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Affine Pieri rule for periodic Macdonald spherical functions and fusion rings ☆

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A B S T R A C T

Let $\hat{\mathfrak{g}}$ be an untwisted affine Lie algebra or the twisted counterpart thereof (which excludes the affine Lie algebras of type $\widehat{BC}_n = A^{(2)}_{2n}$). We present an affine Pieri rule for a basis of periodic Macdonald spherical functions associated with $\hat{\mathfrak{g}}$. In type $\hat{A}_{n-1} = A^{(1)}_{n-1}$ the formula in question reproduces an affine Pieri rule for cylindric Hall-Littlewood polynomials due to Korff, which at $t = 0$ specializes in turn to a well-known Pieri formula in the fusion ring of genus zero $\widehat{\mathfrak{sl}}(n)_c$-Wess-Zumino-Witten conformal field theories.

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1. Introduction

The Hall polynomials form a generalization of the Littlewood-Richardson coefficients that provide the structure constants of the classical Hall algebra in the basis of Hall-Littlewood polynomials; these structure constants (which are polynomial in the Hall-Littlewood parameter) are known to enjoy a very intricate combinatorics [36, Chapters II, III]. Indeed, the Hall algebra and its generalizations in terms of quivers turn out to encode a host of combinatorial, algebra-geometric, and representation-theoretic data [4,36,48,55]. Recently, Korff introduced an affine analog of the Hall polynomials; these arise as structure constants of a t-deformation of the fusion ring (a.k.a. Verlinde algebra) for $\widehat{\mathfrak{sl}}(n)_c$-Wess-Zumino-Witten conformal field theories with respect to a natural basis built from cylindric Hall-Littlewood polynomials [28]. While to date the precise geometric and/or representation-theoretic interpretation of this t-deformed fusion ring has yet to be disclosed, indications of an intimate relation with the deformed Verlinde algebras in [52,53] have been noticed [21,28,43].

At $t = 0$ the Hall-Littlewood polynomials become Schur polynomials. The corresponding Littlewood-Richardson coefficients [36, Chapter I.9] and their affine counterparts, which arise as fusion coefficients for $\widehat{\mathfrak{sl}}(n)_c$-Wess-Zumino-Witten conformal field theories [14, Chapter 16], have received massive attention across the mathematics literature because of their rich combinatorics and profound applications in representation theory and Schubert calculus, cf. e.g. [17] and [20,18,29,40] as well as further references therein. Korff's t-deformation is different from the q-deformed fusion ring in [16], which recovers the $\widehat{\mathfrak{sl}}(n)_c$-Wess-Zumino-Witten fusion ring at the value $q = 1$. Of special interest is in this connection the well-known fact that the closely related $\widehat{\mathfrak{gl}}(n)_c$-fusion ring amounts to a $q = 1$ degeneration of the small quantum cohomology ring of the Grassmannian of n-dimensional linear subspaces in $\mathbb{C}^{n+c}$ [3,6]. The structure coefficients of this small quantum cohomology ring in the basis of Schubert classes, the genus zero 3-point Gromov-Witten invariants, can be computed as quantum counterparts of the Littlewood-Richardson coefficients for Schur polynomials [5,18,25,46,50,51,54,56]. Various other combinatorial constructions related to the computation of genus zero 3-point Gromov-Witten invariants have been considered in the literature, e.g. via the structure constants of algebras of symmetric polynomials in bases of cylindric Schur polynomials [45,39], in bases of k-Schur polynomials [32,30,31], or in bases of noncommutative Schur polynomials in variables from a plactic algebra [29], respectively.

If at least one of the two factors in the Littlewood-Richardson product consists of a Hall-Littlewood polynomial attached to a partition with only a single column, then the explicit form of the pertinent Hall polynomials is given by the Pieri formula [36, Chapter III.3]. The affine analog of this Pieri formula for cylindric Hall-Littlewood polynomials can be found in [28, Corollary 7.4]. The purpose of the present work is to generalize the affine Pieri formula in question from $\widehat{\mathfrak{sl}}(n)_c$ to the case of an arbitrary affine Lie algebra $\hat{\mathfrak{g}}$ [26], excluding those of type $\widehat{BC}_n = A^{(2)}_{2n}$. In other words, $\hat{\mathfrak{g}}$ is assumed to be untwisted or to be the twisted counterpart of an untwisted affine Lie algebra.

Let us recall at this point that from the perspective of Lie algebras the Hall-Littlewood polynomials in n variables are associated with $\mathfrak{sl}(n)$. The corresponding generalization of these polynomials to simple Lie algebras of arbitrary type is given by the Macdonald spherical functions [37,42,44,49], which were constructed originally by Macdonald as spherical functions on p-adic symmetric spaces [34]. With the aid of suitable representations of the affine Hecke algebra, the Pieri formula for the Hall-Littlewood polynomials was generalized to a Pieri formula for Macdonald spherical functions of arbitrary simple Lie type in [11]. The key to achieve an analogous generalization of the affine Pieri formula in [28] is to connect with the work in [9]. To this end, we will detail briefly how affine Pieri formulas arise in the context of [9], while also emphasizing in which sense these differ from the usual Pieri formulas for the Hall-Littlewood polynomials in [36].

Associated with the standard unit basis $e_1, \ldots, e_n$ for $\mathbb{Z}^n \subset \mathbb{R}^n \subset \mathbb{C}^n$, let us denote $\bar{e}_j = e_j - \frac{1}{n}(e_1 + \cdots + e_n)$ ($j = 1, \ldots, n$) and $\omega_r = \bar{e}_1 + \cdots + \bar{e}_r$ ($r = 1, \ldots, n-1$). For $\lambda \in \Lambda^{(n)} = \{m_1\omega_1 + \cdots + m_{n-1}\omega_{n-1} \mid m_1, \ldots, m_{n-1} \in \mathbb{Z}_{\geq 0}\}$ the $\mathfrak{sl}(n)$ Hall-Littlewood polynomial $R_\lambda(x;t)$ with variable $x = (x_1, \ldots, x_n)$ and parameter t is defined by the explicit formula

$$R_\lambda(x;t) = \sum_{\sigma \in S_n} C(x_{\sigma_1}, \ldots, x_{\sigma_n}; t) x_{\sigma_1}^{\lambda_1} \cdots x_{\sigma_n}^{\lambda_n},$$

where

$$C(x_1, \ldots, x_n; t) = \prod_{1 \leq j < k \leq n} \frac{1 - t x_j^{-1} x_k}{1 - x_j^{-1} x_k},$$

and the summation is meant over all permutations $\sigma = \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{smallmatrix}\right)$ of the symmetric group S_n. When $\mu = \omega_r$, the corresponding t-deformed Littlewood-Richardson coefficients

$$R_\lambda R_\mu = \sum_{\nu \in \Lambda^{(n)}} \mathrm{c}_{\lambda,\mu}^{\nu}(t) R_\nu \qquad (\lambda, \mu \in \Lambda^{(n)})$$

are given explicitly by the Pieri rule [36, Chapter III.3]

$$R_\lambda R_{\omega_r} = c_{\omega_r}(t) \sum_{\substack{J \subseteq \{1,\ldots,n\},\, |J|=r \\ \lambda + \bar{e}_J \in \Lambda^{(n)}}} R_{\lambda + \bar{e}_J} \prod_{\substack{1 \leq j < k \leq n \\ j \in J,\, k \notin J \\ \lambda_j = \lambda_k}} \frac{1 - t^{k-j+1}}{1 - t^{k-j}}. \tag{1.1}$$

Here $\bar{e}_J = \sum_{j \in J} \bar{e}_j$, $|J|$ denotes the cardinality of J, and $c_{\omega_r}(t) = S_r(t) S_{n-r}(t)$ with $S_n(t) = \prod_{1 \leq j < k \leq n} \frac{1 - t^{k-j+1}}{1 - t^{k-j}}$.

Given a positive integral level c, an affine analog of the Pieri formula (1.1) valid for $\lambda \in \Lambda^{(n,c)} = \{m_1\omega_1 + \cdots + m_{n-1}\omega_{n-1} \in \Lambda^{(n)} \mid m_1 + \cdots + m_{n-1} \leq c\}$ follows from [9, Theorem 5.1]:

$$R^{(c)}_\lambda R^{(c)}_{\omega_r} = \tag{1.2}$$
$$c_{\omega_r}(t) \sum_{\substack{J\subseteq\{1,\ldots,n\},\,|J|=r\\ \lambda+\bar{e}_J\in\Lambda^{(n,c)}}} R^{(c)}_{\lambda+\bar{e}_J} \prod_{\substack{1\leq j<k\leq n\\ j\in J,\,k\notin J\\ \lambda_j=\lambda_k}} \frac{1-t^{k-j+1}}{1-t^{k-j}} \prod_{\substack{1\leq j<k\leq n\\ j\notin J,\,k\in J\\ \lambda_j=\lambda_k+c}} \frac{1-t^{n+1-k+j}}{1-t^{n-k+j}}.$$

Here $R^{(c)}_\lambda : X^{(n,c)} \to \mathbb{C}$ refers to the Hall-Littlewood polynomial R_λ viewed as a function on a discrete set $X^{(n,c)} = X^{(n,c)}(t) \subset \mathbb{T}^n = \{(x_1,\ldots,x_n) \in \mathbb{C}^n \mid |x_j| = 1, j = 1,\ldots n\}$. This set consists of points $x_\mu(t)$, $\mu \in \Lambda^{(n,c)}$ that depend analytically on the Hall-Littlewood parameter $t \in (-1,1)$. Specifically, for $\mu \in \Lambda^{(n,c)}$ and $t \in (-1,1)$ the point $x_\mu(t)$ is of the form $(e^{i\xi_1},\ldots,e^{i\xi_n})$ with the vector of angle coordinates $\xi = \xi_\mu(t)$ being defined as the unique global minimum of the radially unbounded strictly convex Morse function $\mathcal{V}^{(n,c)}_\mu : \mathbb{R}^n \to \mathbb{R}$

$$\mathcal{V}^{(n,c)}_\mu(\xi) = \sum_{1\leq j<k\leq n} \int_0^{\xi_j-\xi_k} v(\mathrm{x})\mathrm{dx} + \sum_{1\leq j\leq n} \left(\tfrac{c}{2}\xi_j^2 - 2\pi(\rho_j+\mu_j)\xi_j\right), \tag{1.3}$$

where $\rho = \omega_1 + \cdots + \omega_{n-1}$ and $v(\mathrm{x}) = \int_0^{\mathrm{x}} \frac{1-t^2}{1-2t\cos(\mathrm{y})+t^2}\mathrm{dy}$. This Morse function can be loosely thought of as an analog of the fusion potential, cf. [18,8]. It is known that the Hall-Littlewood polynomials $R^{(c)}_\lambda$, $\lambda \in \Lambda^{(n,c)}$ form a linear basis for the algebra of functions $f : X^{(n,c)} \to \mathbb{C}$ (cf. [9, Theorem 5.2]), which gives rise to the following affine analog of the Littlewood-Richardson coefficients for the Hall-Littlewood polynomials:

$$R^{(c)}_\lambda R^{(c)}_\mu = \sum_{\nu\in\Lambda^{(n,c)}} \mathrm{c}^{\nu,(c)}_{\lambda,\mu}(t) R^{(c)}_\nu \qquad (\lambda,\mu \in \Lambda^{(n,c)}). \tag{1.4}$$

For $\mu = \omega_r$, the explicit form of $\mathrm{c}^{\nu,(c)}_{\lambda,\mu}(t)$ is given by the affine Pieri rule in Eq. (1.2).

The structure constants $\mathrm{c}^{\nu,(c)}_{\lambda,\mu}(t)$ (1.4) constitute a t-deformation of the fusion coefficients for the genus zero $\widehat{\mathfrak{sl}}(n)_c$-Wess-Zumino-Witten conformal field theories, which are recovered at $t = 0$. Indeed, $s_\lambda(x) = R_\lambda(x;0)$ is given by the $\mathfrak{sl}(n)$ Schur character and the vector of coordinate angles becomes $\xi_\mu(0) = \frac{2\pi}{n+c}(\rho+\mu)$. The coordinates of the points in $X^{(n,c)} = X^{(n,c)}(0)$ are thus given by explicit roots of unity: $x_\mu(0) = e^{\frac{2\pi i}{n+c}(\rho+\mu)} = (e^{\frac{2\pi i}{n+c}(\rho_1+\mu_1)},\ldots,e^{\frac{2\pi i}{n+c}(\rho_n+\mu_n)})$, $\mu \in \Lambda^{(n,c)}$. The basis functions $s^{(c)}_\lambda : X^{(n,c)} \to \mathbb{C}$, $\lambda \in \Lambda^{(n,c)}$, given by

$$s^{(c)}_\lambda(x_\mu) = s_\lambda(e^{\frac{2\pi i}{n+c}(\rho+\mu)}) \qquad (\lambda,\mu \in \Lambda^{(n,c)}),$$

and the associated structure constants

$$s^{(c)}_\lambda s^{(c)}_\mu = \sum_{\nu\in\Lambda^{(n,c)}} \mathrm{c}^{\nu,(c)}_{\lambda,\mu} s^{(c)}_\nu \qquad (\lambda,\mu \in \Lambda^{(n,c)})$$

for the algebra of functions $f : X^{(n,c)} \to \mathbb{C}$ in this basis, provide a well-studied combinatorial model for the genus zero $\widehat{\mathfrak{sl}}(n)_c$-Wess-Zumino-Witten fusion ring [14,18,20,26,27,29]. In particular, the corresponding $t = 0$ specialization of the affine Pieri formula (1.2):

$$s_\lambda^{(c)} s_{\omega_r}^{(c)} = \sum_{\substack{J \subseteq \{1,\dots,n\},\, |J|=r \\ \lambda+\bar{e}_J \in \Lambda^{(n,c)}}} s_{\lambda+\bar{e}_J}^{(c)} \tag{1.5}$$

is well-known in this context, cf. e.g. [2, Equation (3.2)], [14, Equations (16.112), (16.121)], [18, Equation (3.6)], [20, Proposition 2.6], and [47, Theorem 6.2].

