1892, p. 5)

For the decomposition of a polygon into triangles, Schur employed a method of triangulation developed by Möbius (1886). It consisted in choosing arbitrarily one point inside or on the perimeter of the polygon as a common vertex for all triangles, and the sides of the polygon as their bases. Schur argued then that the fact that this notion of measure of area of polygons is well defined follows immediately from two fundamental properties, namely that *i*) congruent triangles have equal measures of area, and that *ii*) this function of measures of area satisfies the *additive* property. The

---

[38] "Nun ist aber klar, dass bei der obigen scharfen Definition der Gleichheit dieser Satz sich keineswegs so ganz von selbst versteht, und ein erster Versuch des Beweises leitet auf ein Exhaustionsverfahren, das von dem dabei angewandten endlosen Processe abgesehen nicht einmal zum Ziele zu führen scheint. Und doch lässt sich auch hier mit ganz einfachen Mitteln und ohne ein Postulat volle Strenge innerhalb des Rahmens der gegebenen Definition von Flächengleichheit erreichen."

first property was considered trivially evident, so no proof was offered. However, Schur did not manage to prove the additive property, but only made some confusing and merely tentative remarks about how this result could be obtained. As we shall see in Sect. 6.2, this lack of precision was not a coincidence at all, since proving the validity of the additive property constitutes a central challenge in the development of the *elementary* theory of measure of area of polygons.

Some of the gaps in Schur's original "proof" were filled by the Italian mathematician Giovanni Biasi, in an article published two years later in the highly regarded *Periodico di matematica*. According to the author, the short note aimed to provide some important details and clarifications of the alleged proof, *personally communicated* by Schur.[39] As was to be expected, these clarifications concerned mainly the introduction of the notion of measure of area of a polygon, which was now made more precise by resorting to the theory of proportion and similarity. In this regard, Biasi (or Schur) found that the following theorem proved to be particularly useful: if the sides of a rectangle are the extremes of a proportion, and the sides of another rectangle are the middle terms, then the two rectangles are equivalent (i.e., equidecomposable). This theorem suggested that the measure of area of a triangle could be defined as the *rectangle*, which has one side equal to the unit segment and the other equal to the fourth proportional to the unit segment, one side of the triangle and half of the corresponding altitude. It also followed immediately from the latter theorem that this notion was well defined, i.e., that the measure of area of a triangle is independent of the side chosen as the base.

Biasi proposed then to define the measure of area of a polygon as the sum of the measure of areas of triangles which have as bases the sides of the polygon, and as common vertex any point on its plane.[40] The fact that this alternative notion was well defined became now a "fundamental theorem":

Theorem *The algebraic sum of the [measure of] area of the triangles, which have as bases the sides of a polygon and as common vertex a point of its plane, is independent of choice of this point.* (Biasi 1894b, p. 86)

Naturally, the proof of this fundamental theorem consisted in showing that the *additive property* was valid. Biasi sketched an argument only for the case of triangles, based on the method developed by Möbius (1886). A central aspect of this method was that the measure of area of a triangle was endowed with a *sign*, depending on whether the figure was considered in its positive or negative orientation. The proof of the general case could be easily obtained, according to Schur, by repeating the argument for the particular case of triangles.

To sum up, Schur outlined a proof of the comparability of plane polygonal figures by *purely geometrical means*, which consisted in constructing an application from the set of plane polygons to the (linearly ordered) set of straight line segments employing the theory of proportion and similarity. An immediate consequence of this correspondence was that the general principle "the whole is greater than the part" was valid for the case of polygonal areas, although Schur did not make any explicit allusion to De Zolt's postulate. The critical observations made by Schur were accepted by Stolz in a

---

[39] Cf. Biasi (1894a, 1894b).

[40] Cf. Biasi (1894b, pp. 86–87).

later work, where the latter also presented a new proof of the comparability of plane polygons based on the admission of the latter postulate as a *new geometrical axiom*.[41]

Finally, we shall conclude this section with a brief mention of Killing's work. We have seen that Killing seemed to have independently discovered and formulated a version of De Zolt's postulate. But in the second volume of his geometrical treatise *Einführung in die Grundlagen der Geometrie* (1898), he also provided a very detailed proof of this central proposition. Briefly, the general idea of the proof consisted in deriving the geometrical postulate from the fundamental properties of the functions of measure of area of plane polygons, which, however, Killing introduced analytically using *definite integrals*.[42] Thus, he resorted to the standard analytic method of integration and to *infinite processes*, such as the passage to the limit. Although notably rigorous, Killing's proof violated the "purity of the method" requirement laid down by Schur. In his forthcoming investigations, Hilbert will repeatedly emphasize the non-elementary character of this proof and its dependence on continuity assumptions, especially the Archimedean axiom.

5 Hilbert's notes for lecture courses in 1898/1899

In the previous Sects. 3 and 4, we have offered a general picture of the debate on the foundations of the theory of plane area, which took place during the second half of the nineteenth century. These discussions posed important foundational, methodological, and epistemological problems regarding the adequate development of this central part of elementary geometry. These problems were not always stated in a clear and precise way. To a significant extent, the modern axiomatic treatment of the theory of area, especially in Hilbert's works, will contribute to putting some of these problems and claims on a solid footing. In this section, we shall analyze Hilbert's early reception of this debate, as documented in his notes for lecture courses on the foundations of geometry, particularly those immediately prior to the publication of *Foundations*.

Hilbert's first "axiomatic" discussion of problems related to the foundations of the theory of plane area took place in a summer course entitled *Über den Begriff des Unendlichen* (Hilbert 1898), held in the Easter break of 1898. The course was targeted to *Oberlehrer* and aimed at presenting new views of some classical problems and results in "elementary mathematics" that, according to Hilbert, should be part of the mathematical curricula in secondary schools.[43] One of these results concerned the definition of the geometrical operation of segment multiplication. Hilbert claimed, without proof, that if segment multiplication was defined by the standard construction of the fourth proportional (*Elements*, VI.12), then the classical "Theorem of Pascal" (better known as Pappus' theorem) could be used to show that this operation satisfies relevant algebraic properties, such as commutativity and associativity. More precisely,

---

[41] Cf. Stolz (1894).

[42] Cf. Killing (1898, pp. 22–33). For a modern presentation of this proof, see Boltianskii (1978).

[43] For an overview of the content of this course, see the introduction to chapter 3 of Hallett and Majer (2004).

Hilbert referred to a *special case* of Pascal's (or Pappus') theorem on conic section, according to the following *affine version*:

Pascal's theorem (affine version) *Let A, B, C, and A', B', C', be two sets of points on two intersecting lines that are distinct from the point of intersection of the lines. If CB' is parallel to BC' and CA' is parallel to AC', then BA' is parallel to AB'.*[44]

As is well known, the unveiling of deep connections between Pascal's theorem and the algebraic properties of segment multiplication was an original result in *Foundations*; nevertheless, in this course Hilbert raised the question whether this fruitful theorem in the context of projective geometry could be used now to obtain a proof of De Zolt's postulate, which he called here the "Killing-Stolz postulate":

> Thus, it all comes down to the theorem: two equivalent rectangles with an equal side must also have the other side equal or the Killing-Stolz postulate: However, one decomposes a rectangle into n triangles, after removing one of them, one can never cover the rectangle with the remaining $n - 1$ triangles. (So the content is independent of the arrangement of the parts) […]
> *Does the Killing-Stolz postulate follow from [the theorem of] Pascal?* Is the Archimedean axiom a consequence of the Killing-Stolz postulate? [45] (Hilbert 1898, p. 176. My emphasis.)

These two questions contain *in nuce* the general guidelines that Hilbert will follow shortly after in his axiomatic reconstruction of the theory of plane area. On the one hand, Hilbert set himself the goal of exploring the possibility of using a segment arithmetic based on Pascal's theorem to provide a rigorous (and strictly geometrical) proof of De Zolt's Postulate. On the other hand, he asked whether such proof could be carried out without assuming the Archimedean axiom. Both questions will receive a precise answer in *Foundations*. Finally, these lecture notes show that Hilbert's initial engagement was clearly influenced by the works of Stolz (1885, 1894) and Killing (1898).

In the winter semester of 1898/1899, Hilbert offered a new lecture course on the foundations of Euclidean geometry. This lecture course constituted the basis for the first edition of *Foundations*.[46] In these notes, Hilbert pointed out more expressly that the core issue in the development of the theory of equivalence was to guarantee the *existence of a relation of (total) ordering for plane polygons*, and that this was

---

[44] Cf. Hilbert (1898, p. 171). Hilbert did not explicitly provide this formulation in his lecture course, but introduced several diagrams that unequivocally suggest this affine version of Pascal's theorem on conic sections. For the corresponding formulation in *Foundations*, see Hilbert (1971, p. 46).

[45] "Also es kommt auf den Satz an: 2 gleiche Rechtecke mit einer gleichen Seite müssen auch die andere gleich haben oder das Killing-Stolzsche Postulat: Wie man auch ein Rechteck in n Dreicke zerlege, man kann nach Fortnahme eines nie durch die $n - 1$ übrigen das Recheck bedecken. (Also Inhalt von Anordnung der Teile unabhängig.) […]
Folgt das Killing-Stolzsche Postulat aus Pascal? Ist das Archimedische Axiom eine Folge des Killing-Stolzschen Postulates?".

