



Brief paper

Enlarging the domain of attraction of stable MPC controllers, maintaining the output performance[☆]

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ABSTRACT

This work presents an alternative way to formulate the stable Model Predictive Control (MPC) optimization problem that allows the enlargement of the domain of attraction, while preserving the controller performance. Based on the dual MPC that uses the null local controller, it proposed the inclusion of an appropriate set of slacked terminal constraints into the control problem. As a result, the domain of attraction is unlimited for the stable modes of the system, and the largest possible for the non-stable modes. Although this controller does not achieve local optimality, simulations show that the input and output performances may be comparable to the ones obtained with the dual MPC that uses the LQR as a local controller.

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1. Introduction

The usual way of guaranteeing stability of linear MPC is by means of an appropriate selection of three components: a terminal cost which is an associated Lyapunov function, a terminal constraint that forces the terminal states to belong to a positively invariant set for the system, and a local unconstrained controller for predictions beyond the control horizon (dual control). For stable systems, the simplest choice for the local controller is the null controller $K(x) = 0$ (Rawlings & Muske, 1993) which produces a bounded terminal cost when the infinite horizon value function is used. Adopting such a control strategy, the terminal region is limited only by the input definition set, at least for the regulator case. For unstable systems, however, the control optimization problem needs to include a terminal constraint that zeros the unstable modes, as they cannot be steered to the origin by the proposed local null controller, reducing in this way the original terminal set. Another choice for both, stable and unstable systems consists in using a Linear Quadratic Regulator (LQR) as a local controller (Sokaert & Rawlings, 1998). In this case, the controller presents a local optimality, i.e., inside the terminal set, the control action obtained by means of the MPC optimization is the same

as that of the LQR. However, since the terminal LQR control does not consider the constraints, the terminal set, and then, the whole domain of attraction may be rather small.

De Doná, Seron, Mayne, and Goodwing (2002), Magni, De Nicolao, Magnani, and Scattolini (2001) and Limon, Alamo, and Camacho (2005), have presented different methods of enlarging the domain of attraction. In the first two cases, the authors used a saturated local control law in order to enlarge the terminal region. In Magni et al. (2001), the enlargement of the domain of attraction (for nonlinear systems) is obtained by considering a prediction horizon larger than the control horizon. On the other hand, Limon et al. (2005) proposed a contractive terminal set given by a sequence of reachable sets.

Mhaskar, El-Farra, and Christofides (2004), have presented a hybrid control scheme, combining by means of an appropriate switching law a bounded control, for which the region of constrained closed-loop stability is explicitly characterized, with MPC (that minimizes a given objective subject to constraints). The scheme reconciles the stability and optimality of both controllers, and Mhaskar, El-Farra, and Christofides (2005) extended the approach to the case of model uncertainty.

This paper presents a different method to enlarge the domain of attraction of the stable MPC, preserving, whenever possible, the performance properties of the standard dual controller (i.e. the dual MPC that uses a LQR as terminal controller). The main idea consists of including an appropriate set of slacked terminal constraints into the optimization problem of the dual MPC that uses a null local controller.

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2. System description

Let us consider the (controllable and stabilizable) system

$$\begin{bmatrix} x^{nst}(k+1) \\ x^{st}(k+1) \end{bmatrix} = \begin{bmatrix} F^{nst} & 0 \\ 0 & F^{st} \end{bmatrix} \begin{bmatrix} x^{nst}(k) \\ x^{st}(k) \end{bmatrix} + \begin{bmatrix} B^{nst} \\ B^{st} \end{bmatrix} \Delta u(k), \quad (1)$$

$$y(k) = \begin{bmatrix} I_{ny} & \Psi \end{bmatrix} \begin{bmatrix} x^{nst}(k) \\ x^{st}(k) \end{bmatrix},$$

where

$$x^{nst} \in X^{nst} \subseteq R^{nns}, \quad x^{st} \in X^{st} \subseteq R^{ns},$$

$$\Delta u(k) = u(k) - u(k-1) \in R^{nu}.$$

The block diagonal form of model (1) can be obtained from the step response of the transfer function model (see González, Odloak, and Marchetti (2007) and Rodrigues and Odloak (2003)), or by an appropriate similarity transformation of a given state space model. The state component x^{st} represents the stable modes, while x^{nst} represents the non-stable modes of the system, containing both the integrating modes induced by the incremental form of the model, and the original unstable modes of the system. In this way, one can define

$$B^{nst} = \begin{bmatrix} B^{i\top} & B^{un\top} \end{bmatrix}^T, \quad F^{nst} = \text{diag}(I_{ny} \quad F^{un}),$$