It is important to emphasize at this point that the deformation of the genus zero $\widehat{\mathfrak{sl}}(n)_c$-Wess-Zumino-Witten fusion ring stemming from Eq. (1.4) is not constructed in exactly the same manner as in [28, Section 7]. In a nut-shell: both deformations are related via level-rank duality [14,20,41], which has not been established for $t \in (-1, 1) \setminus \{0\}$ and thus a priori gives rise to two dual choices for the Hall-Littlewood deformation of the fusion ring.

In order to generalize the affine Pieri formula (1.2) from $\widehat{\mathfrak{sl}}(n)_c$ to other affine Lie algebras, we present an affine counterpart of the Pieri formula for Macdonald spherical functions of arbitrary simple Lie type from [11], which stems from the implementation of periodic boundary conditions. The underlying representations of the affine Hecke algebra that lead to this affine Pieri formula are inspired by previous constructions for the graded affine Hecke algebra that were developed in the context of the study of quantum integrable particle models, cf. [22,15] and references therein. From this perspective, a partial construction for twisted affine Lie algebras can be found in [12]; here we apply these techniques to present a combinatorial model to compute the structure constants of deformed genus zero Wess-Zumino-Witten fusion rings for both twisted and untwisted affine Lie algebras (excluding those of type $\widehat{BC}_n = A_{2n}^{(2)}$, cf. [10]). In line with was remarked at the end of the first paragraph for $\widehat{\mathfrak{sl}}(n)_c$, we expect that these deformed fusion rings are isomorphic to deformed Verlinde algebras from [52,53]; for $\widehat{\mathfrak{sl}}(2)_c$ this isomorphism is manifest from the explicit construction in [1, Appendices A and B].

The material is organized as follows. Section 2 presents our deformation of the genus zero Wess-Zumino-Witten fusion ring, which is built from a basis of periodic Macdonald spherical functions. The main result is an affine Pieri rule that permits to compute the structure constants for the multiplication in the periodic Macdonald spherical basis by basis elements attached to weights that are either minuscule or quasi-minuscule. After setting up some further notational preliminaries concerning the affine Weyl group in Section 3, the pertinent structure constants are exhibited in Section 4. When the deformation parameter vanishes, one finds a corresponding Pieri formula and structure constants for the genus zero Wess-Zumino-Witten fusion ring itself. The bulk of the paper is devoted to the proof of our Pieri rule via a suitable representation of the Hecke algebra of the affine Weyl group. Specifically, the affine Hecke algebra is first employed in Section 5 to construct an affine intertwining operator acting in the space of complex functions over the weight lattice. Via a standard construction involving the idempotent associated with the trivial representation of the Hecke algebra of the finite Weyl group, the periodic Macdonald spherical functions arise in Section 6 upon acting with the affine intertwining operator. In Section 7 it is shown that the periodic Macdonald spherical functions give rise to a basis for a finite-dimensional algebra of functions supported on critical points of a ‘fusion potential’ of the type in Eq. (1.3). We apply the affine intertwining operator so as to derive a family of difference operators diagonalized by the basis of periodic Macdonald spherical functions. The action of these difference operators permits us to compute the corresponding structure constants associated with this basis. In Section 8 the computation in question is carried out explicitly for the particular case of the Pieri formula, and Section 9 outlines how to recover the structure constants more generally from the action of the difference operators.

2. Affine Pieri rule

2.1. Macdonald spherical functions

Let V be a real finite-dimensional Euclidean vector space with inner product $\langle \cdot, \cdot \rangle$ spanned by an irreducible reduced crystallographic root system R_0. We write Q, P, and W_0, for the root lattice, the weight lattice, and the Weyl group associated with R_0. The semigroup of the root lattice generated by a (fixed) choice of positive roots R_0^+ is denoted by Q^+ whereas P^+ stands for the corresponding cone of dominant weights (see e.g. [7,24] for more details concerning root systems).

The dual root system $R_0^\vee := \{\alpha^\vee \mid \alpha \in R_0\}$ and its positive subsystem $R_0^{\vee,+}$ are obtained from R_0 and R_0^+ by applying the involution

$$x \mapsto x^\vee := 2x/\langle x, x\rangle \qquad (x \in V \setminus \{0\}). \tag{2.1}$$

Definition 2.1. For $\lambda \in P^+$, the Macdonald spherical function $M_\lambda : V \to \mathbb{C}$ is the W_0-invariant trigonometric polynomial given explicitly by

$$M_\lambda(\xi) = \sum_{v \in W_0} C(v\xi) e^{i\langle v\xi, \lambda\rangle} \tag{2.2}$$

with

$$C(\xi) := \prod_{\alpha \in R_0^+} \frac{1 - t_\alpha e^{-i\langle \xi, \alpha\rangle}}{1 - e^{-i\langle \xi, \alpha\rangle}}. \tag{2.3}$$

Here $t : R_0 \longrightarrow \mathbb{C}$ is a root multiplicity function such that $t_{w\alpha} = t_\alpha$ for every $w \in W_0$ and $\alpha \in R_0$.

For our purposes the range of the root multiplicity function will be restricted such that $t : R_0 \to (-1, 1) \setminus \{0\}$.

2.2. Basis of periodic Macdonald spherical functions

Let φ and ϑ denote the highest root and the highest short root of R_0^+, respectively. We fix an admissible pair $(R_0, \hat{R}_0)$ with $\hat{R}_0$ being equal either to $R_0^\vee$ or to $u_\varphi R_0$, where $u_\varphi = \frac{2}{\langle\varphi,\varphi\rangle}$, and with the positive system $\hat{R}_0^+$ obtained from R_0^+. In particular, for simply-laced R_0 we have that $\hat{R}_0 = R_0^\vee$. For $\alpha \in R_0$, let $\hat{\alpha} := \alpha^\vee$ if $\hat{R}_0 = R_0^\vee$ and let $\hat{\alpha} := u_\varphi \alpha$ if $\hat{R}_0 = u_\varphi R_0$. Then $\alpha^\vee = m_\alpha \hat{\alpha}$ with $m_\alpha = 2\langle\alpha, \hat{\alpha}\rangle^{-1}$, i.e. $m_\alpha = 1$ if $\hat{R}_0 = R_0^\vee$ and $m_\alpha = \frac{\langle\varphi,\varphi\rangle}{\langle\alpha,\alpha\rangle}$ if $\hat{R}_0 = u_\varphi R_0$. It follows that $\{m_\alpha\}_{\alpha\in R_0} = \{1, m_\vartheta\}$.

We denote the highest short root of $\hat{R}_0^{\vee,+}$ by $-\alpha_0 = \vartheta(\hat{R}_0^\vee)$, so in particular $\alpha_0 = -\vartheta$ if $\hat{R}_0 = R_0^\vee$ and $\alpha_0 = -\varphi$ if $\hat{R}_0 = u_\varphi R_0$. We also write $\hat{Q}$, $\hat{Q}^\vee$, $\hat{P}$ and $\hat{P}^\vee$ for the root lattice, the co-root lattice, the weight lattice and the co-weight lattice of $\hat{R}_0$, respectively. In this setup it will turn out natural to extend the domain of the root multiplicity function in a straightforward manner: $t : R_0 \cup R_0^\vee \cup \hat{R}_0 \to (-1, 1) \setminus \{0\}$ such that $t_{\hat{\alpha}} = t_{\alpha^\vee} = t_\alpha$ for all $\alpha \in R_0$.

Given a fixed positive integer $c > 1$, we consider two affine alcoves in P and $\hat{P}$:

$$P_c := \{\lambda \in P \mid 0 \le \langle\lambda, \beta\rangle \le c,\ \forall \beta \in \hat{R}_0^+\}, \tag{2.4}$$

$$\hat{P}_c := \{\mu \in \hat{P} \mid 0 \le \langle\mu, \alpha\rangle \le c, \forall \alpha \in R_0^+\}, \tag{2.5}$$

and an associated set of nodes $\mathcal{P}_c := \left\{\xi_\mu \mid \mu \in \hat{P}_c\right\}$. Here $\xi_\mu := \xi_\mu(t)$ $(\mu \in \hat{P})$ is defined as the unique global minimum of a radially unbounded strictly convex Morse function $\mathcal{V}_\mu : V \to \mathbb{R}$ of the form

$$\mathcal{V}_\mu(\xi) = \frac{c}{2}\langle\xi, \xi\rangle - 2\pi\langle\hat{\rho} + \mu, \xi\rangle + \sum_{\alpha\in R_0^+} \frac{2}{\langle\alpha, \hat{\alpha}^\vee\rangle} \int_0^{\langle\xi,\alpha\rangle} v_\alpha(\mathrm{x})\mathrm{dx}, \tag{2.6}$$

where $\hat{\rho} := \rho(\hat{R}_0) = \frac{1}{2}\sum_{\alpha\in\hat{R}_0^+} \alpha$ and $v_\alpha(\mathrm{x}) := (1 - t_\alpha^2)\int_0^{\mathrm{x}} \frac{\mathrm{dy}}{1-2t_\alpha \cos(\mathrm{y})+t_\alpha^2}$.

Let $\mathcal{C}(\mathcal{P}_c)$ denote the algebra of functions $f : \mathcal{P}_c \to \mathbb{C}$. For any $\lambda \in P_c$, the periodic Macdonald spherical function $M_\lambda^{(c)} \in \mathcal{C}(\mathcal{P}_c)$ is given by the restriction of M_λ to the nodes $\mathcal{P}_c$.

Theorem 2.2 *(Basis). The periodic Macdonald spherical functions $M_\lambda^{(c)}$, $\lambda \in P_c$, form a basis of $\mathcal{C}(\mathcal{P}_c)$.*

Remark 2.3. In Sect. 6 we will introduce the W_0-invariant affine Macdonald spherical functions $\Phi_\xi \in \mathcal{C}(P)$. It will be seen that, for $\xi \in \mathcal{P}_c$, the lattice function Φ_ξ is periodic with respect to translations over elements in $c\hat{Q}^\vee \subset P$ and that $M_\lambda^{(c)}(\xi) = \Phi_\xi(\lambda)$ for $\lambda \in P_c$ (see Remark 6.5 for more details).

2.3. Structure constants

Theorem 2.2 gives rise to an affine analog of the (t-deformed) Littlewood-Richardson coefficients:

$$M_\lambda^{(c)} M_\mu^{(c)} = \sum_{\nu \in P_c} \mathrm{c}_{\lambda,\mu}^{\nu,(c)}(t) M_\nu^{(c)} \qquad (\lambda, \mu \in P_c). \tag{2.7}$$

When μ is minuscule or quasi-minuscule we have an explicit expression for the structure constants $\mathrm{c}_{\lambda,\mu}^{\nu,(c)}(t)$. Let us recall in this connection that a weight $\mu \in P$ is called *minuscule* if $0 \leq \langle \mu, \alpha^\vee \rangle \leq 1$ for all $\alpha \in R_0^+$ and *quasi-minuscule* if $0 \leq \langle \mu, \alpha^\vee \rangle \leq 2$ for all $\alpha \in R_0^+$ with the upper bound being realized only once (i.e., the quasi-minuscule weight is unique and equal to the highest short root ϑ).

To formulate this explicit expression for the corresponding structure constants let us put:

$$\mathrm{m}_\lambda(e^{i\xi}) := \sum_{\nu \in W_0 \lambda} e^{i\langle \nu, \xi \rangle} \quad (\lambda \in P,\ \xi \in V),$$

and

$$\hat{h}_t := t_{\alpha_0}\, \hat{e}_t(-\alpha_0^\vee) \tag{2.8}$$

with

$$\hat{e}_t(\eta) := t_\vartheta^{\langle \hat{\rho}_s^\vee, \eta \rangle} t_\varphi^{\langle \rho(\hat{R}_0^\vee) - \hat{\rho}_s^\vee, \eta \rangle} = \prod_{\beta \in \hat{R}_0^+} t_\beta^{\langle \eta, \beta^\vee \rangle / 2} \quad (\eta \in \hat{Q}) \tag{2.9}$$

and where $\hat{\rho}_s^\vee = \frac{1}{2} \sum_{\alpha \in W_0 \vartheta \cap R_0^+} \hat{\alpha}^\vee \in \hat{P}^\vee$ (so $\hat{e}_t(\eta)$ is a Laurent polynomial in t_α, cf. e.g. [35]).