[46] There are two existing notes corresponding to this lecture course, namely *Grundlagen der Euklidischen Geometrie* (Hilbert 1898/1899a) and *Elemente der Euklidischen Geometrie* (Hilbert 1898/1899b). For details, see the introduction to chapter 4 of Hallett and Majer (2004).

precisely the fundamental role of De Zolt's postulate (or Killing-Stolz postulate) in this geometrical theory:

> Indeed everything is correct, but all claims are empty and meaningless, as long as it has not been shown that, first, there are polygons of different areas and, further, that if two rectangles have one side equal and the other different, they are not equal in area. […] This is the proof of the theorem established by Killing.[47] (Hilbert 1898/1899a, p. 279)

Moreover, Hilbert also explained why De Zolt's postulate could *not* be assumed as an axiom in any rigorous axiomatic treatment of the theory of plane area:

> If two triangles with equal bases have equal content, then they also have equal altitudes. Are there at all triangles, which are not equivalent? *Totum parte majus est* is applicable? Not *apriori*, of course, *for this general principle of magnitudes is converted into a geometrical theorem as soon as it is applied to our geometrical concepts*. Stolz believes that this proposition must be either taken as an axiom, and Killing proves it with the help of the Archimedean axiom. Both fail to meet the central point, namely that the theorem is provable without Archimedes.[48] (Hilbert 1898/1899a, p. 279. My emphasis)

Hilbert subscribed here to the general dictum that "in mathematics nothing capable of proof ought to be believed without proof".[49] Admitting De Zolt's postulate as a new geometrical axiom was a clear violation of this critical principle. More interestingly, he also outlined an original view regarding the selection of axioms in modern axiomatic geometry. Specifically, this epistemological conception was related to the understanding of De Zolt's postulate as the precise "geometrical interpretation" of Euclid's general principle of magnitudes "the whole is greater than the part." According to this identification, what the former proposition actually stated was not just another geometrical fact about the equivalence of plane figures, but a crucial *property of polygonal magnitudes*. However, a successful axiomatization of geometry demanded that one must be able to *prove* from the axioms of geometry that polygonal areas satisfy all the relevant properties of magnitudes. Therefore, De Zolt's postulate could not be simply assumed as an axiom, but had to be derived as a geometrical theorem. Put differently, the admission of De Zolt's postulate as a new axiom relied on the fundamental assumption that plane polygons satisfy all the properties of magnitudes, a presupposition that must be adequately justified in any modern axiomatization of geometry. Finally, Hilbert also

---

[47] "Zwar Alles richtig, aber sämtliche Behauptungen sind leer und bedeutungslos, so lange nicht vor Allem gezeigt ist, dass es Polygone verschiedenen Inhaltes giebt und ferner, dass wenn 2 Rechtecke mit gleicher einer und verschiedener anderer Seite nicht inhaltsgleich sind. […] Es handelt sich um den Beweis eines von Killing aufgestellte Satzes."

[48] "Wenn 2 Dreiecke mit gleicher Grundlinie gleichen Inhalt haben, so haben sie auch gleiche Höhe. Giebt es überhaupt Dreiecke, die nicht inhaltsgleich sind? *Totum parte majus est* ist [here] anwendbar? Apriori natürlich nicht, da eben dieser allgemeine Grössensatz sich in einen geometrischen Satz verwandelt, sobald er auf unsere geometrischen Begriffe angewandt wird. Stolz glaubt den Satz entweder als Axiom nehmen zu müssen, und Killing beweist ihn mit Hülfe des Archimedischen Axioms. Beides trifft nicht das Wesentliche, in dem der Satz ohne Archimedisches beweisbar ist."

[49] Dedekind (1888, p. 790).

required that the proof of De Zolt's postulate should be carried out *without* assuming the Archimedean axiom.

One last remark to conclude this section. Although Schur (1892) was mentioned in the bibliography, his contributions were not explicitly acknowledged in either versions of the 1898/1899 lecture course. This might be seen as a significant lack of consideration on Hilbert's part, concerning the works which substantially influenced his axiomatic investigations in *Foundations*.[50]

6 The theory of plane area in *Foundations of geometry*

The axiomatic construction of the theory of plane area is carried out in Chapter IV of *Foundations*. Hilbert stresses that in these investigations only the line and plane axioms of incidence, betweenness, and congruence are assumed; thus, no continuity axioms—especially the Archimedean axiom—are employed. The key idea of this new development of the theory of plane area is summarized as follows:

> The theory of proportion discussed in Chapter III and the segment arithmetic introduced there make it possible to develop Euclid's theory of area with the aid of aforementioned axioms, i.e., to develop it *in the plane independently of the axiom of continuity*.
>
> Since by the development in Chapter III the theory of proportion rests essentially on Pascal's Theorem (Theorem 40) the same is true of the theory of area. This development of the theory of area appears as one of the most remarkable applications of Pascal's Theorem in elementary geometry. (Hilbert 1971, p. 61)

The removal of the dependence on the axiom of Archimedes in the construction of the theory of plane area was an original result in *Foundations*, and a fundamental task in the project of providing a new *independent foundation* for this central part of elementary geometry. It should be noted that Hilbert accomplished this primary objective by means of several technical innovations. First, he put forward a new criterion of geometrical equivalence of polygonal figures, namely the relation of equicomplementability, which allowed to circumvent the admission of continuity conditions. In addition, he proved that the notion of equidecomposition and equicomplementability were equivalent only in the presence of the Archimedean axiom. Second, Hilbert resorted to his previous construction of a segment arithmetic based on Pascal's theorem and to the theory of proportion, based on the former, to introduce a notion of *measure of area* of a plane polygon. This geometrical definition of a measure of area—i.e., an *associated segment*—not only solved the problem of the *strictly geometrical* ordering of polygonal areas, but also yielded a rigorous proof of De Zolt's postulate which did not assume the Archimedean axiom. Hilbert succeeded thus in showing that a solid axiomatic foundation for the theory of plane area is possible independently of continuity assumptions, and therefore, of the concept of real number.

---

[50] Schur was deeply disappointed about the fact that, in his opinion, his contributions to the theory of plane area were not dully recognized by Hilbert in the *Festschirft*. He expressed this disappointment in a letter to Hilbert dated January 5, 1900. This letter has been partially published in Toepell (1985).

In this section we discuss in detail Hilbert's development of theory of area in *Foundations*. Section 6.1 analyzes a series of technical and conceptual clarifications in relation to the central concepts of the theory of equivalence. Section 6.2 focuses on the construction of a theory of area measure and the notable proof of De Zolt's postulate.

6.1 Equivalence, decomposition, and addition of polygons

Hilbert began his exposition of the theory of area by providing precise definitions of the concepts of polygon, decomposition, and addition of polygons. This constituted a remarkable improvement of rigor, for these notions were usually described in a very informal or intuitive way. In particular, Hilbert's analysis and treatment of the notions of decomposition and addition of polygons introduced significant conceptual improvements in the development of the theory.

The definition of *polygon* given by Hilbert corresponds to a definition originally advanced by Poinsot (1810), according to which a polygon consists of a cyclically ordered sequence of points (vertices) together with the segments determined by vertices adjacent in the cyclic sequence. Thus, this characterization of a polygon is based on the idea of a *closed* polygonal segment or broken line:

Definition 1 A set of segments AB, BC, CD, ..., KL is called a *polygonal segment* that connects the points A and L. Such a segment will also be briefly denoted by $ABCD...KL$. The points inside the segments AB, BC, CD, ..., KL as well as the points A, B, C, D, E, ..., K, L are collectively called the *points of the polygonal segment*. If the points A, B, C, D, E, ..., K, L all lie in a plane and the point A coincides with the point L, then the polygonal segment is called a *polygon* and is denoted as the polygon $ABCD...K$. The segments AB, BC, CD,
..., KA are also called the *sides of the polygon*. The points A, B, C, D, ..., K are called the *vertices of the polygon*. Polygons of 3, 4, …, *n vertices* are called *triangles, quadrilateral, …, n-gons*. (Hilbert 1971, pp. 8–9)

It is worth noting that, while this definition stipulates that all vertices of a polygon must lie in one plane[51], it remains silent on whether all vertices *needs to be distinct*. Nor does this definition make explicit that no two intermediate segments must be collinear, that is, that no three consecutive intermediate vertices must lie on the same line. However, this does not pose a problem for the development of the theory, for Hilbert immediately restricted his study to polygons which have distinct points as vertices.[52]

Hilbert then restricted his theory of polygonal area to the particular case of *simple polygons*, which he defined as follows:

---

[51] If all the vertices of a polygon lie in one plane, then we speak of a *plane polygon*. If the polygon vertices are not all coplanar, then the polygon is said to be *skew*. For a study of skew polygons, see Gerretsen and Verdenduin (1983).

[52] A definition of plane polygons which does not demand that all their vertices are distinct can be found in Meister (1771). For analysis of the consequences of adopting this definition, see Grünbaum (2012). In his classical *Proof and Refutations*, Imre Lakatos (1976) presents a rich philosophical discussion around these two confronting definitions of polygons.

Fig. 7 Non-juxtaposable polygons
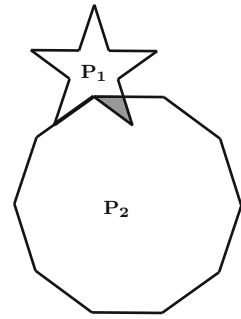
Definition 2 If the vertices of a polygon are all distinct, none of them falls on a side and no two of its nonadjacent sides have a point in common, the polygon is called *simple*. (Hilbert 1971, p. 9)

An important technical innovation was introduced by Hilbert in his definition of decomposition and addition of simple polygons:

Definition 3 If two points of a simple polygon **P** are joined by some polygonal segment that lies entirely in the interior of the polygon and which has no double point, two new simple polygons **P₁** and **P₂** are formed whose interior points lie in the interior of **P**. **P** is then said to **decompose** into **P₁** and **P₂** or **P** is **decomposed into P₁ and P₂** or **P₁** and **P₂** compose **P** [*setzen* **P** *zusammen*]. (Hilbert 1971, p. 60)

In a strict sense, this definition stipulates that a polygon can be decomposed by a polygonal segment into *two* other polygons. For a more precise formulation, one needs to incorporate a *recursive* definition of decomposition of a polygon into several polygons. More interestingly, Hilbert introduced a conceptual clarification regarding the operation of composition or *addition* of two polygons, by characterizing it by means of the concept of decomposition.

The precise definition and the adequate treatment of the operation of addition of *simple* polygons were a central challenge in the modern development of the theory of equivalence. The usual standpoint in nineteenth-century geometry treatises was to characterize this notion informally as the *juxtaposition* or the *nonoverlapping union* of two polygons at a common edge. However, this definition runs into important difficulties, for it is not the case that *any* two polygons can always be juxtaposed. Consider, for example, a regular star pentagon and a regular decagon with sides equal or greater to the distance of two consecutive vertices of the pentagon (Fig. 7)[53]. These two polygons cannot have two points in common at their edges, without also having common points in their *interiors*; therefore, they cannot be *directly* "added."

Hilbert's original solution to this problem consisted in introducing the notion of addition by means of the equality $\mathbf{P} = \mathbf{P_1} + \mathbf{P_2}$, instead of simply defining the operation $\mathbf{P_1} + \mathbf{P_2}$ for any two polygons. But this amounted to imposing a *key restriction* on the operation: Hilbert's definition of addition presupposed the *existence* of the sum polygon, which means that one must first prove or admit the existence of a polygon $\mathbf{P_1} + \mathbf{P_2}$,
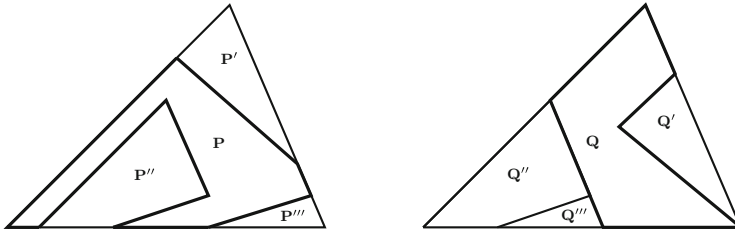
---

[53] Cf. Puig Adam (1980)

Fig. 8 Equicomplementable polygons. (Hilbert 1971, p. 60)

before one can reason about that sum. Stated differently, by adopting Hilbert's definition, the operation of addition was restricted to the case of "compatible" polygons, that is, to polygons that always have *segments* as common boundaries. The operation of addition consisted then in removing the common segment, not necessarily a side of a polygon.[54]

As we have mentioned, Hilbert distinguished then between two different notions or criteria of geometrical equivalence, namely the usual relation of "equidecomposition" [*zerlegungsgleichheit*] and the novel relation of "equicomplementability" [*ergänzungsgleichheit*]:[55]

Definition 4 Two simple polygons are called *equidecomposable* if they can be decomposed into a finite number of triangles that are congruent in pairs.