$$x^{nst} = \begin{bmatrix} x^i\top & x^{un\top} \end{bmatrix}^T,$$

where I_{ny} is the identity matrix of dimension ny , $x^i \in X^i \subseteq R^{ny}$ are the integrating states, $x^{un} \in X^{un} \subseteq R^{nun}$ are the original unstable modes of the system, and $X^{nst} = X^i \times X^{un}$. For systems with non-repeated poles F^{nst} and F^{st} are diagonal matrices with components of the form $e^{r_i T}$ where r_i is a pole of the continuous system, and T is the sampling period. The system has nns and ns non-stable and stable poles, respectively. In addition, $nns = nun + ny$, where nun is the number of unstable modes, and ny is the number of integrating modes (introduced by the incremental form of the model) that is equal to the number of system outputs. Matrix Ψ accommodates the states into the output. The input set U is defined as follows:

$$U = \{ \Delta u : -\Delta u_{\max} \leq \Delta u \leq \Delta u_{\max} \text{ and} \\ u_{\min} \leq u(k-1) + \Delta u \leq u_{\max} \},$$

where $u(k-1)$ is the past value of the input u . In addition, it is assumed that the states are constrained to belong to a set X , given by $X = X^{st} \times X^{nst}$. Here, this set is defined by the operating window of the process. Set X^i must satisfy the input constraints, as follows: $B^i u_{\min} \leq x^i \leq B^i u_{\max}$.

3. Stabilizable sets for the non-stable states

Consider the following j -steps stabilizable set to the set Ω :

$$St_j^{nst}(X^{nst}, \Omega) = \{ x^{nst}(0) \in X^{nst} : \text{for all } k = 0, \dots, j-1, \\ \exists \Delta u(k) \in U \text{ such that } x^{nst}(k) \in X^{nst} \text{ and } x^{nst}(j) \in \Omega \},$$

where Ω (the equilibrium set) is given by

$$\Omega = \{ x^{nst} \in X^{nst} : x^i \in X^i \text{ and } x^{un} \in \{0\} \}.$$

Lemma 1. Set $St_j^{nst}(X^{nst}, \Omega)$ is a control invariant set for states x^{nst} , for all $j \geq 1$.

Proof of Lemma 1. Since for any $x^{nst} \in \Omega$, we have $x^{un} \equiv \{0\}$, then, if we chose the feasible input increment $\Delta u = 0$, the integrating state x^i will remain unmodified, which implies that the state x^{nst} will remain inside Ω . Then, Ω is a control invariant set, which implies that $St_j^{nst}(X^{nst}, \Omega)$ is a control invariant set for all $j \geq 1$. This is so because the stabilizable sets have the following property: $St_j^{nst}(X^{nst}, \Omega) \subseteq St_{j+1}^{nst}(X^{nst}, \Omega)$ for all $j \geq 1$ (see (Kerrigan & Maciejowski, 2000)). \square

Remark 1. Since we are explicitly considering both input and input increment constraints, then there exists a limited set of “non-stable” states that can be steered to the equilibrium set by means of a “feasible” sequence of control actions (see Hu, Miller, and Qiu (2002) and Zhao and Xue (2006)). This is so, since large initial non-stable states require large control actions, or large control moves, to be stabilized. Notice that this is not true for the pure stable modes, that can be steered to the equilibrium by the null control law $\Delta u = 0$. Observe that any non-stable state in Ω is an equilibrium state (since $x^{nst} = F^{nst} x^{nst}$). Thus, by the definition of the j -step stabilizable set, the set of non-stable states that can be steered to the equilibrium set by means of a feasible (arbitrary large) sequence of control actions can be called $St_{\infty}^{nst}(X^{nst}, \Omega)$. Given that $St_{\infty}^{nst}(X^{nst}, \Omega) \subseteq X^{nst}$ and it is limited, one way to explicitly find this set is by increasing the index j of $St_j^{nst}(X^{nst}, \Omega)$ up to N , such that $St_{N+1}^{nst}(X^{nst}, \Omega) \approx St_N^{nst}(X^{nst}, \Omega)$. Then, the so defined largest possible domain of attraction for the non-stable states, Θ , is given by: $\Theta = St_N^{nst}(X^{nst}, \Omega) = St_{\infty}^{nst}(X^{nst}, \Omega)$.