Theorem 2.4 *(Affine Pieri rule). For $\lambda \in P_c$, $\xi \in \mathcal{P}_c$ and $\omega \in P^+$ minuscule or quasi-minuscule, we have that*

$$\mathrm{m}_\omega(e^{i\xi}) M_\lambda^{(c)}(\xi) = U_{\lambda,\omega}(t) M_\lambda^{(c)}(\xi) + \sum_{\substack{\nu \in W_0 \omega \\ \lambda + \nu \in P_c}} V_{\lambda,\nu}(t) M_{\lambda+\nu}^{(c)}(\xi). \tag{2.10}$$

Here

$$V_{\lambda,\nu}(t) := \prod_{\substack{\beta \in \hat{R}_0^+ \\ \langle \lambda, \beta \rangle = 0 \\ \langle \nu, \beta \rangle > 0}} \frac{1 - t_\beta \hat{e}_t(\beta)}{1 - \hat{e}_t(\beta)} \prod_{\substack{\beta \in \hat{R}_0^+ \\ \langle \lambda, \beta \rangle = c \\ \langle \nu, \beta \rangle < 0}} \frac{1 - t_\beta \hat{h}_t \hat{e}_t(-\beta)}{1 - \hat{h}_t \hat{e}_t(-\beta)} \tag{2.11}$$

and the coefficient $U_{\lambda,\omega}(t)$ is given by Eq. (4.1) below (which implies in particular that $U_{\lambda,\omega}(t)$ vanishes when ω is minuscule).

To state the exact expressions for the coefficient $U_{\lambda,\omega}(t)$ in (2.10) and for $\mathrm{c}_{\lambda,\mu}^{\nu,(c)}(t)$ in Eq. (2.7) when μ is (quasi)-minuscule, some more notation regarding the underlying affine Weyl group and affine root system is required. (A more thorough discussion can be found e.g. in [7,24,38]).

3. The affine Weyl group

The *affine root system* R associated with the admissible pair $(R_0, \hat{R}_0)$ is the set of all *affine roots* $\alpha^\vee + m_\alpha rc = m_\alpha(\hat{\alpha} + rc)$ ($\alpha \in R_0, r \in \mathbb{Z}$). An affine root $a = \alpha^\vee + m_\alpha rc \in R$ will be regarded as an affine linear function $a : V \to \mathbb{R}$ of the form $a(x) = \langle x, \alpha^\vee \rangle + rc$ ($x \in V$, $\alpha \in R$, $r \in \mathbb{Z}$), and gives rise to an affine reflection $s_a : V \to V$ across the hyperplane $V_a := \{x \in V \mid a(x) = 0\}$ given by $s_a(x) := x - a(x)\alpha$. The choice of positive roots R_0^+, with a simple basis $\alpha_1, \ldots, \alpha_n$, determines the set of affine positive roots $R^+ := R_0^{\vee,+} \cup \{\alpha^\vee + m_\alpha rc \mid \alpha \in R_0, r \in \mathbb{N}\}$ and a corresponding basis of affine simple roots $a_0, \ldots, a_n$ of the form $a_0 := \alpha_0^\vee + c$ and $a_j := \alpha_j^\vee$ for $j = 1, \ldots, n$. Here n denotes the rank of R_0 ($= \dim V$). Notice that these conventions imply that the affine root system R is of *twisted* type iff $\hat{R}_0 = u_\varphi R_0$ is not simply-laced and of *untwisted* type otherwise.

The *affine Weyl group* W is defined as the group generated by the affine reflections s_a, $a \in R$ and contains the finite Weyl group W_0 as the subgroup fixing the origin. It is an infinite Coxeter group with the simple affine reflections $s_j := s_{a_j}$ ($j = 0, 1, \ldots, n$) as generators and subject to the relations

$$(s_j s_k)^{m_{jk}} = 1, \qquad j, k \in \{0, \ldots, n\} \tag{3.1}$$

Here $m_{jk} = 1$ if $j = k$ and $m_{jk} \in \{2, 3, 4, 6\}$ if $j \neq k$ (and the provision that for $n = 1$ the order $m_{10} = m_{01} = \infty$). In particular, any $w \in W$ can be decomposed as

$$w = s_{j_1} \cdots s_{j_\ell}, \tag{3.2}$$

with $j_1, \ldots, j_\ell \in \{0, \ldots, n\}$. The *length* $\ell(w)$ is defined as the minimum number of reflections s_j ($j = 0, 1, \ldots, n$) involved in any decomposition (3.2) of w. Any decomposition (3.2) with $\ell = \ell(w)$ is called a *reduced expression* of w.

A fundamental domain for the action of W on V is given by the *dominant Weyl alcove*

$$A_c = \{x \in V \mid 0 \leq \langle x, \beta \rangle \leq c, \ \forall \beta \in \hat{R}_0^+\}. \tag{3.3}$$

Furthermore, since our positive scale parameter c is integral-valued the weight lattice $P \subset V$ is stable with respect to the action of W and P_c (2.4) is a fundamental domain for this restriction. Given $x \in V$, we will also write $w_x \in W$ for the *unique* shortest affine Weyl group element such that

$$x_+ := w_x x \in A_c. \tag{3.4}$$

For any $\lambda \in P$ let

$$t[\lambda] := \prod_{a \in R[\lambda]} t_{a'} \tag{3.5}$$

where $(\alpha^\vee + m_\alpha rc)' := \alpha^\vee$ denotes the differential and

$$R[\lambda] := \{a \in R^+ \mid a(\lambda) < 0\}. \tag{3.6}$$

The action of $w \in W$ on V induces a dual action on the space $\mathcal{C}(V)$ of functions $f : V \to \mathbb{C}$ given by

$$(wf)(x) := f(w^{-1}x) \qquad (w \in W,\, f \in \mathcal{C}(V),\, x \in V). \tag{3.7}$$

In Section 4 we will use that $\mathcal{C}(P)$ is an invariant subspace under this action.

4. Affine Littlewood-Richardson coefficients and fusion rules

We are now in the position to make the coefficient $U_{\lambda,\omega}(t)$ in Theorem 2.4 explicit:

$$U_{\lambda,\omega}(t) = \sum_{\substack{\nu \in W_0\omega \\ (\lambda+\nu)_+ = \lambda}} t[\lambda+\nu] + (1 - t_\vartheta^{-1}) \sum_{\substack{\nu \in W_0\omega \\ w_{\lambda+\nu}\lambda = \lambda}} d_{\lambda,\nu}, \tag{4.1}$$

where

$$d_{\lambda,\nu} := \begin{cases} \theta(\lambda+\nu) e_t(-\nu) h_t^{\mathrm{sign}(\langle \lambda, \hat{\nu} \rangle)} & \text{if } \nu \in W_0\vartheta \\ 0 & \text{otherwise} \end{cases} \tag{4.2a}$$

(with the convention that $\mathrm{sign}(0) := 0$) and

$$h_t := t_\vartheta\, e_t(-\alpha_0) \quad \text{with} \quad e_t(\nu) := \prod_{\alpha \in R_0^+} t_\alpha^{\langle \nu, \alpha^\vee \rangle / 2}, \tag{4.2b}$$

(cf. Eq. (2.9)). Here $\theta : P \to \mathbb{N} \cup \{0\}$ denotes the function

$$\theta(\lambda) := \left| \{a \in R^+ \mid a(\lambda) = -2\} \right|. \tag{4.2c}$$

Remark 4.1. Observe that $d_{\lambda,\nu}$ is also a Laurent polynomial in t_α, $\alpha \in R_0$. We will also see in Lemma 8.2 that $\theta(\lambda+\nu) = 0$ if ν is in the orbit of a minuscule weight and therefore it is also possible to write $d_{\lambda,\nu} = \theta(\lambda+\nu) e_t(-\nu) h_t^{\mathrm{sign}(\langle \lambda, \nu^\vee \rangle)}$.

A function $t : W \to (-1, 1) \setminus \{0\}$ satisfying $t_{w\tilde{w}} = t_w t_{\tilde{w}}$ if $\ell(w\tilde{w}) = \ell(w) + \ell(\tilde{w})$ is called a length multiplicative function. We compatibilize this function with the root multiplicative function by setting $t_j := t_{s_j} = t_{\alpha_j}$ for $j = 0, 1, \ldots, n$. For any finite subgroup $G \subset W$ we consider the generalized Poincaré series

$$G(t) = \sum_{w \in G} t_w \tag{4.3}$$

of G associated with the length multiplicative function t.

Corollary 4.2 *(Affine Littlewood-Richardson coefficients). If ω is a (quasi)-minuscule weight and $\lambda, \nu \in P_c$, then the affine Littlewood-Richardson coefficients in (2.7) are given by*

$$c^{\nu,(c)}_{\lambda,\omega}(t) = \begin{cases} W_{0;\omega}(t)\big(U_{\lambda,\omega}(t) - U_{0,\omega}(t)\big) & \textit{if } \lambda = \nu, \\ W_{0;\omega}(t) V_{\lambda,\nu-\lambda}(t) & \textit{if } \nu - \lambda \in W_0\omega, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.4}$$

Here $W_{0;\omega}(t)$ refers to generalized Poincaré series (4.3) of $W_{0;\omega} = \{w \in W_0 \mid w\omega = \omega\}$.

Proof. Applying Theorem 2.4 with $\lambda = 0$ and using that $M_0(\xi) = W_0(t)$ and $V_{0,\theta}(t) = W_0(t)/W_{0;\vartheta}(t)$ yields the identity $M^{(c)}_\omega(\xi) = W_{0;\omega}(t)\big(\mathrm{m}_\omega(e^{i\xi}) - U_{0,\omega}(t)\big)$. Combining this with Theorem 2.4 entails the desired result. □

When R_0 is of type A_{n-1} and ω is minuscule, Corollary 4.2 reproduces the affine Pieri rule in Eq. (1.2).

At $t_\alpha = 0$ ($\alpha \in R_0$) the Macdonald spherical functions M_λ (2.2) specialize to the Weyl characters

$$\chi_\lambda(\xi) = \delta(\xi)^{-1} \sum_{w \in W_0} (-1)^{\ell(w)} e^{i\langle w\xi, \lambda+\rho\rangle} \qquad (\lambda \in P^+), \tag{4.5}$$

where $\rho = \rho(R_0)$ and $\delta(\xi)$ denotes the Weyl denominator

$$\delta(\xi) = \sum_{w \in W_0} (-1)^{\ell(w)} e^{i\langle w\xi, \rho\rangle} = \prod_{\alpha \in R_0^+} (e^{i\langle \xi,\alpha\rangle/2} - e^{-i\langle \xi,\alpha\rangle/2}). \tag{4.6}$$

The nodes $\mathcal{P}_c$ are in this situation given explicitly by

$$\xi_\mu(0) := \frac{2\pi}{h+c}(\hat{\rho} + \mu) \qquad (\mu \in \hat{P}_c), \tag{4.7}$$

where $h = h(R) = 1 - \langle \rho, \alpha_0^\vee\rangle$ denotes the Coxeter number of the affine root system R. Indeed, $\lim_{t\to 0} \xi_\mu(t) = \xi_\mu(0)$ by Lemma 6.2 and Eq. (6.8) below, since $\hat{\rho}_v(\xi) = h\xi$ for $t_\alpha = 0$ in view of Schur's lemma.

The corresponding parameter degeneration of the structure constants in $\mathcal{C}(\mathcal{P}_c)$,

$$\chi_\lambda(\xi)\chi_\mu(\xi) = \sum_{\nu\in P_c} \mathrm{c}_{\lambda,\mu}^{\nu,(c)}(0)\chi_\nu(\xi) \qquad (\lambda,\mu\in P_c,\ \xi\in\mathcal{P}_c), \tag{4.8}$$

model the fusion rules of the genus-zero Wess-Zumino-Witten conformal field theories associated with the affine Lie algebra $\hat{\mathfrak{g}}$ of type $R^\vee = \{2a/\langle a', a'\rangle \mid a\in R\}$ [14,23,26] (cf. also Remark 4.3 below). Corollary 4.2 gives rise to the following Pieri rule in the fusion ring for $\mu=\omega\in P^+$ minuscule or quasi-minuscule:

$$\boxed{\chi_\omega(\xi)\chi_\lambda(\xi) = (N_{0,\omega}-N_{\lambda,\omega})\chi_\lambda(\xi) + \sum_{\substack{\nu\in W_0\omega\\ \lambda+\nu\in P_c}} \chi_{\lambda+\nu}(\xi) \qquad (\lambda\in P_c,\ \xi\in\mathcal{P}_c),}$$

with

$$N_{\lambda,\omega} = |\{a_j \mid a_j(\lambda)=0 \text{ and } \alpha_j\in W_0\omega\}|$$

(so $N_{\lambda,\omega}=0$ if ω is minuscule). When R_0 is of type A_{n-1} and ω is minuscule, this Pieri rule amounts to Eq. (1.5).

