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Algebraic treatment of the Bateman Hamiltonian

Francisco M. Fernández

Abstract: We apply the algebraic method to the Bateman Hamiltonian and obtain its natural frequencies and ladder operators from the adjoint or regular matrix representation of that operator. Present analysis shows that the eigenfunctions compatible with the complex eigenvalues obtained earlier by other authors are not square integrable. In addition to this, the ladder operators annihilate an infinite number of such eigenfunctions.

Key words: Bateman Hamiltonian, algebraic method, natural frequencies, ladder operators, regular matrix representation.

Résumé : Nous appliquons la méthode algébrique au hamiltonien de Bateman et nous obtenons les fréquences naturelles et les opérateurs d'échelle à partir de la représentation matricielle adjointe ou régulière de cet opérateur. Notre analyse montre que les fonctions propres compatibles avec les valeurs propres complexes obtenues plus tôt par d'autres auteurs ne sont pas de carré sommable. De plus, les opérateurs d'échelle annihilent un nombre infini de telles fonctions propres. [Traduit par la Rédaction]

Mots-clés : hamiltonien de Bateman, méthode algébrique, fréquences naturelles, opérateurs d'échelle, représentation matricielle régulière.

1. Introduction

There has recently been a controversy about the eigenvectors and eigenvalues of the Bateman Hamiltonian [1, 2]. On one side, Deguchi et al. [3, 4] discussed two quantization approaches for the Bateman Hamiltonian based on ladder operators, on the other, Bagarello et al. [5, 6] argued that there is no square integrable vacuum for the natural ladder operators.

The purpose of this paper is to study the problem by means of the algebraic method [7] that has proved suitable for the treatment of a variety of quadratic Hamiltonian operators [8–13].

In Section 2 we briefly discuss the model and derive a simpler dimensionless version of the Hamiltonian operator, in Section 3 we outline the main ideas about the algebraic method, in Section 4 we apply this approach to the Bateman Hamiltonian, and, finally, in Section 5 we summarize the main results and draw conclusions.

2. Model

The Hamiltonian operator for the so-called Bateman oscillator model is [3–6]

$$H = \frac{1}{2m}(p_1^2 - p_2^2) + \frac{m\omega^2}{2}(x_1^2 - x_2^2) - \frac{\gamma}{2m}(x_1 p_2 + x_2 p_1) \tag{1}$$

where m, γ, $\omega^2 > 0$, and $[x_j, p_k] = i\hbar\delta_{jk}$.

It is convenient to transform the Hamiltonian operator (1) into a dimensionless one. To this end we define the new variables $(x_1, x_2, p_1, p_2) \to (\alpha x, \alpha y, \hbar\alpha^{-1} p_x, \hbar\alpha^{-1} p_y)$, so that $[u, p_v] = i\delta_{uv}$ and $u, v = x, y$. On choosing $\alpha^2 = \hbar/(m\omega)$ we are left with the one-parameter Hamiltonian

$$H_b = \frac{H}{\hbar\omega} = \frac{1}{2}(p_x^2 - p_y^2) + \frac{1}{2}(x^2 - y^2) - \frac{b}{2}(xp_y + yp_x) \tag{2}$$

where $b = \gamma/(m\omega)$. It is worth noting that $[H_b, H_0] = [H_b, H_1] = [H_0, H_1] = 0$, where $H_0 = H_b(b = 0)$ and $H_1 = H - H_0$.

The operator (2) satisfies

$$\langle f|H_b|g\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)^* H_b g(x,y)\,dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [H_b f(x,y)]^* g(x,y)\,dxdy = \langle H_b f|g\rangle \tag{3}$$

for every pair of square-integrable functions $f(x, y)$ and $g(x, y)$. Therefore, from now on we use the standard quantum-mechanical Hermitian notation $H_b^\dagger = H_b$ and restrict ourselves to the scalar product

$$\langle f|g\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)^* g(x,y)\,dxdy \tag{4}$$

3. Algebraic method

In this section we outline the algebraic method discussed in previous papers [8–13]. The model discussed in Section 2 is a particular case of a quadratic Hamiltonian of the form

$$H = \sum_{i=1}^{2K}\sum_{j=1}^{2K} \gamma_{ij} O_i O_j \tag{5}$$

where $\{O_1, O_2, \ldots, O_{2K}\} = \{x_1, x_2, \ldots, x_K, p_1, p_2, \ldots, p_K\}$, $[x_m, p_n] = i\delta_{mn}$, and $[x_m, x_n] = [p_m, p_n] = 0$. Here we restrict ourselves to operators that satisfy $H^\dagger = H$ but the approach also applies to other cases [10]. The algebraic method is particularly useful for the