Definition 5 Two simple polygons **P** and **Q** are called *equicomplementable* if it is possible to adjoin to them a finite number of pairs of equidecomposable polygons **P**′, **Q**′; **P**″, **Q**″; . . . ; **P**‴, **Q**‴ such that the composed polygons **P** + **P**′ + **P**″ + . . . + **P**‴ and **Q** + **Q**′ + **Q**″ + . . . + **Q**‴ are equidecomposable with each other. (Hilbert 1971, p. 60. Figure 8)

Up to the time of the appearance of Hilbert's *Foundations*, all *modern* reconstructions of the geometrical theory of equivalence were exclusively based on the notion of *equidecomposition*. To a significant extent, this was related to a methodological requirement of "purity," first introduced by Duhamel (1866), which consisted in demanding that the equivalence of two plane polygonal figures were to be established by means of a *unique criterion*. Recall that in the case of the relation of equidecomposition, the equality of area was established by applying only the property originally expressed in Euclid's CN2. In turn, with his notion of equicomplementability, Hilbert incorporated a second criterion of equality of area, grounded now also on

---

[54] Intuitively, a polygon **P** is *compatible* with a polygon **Q** if and only if, for some polygon **R**, we have that **P** + **Q** = **R**. For a discussion of the notion of "compatible" (geometrical) magnitudes in an abstract setting, and in connection to De Zolt's postulate, see Giovannini, Haeusler et al. (2019).

[55] The expression "equivalence by decomposition" and "equivalence by complementation" are alternative translations for the terms "zerlegungsgleichheit" and "ergänzungsgleichheit," respectively. Hilbert introduced the term "equidecomposition" [*Zerlegungsgleichheit*] in the second German edition of *Foundations*, in 1903; in turn, in the first edition he used the term "equality of area" [*Flächengleichheit*]. On the other hand, the expression "equicomplementability" [*Ergänzungsgleichheit*] occurred for the first time in the seventh edition, in 1930. In all previous editions, Hilbert employed the term "equality of content" [*Inhaltsgleichheit*].

CN3, for two equicomplementable polygons results from subtracting pairs of, respectively, congruent polygons (viz. triangles) to a pair of equidecomposable polygons. It is worth noting that Hilbert explicitly suggested that, by incorporating the notion of equicomplementability, he was actually retrieving a central element of Euclid's classical geometrical practice:

> If we proceed with these definitions to consider the theorems in elementary geometry about the equality of area and the related construction problems, we find that it is always here a matter of the equicomplementability of the figures. The theorems, for example, that two parallelograms and also two triangles with the same base and height are equal to each other, that for every polygon one can determine a triangle of equal area, as well as the Pythagorean theorem, are all proved in the sense that the *equicomplementability* of the polygons in question is recognized. The derivation of all these theorems is done entirely without the use of continuity considerations.[56] (Hilbert 1917, pp. 97–98)

Hilbert's next task was to *prove* that these relations of equidecomposition and equicomplementability satisfy the basic properties of the equivalence, comparison, and addition of magnitudes. As we have seen in Sect. 5, this was one of the most fundamental requirements in the modern reconstruction of the theory of equivalence. In his lecture notes, Hilbert stated this point very clearly regarding the transitive property of equidecomposition:

> Now we first prove the theorem: if two polygons are equidecomposable to a third, then they are equidecomposable to one another. (Euclid has this theorem too; but he proves it by invoking a general principle about magnitudes–a misconception we have already mentioned several times). (Hilbert 1898/1899b, p. 369)

Accordingly, Hilbert formulated the following two important properties of equivalence: 1) the combination of equidecomposable polygons results in equidecomposable polygons; and 2) if equidecomposable polygons are removed from equidecomposable polygons the remaining polygons are equicomplementable. These propositions correspond to the *additive* and *subtraction* properties of the relation of equidecomposition, respectively. Hilbert did not prove these properties, but considered them to be trivial corollaries of the definitions. It should be noted that Hilbert neither proved nor formulated the additive property of the relation of equicomplementability. This might respond to the fact that there are important difficulties with this proof, which are related to the definition of the operation of addition.

On the contrary, Hilbert formulated (a version of) the transitive property as Theorem 43:

---

[56] "Gehen wir mit diesen Begriffsbildungen an die Betrachtung der elementargeometrischen Sätze über Flächengleichheit und der damit zusammenhängenden Konstruktions-Aufgaben, so finden wir, dass es sich hier immer um die Ergänzungsgleichheit der Figuren handelt. Die Sätze z. B., dass zwei Parallelogramme und ebenso zwei Dreiecke mit gleicher Grundlinie und Höhe einander gleich sind, dass sich zu jedem Polygon ein Dreieck von gleicher Fläche bestimmen lässt, sowie auch der Pythagoräische Lehrsatz werden alle in dem Sinne bewiesen, dass die *Ergänzungsgleichheit* der betreffenden Polygone erkannt wird. Die Herleitung aller dieser Sätze geschieht vollkommen ohne Anwendung von Stetigkeits-Betrachtungen."
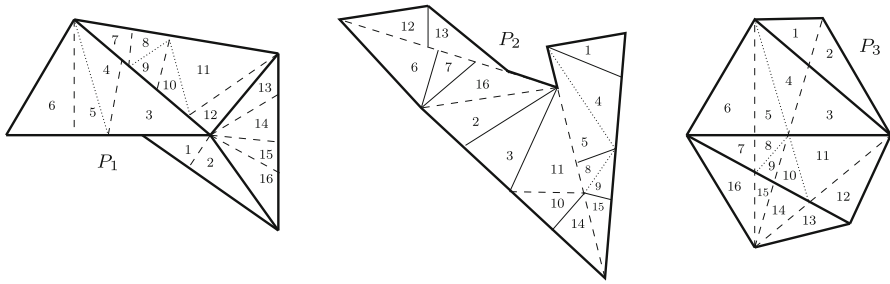
Fig. 9 Transitivity property of equidecomposition. Adapted from (Hilbert 1971, p. 60)

Theorem 43 *If two polygons* **P₁** *and* **P₂** *are equidecomposable with a third polygon* **P₃**, *then they are equidecomposable with each other. If two polygons are equicomplementable with a third one, then they are equicomplementable with each other.* (Hilbert 1971, p. 61)

Hilbert outlined a proof of the *first part* of this theorem, which corresponds to the relation of equidecomposition. The main idea was to consider simultaneously in **P₃** two nets of polygonal segments that decomposed **P₁** and **P₂**, respectively, into congruent triangles. The intertwining of these two nets in **P₃** decomposes the triangles that compose **P₁** and **P₂** into other polygons, which in turn can also be decomposed into triangles through diagonals. These triangles can then be rearranged in suitable ways so that they compose as partial sums the triangles of **P₁**, but also the triangles which form **P₂**. Hence, by definition, **P₁** and **P₂** are equidecomposable (Fig. 9).

Hilbert's proof of the transitivity of the relation of equidecomposition was thus grounded on the accompanying figure or diagram. A more rigorous proof would require a precise derivation of the mutual division into triangles of both nets of polygonal segments, as a consequence of the axioms of betweenness or Jordan's theorems for simple polygons.[57] However, this proof would be rather long and tedious. More importantly, Hilbert did not attempt to prove the *second part* of the theorem, dealing with the relation of equicomplementability, and limits himself to point out that "the second assertion of Theorem 43 follows now with no difficulty" (Hilbert 1971, p. 61). Contrary to Hilbert's opinion, this proof is problematic given the restriction imposed on the notion of addition of polygons. This might be a reason why he chose not to provide this proof.[58]

Hilbert proceeded then to examine the application of these notions of "equidecomposition" and "equicomplementability" to the basic theorems about the equality of area of plane figures. As already mentioned, the main concern was to establish whether,

---

[57] See Theorem 9 and Theorem 10 in (Hilbert 1971).

[58] In a Supplement to the tenth edition of *Foundations*, Bernays (1971) identified this problem with Hilbert's notions equicomplementability and addition of polygons, and provides a solution based on the more general notion of a *polygonal*, i.e., a collection of a finite number of triangles in the plane which satisfies the triangulation property. A similar strategy is applied by Hartshorne (2000), who gives an alternative proof of the transitivity property of "equicomplementability" using set-theoretic tools. On this issue, see also Robering (2016).

and under which conditions, these two notions were equivalent. Accordingly, Hilbert formulated the following theorems:

Theorem 44 *Two parallelograms with the same bases and with the same altitudes are equicomplementable with each other.*[59]

Theorem 45 *Every triangle ABC is equidecomposable with a parallelogram of an equal base and of half the altitude.*[60]

Theorem 46 *Two triangles with equal bases and altitudes are **equicomplementable**.*[61]

The key result here is the proof of the theorem that two parallelograms (or two triangles) with equal bases and altitudes are *equidecomposable*, that is, the corresponding version of Theorem 46 for the relation of equidecomposition. In Sect. 3.1, we have seen that the standard proof of this theorem, due by Duhamel (1866), was grounded on the Archimedean property of line segments (Sect. 3.1). Hilbert's original contribution consisted in providing a *proof of the impossibility* of proving this theorem without the admission of the axiom of Archimedes. More precisely, he showed that in every non-Archimedean geometry it is possible to specify triangles which equal bases and altitude which are equicomplementable, but which are not *equidecomposable*. Since this proof constitutes a central result in *Foundations* in relation to the theory of area, it is worthwhile to present Hilbert's geometrical argument briefly.