To infer the boxed Pieri rule, one first observes that $\lim_{t\to 0} V_{\lambda,\nu-\lambda}(t) = 1$, $\lim_{t\to 0} W_{0;\omega}(t)=1$, and

$$\lim_{t\to 0}(1-t_\vartheta^{-1})d_{\lambda,\nu} = \begin{cases} -\lim_{t\to 0} t_\vartheta^{-1}e_t(-\nu) & \text{if } \nu\in R_0^-\cap W_0\vartheta \text{ and } \langle\lambda,\hat{\nu}\rangle = 0,\\ -\lim_{t\to 0} e_t(-\alpha_0-\nu) & \text{if } \nu\in R_0^+\cap W_0\vartheta \text{ and } \langle\lambda,\hat{\nu}\rangle = c,\end{cases}$$

and where $R_0^- = -R_0^+ = R_0\backslash R_0^+$. Since $\lim_{t\to 0} t_\vartheta^{-1}e_t(\alpha)=1$ if $\alpha\in R_0^+$ is a simple root and 0 otherwise, this shows that $\lim_{t\to 0} U_{\lambda,\omega}(t) = -N_{\lambda,\omega}$ in view of Lemma 8.2 below. The asserted Pieri rule in the fusion ring is now immediate from Corollary 4.2.

Remark 4.3. If, following [38, Chapter I], we denote by $S(R_0;c) = R_0 + c\mathbb{Z}$ the (untwisted) affine root system associated with R_0. Then the following table identifies the Dynkin type of the affine root system R and of the affine Lie algebra $\hat{\mathfrak{g}}$ in terms of the admissible pair $(R_0,\hat{R}_0)$, via the classification in [38, Chapter I.3]:

$(R_0,\hat{R}_0)$	R	$\hat{\mathfrak{g}}$
$(R_0,R_0^\vee)$	$S(R_0^\vee;c)$	$S(R_0^\vee;c)^\vee$
$(R_0,u_\varphi R_0)$	$S(R_0;c/u_\varphi)^\vee$	$S(R_0;c/u_\varphi)$

Remark 4.4. If $\hat{\mathfrak{g}}$ is untwisted (i.e. $(R_0,\hat{R}_0) = (R_0,u_\varphi R_0)$), then $\mathrm{c}_{\lambda,\vartheta}^{\nu,(c)}(0)$ is a nonnegative integer. The same is true when $\hat{\mathfrak{g}}$ is twisted (i.e. $(R_0,\hat{R}_0) = (R_0,R_0^\vee)$ with R_0 not simply-laced) provided c is not an integer multiple of $\frac{\langle\varphi,\varphi\rangle}{\langle\vartheta,\vartheta\rangle}\in\{2,3\}$. If $\hat{\mathfrak{g}}$ is twisted and c is an

integer multiple of $\frac{\langle\varphi,\varphi\rangle}{\langle\vartheta,\vartheta\rangle} \in \{2,3\}$, however, then $\mathrm{c}_{\lambda,\vartheta}^{\lambda,(c)}(0) = N_{0,\vartheta} - N_{\lambda,\vartheta} = -1$ when $\lambda \in P_c$ is chosen such that $a_j(\lambda) = 0$ for all $j \in \{0,1,2,\ldots,n\}$ with $\alpha_j \in W_0\vartheta$. This state of affairs is in agreement with prior observations in [19] regarding the occurrence of negative structure constants in the genus-zero Wess-Zumino-Witten fusion ring when the underlying affine Lie algebra $\hat{\mathfrak{g}}$ is twisted.

Remark 4.5. Recently in [10] a deformation of the Wess-Zumino-Witten fusion ring of type $\widehat{BC}_n = A_{2n}^{(2)}$ was derived, based on a diagonalization of a finite q-boson model with diagonal open end boundary conditions obtained in [13]. The approach in [13] does not use affine Hecke algebras and hinges instead on Sklyanin's quantum inverse scattering method (a.k.a. the algebraic Bethe Ansatz method) for the diagonalization of quantum integrable eigenvalue problems with boundary conditions.

5. Affine intertwining operator

The remainder of the paper is devoted to the proofs of Theorems 2.2 and 2.4 with the aid of the *affine Hecke algebra H*. By definition, H is the unital associative algebra over $\mathbb{C}$ with invertible generators $T_0, T_1 \ldots, T_n$ such that the following relations are satisfied

$$(T_j - t_j)(T_j + 1) = 0 \qquad (0 \le j \le n), \tag{5.1}$$

$$\underbrace{T_jT_kT_j\cdots}_{m_{jk}\ \text{factors}} = \underbrace{T_kT_jT_k\cdots}_{m_{jk}\ \text{factors}} \qquad (0 \le j \ne k \le n), \tag{5.2}$$

where the number of factors m_{jk} on both sides of the braid relation (5.2) is the same as the order of the corresponding braid relation (3.1) for W (see e.g. [24,38]).

For a reduced expression $w = s_{j_1}\cdots s_{j_\ell}$, let $T_w := T_{j_1}\ldots T_{j_\ell}$ (which does not depend on the choice of the reduced expression by virtue of the braid relations). It is known that the elements T_w, with $w \in W$, form a basis for H over $\mathbb{C}$.

The subalgebra of H generated by $T_1, \ldots T_n$ is referred to as the *finite Hecke algebra* H_0 (associated with W_0 and t).

To define the affine intertwining operator we need the following integral-reflection representation of H.

Proposition 5.1. *The following defines an action of H on $\mathcal{C}(P)$:*

$$T_jf = (t_js_j + (t_j - 1)J_j)f \qquad (f \in \mathcal{C}(P), j = 0,\ldots,n), \tag{5.3}$$

where $J_j : \mathcal{C}(P) \to \mathcal{C}(P)$ denotes the operator given by

$$(J_jf)(\lambda) := \begin{cases} -\sum_{k=1}^{a_j(\lambda)} f(\lambda - k\alpha_j) & \text{if } a_j(\lambda) > 0, \\ 0 & \text{if } a_j(\lambda) = 0, \\ \sum_{k=0}^{-a_j(\lambda)-1} f(\lambda + k\alpha_j) & \text{if } a_j(\lambda) < 0, \end{cases} \tag{5.4}$$

Proof. For any $k \in \{0, 1, \dots, n\}$ consider the (finite dimensional) parabolic subalgebra H_k of H generated by $T_0, T_1, \dots, T_{k-1}, T_{k+1}, \dots, T_n$. The idea of the proof is to show that, for any (fixed) k, $T_j \mapsto I_j$, $j \neq k$ extends to a representation of H_k on $\mathcal{C}(P)$. Here $I_j := t_j s_j + (t_j - 1)J_j$ denote the operator on the right-hand side of Eq. (5.3). The fact that H is generated by $T_0, \dots, T_n$ subjected to the braid relations and quadratic relations implies then the proposition. For this let us introduce the vertices $v_0 = 0, v_1, \dots, v_n$ of the alcove A_c (3.3), so in particular

$$a_j(v_k) = \delta_{j,k}, \qquad \text{for all } j \neq k. \tag{5.5}$$

We fix a $k \in \{0, 1, \dots, n\}$ and consider the finite subsystem obtained from R by considering the vertex v_k as the "new origin": $R_k = \{(a')^\vee \mid a \in R, a(v_k) = 0\}$. Then $a \mapsto a'$ defines a root system isomorphism from the parabolic subsystem $\{a \in R \mid a(v_k) = 0\}$ of R onto $R_k^\vee$. The root system R_k is a finite root system of rank n in V, although not necessarily irreducible. A basis of simple roots for R_k is given by α_j, $j \neq k$. The map $s_j \mapsto s_j' = s_{\alpha_j^\vee}$, $j \neq k$ defines a group isomorphism from the parabolic subgroup $W_k = \langle s_j \mid j \neq k\rangle$ of W to the finite Weyl group $W_0(R_k)$ of R_k. This isomorphism induces a natural isomorphism from the parabolic subalgebra $H_k = \langle T_j \mid j \neq k\rangle$ of H to the finite Hecke algebra $H_0(W_0(R_k))$ associated with $W_0(R_k)$ and t_j, $j \neq k$.

From $R_k \subset R_0$ follows that $P \subset P(R_k)$, where $P(R_k)$ denotes the weight lattice of R_k. We will also need the $(-v_k)$-translation $P_k' := -v_k + P \subset P(R_k)$ of P. For any $j \neq k$ consider the integral-reflection operator $I_j' : \mathcal{C}(P(R_k)) \to \mathcal{C}(P(R_k))$ associated to the finite root system R_k, i.e. $I_j' = t_j s_j' + (t_j - 1)J_j'$ where J_j' is given by the same expression as (5.4) but with $a_j(\lambda)$ replaced mechanically by $\langle \alpha_j^\vee, \lambda\rangle$. Observe that $\mathcal{C}(P_k')$ is invariant under the operators J_j' and I_j', $j \neq k$. The linear isomorphism $\ell_k : \mathcal{C}(P) \to \mathcal{C}(P_k')$ defined by $(\ell_k f)(y) = f(v_k + y)$ satisfies $\ell_k(s_j f) = s_j' \ell_k(f)$, $\ell_k(J_j f) = J_j' \ell_k(f)$, and therefore also $\ell_k(I_j f) = I_j' \ell_k(f)$, for all $j \neq k$.

By [11, Lem. 4.2], applied to the finite root system R_k and restricted to the finite Hecke algebra part, it follows that $T_j \mapsto I_j'$, $j \neq k$, defines a representation of $H_0(W_0(R_k))$ on $\mathcal{C}(P(R_k))$ (see also Remark 5.2 below). Since $\mathcal{C}(P_k')$ is an invariant subspace under this representation it follows that $T_j \mapsto I_j'$, $j \neq k$ extends to a representation of $H_0(W_0(R_k))$ on $\mathcal{C}(P_k')$. Using the above mentioned isomorphisms from H_k to $H_0(W_0(R_k))$ we deduce that $T_j \mapsto I_j'$, $j \neq k$ extends to a representation of H_k on $\mathcal{C}(P_k')$. By taking the pullback of the linear isomorphism ℓ_k we conclude that $T_j \mapsto I_j$, $j \neq k$ extends to representation of H_k on $\mathcal{C}(P)$. Since k was arbitrary this finishes the proof, as indicated in the beginning. □

Remark 5.2. In [11, Lem. 4.2] it was assumed that the underlying finite crystallographic root system R was irreducible and of full rank. However, the proposition holds for *all* finite crystallographic root systems of full rank. If R is decomposed into disjoint, irreducible and orthogonal subsystems $R_1 \cup \cdots \cup R_\ell$, then $H_0(W_0(R)) \simeq H_0(W_0(R_1)) \otimes \cdots \otimes$

$H_0(W_0(R_\ell))$ and the action of the integral-reflection representation also decomposes on $\mathcal{C}(P) \simeq \mathcal{C}(P(R_1)) \otimes \cdots \otimes \mathcal{C}(P(R_\ell))$, which yields immediately the result.

Alternatively, the proof of [11, Lem. 4.2] works verbatim if the requirement that R be irreducible is dropped.

The affine intertwining operator $\mathcal{J} : \mathcal{C}(P) \to \mathcal{C}(P)$ is now defined as follows:

$$(\mathcal{J} f)(\lambda) := t[\lambda]^{-1}(T_{w_\lambda} f)(\lambda_+). \tag{5.6}$$

Proposition 5.3. *The operator $\mathcal{J}$ is invertible.*

Proposition 5.3 is a direct consequence of Lemma 5.4 below. Given $v, w \in W$ we recall that $v \leq w$ in the *Bruhat partial order* on W if v may be obtained by deleting simple reflections from the reduced expression of w (see [38, Sec. 2.3]). For $x \in V$, we denote by $[x]$ the finite set $\{y \in V \mid y_+ = x_+ \text{ and } w_y \leq w_x\}$ and by $\mathrm{Conv}\,[x]$ the convex hull of $[x]$. Now we consider the following partial order $\preceq$ on P:

$$\forall \mu, \lambda \in P \quad \mu \preceq \lambda \text{ iff (i) } \lambda - \mu \in Q \text{ and (ii) } \mathrm{Conv}\,[\mu] \subseteq \mathrm{Conv}\,[\lambda]. \tag{5.7}$$

Lemma 5.4. *The action of $\mathcal{J}$ is triangular with respect to the above partial order:*

$$(\mathcal{J} f)(\lambda) = \sum_{\mu \in P,\, \mu \preceq \lambda} J_{\lambda,\mu} f(\mu), \qquad (f \in \mathcal{C}(P), \lambda \in P) \tag{5.8}$$

for some coefficients $J_{\lambda,\mu} \in \mathbb{C}$ and with $J_{\lambda,\lambda} = t[\lambda]^{-1}$.

Proof. We observe that $t[\lambda] = t_{w_\lambda}$ and proceed inductively in the length of w_λ. For $\ell(w_\lambda) = 0$ clearly $(\mathcal{J} f)(\lambda) = f(\lambda)$. Next, assuming $\ell(w_\lambda) > 0$ let j be such that $w_\lambda s_j < w_\lambda$. Hence

$$\begin{aligned}(\mathcal{J} f)(\lambda) &= t[\lambda]^{-1}(T_{w_\lambda} f)(\lambda_+) = t_j^{-1} t[s_j\lambda]^{-1}(T_{w_{s_j\lambda}} T_j f)((s_j\lambda)_+) \\ &\overset{\text{IH}}{=} t_j^{-1} \sum_{\mu \in P,\, \mu \preceq s_j\lambda} J_{s_j\lambda,\mu}(T_j f)(\mu) = \sum_{\mu \in P,\, \mu \preceq \lambda} J_{\lambda,\mu} f(\mu),\end{aligned} \tag{5.9}$$

where the step IH hinges on the induction hypothesis and the last equality is due to the fact that the convex hull of $\mathrm{Conv}\,[s_j\lambda]$ and $s_j(\mathrm{Conv}\,[s_j\lambda])$ is contained in $\mathrm{Conv}\,[\lambda]$ since $[s_j\lambda] \cup s_j([s_j\lambda]) \subseteq [\lambda]$.