Note that the set Θ (it will be shown later that this set defines the domain of attraction of the non-stable states of the proposed controller) does not depend on the selected control law, but on the nature of the system and the states, as well as on the input constraints. In addition, in the case of box constraints, that is, upper and lower bounds, the set $St_j^{nst}(X^{nst}, \Omega)$ can be easily computed (*a priori*) using available tools (e.g. set invariance toolbox for LTI systems, (Kerrigan, 2000)), and it has the form: $\forall x \leq v$.

4. MPC controller

Rodrigues and Odloak (2003) proposed an infinite horizon MPC explicitly designed for model (1). They define an appropriate set of slacked terminal constraints that assure stability while preserving recursive feasibility. To extend the method to an unstable system, consider the problem:

Problem 1.

$$\min_{\Delta u_k, \delta_k^{nst}} V_k = \sum_{j=0}^{\infty} (x(k+j|k) + \bar{\delta}(k,j))^T Q (x(k+j|k) + \bar{\delta}(k,j)) \\ + \sum_{j=0}^{m-1} \Delta u(k+j|k)^T R \Delta u(k+j|k) + \delta_k^{nst\top} S^{nst} \delta_k^{nst} \quad (2)$$

subject to:

$$\Delta u(k+j|k) \in U, \quad j = 1, \dots, m-1, \quad (\text{input const.}) \quad (3)$$

$$F^{nst\top m} (x^{nst}(k) - x^{sp}) + C^{nst} \Delta u_k + F^{nst\top m} \delta_k^{nst} = 0 \\ (\text{term. constr.}) \quad (4)$$

where

$$x(k+j|k) = \left[(x^{nst}(k+j|k) - x^{sp})^T \quad x^{st}(k+j|k)^T \right]^T,$$

$$x^{sp} = \begin{bmatrix} y^{sp\top} & 0 \end{bmatrix}^T, \quad x(k|k) = x(k), \quad \Delta u(k+j|k)$$

is the control move computed at time k to be applied at time $k+j$, m is the control horizon, Q , R and S^{nst} are positive weighting matrices of appropriate dimension, x^{sp} is the output reference,

$$\Delta u_k = [\Delta u(k|k)^T \quad \dots \quad \Delta u(k+m-1|k)^T]^T,$$

$$C^{nst} = [F^{nst\ m-1} B^{nst} \quad F^{nst\ m-2} B^{nst} \quad \dots \quad B^{nst}],$$

$$\bar{\delta}(k, j) = \left[\left(F^{nst\ j} \delta_k^{nst} \right)^T \quad 0 \right]^T, \quad \text{and} \quad \delta_k^{nst} = \left[\delta_k^{st} \quad \delta_k^{un} \right]^T \in R^{nns}$$

is a vector of slack variables. The corresponding penalization can be written as $S^{nst} = \text{diag}(S^i \ S^{un})$. The inclusion of the slack variables enlarges the domain of attraction, but it may deteriorate the convergence of the cost (with the slack variables, the unstable states do not necessarily go to zero).

Remark 2. It can be shown that for a positive matrix S^{un} , the slack δ_k^{un} resulting from the solution to **Problem 1** will be different from zero only if $x^{nst}(k) \notin St_m^{nst}(X^{nst}, \Omega)$. This means that if the initial states are such that $x^{nst}(k) \in St_m^{nst}(X^{nst}, \Omega)$, then the solution to **Problem 1** will produce the same control action as when the slack vector δ_k^{un} is not included in the control problem.

4.1. The stable MPC

Now, assume that the initial non-stable state is such that $x^{nst}(k) \in (St_j^{nst}(X^{nst}, \Omega) \setminus St_{j-1}^{nst}(X^{nst}, \Omega))$, for some $1 \leq j \leq N$, where N represents an integer such that $St_{N+1}^{nst}(X^{nst}, \Omega) \approx St_N^{nst}(X^{nst}, \Omega)$. Then, one may define the optimization problem that produces the stable MPC as:

Problem 2.

$$\min_{\Delta u_k, \delta_k^{nst}} V_k$$

subject to: (3), (4) and

$$x^{nst}(k+1|k) \in St_{index}^{nst}(X^{nst}, \Omega), \quad index = \max(j-1, m). \quad (5)$$

Remark 3. Constraint (5) forces the non-stable state to go from one stabilizable set to the next one, until the state reaches $St_m^{nst}(X^{nst}, \Omega)$. Once this set is reached, the slack corresponding to the unstable states, δ_k^{un} , will be zeroed.