Let e be an infinite element and a a finite element of a non-Archimedean geometry.[62] On a ray of this non-Archimedean geometry construct the segments $AB = e$ and $AD = a$. Thus, there is no integer n such that $n \cdot e \geq a$. By Theorem 46, the triangles ABC and ABC' are *equicomplementable* (Fig. 10). Consider now the triangle ABC. Since in every triangle the sum of any two of its sides is greater than the third side[63], it follows that $BC < 2e$. Moreover, every segment lying in the interior of ABC is also less than $2e$. Assume now that there are decompositions of ABC and ABC' into a finite number k of triangles congruent in pairs, i.e., ABC and ABC' are equidecomposable. Every side of a partial triangle in the decomposition of ABC is less than $2e$. Therefore, the perimeter of this triangle is less than $6e$, and the sum of the perimeters of all these k triangles is less than $6k \cdot e$. From the supposition that ABC and ABC' are

---

[59] Hilbert (1971, p. 62).

[60] Hilbert (1971, p. 62).

[61] Hilbert (1971, p. 62).

[62] As an example of a non-Archimedean geometry, Hilbert refers to the model provided in §12 of *Foundations* (Hilbert 1971). Schematically, this "analytical model" consists of the set $\Omega(t)$ of all algebraic functions of one variable t obtained by finitely many applications of the operations of addition, subtraction, multiplication, division, and the fifth operation $\sqrt{1 + \omega^2}$, where ω denotes a function which is obtained by these five operations. A relation of ordering on the functions in $\Omega(t)$ is defined as follows: a is said greater than b if $a - b$ is always positive for a sufficiently large t. Likewise, a is said lesser than b if $a - b$ is always negative for a sufficiently large t. Clearly, in an analytic geometry constructed over the "complex number system" $\Omega(t)$, the Archimedean axiom is not valid. On the one hand, every constant function c in $\Omega(t)$ is lesser than the function $\omega(t) = t$. On the other hand, there is no natural number n such that the relation $n \cdot c > \omega(t)$ holds. For more details on this model, see Volkert (2015).

[63] Hilbert observes that the property of triangle inequality is an immediate consequence of his Theorem 23: "In every triangle the greater angle lies opposite to the greater side" (Hilbert 1971, p. 22).
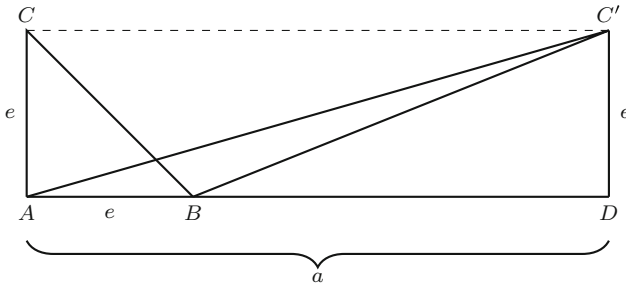
Fig. 10 Equidecomposition in non-Archimedean geometries (adapted from (Hilbert 1971, p. 63))

equidecomposable, it follows that the sums of the perimeters of the k triangles which decompose ABC' must also be less than $6k \cdot e$. But the side AC' is evidently a summand in the latter summation, that is, $AC' < 6k \cdot e$. Then, since $a < AC'$ (by Theorem 23), it results that $a < 6k \cdot e$. This contradicts the initial hypothesis about the relation between the segments e and a. Hence, the triangles ABC and ABC' are not equidecomposable. QED.

Hilbert proved thus that the relations of equidecomposition and equicomplementability are equivalent only if the axiom of Archimedes is assumed. This follows from the fact that the Wallace–Bolyai–Gerwien theorem can only be proved with the aid of the Archimedean axiom. Indeed, in the proof just given, the triangles ABC and ABC' have the same measure of area (since they have a common base and equal altitudes), but *are not equidecomposable*. Consequently, the construction of the theory of plane area independently of Archimedean axiom is only possible on the basis of the relation of equicomplementability.

From the seventh edition of *Foundations*, published in 1930, Hilbert also included the following theorem about the equicomplementability of polygons:

Theorem 47 *For every triangle and hence for every simple polygon it is always possible to construct a right-angled triangle, one of whose legs is* 1 *and which is equicomplementable with the triangle or polygon.*[64]

It is striking that Hilbert did not include this theorem in previous editions of *Foundations*, since this result had played a key role in the historical development of the theory of equivalence. In particular, the standard method to compare polygonal figures, as implemented, for instance, by Stolz (1885) and Schur (1892), was essentially grounded on this theorem. We surmise that this significant omission was related to two main reasons. First, Hilbert appealed to a geometrical notion of measure of area to compare any two polygonal figures; nevertheless, to introduce measures of area he resorted to the arithmetic of segments, and not to the usual method of transformation of polygons into equivalent parallelograms with a given base. Hence, this theorem did not play such a significant role in his reconstruction of the theory of equivalence. Second, the problems we have mentioned concerning the proofs of the main properties (viz. additivity and transitivity) of the relation equicomplementability made the proof

---

[64] Hilbert (1971, p. 63). We have introduced some minor modifications in this translation.

of Theorem 47 particularly challenging. In fact, Hilbert did not provide a detailed proof of this theorem in *Foundations*.

Hilbert then reached the central problem in the development of the theory of equivalence, namely to prove that plane polygons can be *totally* ordered based on the relation of equicomplementability or, as he alternatively put it, to show that "not *all* polygons are equicomplementable." As we have seen, this boiled down to prove the proposition I.39 of Euclid's *Elements*, which he reformulated in terms of the relation of equicomplementability:

Theorem 48 *If two equicomplementable triangles have the same bases, then they also have the same altitudes.*[65]

Hilbert added the following observation about this theorem:

This fundamental theorem is found in the first book of Euclid's *Elements* as Theorem 39. In the proof Euclid appeals to the general theorem of magnitudes "Καὶ τὸ ὅλου τοῦ μέρουε μεῖζόν ἐστω" (The whole is greater than any of its parts), a method that is equivalent to the introduction of a new geometric axiom of equicomplementability.
However, it is possible to establish Theorem 48 and also the theory of area in the manner proposed, i.e., with the aid of the plane axioms alone and without the use of the Archimedean axiom. In order to see this, *one needs the concept of measure of area* [*Inhaltmaßes*]. (Hilbert 1971, p. 64. My emphasis)

Thus, Hilbert appealed to the introduction of a measure of area of plane polygons to prove De Zolt's postulate and the key Theorem 48. This means that the problem of the total ordering of polygonal areas was solved by developing an (elementary) theory of measure of area, or more precisely, by proving that there was a *correspondence* between the concept(s) of geometrical equivalence (viz. equicomplementability) and the concept of measure of area. It is worth noting then that Hilbert's development of the theory of geometrical equivalence differed from Euclid's theory in (at least) one crucial respect, for the comparability of polygonal areas was grounded ultimately on a geometrical notion of measure of area. In the next section, we focus on Hilbert's careful construction of a theory of area measure, which resulted in his novel and rigorous proof of De Zolt's postulate.

6.2 The proof of De Zolt's postulate

Hilbert's proof of De Zolt's postulate in *Foundations* constituted a landmark in the modern theory of area, in the sense that it became almost immediately the "canonical" or "standard" proof of the geometrical postulate. This might be credited not only to the notable success of the epochal monograph, but also to the very character or structure of the proof. Since the proof did not depend on any continuity assumption—specifically, the Archimedean axiom—but made an essential appeal to a *purely geometrical* notion of measure of area, it could well be considered as *elementary*. However, this elementary

---

[65] Hilbert (1971, p. 64).

character did not prevent that Hilbert's proof could be still regarded as complex and involved, at least for an exposition of the theory of equivalence in elementary geometry textbooks.[66]

It is worth noting that the proof of De Zolt's postulate underwent significant changes over the several editions of *Foundations*. In particular, the seventh edition of 1930 introduced completely reworked proofs of a pair of auxiliary theorems, which constitute the core of the proof of the central geometrical postulate. Notwithstanding, these changes did not modify the general idea or strategy of the geometrical argument. As mentioned earlier, the central idea of Hilbert's elementary proof consisted in deriving De Zolt's postulate as an immediate consequence of the existence of a function of area measure of plane polygons, introduced in a purely geometrical fashion. Hilbert's functions of measure of area did not take numerical values (i.e., positive real numbers) as usual; in other words, they did not rest on the possibility of measuring the length of line segments by means of real numbers. On the contrary, Hilbert defined the measure of area of a plane polygon as a *characteristic segment* or, more precisely, as an element of the ordered field generated by his arithmetic of segments or *Streckenrechnung*, in German. The main strategy of the proof was to obtain, by means of strictly geometrical arguments, the standard properties of area measures and to prove that equivalent polygons (viz. equicomplementable polygons) have equal measures of area. De Zolt's postulate was then a corollary of the latter geometrical fact.

Hilbert's axiomatic construction of a theory of measure of area was thus grounded on the arithmetic of segments and the theory of proportion and similar triangles, based on the former, developed in Chapter III of *Foundations*. These important geometrical results are very well known, but let us recall them briefly.[67] Hilbert's construction of a segment arithmetic consisted in defining pure geometrically the operations of addition and multiplication of segments and then in proving that these operations satisfy the relevant algebraic properties. In particular, segment multiplication was defined by appealing to the standard geometric construction of the fourth proportional (*Elements*, VI.12), which Descartes had used for the first time to define the product of two line segments as another segment. This definition of segment multiplication required fixing a *unit segment* and the validity of the parallel axiom. Hilbert's key realization was that the classical theorems of Desargues and Pascal could be used to prove that these operations satisfy all the properties of an *ordered field*.[68] In particular, he showed that

---

[66] This observation was made, for example, by Tarski in his work "On the equivalence of polygons" (1924):

As is well known, David Hilbert showed that the preceding statement [i.e., De Zolt's postulate] can be proved with the help of axioms usually cited in elementary geometry textbooks. Because of the difficulty of that proof, however, one does not make use of it in a secondary-school class. (Tarski 1924, p. 79)

[67] The development of a geometrical calculus of segments is often mentioned as one of the most important contributions of Hilbert's early axiomatic work and has been studied at length in the literature. For some recent studies, see, for example, Hallett (2008); Giovannini (2016), and Baldwin (2018a).