The diagonal coefficient of

$$\sum_{\mu \in P,\, \mu \preceq s_j\lambda} J_{s_j\lambda,\mu}(T_j f)(\mu)$$

corresponds to term with $\mu = s_j\lambda$ and the coefficient of $f(\lambda)$ in $(T_j f)(s_j\lambda)$ is equal to 1 when $s_j\lambda \prec \lambda$ (by Eqs. (5.3), (5.4)). Hence, upon comparing the coefficients of $f(\lambda)$ on both sides of (5.9) it is seen that $J_{\lambda,\lambda} = t_j^{-1} J_{s_j\lambda,s_j\lambda}$, which proves the lemma. □

To prepare for the next section, we finish with a convenient characterization of the W-invariant subspace of $\mathcal{C}(P)$ in terms of H and $\mathcal{J}$.

Lemma 5.5. *The W-invariant subspace*

$$\mathcal{C}(P)^W := \{f \in \mathcal{C}(P) \mid wf = f,\ w \in W\} \tag{5.10}$$

consists of the functions $f : \mathcal{C}(P) \to \mathbb{C}$ that satisfy

$$\mathcal{J}T_j\mathcal{J}^{-1}f = t_j f \qquad (j = 0, \ldots, n).$$

Proof. For any $f \in \mathcal{C}(P)$, $j \in \{0, \ldots n\}$ and $\lambda \in P$ we have

$$\begin{aligned}
(\mathcal{J}T_j\mathcal{J}^{-1}f)(\lambda) =& t[\lambda]^{-1}(T_{w_\lambda}T_j\mathcal{J}^{-1}f)(\lambda_+)\\
=& \begin{cases} t[\lambda]^{-1}(T_{w_\lambda s_j}\mathcal{J}^{-1}f)(\lambda_+) & \text{if } a_j(\lambda) \geq 0\\ t[\lambda]^{-1}\left(t_j(T_{w_\lambda s_j}\mathcal{J}^{-1}f)(\lambda_+) + (t_j-1)(T_{w_\lambda}\mathcal{J}^{-1}f)(\lambda_+)\right) & \text{if } a_j(\lambda) < 0\end{cases}\\
=& t_j t[\lambda]^{-1}(T_{w_\lambda}\mathcal{J}^{-1}f)(\lambda_+)\\
&+ t_j^{\chi(a_j(\lambda))}\left(t_{w_\lambda s_j}^{-1}(T_{w_\lambda s_j}\mathcal{J}^{-1}f)(\lambda_+) - t[\lambda]^{-1}(T_{w_\lambda}\mathcal{J}^{-1}f)(\lambda_+)\right)\\
=& t_j f(\lambda) + t_j^{\chi(a_j(\lambda))}\left(f(s_j\lambda) - f(\lambda)\right).
\end{aligned}$$

Here χ denotes the characteristic function of $[0, \infty)$ and we also used that

$$T_wT_j = \begin{cases} T_{ws_j} & \text{if } \ell(ws_j) = \ell(w) + 1,\\ t_jT_{ws_j} + (t_j - 1)T_w & \text{if } \ell(ws_j) = \ell(w) - 1,\end{cases}$$

the relation $\ell(w_\lambda s_j) = \ell(w_\lambda) + 1$ if $a_j(\lambda) \geq 0$, $\ell(w_\lambda s_j) = \ell(w_\lambda) - 1$ if $a_j(\lambda) < 0$, and the observation that

$$t_{w_\lambda s_j}^{-1}(T_{w_\lambda s_j}f)(\lambda_+) = t[s_j\lambda]^{-1}(T_{w_{s_j\lambda}}f)(\lambda_+).$$

Hence, f is W-invariant if and only if $\mathcal{J}T_j\mathcal{J}^{-1}f = t_j f$. □

6. Periodic Macdonald spherical functions

For a $\xi \in V$ we define the *affine Macdonald spherical functions* function in $\mathcal{C}(P)$:

$$\Phi_\xi := \mathcal{J}\phi_\xi \quad \text{with} \quad \phi_\xi := \sum_{v \in W_0} T_v e^{i\xi}, \tag{6.1}$$

where $e^{i\xi}$ denotes the plane wave function $e^{i\xi}(\lambda) := e^{i\langle\lambda,\xi\rangle}$ $(\lambda \in P)$.

The plane waves decomposition for ϕ_ξ (6.1) in the next theorem is a known result, see [33, Thm. 1] and (with more details) [34, (4.1.2)] or also [42, Thm. 2.9(a)] and [44, Thm. 6.9]). To keep our presentation self contained we include a brief verification based on the representation from Proposition 5.1. For any $w \in W$ we define the finite set

$$R(w) := R^+ \cap w^{-1}(R^-)$$

with $R^- = -R^+ = R \backslash R^+$. Let us observe that the cardinality of $R(w)$ is equal to the length of w and that for any $\lambda \in P$ we have that $R(w_\lambda) = R[\lambda]$ (the reader may consult [38, Section 2.2] and [38, (2.4.4)], respectively). It is also clear that for any $w \in W_0$ one has $R(w) = R_0^+ \cap w^{-1}(R_0^-)$ and where (recall) $R_0^- = -R_0^+$.

Proposition 6.1. *The function ϕ_ξ, for*

$$\xi \in V_{reg} := \{\xi \in V \mid \langle \xi, \alpha \rangle \notin 2\pi\mathbb{Z}, \, \forall \alpha \in R_0^+\}, \tag{6.2}$$

decomposes as the following linear combination of plane waves

$$\phi_\xi = \sum_{w \in W_0} C(w\xi) \mathbf{e}^{iw\xi}, \tag{6.3}$$

with

$$C(\xi) := \prod_{\alpha \in R_0^+} \frac{1 - t_\alpha e^{-i\langle \xi, \alpha \rangle}}{1 - e^{-i\langle \xi, \alpha \rangle}}. \tag{6.4}$$

In particular, $\phi_\xi(\lambda) = M_\lambda(\xi)$ $(\lambda \in P^+, \xi \in V_{reg})$.

Proof. From the action of T_j $(j = 1, \ldots, n)$ (5.3) we have that for any $\boldsymbol{\xi} \in V_{\text{reg}}$

$$T_j \mathbf{e}^{i\xi} = \mathrm{b}_j(s_j\boldsymbol{\xi}) \mathbf{e}^{i\boldsymbol{\xi}} + \mathrm{c}_j(s_j\boldsymbol{\xi}) \mathbf{e}^{is_j\boldsymbol{\xi}} = \mathrm{b}_j(-\boldsymbol{\xi}) \mathbf{e}^{i\boldsymbol{\xi}} + \mathrm{c}_j(-\boldsymbol{\xi}) \mathbf{e}^{is_j\boldsymbol{\xi}}, \tag{6.5a}$$

with

$$\mathrm{c}_j(\boldsymbol{\xi}) = \frac{1 - t_j e^{-i\langle \xi, \alpha_j \rangle}}{1 - e^{-i\langle \xi, \alpha_j \rangle}}, \qquad \mathrm{b}_j(\boldsymbol{\xi}) = t_j - \mathrm{c}_j(\boldsymbol{\xi}) = \mathrm{c}_j(-\boldsymbol{\xi}) - 1.$$

Since the stabilizer of $\xi \in V_{\text{reg}}$ for the action of $W_0 \ltimes 2\pi Q^\vee$ is trivial, all the vectors $w\xi$, for $w \in W_0$, are different to each other modulo $2\pi Q^\vee$. Then, for any $\xi \in V_{\text{reg}}$ the plane waves $e^{iw\boldsymbol{\xi}}$, $w \in W_0$, are linearly independent in $\mathcal{C}(P)$. Therefore, the function ϕ_ξ may be written as

$$\phi_{\boldsymbol{\xi}} = \sum_{w \in W_0} C_w(\boldsymbol{\xi}) \mathbf{e}^{iw\xi}, \tag{6.5b}$$

for some unique coefficients $C_w(\boldsymbol{\xi}) \in \mathbb{C}$.

It follows from Eq. (6.5a) that for any reduced expression $w = s_{j_\ell} \dots s_{j_1} \in W_0$ the action of T_w on $e^{i\xi}$ is of the form

$$T_w \mathbf{e}^{i\xi} = \left(\prod_{1 \leq k \leq \ell} c_{j_k}(s_{j_k} \cdots s_{j_2} s_{j_1} \xi) \right) e^{iw\xi} \quad + \text{l.o.}, \tag{6.5c}$$

for some coefficients c_{j_k} and l.o. is a linear combination of plane waves $\mathbf{e}^{iv\xi}$ with $v < w$ in the Bruhat partial order on W_0.

Let w_0 be the longest element of W_0. Applying the above identity to a reduced expression $w_0 = s_{j_\ell} \dots s_{j_1}$ (so $\ell = \# R_0^+$) we conclude that

$$\begin{aligned} C_{w_0}(\xi) &= \prod_{1 \leq k \leq \ell} c_{j_k}(-s_{j_{k-1}} \cdots s_{j_2} s_{j_1} \boldsymbol{\xi}) = \prod_{1 \leq k \leq \ell} \frac{1 - t_{j_k} e^{i\langle \xi, s_{j_1} s_{j_2} \cdots s_{j_{k-1}} \alpha_{j_k} \rangle}}{1 - e^{i\langle \xi, s_{j_1} s_{j_2} \cdots s_{j_{k-1}} \alpha_{j_k} \rangle}} \\ &= \prod_{\alpha \in R_0^+} \frac{1 - t_\alpha e^{i\langle \xi, \alpha \rangle}}{1 - e^{i\langle \xi, \alpha \rangle}} = C(-\xi) = C(w_0 \xi). \end{aligned}$$

In the last equality we have used that $R_0^+ = R(w_0)$ and in the third that (see e.g. [38, (2.2.9)]) $R_0^+ = R(w_0) = \{ s_{j_1} s_{j_2} \cdots s_{j_{k-1}} \alpha_{j_k} \mid k = 1, 2, \dots, \ell \}$.

Let us denote the trivial idempotent

$$\imath_{\mathbf{0}} := \sum_{v \in W_0} T_v, \tag{6.6}$$

having then $\phi_\xi = \imath_{\mathbf{0}}\, e^{i\xi}$. Since $T_j \imath_{\mathbf{0}} = t_j \imath_{\mathbf{0}}$ we have that $T_j \phi_\xi = t_j \phi_\xi$ for $j = 1, \dots, n$. It follows from Eq. (6.5a) and the linear independence of the plane waves that for $\xi \in V_{\text{reg}}$

$$C_{s_j w}(\boldsymbol{\xi}) \mathrm{c}_j(w\boldsymbol{\xi}) = C_w(\boldsymbol{\xi}) \mathrm{c}_j(-w\boldsymbol{\xi}) \quad \text{for all } w \in W_0,\ j \in \{1, \dots, n\}. \tag{6.7}$$

On the other hand, from the product formula in Eq. (6.4) it follows that for any $\xi \in V_{\text{reg}}$

$$C(s_j \boldsymbol{\xi}) \mathrm{c}_j(\boldsymbol{\xi}) = C(\boldsymbol{\xi}) \mathrm{c}_j(-\boldsymbol{\xi}) \quad \text{for all } j \in \{1, \dots, n\}.$$

Hence, $C(w\xi)$ also satisfies the recurrence relation in Eq. (6.7). Finally, by downward induction with respect to the Bruhat order starting from the initial condition $C_{w_0}(\xi) = C(w_0 \xi)$ (and using that $c_j(\xi) \neq 0$), we conclude that $C_w(\xi) = C(w\xi)$ for all $w \in W_0$ and any $\xi \in V_{\text{reg}}$. $\square$

Before stating the next results, let us recall that the nodes $\mathcal{P}_c$ are given by the unique global minima stemming from the strictly convex Morse functions $\mathcal{V}_\mu$ (2.6), $\mu \in \hat{P}_c$ (2.5).

Given μ, the existence of the global minimum is guaranteed because $\mathcal{V}_\mu(\boldsymbol{\xi})$ is smooth and $\mathcal{V}_\mu(\xi) \to +\infty$ for $\xi \to \infty$. Since ξ_μ is a minimum of $\mathcal{V}_\mu$, it is a solution for $\nabla \mathcal{V}_\mu = 0$:

$$c\xi_\mu + \hat{\rho}_v(\xi_\mu) = 2\pi(\hat{\rho} + \mu) \qquad \text{where} \quad \hat{\rho}_v(\xi) := \sum_{\alpha \in R_0^+} v_\alpha(\langle \xi, \alpha \rangle)\hat{\alpha}. \tag{6.8}$$

Lemma 6.2. *The critical points ξ_μ, $\mu \in \hat{P}_c$ are all distinct and belong to the open alcove (with respect to the affine action of $W_0 \ltimes 2\pi Q^\vee$ on V)*

$$A = \{\xi \in V \mid 0 < \langle \xi, \alpha \rangle < 2\pi, \, \forall \alpha \in R_0^+\}. \tag{6.9}$$

Moreover, the position of ξ_μ depends analytically on the parameters $t_\alpha \in (-1, 1)$.

