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Comment

Comment on "Generalized differential transform method to differential-difference equation" by L. Zou et al.

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ABSTRACT

We show that the differential transform is the well known Taylor series (contrary to what the authors claim) and that the differential transform method (DTM) is the well known Frobenius method, commonly used to obtain exact or approximate solutions to differential equations. The generalized transform method (GTM) is merely the application of Padé approximants to the Taylor series produced by the Frobenius method. We argue that the particular solutions to the differential-difference nonlinear equations are of scarce utility because the initial conditions are not flexible enough for practical physical applications.

Some time ago Zou et al. [1] (ZWZ from now on) solved differential-difference equations by means of the so-called differential transform method (DTM) and proposed an improvement that they termed generalized differential transform method (GDTM) and also differential transform-Padé technique or generalized differential-difference Padé approximation. They argued that it is simple and easy to apply and showed results for several models with exact particular solutions. The purpose of this Comment is the analysis of the DTM and the GTM.

According to ZWZ the *differential transform* of a function $u_n(t)$ is given by

$$U_n(k) = \frac{1}{k!} \left. \frac{d^k u_n(t)}{dt^k} \right|_{t=t_0}, \tag{1}$$

whereas the *differential inverse transform function* is given by

$$u_n(t) = \sum_{k=0}^{\infty} U_n(k) \left(t - t_0\right)^k. \tag{2}$$

ZWZ forgot to add that the function $u_n(t)$ should be analytic in a neighbourhood of $t = t_0$. Anybody familiar with Calculus realizes that equations (1) and (2) give us the Taylor series of $u_n(t)$ about $t = t_0$ [2]. Curiously, ZWZ stated that "It is different from the traditional high order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders." All the expressions shown in their Table 1 are already well known mathematical properties of the derivative of a function and the coefficients of Taylor series. The fact is that the differential transform is the well known Taylor series and the DTM is the well known Frobenius method for the treatment of differential equations [3].

The GTM is merely the improvement of the Taylor series by means of Padé approximants [3]. If we have a finite number of coefficients of the Taylor series

$$f(x) = \sum_{j=0}^{\infty} f_j x^j, \tag{3}$$

then we can improve the results of the partial sums by means of Padé approximants of the form

$$[M/N](x) = \frac{\sum_{j=0}^{M} a_j x^j}{\sum_{j=0}^{N} b_j x^j} = \sum_{j=0}^{M+N} f_j x^j + \mathcal{O}\left(x^{M+N+1}\right). \tag{4}$$

ZWZ applied the approach to four examples of differential-difference equations; the first one is

$$u_n' = u_n \left(u_{n+1} - u_{n-1}\right). \tag{5}$$

If the initial condition is $u_n(0) = n$ then there is a particular exact solution

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$$u_n(t) = \frac{n}{1-2t},\ n = 1, 2, \ldots . \tag{6}$$

It seems that, in this as well as in the other examples, the initial condition is chosen according to the available particular solution. For example, $u_n(t) = (\alpha + n)/(\beta - 2t)$, where α and β are real parameters, is a somewhat more general particular solution with a more flexible initial condition $u_n(0) = (\alpha + n)/\beta$. We obtain ZWZ's example when $\alpha = 0$ and $\beta = 1$. The physical utility of this model appears to be highly dubious if restricted to this kind of particular exact solutions. ZWZ compared the Taylor series of $u_n(t)$ and the [2, 3] Padé approximant with the exact solution (6) in their Figs. 1 and 2, respectively. The fact is that any Padé approximant with $M \geq 1$ yields the exact solution; therefore, Fig. 2 compares the exact solution with itself.

The second example is

$$u_n' = u_n\left(u_{n-1} - u_{n+1} + u_{n-2} - u_{n+2}\right). \tag{7}$$

A particular solution with the initial condition $u_n(0) = n$ is

$$u_n(t) = \frac{n}{1+6t},\ n = 1, 2, \ldots . \tag{8}$$

A somewhat more general particular solution is $u_n(t) = (\alpha + n)/(\beta + 6t)$, where α and β are real parameters. Obviously, in this case the initial condition should be $u_n(0) = (\alpha + n)/\beta$ and we obtain ZWZ's example when $\alpha = 0$ and $\beta = 1$. All the conclusions about the preceding toy problem apply to this one. In particular, ZWZ's Fig. 4 compares the exact solution with itself.

The third example

$$u_n' = u_n^2\left(u_{n+1} - u_{n-1}\right), \tag{9}$$

does not admit rational solutions and the Padé approximants do not yield the exact result. ZWZ chose the particular exact solution

$$u_n(t) = 1 + \frac{\cosh(d) - 1}{1 + \cosh\left(d\,n + 2\sinh(d)\,t - 2\right)}, \tag{10}$$

with the initial condition $u_n(0)$ given by setting $t = 0$ in this equation. ZWZ's Fig. 5 shows that the [4, 4] Padé approximant is by far more accurate than the 10th-order Taylor series. This fact is hardly surprising because the radius of convergence of the Taylor series for $u_n(t)$ is determined by the poles of this function closest to the origin in the complex t-plane, which in this case are given by

$$t_c = -\frac{d\,n - 2 \pm i\pi}{2\sinh(d)}. \tag{11}$$

Therefore, the Taylor series diverges for all $|t| > 5.160877398$ when $n = 7$ and $d = 0.3$ (as roughly suggested by ZWZ's Fig. 5). On the other hand, sequences of Padé approximants converge towards $u_n(t)$ for all values of t and the poles of the latter function appear as roots of the denominators of the former ones [3]. For example, from the Padé approximant [14/14] we estimated $t_c \approx -0.1641926696 \pm 5.158264850\,i$ in good agreement with the exact result $t_c = -0.1641926698 \pm 5.158264853\,i$.

The fourth example is a slight generalization of the third one and we do not deem it necessary to discuss it here.

Summarizing: the differential transform is the Taylor series of the (supposedly unknown) solution to a differential equation. The DTM is the well known Frobenius method in disguise. This method, known since long ago, is commonly used to obtain approximate or exact solutions to differential equations [3]. The GDTM is merely the improvement, by means of Padé approximants [3], of the results provided by the partial sums of the series obtained by the Frobenius method. ZWZ applied this approach to extremely simple particular solutions to specially chosen differential-difference equations. Such particular solutions are of scarce utility because their initial conditions are too restricted to describe actual physical problems.

CRediT authorship contribution statement

Francisco M. Fernández: Conceptualization, Formal analysis, Software, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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