[68] The importance of the theorems of Desargues and Pascal (or, better, Pappus) in the context of projective geometry was well known by the last decade of the nineteenth century. Particularly, Wiener (1893) and Schur (1898) showed that these theorems were essential to von Staudt's method to introduce coordinates into projective geometry—i.e., his famous *Wurfrechnung*—, for they could be used to prove the fundamental theorem of projective geometry *without assuming any continuity axiom*. Hilbert's novel insight was to explore the significance of these theorems in the context of Euclidean geometry. For an excellent and

while the former theorem was essential to prove the associative law under multiplication, the latter warranted the commutative property of the same operation. In modern terminology, Hilbert proved that while any plane where Desargues's theorem holds can be coordinatized by a division ring or skew field, Pascal's theorem guarantees that the plane can be coordinatized by an ordered field. Hilbert accomplished then a "purely geometrical" or "internal" introduction of number into geometry, in the sense that the *coordinates* were now elements of the field generated by his segment arithmetic. As is well known, the adequate introduction of number into geometry was an overwhelming aim in *Foundations*[69]:

> But, lest science should fall prey to an *unfruitful formalism*, it will have to *reflect on itself* in a later phase of development and at least examine the grounds upon which it arrived at the *introduction of number*.[70] (Hilbert 1898/1899a, p. 222; Emphasis in original)

The geometrical derivation of the structure of an ordered field from the structure of the Euclidean plane also allowed Hilbert to provide an adequate definition of proportionality for line segments and to reconstruct the theory of similar triangles. More specifically, the proportionality of line segments was defined as the equality of the product of two pairs of line segments:

Definition 6 If a, b, a', b' are any four segments let the proportion $a : b = a' : b'$ denote nothing else but the segment equation $ab' = ba'$. (Hilbert 1971, p. 55)

A crucial aspect of this approach was that, starting from a definition of segment multiplication, Hilbert regained *directly* the notion of proportionality for line segments, thereby avoiding the reference to the axiom of Archimedes.[71] Moreover, this definition rested essentially on the commutative property of the multiplication of line segments and, therefore, on Pascal's theorem. The fact that the construction of a segment arithmetic—but also his new original proof of Pascal's theorem—did not assume the Archimedean axiom was then essential for the general aim of providing a rigorous foundation of the theory of plane area independently of any continuity assumption, and therefore, of the concept of real number.

Back to the construction of the elementary theory of measure of area, Hilbert started as customary by defining a measure of area for triangles. The measure of area of a triangle was defined as a *characteristic segment s*, which was obtained as the semi-product of the base by the corresponding altitude, in symbols, $\frac{1}{2}bh$. This

---

detailed study of the significance of the theorems Desargues and Pappus in modern axiomatic geometry, see Pambuccian and Schacht (2019).

[69] On the chief significance of the introduction of number into geometry for Hilbert's axiomatic project, see Pambuccian (2013). I would like to thank an anonymous reviewer for calling my attention to this point.

[70] "Aber, wenn die Wissenschaft nicht einem *unfruchtbaren Formalismus* anheimfallen soll, so wird sie auf einem späteren Stadium der Entwicklung sich wieder auf sich *selbst besinnen müssen* und mindestens die Grundlagen prüfen, auf denen sie zur *Einführung der Zahl* gekommen ist."

[71] As is well known, Descartes derived his definition of segment multiplication from Proposition VI.2 of the *Elements* on the proportionality of similar triangles; consequently, he assumed not only the classical theory of proportion of Book V in its entirety but also the validity of the Archimedean axiom. This constitutes a crucial difference with Hilbert's treatment of segment multiplication in *Foundations*.

Fig. 11 Measure of area of a triangle
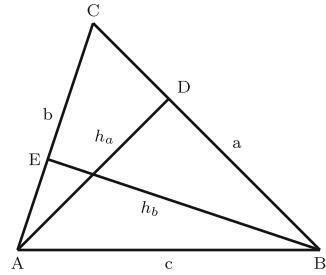
associated segment *s* should be taken as an element of the ordered field generated by the segment arithmetic. Thus, a main innovation of Hilbert's approach in relation to Schur's strategy discussed in Sect. 4.2, which also associated line segments to polygonal areas, was that this set of segments was now endowed with the algebraic structure of an ordered field. In other words, Hilbert's elementary theory of measure of area did not consist in just building an application or mapping from the set of plane polygons to the linearly ordered set of line segments, but to the richer algebraic structure of an ordered field. This was especially important because some basic algebraic properties of multiplication, such as commutativity and distributivity over addition, were essential to prove the fundamental properties of measure of area functions, and then to provide a rigorous foundation for the theory of plane area.

The main problem in the introduction of a measure of area function consisted in proving that these functions are *well defined*, in the sense that the measure of area of a triangle is independent of the side chosen as the base and of the corresponding altitude. This fundamental property was proved based on above definition of proportionality of line segments and the following central theorem about the triangle similarity:

Theorem 41 *If a, b and a', b' are corresponding sides of two similar triangles, then the proportion $a : b = a' : b'$ holds.* (Hilbert 1971, p. 55)

The independence of the measure of area of a triangle from the side chosen as the base follows immediately from this theorem. In fact, consider a given triangle ABC and draw the corresponding altitudes $h_a = AD$ and $h_b = BE$ (Fig. 11). Then, from the similarity of the triangles BCE and ACD, one obtains (by Theorem 41) the following proportion: $a : h_b = b : h_a$, that is, $a.h_a = b.h_b$. The same argument can be applied to the side c and the corresponding altitude h_c. Hence, one can conclude that $a.h_a = b.h_b = c.h_c$.[72]

Hilbert assigned a *sign* to the measure of area of a triangle, depending on whether one considers its positive or the negative *orientation*. More precisely, if all the points in the interior of triangle ABC lie to the left of the sides AB, BC, CA, then ABC is

---

[72] In the Appendix II of *Foundations*, Hilbert shows that if the standard triangle congruence axiom (III.5) is replaced by a *weaker* version, which restricts its application only to triangles with the *same orientation*, this central property of a measure of area of triangles fails. Moreover, in this "non-Pythagorean geometry," the proposition I.39 of the *Elements*, as well as De Zolt's postulate, do not generally hold. See Hilbert (1902). In addition, it should be noted that in order to define a measure of area of triangles in this prescribed way, one must also show that every triangle has at least one height which is completely in its interior. I would like to thank Klaus Volkert for this observation.

called the *positive orientation* of the triangle. In turn, if all the interior points of ABC lie to the left of the sides CB, BA, AC, then CBA is said to be the *negative orientation*. In other words, the orientation of a triangle is established by considering the order of the corresponding vertices in a clockwise (i.e., negative) or counterclockwise (i.e., positive) direction. Thus, the measure of area of a triangle ABC, positively oriented, is a *positively directed segment s*. In symbols, the measure of a positively oriented triangle ABC is denoted as $[ABC]$, from which it follows that $[CBA] = -[ABC]$. The orientation assigned to triangles is essential to guarantee a *fundamental property* of a measure of area function, namely for any triangle T, positively oriented, the measure of area of T is always > 0. Moreover, from the above definition of measure of area of a triangle it follows that if T and T' are congruent triangles, then T and T' have the same measure of area. This is the second fundamental property of the measure of area.

The possibility of decomposing any polygon into triangles in an entirely determined way naturally suggested the definition of its measure of area: the measure of area of a polygon (positively oriented) is the sum of the measure of area of the triangles (positively oriented) in which it is decomposed under a given triangulation. The most critical task and the central challenge in the development of a theory of measure of area of plane polygons was to prove that this function is *well defined*, i.e., that the measure of area is uniquely determined by the polygon or, equivalently, that it is independent of the triangulation which is used for its calculation. This can be appreciated in the fact that this is the part of Hilbert's proof that underwent the most substantial changes in the several editions of *Foundations*.

More specifically, Hilbert advanced two *different* geometrical arguments to prove that his functions of area measure of polygons are well defined. The first argument was presented, with minor modifications, from the *Festschrift* to the sixth edition of *Foundations*, published in 1923. In turn, the second argument appeared for the first time in print in the seventh edition of 1930. However, these two different arguments followed the same *general strategy*, which consisted in proving first that the function of measure of area of *triangles* satisfies the *additive property*, that is, that if a triangle is decomposed into a (finite) number k of triangles, then the sum of the measure of area of the k triangles (positively oriented) is equal to the measure of area of the original triangle (also positively oriented). From this fundamental property of area measure of triangles, he quickly derived that every polygon uniquely determines its measure of area independently of the triangulation used for its calculation.

To be more precise, Hilbert's original proof that the measure of area of triangles satisfies additivity was based significantly on the work of Louis Gérard, particularly on (Gérard 1898). In general terms, the proof strategy was built on the notion of *transversal decomposition of a triangle*, that is, the decomposition which results from a segment joining a vertex of a triangle with a point on the opposite side. From the distributive law of the segment arithmetic, it followed immediately that the area measure of an arbitrary triangle is equal to the sum of the measures of area of the two triangles which are obtained from a transversal decomposition. Moreover, by repeating the same reasoning, one could show that in general the measure of area of any triangle is equal to the sum of the measures of area of the triangles which arise by applying successively transversal decompositions of the given triangle in any (finite) number

Fig. 12 Theorem 49 of (Hilbert 1971)
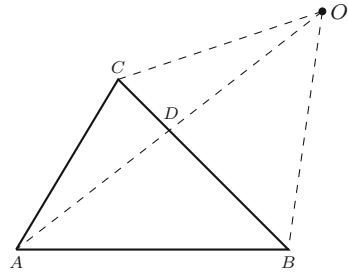
of times. Then, by means of a simple geometrical argument, Hilbert showed that any arbitrary decomposition of a triangle into partial triangles is *reducible* to transversal decompositions.[73]

In turn, Hilbert's second proof made a fundamental appeal to a notion of *oriented* measure of area of a triangle. This strategy had its conceptual roots in a method developed by August Möbius in an influential paper on the theory of content of polyhedra.[74] However, to our knowledge, Hilbert did not make any explicit reference to Möbius' method neither in the several editions of *Foundations* nor in unpublished sources. We will focus now on this second proof strategy, which Hilbert considered more clear and illuminating for the grounding of a theory of measure of area.

Hilbert's geometrical argument proceeded in two steps. First, he formulates and proves the following auxiliary theorem:

Theorem 49 *If O is a point outside a triangle ABC, then the relation*

$$[ABC] = [OAB] + [OBC] + [OCA]$$

*holds for the area [*Inhaltmaß*] of the triangle. (Hilbert 1971, p. 65)*

The complete geometrical proof of this theorem demands considering several cases concerning the position of the point O with respect to the sides of the triangle. Hilbert only proved the case where O lies in the exterior of the triangle, but in the interior of one of its angles (Fig. 12). Schematically, the argument runs as follows: let the segments AO and BC meet at a point D. By resorting to the distributive law of multiplication over addition of the segment arithmetic, one obtains the following relations:

$$[OAB] = [ODB] + [DAB],$$
$$[OBC] = -[OCD] - [ODB],$$
$$[OCA] = [OCD] + [DCA].$$

By adding the left and right terms of the equalities, respectively, one obtains the equality $[OAB] + [OBC] + [OCA] = [DAB] + [DCA]$. But from the application of

---

[73] For an detailed presentation of Hilbert's original proof strategy, see Zacharias (1930) and Hessenberg (1967). A modern reconstruction can be found in Hartshorne (2000).