Proof. From Eq. (6.8) it is clear that one can recover μ from the value of ξ_μ, thus $\xi_\mu \neq \xi_\lambda$ if $\mu \neq \lambda$. Also, for any $\beta \in R_0^+$ we have that

$$c\langle \xi_\mu, \beta \rangle + \langle \hat{\rho}_v(\xi_\mu), \beta \rangle = 2\pi \langle \hat{\rho} + \mu, \beta \rangle. \tag{6.10a}$$

Since $v_\alpha(\mathrm{x})$ is an odd function it follows moreover that

$$\langle \hat{\rho}_v(\xi_\mu), \beta \rangle = \frac{1}{2} \sum_{\substack{\alpha \in R_0 \\ \langle \alpha^\vee, \beta \rangle > 0}} \big(v_\alpha(\langle \xi_\mu, \alpha \rangle) - v_\alpha(\langle s_{\beta^\vee} \xi_\mu, \alpha \rangle) \big) \langle \hat{\alpha}, \beta \rangle. \tag{6.10b}$$

From Eqs. (6.10a), (6.10b) one deduces that $\langle \xi_\mu, \beta \rangle > 0$ for $\mu \in \hat{P}_c$. Here one exploits that $v_\alpha(\mathrm{x})$ is strictly monotonously increasing and that

$$\langle \xi_\mu, \alpha \rangle - \langle s_{\beta^\vee} \xi_\mu, \alpha \rangle = \langle \xi_\mu, \beta \rangle \langle \alpha, \beta^\vee \rangle. \tag{6.10c}$$

Moreover, from Eqs. (6.10b), (6.10c) with $\beta = \varphi$ and the quasi-periodicity of the function $v_\alpha(\mathrm{x})$ one deduces that for $\langle \xi_\mu, \varphi \rangle \geq 2\pi$ we would have

$$c\langle \xi_\mu, \varphi \rangle + \langle \hat{\rho}_v(\xi_\mu), \varphi \rangle \geq 2\pi c + \pi \sum_{\alpha \in R_0^+} \langle \alpha, \varphi^\vee \rangle \langle \varphi, \frac{\alpha^\vee}{m_\alpha} \rangle = 2\pi(1 + c + \langle \hat{\rho}, \varphi \rangle), \tag{6.10d}$$

where in the last connection we used that for *any* root multiplicity function $t : R_0 \to \mathbb{C}$ and root $\beta \in R_0$ we have

$$\sum_{\alpha \in R_0^+} t_\alpha \langle \beta, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle = \frac{2}{n} \sum_{\alpha \in R_0} t_\alpha \tag{6.10e}$$

(which follows from the Schur's lemma, cf. the proof of [15, Lem. 10.1] and [12, Rem. 7.4]) and that

$$\langle \hat{\rho}, \varphi \rangle + 1 = \frac{1}{n} \sum_{\alpha \in R_0} \frac{1}{m_\alpha}.$$

Now by combining the Eq. (6.10d) with Eq. (6.10a) for $\beta = \varphi$, we would have

$$c + \langle \hat{\rho}, \varphi \rangle + 1 \leq \langle \hat{\rho} + \mu, \varphi \rangle = \langle \mu, \varphi \rangle + \langle \hat{\rho}, \varphi \rangle,$$

which contradicts our assumption that $\mu \in \hat{P}_c$. Hence, one must have that $\langle \xi_\mu, \varphi \rangle < 2\pi$, i.e. $\xi_\mu \in A$.

Finally, it is clear that the critical equation (6.8) is analytic in the parameters $t_\alpha \in (-1, 1)$. Since the Jacobian of the critical equation is invertible, the implicit function theorem now ensures that the dependence of the critical point ξ_μ is also analytic in these parameters. □

Proposition 6.3 *(Periodic Macdonald Spherical Function). For every $\mu \in \hat{P}_c$ the function Φ_{ξ_μ} belongs to the W-invariant subspace $\mathcal{C}(P)^W$.*

Proof. Since $T_j \imath_{\mathbf{0}} = t_j \imath_{\mathbf{0}}$ for $j = 1, \ldots, n$ (see Eq. (6.6)), we have that

$$\mathcal{J} T_j \mathcal{J}^{-1} \Phi_\xi = \mathcal{J} T_j \phi_\xi = t_j \mathcal{J} \phi_\xi = t_j \Phi_\xi. \tag{6.11}$$

Hence, by Lemma 5.5, we only need to prove that $\mathcal{J} T_0 \mathcal{J}^{-1} \Phi_\xi = t_0 \Phi_\xi$, or equivalently $T_0 \phi_\xi = t_0 \phi_\xi$. For $\xi \in V_{\text{reg}}$, the decomposition in Eq. (6.3) together with the explicit action of T_0 on $e^{i\xi}$ (cf. Eqs. (5.3)–(5.4)) gives us that

$$T_0 \phi_\xi = \sum_{v \in W_0} \frac{t_0 - 1}{1 - e^{i \langle v\xi, \alpha_0 \rangle}} C(v\xi) e^{iv\xi} + \sum_{v \in W_0} \frac{1 - t_0 e^{i\langle v\xi, -\alpha_0 \rangle}}{1 - e^{i \langle v\xi, -\alpha_0 \rangle}} C(s_0' v\xi) e^{ic\langle v\xi, \alpha_0 \rangle} e^{iv\xi}.$$

Now, by comparing with the corresponding decomposition of $t_0 \phi_\xi$, we have that $T_0 \phi_\xi = t_0 \phi_\xi$ if for $\xi \in V_{reg}$

$$e^{ic\langle v\xi, -\alpha_0 \rangle} = \frac{C(s_0' v\xi)}{C(v\xi)} \frac{1 - t_0 e^{i\langle v\xi, -\alpha_0 \rangle}}{t_0 - e^{i\langle v\xi, -\alpha_0 \rangle}}, \qquad \forall v \in W_0.$$

By substituting the product expansion for $C(\cdot)$ over R_0^+ (cf. Eq. (6.4)) we have

$$\begin{aligned} \frac{C(s_0'\xi)}{C(\xi)} &= \prod_{\substack{\alpha \in R_0^+ \\ \langle -\alpha_0, \alpha^\vee \rangle > 0}} \frac{1 - t_\alpha e^{i\langle \xi, \alpha \rangle}}{t_\alpha - e^{i\langle \xi, \alpha \rangle}} \\ &= \frac{1 - t_0 e^{i\langle \xi, -\alpha_0 \rangle}}{t_0 - e^{i\langle \xi, -\alpha_0 \rangle}} \prod_{\alpha \in R_0^+ \setminus \{-\alpha_0\}} \left(\frac{1 - t_\alpha e^{i\langle \xi, \alpha \rangle}}{t_\alpha - e^{i\langle \xi, \alpha \rangle}} \right)^{\langle -\alpha_0, \hat{\alpha} \rangle}, \end{aligned}$$

where we used that $\langle -\alpha_0, \hat{\alpha}\rangle \in \{0,1\}$ for all $\alpha \in R_0^+ \setminus \{-\alpha_0\}$. The relation now may be written as

$$e^{ic\langle \xi, -v\alpha_0\rangle} = \prod_{\alpha\in R_0^+} \left(\frac{1-t_\alpha e^{i\langle \xi,\alpha\rangle}}{e^{i\langle \xi,\alpha\rangle}-t_\alpha}\right)^{\langle -v\alpha_0,\hat{\alpha}\rangle}, \qquad \forall v\in W_0, \tag{6.12}$$

by using that an overall flip of the signs in the factors at the right-hand side cancels out because $\prod_{\alpha\in\hat{R}_0^+}(-1)^{\langle\beta^\vee,\alpha\rangle} = (-1)^{\langle \beta^\vee, 2\hat{\rho}\rangle} = 1$ for all $\beta \in \hat{R}_0$.

To finish this proof, let us observe that if we multiply Eq. (6.10a) by the imaginary unit and exponentiate both sides, using that

$$v_\alpha(\mathrm{x}) = 2\arctan\Bigl(\frac{1+t_\alpha}{1-t_\alpha}\tan\left(\frac{\mathrm{x}}{2}\right)\Bigr) = i\log\Bigl(\frac{1-t_\alpha e^{i\mathrm{x}}}{e^{i\mathrm{x}}-t_\alpha}\Bigr),$$

then it follows that ξ_μ is indeed a solution for Eq. (6.12). □

Remark 6.4. In [11, Eqn. (5.12a)] a Macdonald spherical function $\Phi_\xi \in \mathcal{C}(P)^{W_0}$ was introduced in terms of an intertwiner operator built up (essentially) from an integral-reflection representation of the finite Hecke algebra H_0. In contrast, the affine Macdonald spherical function $\Phi_\xi \in \mathcal{C}(P)^{W_0}$ (6.1) is based on the affine intertwiner operator $\mathcal{J}$ (5.6), built up from the integral-reflection representation of the affine Hecke algebra H.

Remark 6.5. For $y \in V$, let us denote by $\tau_y : V \to V$ the translation determined by the action $\tau_y(x) := x + y$. Then the affine Weyl group admits the alternative presentation $W = W_0 \ltimes \tau(c\hat{Q}^\vee)$ because $s_{\alpha^\vee} s_{\alpha^\vee + m_\alpha rc} = \tau_{cr\hat{\alpha}^\vee}$ for $\alpha \in R_0$, $r \in \mathbb{Z}$. Because of the above proposition it follows that Φ_{ξ_μ} ($\mu \in \hat{P}_c$) is W_0-invariant and $c\hat{Q}^\vee$-periodic, explaining the name *periodic* Macdonald spherical function for Φ_{ξ_μ}.

Remark 6.6. From the proof of Proposition 6.3, it is clear that for every $\mu \in \hat{P}$ the vector $\xi = \xi_\mu$ solves the following algebraic system of equations of Bethe type

$$e^{ic\langle \xi,\beta^\vee\rangle} = \prod_{\alpha\in R_0^+}\Bigl(\frac{1-t_\alpha e^{i\langle\xi,\alpha\rangle}}{e^{i\langle\xi,\alpha\rangle}-t_\alpha}\Bigr)^{\langle\hat{\alpha},\beta^\vee\rangle}, \qquad \forall\beta\in\hat{R}_0.$$

Indeed, at $\xi = \xi_\mu$ Eq. (6.12) is satisfied and the short roots of $\hat{R}_0^\vee$ generate $\hat{R}_0^\vee$ over $\mathbb{Z}$.

7. Proof of Theorem 2.2 (basis)

For any $\omega \in P^+$ we consider the *free* operator $L_{\omega;1} : \mathcal{C}(P) \to \mathcal{C}(P)$ given by

$$(\mathrm{L}_{\omega;1} f)(\lambda) := \sum_{\nu\in W_0\omega} f(\lambda+\nu) \tag{7.1}$$

and the operator $L_\omega : \mathcal{C}(P) \to \mathcal{C}(P)$ given by

$$L_\omega = \mathcal{J} L_{\omega;1} \mathcal{J}^{-1}. \tag{7.2}$$

For the isotropy group W_λ of $\lambda \in P_c$ in W, Macdonald's product formula for the generalized Poincaré series of the Coxeter group associated with the length multiplicative function t [35] tells us that

$$W_\lambda(t) = \sum_{w \in W_\lambda} t_w = \prod_{\substack{\alpha \in \hat{R}_0^+ \\ \langle \lambda, \alpha \rangle = 0}} \frac{1 - t_\alpha \hat{e}_t(\alpha)}{1 - \hat{e}_t(\alpha)} \prod_{\substack{\alpha \in \hat{R}_0^+ \\ \langle \lambda, \alpha \rangle = c}} \frac{1 - t_\alpha \hat{h}_t \hat{e}_t(-\alpha)}{1 - \hat{h}_t \hat{e}_t(-\alpha)}, \tag{7.3}$$

with $\hat{h}_t$ and $\hat{e}_t$ given by (2.8), (2.9). Armed with this identity, we can readily infer that for any $\mu \in \hat{P}_c$ the function Φ_{ξ_μ} is nonzero in $\mathcal{C}(P_c) \cong \mathcal{C}(P)^W$. Indeed, from Proposition 6.1, Lemma 6.2, Proposition 6.3 and the trivial action of $\mathcal{J}$ on $\mathcal{C}(P_c)$, it follows that

$$\Phi_\xi(\lambda) = M_\lambda(\xi) = M_\lambda^{(c)}(\xi) \quad \text{for any } \xi \in \mathcal{P}_c \text{ and } \lambda \in P_c. \tag{7.4}$$

In particular, at $\lambda = 0$ this yields that

$$\begin{aligned} \Phi_\xi(0) &= \sum_{v \in W_0} C(v\xi) = \sum_{v \in W_0} \prod_{\alpha \in R_0^+} \frac{1 - t_\alpha e^{-i\langle v\xi, \alpha \rangle}}{1 - e^{-i\langle v\xi, \alpha\rangle}} \\ &\overset{\star}{=} \sum_{v \in W_0} t_v = \prod_{\alpha \in R_0^+} \frac{1 - t_\alpha e_t(\alpha)}{1 - e_t(\alpha)} > 0, \end{aligned}$$

where we used Macdonald's identity from Ref. [35, Thm. (2.8)] for the $\star$ equality.