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analysis of the spectrum of H because it satisfies the commutation relations

$$[H, O_i] = \sum_{j=1}^{2K} H_{ji} O_j \tag{6}$$

For this reason, it is possible to obtain an operator of the form

$$Z = \sum_{i=1}^{2K} c_i O_i \tag{7}$$

such that

$$[H, Z] = \lambda Z \tag{8}$$

The operator Z is important for our purposes because if $H|\psi\rangle = E|\psi\rangle$ then

$$HZ|\psi\rangle = (E + \lambda)Z|\psi\rangle \tag{9}$$

It follows from eqs. (6), (7), and (8) that

$$(\mathbf{H} - \lambda\mathbf{I})\mathbf{C} = 0 \tag{10}$$

where $\mathbf{H}$ is a $2K \times 2K$ matrix with elements H_{ij}, $\mathbf{I}$ is the $2K \times 2K$ identity matrix, and $\mathbf{C}$ is a $2K \times 1$ column matrix with elements c_i. There are nontrivial solutions only for those values of λ that are roots of the characteristic polynomial $p(\lambda) = \det(\mathbf{H} - \lambda\mathbf{I})$. $\mathbf{H}$ is called the adjoint or regular matrix representation of H in the operator basis set $\{O_1, O_2, \ldots, O_{2K}\}$ [7]. This matrix is closely related to the fundamental matrix that proved to be useful in determining the conditions under which a PT-symmetric elliptic quadratic differential operator with real spectrum is similar to a self-adjoint operator [14].

In the case of a Hermitian operator we expect all the roots λ_i, $i = 1, 2, \ldots, 2K$ of the characteristic polynomial $p(\lambda)$ to be real. These roots are obviously the natural frequencies of the quantum-mechanical system (the actual quantum-mechanical frequencies being linear combinations of them). However, in the present case we do not assume this condition. Therefore, it follows from eq. (8) that

$$[H, Z^\dagger] = -\lambda^* Z^\dagger \tag{11}$$

where $Z^\dagger$ is a linear combination like eq. (7) with coefficients c_i^*. This equation tells us that if λ is a real root of $p(\lambda) = 0$, then $-\lambda^*$ is also a root. Obviously, Z and $Z^\dagger$ look like a pair of annihilation–creation or ladder operators because, in addition to eq. (9), we also have

$$HZ^\dagger|\psi\rangle = (E - \lambda^*)Z^\dagger|\psi\rangle \tag{12}$$

If A is a quadratic Hermitian operator that satisfies

$$[H, A] = 0 \qquad [A, O_i] = \sum_{j=1}^{2K} A_{ji} O_j \tag{13}$$

then it follows from the Jacobi identity

$$[H, [A, O_i]] + [O_i, [H, A]] + [A, [O_i, H]] \tag{14}$$

that

$$\mathbf{HA} - \mathbf{AH} = 0 \tag{15}$$

where $\mathbf{A}$ is the adjoint matrix representation of A.

4. Results

The eigenvalues of the matrix representation $\mathbf{H}$ of H_b

$$\mathbf{H} = \frac{i}{2}\begin{pmatrix} 0 & b & 2 & 0 \\ b & 0 & 0 & -2 \\ -2 & 0 & 0 & -b \\ 0 & 2 & -b & 0 \end{pmatrix} \tag{16}$$

are

$$\lambda_1 = -1 - i\frac{b}{2} \qquad \lambda_2 = -1 + i\frac{b}{2} \qquad \lambda_3 = 1 - i\frac{b}{2} \qquad \lambda_4 = 1 + i\frac{b}{2} \tag{17}$$

with the associated ladder operators

$$Z_1 = x - y + i(p_x + p_y) \qquad Z_2 = x + y + i(p_x - p_y)$$
$$Z_3 = Z_1^\dagger \qquad Z_4 = Z_2^\dagger \tag{18}$$

The only nonzero commutators are $[Z_1, Z_4] = [Z_2, Z_3] = 4$. Every one of these ladder operators looks like the sum of a creation operator for one vibrational mode and an annihilation operator for the other, which casts doubts on the existence of a square integrable vacuum.

In the present case, the adjoint matrix representations $\mathbf{H}$ and $\mathbf{H}_1$ of the operators H_b and H_1, respectively, commute in accordance with eq. (15).