[74] Cf. Möbius (1886). The basic ideas of this method were already developed by Möbius in his book *Der barycentrische Calcul* Möbius (1885).

Fig. 13 Theorem 50. Adapted
from (Hilbert 1971)
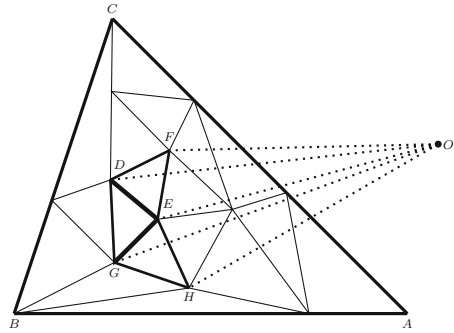
the distributive law of multiplication over addition it results that $[DAB] + [DCA] = [ABC]$. Hence, $[OAB] + [OBC] + [OCA] = [ABC]$, namely the desired result. The remaining cases can be proved easily by repeating a similar argument.

The second step consists in proving properly that the additive property of the area measure of triangles is valid for *any* arbitrary decomposition of a triangle into partial triangles. This is formulated as Theorem 50:

Theorem 50 *If a triangle ABC is decomposed into a finite number of triangles $\triangle_k$, then the area [*Inhaltmaß*] of the positively oriented triangle ABC is equal to the sum of the areas [*Inhaltmaße*] of all positively oriented triangles $\triangle_k$.*

Hilbert's proof of this key theorem in the elementary theory of measure of area can be reconstructed as follows. Consider a given decomposition into triangles of a triangle ABC and let ABC be its positive orientation. Let DEF and DEG be two adjacent triangles in this decomposition, such that the common side DE lies in the interior of ABC. Choose a point O in the plane outside the triangle ABC (Fig. 13). By Theorem 49, the following equalities hold:

$$[DEF] = [ODE] + [OEF] + [OFD]$$
$$[GED] = [OGE] + [OED] + [ODG].$$

Adding the right and left terms of these segment equalities, respectively, the terms $[ODE]$ and $[OED]$ will cancel each other out (since $[OED] = -[ODE]$), that is, the measure of area of the triangle defined by O and the common side DE will be canceled out on the right-hand side of this equality. Similarly, adding the measure of area of another adjacent triangle (positively oriented)

$$[EGH] = [OEG] + [OGH] + [OHE],$$

the term $[OGE]$ will also cancel out on the right-hand side of the equality. Repeat now this process and add in the same manner the measure of area of all positively oriented triangles $\triangle_k$ which form the given decomposition of the triangle ABC. It is clear that for *every segment DE*, which lies in the interior of the triangle ABC, the measure of area $[ODE]$ will cancel out on the right-hand side of the equality.

Fig. 14 Additive property of measure of area of triangles
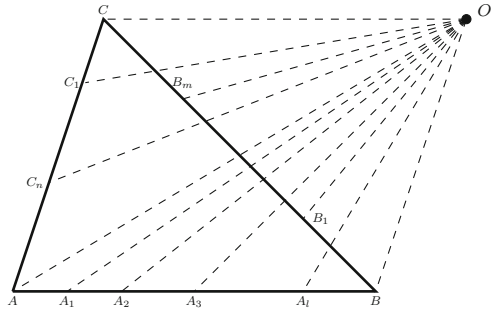
In other words, the measure of area of triangles defined by O and all segments of the triangle net which lie *in the interior* of the triangle ABC will cancel out. Thus, what remains is the sum of the measure of areas of the triangles defined by O and the points used for the decomposition of the triangle ABC *lying on its sides*, i.e., $AA_1, \ldots, A_lB, BB_1, \ldots, B_mC, CC_1, \ldots, C_nA$ (Fig. 14). Denoting $\sum_{k=1}^{n}[\triangle_k]$ as the sum of the measure of areas of all positively oriented triangles $\triangle_k$, one obtains that

$$
\sum_{k=1}^{n}[\triangle_k] = [OAA_1] + \cdots + [OA_lB]
$$
$$
+ [OBB_1] + \cdots + [OB_mC]
$$
$$
+ [OCC_1] + \cdots + [OC_nA]
$$

Hence, by theorem 49, $\sum_{k=1}^{n}[\triangle_k] = [OAB] + [OBC] + [OCA] = [ABC]$. QED.

The final step to prove that the function of measure of area for plane polygons is *well defined* was just hinted by Hilbert. In fact, he just restricted himself to provide the following definition, accompanied by an informal remark:

Definition 7 Let the area [P] of a positively oriented simple polygon be defined as the sum of the areas [*Inhaltmaße*] of all positively oriented triangles into which the polygon splits in some definite decomposition. By an argument similar to the one used in Section 18 for the proof of the Theorem 43, it becomes apparent that the area [*Inhaltmaß*] of [P] is independent of the manner of decomposition into triangles and thus it is uniquely determined only by the polygon. (Hilbert 1971, p. 67)

From the above definition of measure of area of polygons, it follows immediately that congruent polygons have equal measure of area. Then, by Theorem 50, it follows that equidecomposable polygons have equal measure of area. From this one can also show by a very simple argument that *equicomplementable polygons have equal measure of area*. Hilbert used this relation of implication between the concepts of equidecomposition and equicomplementary and the notion of measure of area to provide the desired proof of central proposition I.39 of the *Elements*, which in his reconstruction asserts that if two equicomplementary triangles have the same base, they also have the same altitude (Theorem 48). More precisely, let b be the base of the triangles and call h and h' the corresponding altitudes. Then, from the assumption

that the two triangles are equicomplementary, one deduces that they must have equal measure of area, that is,

$$\frac{1}{2}bh = \frac{1}{2}bh'.$$

From this equality it follows immediately that h and h' are equal, that is, the triangles will necessarily have the same altitude.[75]

Hilbert also provided a proof of a version of the Wallace–Bolyai–Gerwien theorem corresponding to his notion of equicomplementability, namely that if two polygons have equal measures of area, then they are equicomplementable. This theorem is obtained easily by transforming the two polygons with equal measures of area into two equicomplementable right triangles with a unit leg (by Theorem 47), and then showing that these triangles must be congruent and, therefore, equicomplementable. These results regarding the co-implication between equicomplementability and measure of area are gathered together in the following theorem:

Theorem 51 *Two equicomplementable polygons have the same measure of areas [*Inhaltmaße*] and two polygons with the same area [*Inhaltmaß*] are equicomplementable.* (Hilbert 1971, p. 69)

Theorem 51 ensures that if two equicomplementable rectangles have a common side, then their other side must also coincide. Moreover, this theorem is also often expressed by means of its contrapositive, namely that if two polygons do not have equal measures of area, then they are not equicomplementable. Thus, De Zolt's postulate becomes just a corollary of the latter theorem. Hilbert formulates the fundamental geometrical postulate according to the following version:

Theorem 52 (De Zolt's postulate) *If a rectangle is decomposed by lines into several triangles and one of these triangles is omitted, then it is impossible to fill out the rectangle with the remaining triangles. (Hilbert 1971, p. 69)*

To prove this central theorem, one only needs to show that a polygon can never be equicomplementable to a proper polygonal component. Let a given polygon $\mathbf{P}$ be decomposed into several polygonal parts $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$. By the additive property of measure of area, it follows that:

$$[\mathbf{P}] = [\mathbf{P}_1] + [\mathbf{P}_2] + \ldots + [\mathbf{P}_n]$$

But since the measure of area of *each one* of the polygonal parts $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ is greater than 0, the measure of area of the polygon $\mathbf{P}$ is *greater than* any of its polygonal components, such as, for example, $\mathbf{P}_1$. Hence, by Theorem 51, $\mathbf{P}$ cannot be equicomplementable to $\mathbf{P}_1$. De Zolt's postulate is then a special case of this result.

---

[75] Note that Hilbert's proof of the key Theorem 48 relies essentially on a notion of measure of area of a triangle, particularly, on *Theorem 50*. This reveals that, in certain (important) cases, the numbering or counting of theorems in Chapter IV of *Foundations* does not necessarily follow a "logical order" of justification. I would like to thank Günther Eder for this observation.

With the proof of De Zolt's postulate, Hilbert achieved the goal of providing a solid foundation of the theory of plane area. The problem of the comparability of plane polygons with respect to the relation of equicomplementability was thus solved by resorting to a notion of measure of area. This conclusion followed essentially from the fact that there is a perfect correspondence between his notions of equicomplementability and measure of area, as proved in Theorem 51.[76] Nevertheless, this appeal to a "metrical" concept of area was not problematic from the standpoint of his axiomatic project, for this notion was defined in pure geometric fashion without relying on the concept of real number or on any continuity assumption. [77]

7 Conclusions

The central aim of this paper was to provide a detailed historical account of Hilbert's axiomatization of the theory of plane area. More specifically, our goal was *two-fold*: first, to examine and assess this geometrical theory against its historical background; second, to elucidate from a more conceptual perspective its role and significance for the general foundational program carried out in *Foundations*. From a historical perspective, we have seen that Hilbert's central contribution was to elevate the construction of the elementary theory of area to an unprecedented level of rigor. This specific contribution involved several elements. Hilbert's treatment of the central notions of decomposition and addition of polygons introduced important conceptual clarifications, and resulted in more rigorous proofs of the fundamental properties of geometrical equivalence. Moreover, the original distinction between the relations of equidecomposition and equicomplementability was essential for the principal objective of removing the dependence on the Archimedean axiom from the development of the theory of plane area. These conceptual clarifications impacted on Hilbert's notable proof of De Zolt's postulate, an important contribution to modern axiomatic geometry.

From a conceptual viewpoint, our examination of Hilbert's theory of plane area has contributed to clarifying its significance in the geometrical program executed in *Foundations*. More precisely, we have seen how Hilbert's two key technical innovations, i.e., his calculus of segments and his definition of proportionality, were structurally connected to the axiomatic reconstruction of the theory of plane area. As clearly revealed

---

[76] Let a relation of *greater* ("<") and *lesser* (">") in area be introduced as follows: a polygon **P** is called greater than a polygon **Q** (and **Q** lesser than **P**), if there is a polygon **P′** properly contained in **P** such that **P′** and **Q** are *equicomplementable*. Then, from Theorem 51, it follows that $\mathbf{P} \lesseqgtr \mathbf{Q}$ if and only if $[\mathbf{P}] \lesseqgtr [\mathbf{Q}]$. As mentioned, Hilbert also proved that the Archimedean axiom required to prove that the same co-implication is valid for the relation of equidecomposition.