Proposition 7.1 *(Completeness of the periodic Macdonald spherical functions). The restriction of the functions Φ_{ξ_μ}, $\mu \in \hat{P}_c$, constitutes a basis for $\mathcal{C}(P_c)$ that diagonalizes the commuting operators L_ω simultaneously:*

$$L_\omega \Phi_{\xi_\mu} = \mathrm{m}_\omega(e^{i\xi_\mu}) \Phi_{\xi_\mu} \qquad (\omega \in P^+, \mu \in \hat{P}_c). \tag{7.5}$$

Proof. For any $\omega \in P^+$ the action of $L_{\omega;1}$ on a plane wave yields

$$L_{\omega;1} e^{iv\xi} = \mathrm{m}_\omega(e^{i\xi}) e^{iv\xi} \qquad (v \in W_0).$$

Hence, given $\mu \in \hat{P}_c$ it follows that at $\xi = \xi_\mu$:

$$L_\omega \Phi_\xi = \mathcal{J} L_{\omega;1} \mathcal{J}^{-1} \mathcal{J} \phi_\xi = \mathcal{J} L_{\omega;1} \phi_\xi = \mathrm{m}_\omega(e^{i\xi}) \Phi_\xi.$$

The upshot is that the nontrivial eigensolutions Φ_{ξ_μ}, $\mu \in \hat{P}_c$ in Eq. (7.5) must be linearly independent in $\mathcal{C}(P_c)$, in view of Lemma 6.2 and the well-known fact that the

W_0-invariant trigonometric polynomials $m_\omega(e^{i\xi})$, $\omega \in P^+$, separate the points of the fundamental alcove A.

To finish the proof, it suffices to verify that $\dim \mathcal{C}(P_c) = |P_c| = |\hat{P}_c|$, for this confirms that the eigenfunctions Φ_{ξ_μ} ($\mu \in \hat{P}_c$) form a basis of $\mathcal{C}(P_c)$. To this end we first observe that P_c consists of all nonnegative integral combinations $c_1\omega_1 + \cdots + c_n\omega_n$ of the fundamental weights of R_0 satisfying $c_1 m_1 + \cdots + c_n m_n \leq c$, where the positive integers $m_1, \ldots, m_n$ refer to the coefficients of the highest root $-\alpha_0^\vee$ of $\hat{R}_0$ in the simple basis $\alpha_1^\vee, \ldots, \alpha_n^\vee$ of $R_0^\vee$. Similarly, $\hat{P}_c$ consists of all nonnegative integral combinations $c_1\hat{\omega}_1 + \cdots + c_n\hat{\omega}_n$ of the fundamental weights of $\hat{R}_0$ satisfying $c_1\hat{m}_1 + \cdots + c_n\hat{m}_n \leq c$ and where the positive integers $\hat{m}_1, \ldots, \hat{m}_n$ now refer to the coefficients of φ in the simple basis $\hat{\alpha}_1^\vee, \ldots, \hat{\alpha}_n^\vee$ of $\hat{R}_0^\vee$. If $\hat{R}_0 = u_\varphi R_0$ then clearly $\hat{m}_j = m_j$ for all j and if $\hat{R}_0 = R_0^\vee$ then the $\hat{m}_1, \ldots, \hat{m}_n$ are a permutation of the $m_1, \ldots, m_n$. So in both cases $|P_c| = |\hat{P}_c|$, which completes the proof of the proposition. □

Proposition 7.1 guarantees that the square matrix $\left[M_\lambda^{(c)}(\xi_\mu)\right]_{\lambda \in P_c, \mu \in \hat{P}_c}$ is of full rank, which finishes the proof of Theorem 2.2.

8. Proof of Theorem 2.4 (affine Pieri rule)

For any function $f \in \mathcal{C}(P)^W$, $\omega \in P^+$ (quasi)-minuscule and $\lambda \in P_c$ we have that

$$
\begin{aligned}
(L_\omega f)(\lambda) &= (\mathcal{J} L_{\omega;1} \mathcal{J}^{-1} f)(\lambda) = (L_{\omega;1} \mathcal{J}^{-1} f)(\lambda) \\
&= \sum_{\nu \in W_0\omega} (\mathcal{J}^{-1} f)(\lambda + \nu) \\
&= \sum_{\nu \in W_0\omega} t[\lambda + \nu] f(\lambda + \nu) + d_{\lambda,\nu}(1 - t_\vartheta^{-1}) f(\lambda).
\end{aligned}
$$

The last equality hinges on the following lemma (whose proof is delayed until subsection 8.2):

Lemma 8.1. *For any $f \in \mathcal{C}(P)$, $\lambda \in P_c$ and $\nu \in P_\vartheta^\star := \{w\eta \mid w \in W_0$ and $\eta \in P$ is a minuscule or quasi-minuscule weight$\}$, one has that*

$$
(\mathcal{J}^{-1} f)(\lambda + \nu) = t[\lambda + \nu] f(\lambda + \nu) + d_{\lambda,\nu}(1 - t_\vartheta^{-1}) f(\lambda),
$$

where $d_{\lambda,\nu}$ is taken from (4.2a).

Since f is W-invariant, it follows that

$$
(L_\omega f)(\lambda) = \left(\sum_{\substack{\nu \in W_0\omega \\ (\lambda+\nu)_+ = \lambda}} t[\lambda + \nu] + (1 - t_\vartheta^{-1}) \sum_{\nu \in W_0\omega} d_{\lambda,\nu} \right) f(\lambda) \tag{8.1}
$$

$$+\sum_{\substack{\nu\in W_0\omega\\ \lambda+\nu\in P_c}}\ \sum_{\substack{\eta\in W_0\omega\\ (\lambda+\eta)_+=\lambda+\nu}} t[\lambda+\eta]\, f(\lambda+\nu).$$

The action of L_ω on f is therefore of the form

$$(L_\omega f)(\lambda)=U_{\lambda,\omega}(t)f(\lambda)+\sum_{\substack{\nu\in W_0\omega\\ \lambda+\nu\in P_c}} V_{\lambda,\nu}(t)f(\lambda+\nu).$$

The computation of the coefficients $U_{\lambda,\omega}(t)$ and $V_{\lambda,\nu}(t)$ hinges on the following lemma (whose proof is relegated in turn to subsection 8.1):

Lemma 8.2. *For $\lambda\in P_c$ and $\nu\in P_\vartheta^\star$, we are in either one of the following two situations: i) When $(\lambda+\nu)_+=\lambda$, then $w'_{\lambda+\nu}\nu=\alpha_j$ for some $j\in\{0,\ldots,n\}$ with $t_j=t_0$ and $\theta(\lambda+\nu)=0$.*

ii) When $(\lambda+\nu)_+\neq\lambda$, then $w_{\lambda+\nu}\in W_\lambda$ and $\theta(\lambda+\nu)=1$ if $\nu\in R_0^-\cap W_0\vartheta$ and $\langle\lambda,\hat{\nu}\rangle=0$, or if $\nu\in R_0^+\cap W_0\vartheta$ and $\langle\lambda,\hat{\nu}\rangle=c$, while $\theta(\lambda+\nu)=0$ otherwise.

Indeed, the asserted expression for $U_{\lambda,\omega}$ in (4.1) is immediate from Eq. (8.1) and the lemma, while the coefficient $V_{\lambda,\nu}(t)$ of $f(\lambda+\nu)$ in Eq. (2.11) is retrieved after a short computation:

$$\sum_{\substack{\eta\in W_0\omega\\ (\lambda+\eta)_+=\lambda+\nu}} t[\lambda+\eta]=\sum_{\mu\in W_\lambda(\lambda+\nu)} t[\mu]=W_\lambda(t)/(W_\lambda\cap W_{\lambda+\nu})(t)=V_{\lambda,\nu}(t),$$

where in the last step Macdonald's product formula (7.3) was used.

Upon combining with Eq. (7.5) and recalling that for $\lambda\in P_c$ and $\mu\in\hat{P}_c$: $\Phi_{\xi_\mu}(\lambda)=M_\lambda(\xi_\mu)=M_\lambda^c(\xi_\mu)$ (cf. Eq. (7.4)), the Pieri formula in Eq. (2.10) readily follows.

8.1. Proof of Lemma 8.2

Let $\mu\in P\setminus P_c$ and $j\in\{0,\ldots,n\}$ such that $a_j\in R[\mu]$ (recall (3.6)). Then $w_\mu=w_{s_j\mu}s_j$ with $\ell(w_\mu)=\ell(w_{s_j\mu})+1$, and thus $R[\mu]=s_jR[s_j\mu]\cup\{a_j\}$ (cf. [38, (2.2.4)], although it is only stated for non-twisted types, it is actually true also for twisted types). From (4.2c) it follows that

$$\theta(\mu)=\begin{cases}\theta(s_j\mu)+1 & \text{if } a_j(\mu)=-2,\\ \theta(s_j\mu) & \text{if } a_j(\mu)\neq-2.\end{cases}\tag{8.2}$$

Let us consider the situation $\lambda\in P_c$, $\nu\in P_\vartheta^\star$ and $\lambda+\nu\notin P_c$. If for any $j\in\{0,\ldots,n\}$ such that $a_j\in R[\lambda+\nu]$ we define $\tilde{\nu}:=s_j(\lambda+\nu)-\lambda$ (having $s_j(\lambda+\nu)=\lambda+\tilde{\nu}$), then we are in one of the following cases and thanks to (8.2) we have the corresponding expressions for $\theta(\lambda+\nu)$

(A) $a_j(\lambda) = 0$ and $\langle \nu, \alpha_j^\vee \rangle = -1$ (so $a_j(\lambda+\nu) = -1$). Then $s_j \in W_\lambda$, so $\tilde{\nu} = s'_j\nu$ and $\theta(\lambda+\nu)$=$\theta(\lambda+s'_j\nu)$.

(B) $a_j(\lambda) = 0$ and $\langle \nu, \alpha_j^\vee \rangle = -2$ (so $a_j(\lambda+\nu) = -2$). Then $s_j \in W_\lambda$ and $\nu = -\alpha_j$, so $\tilde{\nu} = s'_j\nu = \alpha_j$ and $\theta(\lambda+\nu)$=$\theta(\lambda+s'_j\nu)+1$.

(C) $a_j(\lambda) = 1$ and $\langle \nu, \alpha_j^\vee \rangle = -2$ (so $a_j(\lambda+\nu) = -1$). Then $\nu = -\alpha_j$ and $\tilde{\nu} = 0$, so $w_{\lambda+\nu} = s_j$ and $\theta(\lambda+\nu)$=$\theta(\lambda) = 0$.

Cases (B) and (C) only occur when $\nu \in W_0\vartheta$. If furthermore $j = 0$, then also $\alpha_0 = -\vartheta$ (so we are necessarily in the untwisted case $\hat{R}_0 = R_0^\vee$).

When $\lambda+\nu \in P_c$ the lemma is trivial. Let $\lambda+\nu \notin P_c$ and a decomposition $w_{\lambda+\nu} = s_{j_\ell}\cdots s_{j_1}$ with $\ell = \ell(w_{\lambda+\nu}) \geq 1$, we define $\nu_0 := \nu$, $\nu_k := s'_{j_k}\nu_{k-1}$ (for $k = 1,\ldots,\ell$), $b_0 := a_{j_1}$ and $b_k = \beta_k^\vee + r_k m_\beta c := s_{j_1}\cdots s_{j_k}a_{j_{k+1}}$ (for $k = 1,\ldots,\ell-1$). This means that $R[\lambda+\nu] = \{b_0,\ldots,b_{\ell-1}\}$ (cf. [38, (2.2.9)]). By considering the three aforementioned possible cases we have

$$\lambda+\nu = \lambda+\nu_0 \xrightarrow{s_{j_1}} \lambda+\nu_1 \xrightarrow{s_{j_2}} \cdots \xrightarrow{s_{j_{\ell-1}}} \lambda+\nu_{\ell-1},$$

where all these steps involve only (A) or (B), while (C) can only occur at the final step

$$\lambda+\nu_{\ell-1} \qquad \xrightarrow{s_{j_\ell}} \lambda = (\lambda+\nu)_+, \tag{8.3}$$

or does not occur at all

$$\lambda+\nu_{\ell-1} \qquad \xrightarrow{s_{j_\ell}} \lambda+\nu_\ell = (\lambda+\nu)_+. \tag{8.4}$$

In situation (8.3) we have that $s_{j_\ell}w_{\lambda+\nu} = s_{j_{\ell-1}}\cdots s_{j_1} \in W_\lambda$, $(\lambda+\nu)_+ = \lambda$, and $t_{j_\ell} = t_0$. Even more, we have $s'_{j_\ell}w'_{\lambda+\nu}\nu = \nu_{\ell-1} = -\alpha_{j_\ell}$, which implies $-\nu = (s'_{j_\ell}w'_{\lambda+\nu})^{-1}\alpha_{j_\ell} = (s'_{j_1}\cdots s'_{j_{\ell-1}}\alpha_{j_\ell}) = \beta_{\ell-1}$, thus

$$\begin{aligned} -\langle \lambda, \nu^\vee \rangle + r_{\ell-1}m_\nu c &= b_{\ell-1}(\lambda) = ((s_{j_\ell}w_{\lambda+\nu})^{-1}a_{j_\ell})(\lambda) \\ &= a_{j_\ell}(s_{j_{\ell-1}}\cdots s_{j_1}\lambda) = a_{j_\ell}(\lambda) = 1, \end{aligned}$$

hence

$$-\nu^\vee + (1+\langle \lambda, \nu^\vee \rangle) = b_{\ell-1} \in R[\lambda+\nu]. \tag{8.5}$$

On the other hand, in situation (8.4) we have that $w_{\lambda+\nu} \in W_\lambda$ and $(\lambda+\nu)_+ \neq \lambda$.