If $|\psi\rangle$ is a square-integrable eigenvector of H_b with eigenvalue E, $H_b|\psi\rangle = E|\psi\rangle$, then it follows from $\langle\psi|H_b|\psi\rangle = \langle H_b\psi|\psi\rangle$ that $(E - E^*)\langle\psi|\psi\rangle = 0$ and the eigenvalue is real. Since in the present case the eigenvalues of H_b are complex we conclude that the corresponding eigenvectors are not square integrable (with respect to the scalar product defined above in eq. (4)).

The function

$$\psi_0(x, y) = e^{-x^2/2 + y^2/2} \tag{19}$$

satisfies $Z_1\psi_0 = Z_2\psi_0 = 0$ and $H\psi_0 = \psi_0$. Besides,

$$HZ_3\psi_0 = \left(2 - i\frac{b}{2}\right)Z_3\psi_0 \qquad Z_3\psi_0 = 2(x - y)\psi_0$$
$$HZ_4\psi_0 = \left(2 + i\frac{b}{2}\right)Z_4\psi_0 \qquad Z_4\psi_0 = 2(x + y)\psi_0$$
$$HZ_3Z_4\psi_0 = 3Z_3Z_4\psi_0 \qquad Z_3Z_4\psi_0 = Z_4Z_3\psi_0 = 4(x^2 - y^2 - 1)\psi_0 \tag{20}$$

in agreement with the general results of Section 3. A curious result is that $Z_1Z_3^n\psi_0 = 0$ and $Z_2Z_4^n\psi_0 = 0$, which follow from $[Z_1, Z_3] = 0$ and $[Z_2, Z_4] = 0$, respectively. In this case we obtain the eigenvalues of Deguchi et al. [3, 4]:

$$H\psi_{nm}^{(0)} = E_{nm}^{(0)}\psi_{nm}^{(0)} \qquad \psi_{nm}^{(0)} = Z_3^n Z_4^m \psi_0$$
$$E_{nm}^{(0)} = n + m + 1 + (m - n)i\frac{b}{2} \qquad n, m = 0, 1, \ldots \tag{21}$$

On the other hand, the function

$$\psi_1(x, y) = e^{x^2/2 - y^2/2} \tag{22}$$

satisfies $Z_3\psi_1 = Z_4\psi_1 = 0$ and $H\psi_1 = -\psi_1$. We also have,

$$HZ_1\psi_1 = \left(-2 - i\frac{b}{2}\right)Z_1\psi_1 \qquad Z_1\psi_1 = 2(x - y)\psi_1$$
$$HZ_2\psi_1 = \left(-2 + i\frac{b}{2}\right)Z_2\psi_1 \qquad Z_2\psi_1 = 2(x + y)\psi_1$$
$$HZ_1Z_2\psi_1 = -3Z_1Z_2\psi_1 \qquad Z_1Z_2\psi_1 = Z_2Z_1\psi_1 = 4(x^2 - y^2 + 1)\psi_1 \tag{23}$$

also in agreement with the general results of Section 3. In this case we have $Z_3Z_1^n\psi_1 = 0$ and $Z_4Z_2^n\psi_1 = 0$ for the same reason

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given above. The eigenvalues are quite different from those shown above

$$H\psi_{nm}^{(1)} = E_{nm}^{(1)} \psi_{nm}^{(1)} \qquad \psi_{nm}^{(1)} = Z_3^n Z_4^m \psi_1$$
$$E_{nm}^{(1)} = -(n+m+1) + (m-n)i\frac{b}{2} \qquad n, m = 0, 1, \ldots \tag{24}$$

Notice that the eigenvalues derived by means of the algebraic method and non-square-integrable eigenfunctions yield those obtained by Deguchi et al. [3, 4] but not exactly those of Feshbach and Tikochinski [2]. It is worth mentioning that Deguchi et al. [3, 4] resorted to a different scalar product that we do not discuss here.

5. Conclusion

The algebraic method is a simple and straightforward way of analyzing the properties of quadratic Hamiltonians, as shown in earlier papers [8–13]. In this paper we have applied it to the Bateman Hamiltonian that has been the origin of some controversy [3–6]. Our results show that the eigenvalues of the Bateman Hamiltonian are complex and the corresponding eigenfunctions are not square integrable with respect to the scalar product (4). In addition to this, we have found that some ladder operators annihilate an infinite number of wavefunctions. These facts reveal the difficulty in obtaining

a suitable vacuum state as argued by Bagarello et al. [5, 6] by means of different arguments.

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