[77] In his modern reconstruction of Hilbert's theory of area, Hartshorne (2000) has suggested that the detour in Hilbert's proof of De Zolt's postulate through a theory of area measure might be unavoidable: "This proof [of De Zolt's postulate] is analytic in that it makes use of the field of segment arithmetic and similar triangles. We do not know any purely geometric proof, for example of (I.39), that triangles on the same base with equal content [i.e., equicomplementable] have the same altitude" (p. 210). The same observation has been made by Volkert (2010, 2015) and Baldwin (2018b). Interestingly, Hilbert's construction of models of non-Pythagorean geometries in the Appendix II of *Foundations* seems also to suggest this connection between De Zolt's postulate and a notion of area measure of polygons (see footnote 71 above). The presumed impossibility of proving De Zolt's postulate with elementary means and without relying on some measure of area raises then interesting and complex technical issue and deserves further investigation.

in his lecture courses, Hilbert's fundamental concern about the foundations of this theory was to guarantee the existence of a relation of *total* order for polygonal areas. More importantly, this problem should be solved in a purely geometrical fashion, by delivering an elementary proof of De Zolt's postulate which did not depend on the Archimedean axiom. Both the construction of the segment arithmetic and the theory of proportion played an essential role in this task.

On the one hand, Hilbert's segment arithmetic allows one to derive the algebraic structure of an ordered field from the axioms for the Euclidean plane, without assuming any continuity axioms. A crucial element in this construction was a new proof of (an affine version of) Pascal's theorem, based exclusively on the *plane* axioms of incidence, betweenness, and congruence. The commutative and distributive laws of segment multiplication, obtained by the latter theorem, were then essential for the rigorous and purely geometrical introduction of measure of area functions, for they delivered central properties such as additivity. On the other hand, Hilbert's original strategy of *starting* from the definition of segment multiplication to regain directly the notion of proportionality for line segments had significant implications for the development of the theory of area, namely: it allowed to define measures of area according to the standard formulas. In sum, we have seen how Hilbert explored the potentiality of these two technical innovations to achieve a rigorous development of the theory of plane area, which was in accordance with the general methodological and epistemological requirements that he laid down for his axiomatization of Euclidean geometry.

Our discussion of the theory of plane area has also proved to be instructive for the understanding of the central requirement of "purity of the method" in modern synthetic geometry. In this specific context, this methodological requirement was usually equated to the demand of avoiding the concept of real number and continuity assumptions when laying down the foundations of geometry. In this paper, we have seen that Hilbert advanced an alternatively, or better complementary, interpretation of "purity," which consisted in the *exclusion* of the concept of "extensive" or "measurable" *magnitude* from the axiomatic reconstruction Euclidean geometry. Specifically, this meant that the usual "general principles of magnitudes" had to be converted into *geometrical theorems* by interpreting the relations and operations of magnitudes as specific geometrical relations and operations for every (relevant) kind of geometrical object. Indeed, simply *assuming* that geometrical objects (such as segments, angles, and plane figures) bear all the fundamental (algebraic) properties of magnitudes was tantamount to accepting without proof that they behave like "numbers." This can be considered an additional reason for the demand of proving De Zolt's postulate as a geometrical theorem in late nineteenth-century geometry.

Finally, Hilbert's construction of the theory of area independently of Archimedes' axiom naturally posed the question of whether an analogous development of the theory of volume in space was possible. Hilbert himself raised the challenge as the third of his famous "Mathematical Problems" in 1900. As is well known, Max Dehn promptly answered the question in the negative by proving that the Wallace–Bolyai–Gerwien theorem is not generally valid in space for polyhedra, or more precisely, that there are polyhedra (specifically, tetrahedra) with equal volume which are nei-

ther equidecomposable nor equicomplementable.[78] This notable result inspired a rich array of investigations on the foundations of the theory of equivalence of polyhedra.[79] Schatunowsky (1903), for instance, developed an elementary theory of volume without resorting to either the concept of limit or continuity axioms, in which a proposition analogous to De Zolt's postulate was valid, but the Wallace–Bolyai–Gerwien did not generally hold.[80] In turn, Süß (1921) provided a rigorous treatment of the theory of equivalence of polyhedra based on Cavalieri's principle on the equality of volume of solids, that is, on non-elementary methods.[81] The formulation of De Zolt's postulate in solid geometry, as well as the specification of the necessary and sufficient conditions to prove it, then poses interesting historical and conceptual problems that have to be investigated in a different paper.

References

Aigner M., Ziegler, G. 2018. Hilbert's Third Problem: decomposing polyhedra. In: *Proofs from The Book*, pp. 67–75. Berlin: Springer. https://doi.org/10.1007/978-3-662-57265-8-10.

Amaldi, U. 1900. Sulla teoria dell'equivalenza. In *Questioni riguardanti la geometria elementare*, ed. F. Enriques, 103–142. Bologna: Ditta Nicola Zanichelli.

Andreotti, A. 1949. Sulla proposizione di De Zolt pei poliedri. *Bollettino dell'Unione Matematica Italiana* 3 (4): 68–75.

---

[78] Cf. Dehn (1902). For a detailed exposition of Hilbert's third problem and modern proofs of Dehn's result, see Boltianskii (1978), Bartocci (2012), and Aigner and Ziegler (2018).

[79] The theory of equivalence of plane polygons was also further extended to non-polygonal surfaces as well as to absolute geometry. For the case of absolute geometry, see Finzel (1912). A historical discussion of these extensions of the theory of equivalence can be found in Volkert (1999).

[80] Schatunowsky admitted then that to guarantee that polyhedra in space are *comparable* one needed to appeal to continuity arguments (Cf. Schatunowsky 1903, p. 507). Schatunowsky's proof of De Zolt's postulate for polyhedra has been criticized by Andreotti (1949), who claims that the Archimedean axiom is still a necessary assumption in the geometrical argument.

[81] It is worth mentioning that Sencer (1938) and Frei (1970) also presented an abstract axiomatic development of the theory of volume in space.

Baldwin, J. 2018a. Axiomatizing Changing Conceptions of the Geometric Continuum I: Euclid-Hilbert. *Philosophia Mathematica* 26 (3): 346–374. https://doi.org/10.1093/philmat/nkx030.

Baldwin, J. 2018b. *Model Theory and the Philosophy of Mathematical Practice*. Cambridge: Cambridge University Press.

Baldwin, J., and A. Mueller. 2019. The Autonomy of Geometry. Annales Universitatis Paedagogicae Cracoviensis. *Studia ad Didacticam Mathematicae Pertinentia* 11: 5–24. https://doi.org/10.24917/20809751.11.1.

Bartocci, C. 2012. *Una piramide di problemi. Storie di geometria da Gauss a Hilbert*. Milano: Raffaelo Cortina Editore.

Bernays, P. 1971. Remarks on the Theory of Plane Areas. Supplement III of (Hilbert 1971), 207–214.

Biasi, G. 1894a. Sulla equivalenza dei poligoni. *Periodico di matematica per l'insegnamento secondario* 9 (1): 19–26. 48-52.

Biasi, G. 1894b. Ancora sulla equivalenza dei poligoni. *Periodico di matematica per l'insegnamento secondario* 9 (3–4): 85–87.

Błaszczyk, P., and A. Petiurenko. 2019. Euclid's Theory of Proportion Revised. *Annales Universitatis Paedagogicae Cracoviensis. Studia ad Didacticam Mathematicae Pertinentia* 11: 37–62. https://doi.org/10.24917/20809751.11.3.

Boltianskii, V. 1978. *Hilbert's Third Problem*. Washington: Winston & Sons.

Corry, L. 2013. Geometry and Arithmetic in the Medieval Traditions of Euclid's Elements: A view from Book II. *Archives for History of Exact Sciences* 67: 637–705. https://doi.org/10.1007/s00407-013-0121-5.

Dedekind, R. 1888. Was sind und was sollen die Zahlen? Braunschweig: Vieweg. English translation by W. W. Beman, revised by W. Ewald 1996: 787–833.

Dehn, M. 1902. Über den Rauminhalt. *Mathematische Annelen* 55: 465–478.

De Paolis, R. 1884. *Elementi di geometria*. Torino: Loescher.

De Paolis, R. 1886. Sopra una proposizione fondamentale della teoria dell'equivalenza. *Periodico di Matematica* 1: 44–46.

De Risi, V. 2020. Euclid's Common Notions and the Theory of Equivalence. *Foundations of Science*. https://doi.org/10.1007/s10699-020-09694-w.

De Zolt, A. 1881. *Principii della eguaglianza di poligoni preceduti da alcuni cenni critici sulla teoria della equivalenza geometrica*. Milano: Briola.

Duhamel, J. 1866. Des méthodes dans les sciences de raisonnement. Deuxième partie. Paris: Gauthier Villars. 2nd edition, 1878.

Ehrlich, P. 2006. The Rise of non-Archimedean Mathematics and the Roots of a Misconception I: The Emergence of non-Archimedean Systems of Magnitudes. *Archive for History of Exact Sciences* 60: 1–121. https://doi.org/10.1007/s00407-005-0102-4.

Enriques, F., and U. Amaldi. 1903. *Elementi di geometrie ad uso delle scuole secondarie superiori*. Bologna: Ditta Nicola Zanichelli.

Euclid. See (Heath 1956).

Ewald, W. 1996. *From Kant to Hilbert: A source book in mathematics*, vol. 2. Oxford: Clarendon Press.

Ewald, W., and W. Sieg. 2013. *David Hilbert's Lectures on the Foundations of Arithmetic and Logic, 1917–1933*. Berlin: Springer.

Faifofer, A. 1878. Elementi di geometria ad uso dei lecei. Venezia: Tipografía Emiliana. 1st edition.

Faifofer, A. 1882. *Elementi di geometria ad uso dei lecei*. Venezia: Tipografía Emiliana. Terza edizione migliorata.

Faifofer, A. 1886. Dimostrazione di una proposizione fondamentale della teoria dell'equivalenza. *Periodico di Matematica* 1: 13–15.

Finzel, A. 1912. Die Lehre vom Flächeninhalt in der allgemeinen Geometrie. *Mathematische Annalen* 72: 262–284.

Frei, G. 1970. Beiträge zur axiomatischen Inhaltstheorie. *Mathematische Annalen* 187: 220–240.

Gérard, L. 1898. Cours de géométrie élémentaire. Vol. I. Géométrie plane. Paris: Carré & Naud.

Gerretsen, J., Verdenduin, P. 1983. Polygons and Polyhedra. In Fundamentals of Mathematics. Vol. II. Geometry, eds. H. Behnke, F. Bachmann, K. Fladt and H. Kunle, pp. 238–290. Cambridge, Massachusetts: MIT Press.

Gerwien, P. 1833a. Zerschneidung jeder beliebigen Anzahl von gleichen geradlinigen Figuren in dieselben Stücke. *Journal für die reine und angewandte Mathematik* 10: 228–234.