In order to compute $\theta(\lambda+\nu)$ let us notice that it has to be the number of times that the case (B) occurs in the steps above, since $\theta((\lambda+\nu)_+) = 0$. In other words, the number of times that $\langle \nu_k, \alpha_{j_{k+1}}^\vee \rangle = -2$ for $k = 0,\ldots,\ell'-1$, with $\ell' = \ell-1$ in situation (8.3) and $\ell' = \ell$ in situation (8.4). Since for $k = 0,\ldots,\ell'-1$:

$$\langle \nu_k, \alpha_{j_{k+1}}^\vee \rangle = -2 \Leftrightarrow \langle \nu, \beta_k^\vee \rangle = -2 \Leftrightarrow \nu = -\beta_k,$$

and

$$\begin{aligned}\langle \lambda, \beta_k^\vee \rangle + m_{\beta_k} r_k c = b_k(\lambda) = (s_{j_1} \cdots s_{j_k} a_{j_{k+1}})(\lambda) \\ = a_{j_{k+1}}(s_{j_k} \cdots s_{j_1} \lambda) = a_{j_{k+1}}(\lambda) = 0,\end{aligned}$$

then

$$\beta_k^\vee - \langle \lambda, \beta_k^\vee \rangle = b_k \in R[\lambda + \nu].$$

Hence, $\theta(\lambda+\nu)$ is equal to 1 or 0 when $-\nu^\vee + \langle \lambda, \nu^\vee \rangle \in R[\lambda+\nu]$ or $-\nu^\vee + \langle \lambda, \nu^\vee \rangle \notin R[\lambda+\nu]$, respectively. In situation (8.3) we have that $\theta(\lambda+\nu) = 0$, because by Eq. (8.5) we have $\langle \lambda, \nu^\vee \rangle + 1 \in m_\nu c\mathbb{Z}$ and if $-\nu^\vee + \langle \lambda, \nu^\vee \rangle \in R[\lambda+\nu] \subset R$ then $\langle \lambda, \nu^\vee \rangle \in m_\nu c\mathbb{Z}$, this would contradict that $c > 1$.

For $\theta(\lambda+\nu) = 1$ we have $-\nu^\vee + \langle \lambda, \nu^\vee \rangle \in R^+$, and therefore $\nu \in R_0 \cap P_\vartheta^\star = W_0\vartheta$ and $\langle \lambda, \nu^\vee \rangle \in m_\vartheta c\mathbb{Z}$. On the other hand we have that $|\langle \lambda, \nu^\vee \rangle| = |\langle \lambda, m_\nu \hat{\nu} \rangle| \leq m_\vartheta c$ for all $\lambda \in P_c$, proving $m_\vartheta \langle \lambda, \hat{\nu} \rangle = \langle \lambda, \nu^\vee \rangle \in \{m_\vartheta c, 0\}$ if $\theta(\lambda+\nu) > 0$, and this concludes the proof of the lemma.

8.2. Proof of Lemma 8.1

It will be more useful to use the following reformulation:

$$(T_{w_{\lambda+\nu}} f)((\lambda+\nu)_+) = f(\lambda+\nu) - d_{\lambda,\nu}(1 - t_\vartheta^{-1}) f(\lambda).$$

For $f \in C(P)$, $\mu \in P$, $j = 0, \ldots, n$ with $0 \leq a_j(\mu) \leq 2$ the action of T_j is given explicitly by

$$(T_j f)(\mu) = \begin{cases} t_j f(\mu) & \text{if } a_j(\mu) = 0 \\ f(s_j\mu) = f(\mu - \alpha_j) & \text{if } a_j(\mu) = 1 \\ f(\mu - 2\alpha_j) - (t_j - 1) f(\mu - \alpha_j) & \text{if } a_j(\mu) = 2 \end{cases}. \tag{8.6}$$

Now we proceed by induction on $\ell(w_{\lambda+\nu})$. For $\lambda+\nu \in P_c$ the result is trivial. Let assume that $\ell(w_{\lambda+\nu}) > 1$ and s_j $(0 \leq j \leq n)$ such that $a_j \in R[\lambda+\nu]$, then $\ell(w_{\lambda+\nu} s_j) = \ell(w_{\lambda+\nu}) - 1$. By the observations made at the beginning of Section 8.1 we have that $w_{\lambda+\nu} s_j = w_{s_j(\lambda+\nu)}$ with either $s_j(\lambda+\nu) = \lambda + s_j'\nu$ (cases (A) and (B)) or $s_j(\lambda+\nu) = \lambda (\in P_c)$ (case (C)). In the case (C) we have $w_{\lambda+\nu} = s_j$ and the statement to prove is just the case $a_j(\mu) = 1$ of Eq. (8.6) with $\mu = \lambda$. Furthermore, for the cases (A) and (B) we have

$$\begin{aligned}(T_{w_{\lambda+\nu}} f)((\lambda+\nu)_+) &= (T_{w_{\lambda+s_j'\nu}} T_j f)((\lambda + s_j'\nu)_+) \\ &= (T_j f)(\lambda + s_j'\nu) - d_{\lambda, s_j'\nu}(1 - t_\vartheta^{-1})(T_j f)(\lambda)\end{aligned}$$

by the induction hypothesis and the fact $(\lambda + s'_j\nu)_+ = (\lambda+\nu)_+$.

If we are in the case (A) then we have $(T_jf)(\lambda+s'_j\nu) = f(\lambda+\nu)$ and $(T_jf)(\lambda) = t_jf(\lambda)$ by the situations $a_j(\mu) = 1$ and $a_j(\mu) = 0$ of Eq. (8.6), respectively. This finishes the induction step since $d_{\lambda,s'_j\nu} = d_{\lambda,\nu}t_j^{-1}$. For $\nu \in P^\star_\vartheta \backslash W_0\vartheta$ this follows from $d_{\lambda,s'_j\nu} = d_{\lambda,\nu} = 0$, while for $\nu \in W_0\vartheta$ it follows from $\theta(\lambda + s'_j\nu) = \theta(\lambda+\nu)$ and that for $j > 0$ we have $e_t(s_j\nu) = e_t(\nu)t_j$ and $\langle\lambda, s_j\hat{\nu}\rangle = \langle s_j\lambda, \hat{\nu}\rangle = \langle\lambda,\hat{\nu}\rangle$, while on the other hand, for $j = 0$ we have $e_t(s'_0\nu) = e_t(\nu+\alpha_0) = e_t(\nu)h_t^{-1}t_\vartheta$ and (if $\theta(\lambda+\nu) > 0$) also $\langle\lambda, s'_0\hat{\nu}\rangle = \langle s'_0\lambda,\hat{\nu}\rangle = \langle\lambda + c\alpha_0,\hat{\nu}\rangle = \langle\lambda,\hat{\nu}\rangle + c\langle\alpha_0,\hat{\nu}\rangle = \langle\lambda,\hat{\nu}\rangle - c$ and therefore $\mathrm{sign}(\langle\lambda, s'_0\hat{\nu}\rangle) = \mathrm{sign}(\langle\lambda,\hat{\nu}\rangle) - 1$ (cf. Lemma. 8.2).

If we are in the case (B) then we have that $\nu \in W_0\vartheta$, $t_j = t_\vartheta$ and $(T_jf)(\lambda + s'_j\nu) = f(\lambda+\nu) - t_\vartheta(1-t_\vartheta^{-1})f(\lambda)$ because of the case $a_j(\mu) = 2$ of Eq. (8.6) for $\mu = \lambda + s'_j\nu$. We also have $d_{\lambda,s'_j\nu} = 0$ because $0 \le \theta(\lambda + s'_j\nu) < \theta(\lambda+\nu) \le 1$. To complete the induction it remains to prove that $d_{\lambda,\nu} = t_\vartheta$. For this observe that $\theta(\lambda+\nu) = 1$ and when $j > 0$ we have $e_t(-\nu) = e_t(\alpha_j) = t_j = t_\vartheta$ (as $e_t(\alpha_j) = e_t(s_j\alpha_j)t_j^{\langle\alpha_j,\alpha_j^\vee\rangle} = e_t(-\alpha_j)t_j^2 = t_j^2/e_t(\alpha_j)$) and $\langle\lambda,\hat{\nu}\rangle = -\langle\lambda,\hat{\alpha}_j\rangle = 0$, while for $j = 0$ we have $e_t(\nu) = e_t(-\alpha_0) = e_t(\vartheta) = h_t/t_\vartheta$ (since in this case $\alpha_0 = -\vartheta$, cf. Lemma 8.2) and $\langle\lambda,\hat{\nu}\rangle = -\langle\lambda,\hat{\alpha}_0\rangle = -\langle\lambda,\alpha_0^\vee\rangle = -a_0(\lambda) + c = c > 0$.

9. The structure constants revisited

The computation in the previous section produces the coefficients of the Pieri rule from the action of L_ω (7.2) in $\mathcal{C}(P_c)$. In principle the same strategy can be followed to compute the structure constants $c^{\nu,(c)}_{\lambda,\mu}(t)$ $(\lambda,\mu,\nu \in P_c)$ more generally. To this end one starts with the monomial expansion of the Macdonald spherical function $M_\lambda(\xi)$, $\lambda \in P^+$:

$$M_\lambda(\xi) = \sum_{\mu\in P^+,\, \mu\le\lambda} n_{\lambda,\mu}(t)m_\mu(e^{i\xi}), \tag{9.1}$$

where we have employed the dominance partial order on P^+: $\mu \le \lambda$ iff $\lambda - \mu \in Q^+$. With the aid of the expansion coefficients $n_{\lambda,\mu}(t)$ one defines the following operator-valued Macdonald spherical function $M_\lambda(L) : \mathcal{C}(P) \to \mathcal{C}(P)$ via the formula:

$$M_\lambda(L) = \sum_{\mu\in P^+,\, \mu\le\lambda} n_{\lambda,\mu}(t)L_\mu \tag{9.2}$$

(cf. Eqs. (7.1), (7.2)). For $\lambda \in P_c$, the operator-valued Macdonald spherical function $M_\lambda(L)$ (9.2) acts as a linear difference operator in the invariant subspace $\mathcal{C}(P)^W \cong \mathcal{C}(P_c)$ with coefficients given by the structure constants $c^{\nu,(c)}_{\lambda,\mu}(t)$ $(\mu,\nu \in P_c)$.

Theorem 9.1 *(Structure constants). For any $\lambda \in P_c$, the action of $M_\lambda(L)$ on $f \in \mathcal{C}(P_c)$ is given by*

$$(M_\lambda(L)f)(\mu) = \sum_{\nu\in P_c} c^{\nu,(c)}_{\lambda,\mu}(t)f(\nu), \tag{9.3}$$

with $c^{\nu,(c)}_{\lambda,\mu}(t)$ *as defined in Eq.* (2.7)*.*

Proof. By linearity, it suffices to verify Eq. (9.3) on the basis of Macdonald spherical functions Φ_ξ, $\xi \in \mathcal{P}_c$. To this end we compute for $\xi \in \mathcal{P}_c$:

$$M_\lambda(L)\Phi_\xi \overset{\text{Eq.(9.2)}}{=} \sum_{\mu\in P^+,\, \mu\leq\lambda} n_{\lambda,\mu}(t) L_\mu \Phi_\xi \overset{\text{Prop.7.1}}{=} \sum_{\mu\in P^+,\, \mu\leq\lambda} n_{\lambda,\mu}(t) m_\mu(e^{i\xi})\Phi_\xi$$
$$\overset{\text{Eq.(9.1)}}{=} M_\lambda(\xi)\Phi_\xi = M^{(c)}_\lambda(\xi)\Phi_\xi .$$

Evaluation of this identity at $\mu \in P_c$ with the aid of Eq. (7.4) entails the desired formula for $f = \Phi_\xi$:

$$(M_\lambda(L)\Phi_\xi)(\mu) = M^{(c)}_\lambda(\xi)\Phi_\xi(\mu) = M^{(c)}_\lambda(\xi)M^{(c)}_\mu(\xi) = \sum_{\nu\in P_c} c^{\nu,(c)}_{\lambda,\mu} M^{(c)}_\nu(\xi)$$
$$= \sum_{\nu\in P_c} c^{\nu,(c)}_{\lambda,\mu}\Phi_\xi(\nu). \quad \square$$

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