Gerwien, P. 1833b. Zerschneidung jeder beliebigen Menge verschieden gestalteter Figuren von gleichem Inhalt auf der Kugelfläche in dieselben Stücke. *Journal für die reine und angewandte Mathematik* 10: 228–234.

Giacardi, L., and R. Scoth. 2014. Secondary School Mathematics Teaching from the Early Nineteenth Century to the Mid-Twentieth Century in Italy. In *Handbook on the History of Mathematics Education*, ed. A. Karp and G. Schubring, 201–228. Berlin: Springer.

Giovannini, E. 2016. Bridging the Gap between Analytic and Synthetic Geometry: Hilbert's Axiomatic Approach. *Synthese* 193: 31–70. https://doi.org/10.1007/s11229-015-0743-z.

Giovannini, E., E. Haeusler, A. Lassalle-Casanave, and P. Veloso. 2019. *De Zolt's Postulate: An Abstract Approach*. The Review of Symbolic Logic: First view. https://doi.org/10.1017/S1755020319000339.

Grünbaum, B. 2012. Polygons: Meister was right and Poinsot was wrong but prevailed. *Beiträge zur Algebra und Geometrie* 53: 57–71. https://doi.org/10.1007/s13366-011-0047-5.

Hallett, M. 2008. Reflections on the Purity of Method in Hilbert's Grundlagen der Geometrie. In *The Philosophy of Mathematical Practice*, ed. P. Mancosu, 198–255. New York: Oxford University Press.

Hallett, M., and U. Majer eds. 2004. *David Hilbert's Lectures on the Foundations of Geometry, 1891–1902*. Berlin: Springer.

Hartshorne, R. 2000. *Geometry: Euclid and Beyond*. New York: Springer.

Heath, Th. 1956. The Thirteen Books of Euclid's Elements (second edition, vols. I–III). New York: Dover Publications. Translation with introduction and commentary from the text of Heiberg.

Hessenberg, G., and J. Diller. 1967. *Grundlagen der Geometrie*. Berlin and New York: De Gruyter.

Hilbert, D. 1898. Feriencursus: Über den Begriff des Unendlichen. Ostern 1898. - Niedersächsische Staats- und Universitätsbibliotheck Göttingen, Handschriftenabteilung, Cod. Ms. Hilbert 597. In (Hallett & Majer 2004, pp. 160–184).

Hilbert, D. 1898/1899a. Grundlagen der Euklidischen Geometrie. In (Hallett & Majer 2004, pp. 221–301).

Hilbert, D. 1898/1899b. Elemente der Euklidischen Geometrie. In (Hallett & Majer 2004, pp. 302–406).

Hilbert, D. 1899. Grundlagen der Geometrie. Festschrift zur Feier der Enthüllung des Gauss-Weber Denkmals in Göttingen. In Hallett & Majer (2004).

Hilbert, D. 1902/03 Über den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck. Proceedings of the London Mathematical Society 35: 50-68. Reprinted as Appendix II in (Hilbert 1971, pp. 113–132).

Hilbert, D. 1903. *Grundlagen der Geometrie*, 2nd ed. Leipzig: Teubner.

Hilbert, D. 1917. Prinzipien der Mathematik; (Vorlesung, WS 1917/18). Ausgearbeitet von P. Bernays. Georg–August–Universität Göttingen, Mathematisches Institut, Lesesaal. Published in (Hilbert 2013, pp. 59–214) Ewald & Sieg 2013, (pp. 59-214).

Hilbert, D. 1930. *Grundlagen der Geometrie*, 7th ed. Leipzig, Berlin: Teubner.

Hilbert, D. 1971. *Foundations of Geometry*. La Salle: Open Court. Translated by L. Unger from the 10th German Edition.

Hilbert, D. 1999. *Grundlagen der Geometrie. Mit Supplementen von Paul Bernays. Herausgegeben und mit Anhängen versehen von Michael Toepell*, 14th ed. Leipzig: Teubner.

Jackson, W.H. 1912. Wallace's Theorem Concerning Plane Polygons of the Same Area. *American Journal of Mathematics* 34 (4): 383–390.

Killing, W. 1886. Rezension: Otto Stolz, Volesungen über allgemeine Arithmetic. Erster Theil. Zeitschrift für Mathematik und Physik. *Historisch-literarische Abtheilung*. 31: 182–187.

Killing, W. 1898. *Einführung in die Grundlagen der Geometrie. Zweiter Band*. Paderborn: Ferdinand Schöningh.

Lakatos, I. 1976. *Proofs and Refutations. The Logic of Mathematical Discovery*. Cambridge: Cambridge University Press.

Lazzeri, G. 1895. Sulle teoría della equivalenza geometrica. Periodico di matematica 10 (3-4): 77-93 and (5-6): 133–141.

Lazzeri, G., and A. Bassani. 1891. *Elementi di geometria*. Livorno: Tipografia di Raffaello Giusti.

Legendre, A.M. 1806. *Eléments de géométrie*, 6th ed. Paris: Didot.

McFarland, A., J. McFarland, and J.T. Smith eds. 2014. *Alfred Tarski: Early Work in Poland - Geometry and Teaching*. Berlin: Birkhäuser.

Mancosu, P. 2016. *Abstraction and Infinity*. Oxford: Oxford University Press.

Meister, A. L. F. 1771. Generalia de Genesi Figurarum Planarum et inde Pendentibus Earum Affectionibus. Novi Commentarii Societatis Reglae Scientiarum Gottingensis 1: 144–180.

Möbius, A. 1827. Der barycentrische Calcul. In: Gesammelte Werke. Band I, ed. R. Baltzer, 1–138. Leipzig: Hirzel, 1885.

Möbius, A. 1865. Über die Bestimmung des Inhaltes eines Polyeders. In: Gesammelte Werke. Band II, ed. F. Klein, 473-512. Leipzig: Hirzel, 1886.

Mueller, I. 1981. *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. New York: Dover.

Pambuccian, V. 2013. [Review]: David Hilbert's Lectures on the Foundations of Geometry, 1891–1902. *Philosophia Mathematica* 21 (2): 255–275.

Pambuccian V., Schacht C. 2019. The Axiomatic Destiny of the Theorems of Pappus and Desargues. In: *Geometry in History*, eds. S. G. Dani and A. Papadopoulos, pp. 355–399. Berlin: Springer. https://doi.org/10.1007/978-3-030-13609-3_8

Poinsot, L. 1810. Mémoire sur les polygones et les polyèdres. *J. Ècole Polytech.* 10: 16–48. German translation by R. Haußner, with comments, as "Abhandlung über die Vielecke und Vielflache", pp. 3–48, 105–120 in *Abhandlungen über die regelmäßigen Sternkörper, Ostwald's Klassiker der exakten Wissenschaften*, Nr. 151. Engelmann, Leipzig, 1906.

Puig Adam, P. 1980. *Curso de geometría métrica*, 15th ed. Madrid: Editorial Euler.

Rausenberger, O. 1893. Das Grundproblem der Flächen- und Rauminhaltlehre. *Mathematische Annalen* 43: 275–284.

Réthy, O. 1891. Endlich-gleiche Flächen. *Mathematische Annalen* 38: 405–428.

Robering, K. 2016. The whole is greater than the part. Mereology in Euclid's Elements. *Logic and Logical Philosophy* 25(3): 371–409. https://doi.org/10.12775/LLP.2016.011.

Schatunowsky, S. 1903. Über den der Rauminhalt Polyeder. *Mathematische Annalen* 57: 496–508.

Schur, F. 1892. Über den Flächeninhalt geradlinig begrenzter ebener Figuren. *Sitzungsberichte der Dorpater Naturforschenden Gesellschaft* 10: 2–6.

Schur, F. 1898. Über den Fundamentalsatz der projectiven Geometrie. *Mathematische Annalen* 51: 401–409.

Sencer, S. 1938. *Axiomatische Theorie der Rauminhalte. Dissertation an der Universität Leipzig bei van der Waerden*. Leipzig: Moltzen.

Simon, M. 1906. *Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert*. Leipzig: Teubner.

Stolz, O. 1885. *Vorlesungen über allgemeine Arithmetik. Erster Theil: Allgemeines und Arithmetik der reellen Zahlen*. Leipzig: Teubner.

Stolz, O. 1894. Die eben Vielecke und die Winkel mit Einschluss der Berührung-Winkel als Systeme von absoluten Grßsen. *Monatshefte für Mathematik und Physik* 5: 233–240.

Süß, W. 1921. Begründung der Lehre vom Polyederinhalt. *Mathematische Annalen* 82: 297–305.

Tarski, A. 1924. On the Equivalence of Polygons. In McFarland, A. et. al (2014), pp. 70-91.

Toepell, M. 1985. Zur Schlüsselrolle Friedrich Schurs bei der Entstehung von David Hilberts "Grundlagen der Geometrie". In: *Mathemata. Festschrift für Helmuth Gericke*, ed. M. Folkerts, pp. 637–649. Stuttgart: Franz Steiner Verlag.

Unguru, S., and D. Rowe. 1981. Does the Quadratic Equation have Greek Roots? *Libertas Mathematica* 1: 1–49.

Unguru, S., and D. Rowe. 1982. Does the Quadratic Equation have Greek Roots? *Libertas Mathematica* 2: 1–62.

Vecchi, M. 1915. On the characteristics of the principal manuals of elementary geometry published in Italy in the course of the last fifty years. *Bulletin of the American Mathematical Society* 21: 188–191.

Veronese, G. 1894/1895. Dimostrazione della proposizione fondamentale dell'equivalenza delle figure. Atti del Regio Istituto Veneto di Scienze, Lettere ed Arti 7(6): 421–437.

Veronese, G., and P. Gazzinaga. 1900. *Elementi di geometria*. Verona: Fratelli Drucker.

Volkert, K. 1999. Die Lehre vom Flächeninhalt ebener Polygone: einige Schritte in der Mathematisierung eines anschaulichen Konzeptes. *Mathematische Semesterberichte* 46: 1–28.

Volkert, K. 2010. Le tout est-il toujours plus grand que la partie? *Revue d'histoire des mathématiques* 16 (2): 287–306.

Volkert, K. 2015. *David Hilbert. Grundlagen der Geometrie*. Berlin: Springer.

Wallace, W., Lowry, J. 1814. Question 269. New Series of the Mathematical Repository 3: 44–46. Ed. by Thomas Leybourn.

Wiener, H. 1893. Weiteres über Grundlagen und Aufbau der Geometrie. *Jahresbericht der Deutschen Mathematiker-Vereinigung* 3: 70–90.

Zacharias, M. 1930. *Elementargeometrie der Ebene und des Raumes*, First ed. Berlin: de Gruyter.