Remark 4. Given that the effect of the (slacked) non-stable modes on the cost is zeroed at the end of the control horizon (by imposing constraint (4)), the cost defined in (2) can be simplified. First, consider the error on the non-stable state predictions beyond the control horizon:

$$x^{nst}(k+m+j|k) - x^{sp} + F^{nst\ (m+j)} \delta_k^{nst}$$

$$= F^{nst\ j} \left(x^{nst}(k+m|k) - x^{sp} + F^{nst\ m} \delta_k^{nst} \right) = 0, \quad j = 1, \dots$$

Then, the cost can be written as follows:

$$V_k = \sum_{j=0}^{m-1} \left(x(k+j|k) + \bar{\delta}(k, j) \right)^T Q \left(x(k+j|k) + \bar{\delta}(k, j) \right)$$

$$+ \delta_k^{nst\ T} S^{nst} \delta_k^{nst} + \sum_{j=0}^{m-1} \Delta u(k+j|k)^T R \Delta u(k+j|k)$$

$$+ x^{st}(k+m|k)^T P x^{st}(k+m|k)$$

where matrix P is computed using the following Lyapunov equation: $P = Q + F^{st\ T} P F^{st}$.

The convergence of the closed loop system with the controller defined by **Problem 2** is assured by the theorem:

Theorem 1. For systems with stable and non-stable modes that remain controllable at the steady state corresponding to the desired output reference, **Problem 2** is always feasible with a domain of attraction given by $X^{at} = \{x \in X : x^{nst} \in \Theta\}$, where Θ is obtained, as in **Remark 1**. Also, if weight S^i is sufficiently large, then the control sequence obtained from the solution to **Problem 2** at successive time steps drives the output of the closed loop system asymptotically to the reference value.

Proof. Since the slack corresponding to the non-stable state is unbounded, in **Problem 2**, constraint (4) together with constraint (3), are always feasible (independent of other constraints). Then, assume first that the initial state is such that $x^{nst}(k) \in St_m^{nst}(X^{nst}, \Omega)$, which means that $\delta_k^{un} = 0$. Then, since $St_m^{nst}(X^{nst}, \Omega)$ is a control invariant set (**Lemma 1**), there exists a control action such that $x^{nst}(k+1|k) \in St_m^{nst}(X^{nst}, \Omega)$. Therefore, for the undisturbed nominal system, it results that $x^{nst}(k+1) \in St_m^{nst}(X^{nst}, \Omega)$, which means that constraint (5) will be feasible at $k+1$. Then, by induction, constraint (5) will be feasible at any future time step.

On the other hand, if the initial state is such that $x^{nst}(k) \notin St_m^{nst}(X^{nst}, \Omega)$, the slacks corresponding to the non-stable states will be necessarily different from zero. Since we assume that the initial non-stable state, $x^{nst}(k)$, is such that $x^{nst}(k) \in St_j^{nst}(X^{nst}, \Omega)$ for a finite $j \leq N$, then, there exists a feasible input increment such that $x^{nst}(k+1|k) \in St_{j-1}^{nst}(X^{nst}, \Omega)$. Therefore, for the undisturbed nominal system $x^{nst}(k+1) \in St_{j-1}^{nst}(X^{nst}, \Omega)$. Then, there exists a feasible input increment such that $x^{nst}(k+2|k+1) \in St_{j-2}^{nst}(X^{nst}, \Omega)$, which means that constraint (5) will be feasible at $k+1$. Then, by induction, constraint (5) will be feasible at any future time step, up to $k+j-m$, where the non-stable state reaches $St_m^{nst}(X^{nst}, \Omega)$. This means that **Problem 2** has recursive feasibility.

Because of the recursive feasibility of **Problem 2**, slack variables δ_k^{un} will be null after j time steps. Once the unstable slack variables are zeroed, the controller becomes equivalent to the one presented in **González et al. (2007)**, and the convergence of the states (including the non-stable states), the input increment and the integrating slacks δ_k^i to zero is assured (in that work is presented a lower bound to S^i , that assures the cost convergence). \square

Remark 5. This controller is not optimal, in the sense that it does not use an optimal controller as the terminal controller (even for the unconstrained case). However, it will be shown in the simulation section that the performance of the proposed controller may be similar to the one of the standard MPC + LQR dual controller.

4.2. Case where the system has only one unstable state per input ($nun \leq nu$)

Consider now the case that does occur in most real continuous process systems, where each input affects at most only one unstable state x^{un} . For this case, the slack reduction operated by constraint (5) can be obtained by means of a simple norm minimization. For the one-dimensional case, the successive non-stable stabilizable sets can be associated with the norm of the slacks of the unstable states, and then, to minimize this norm is equivalent to steering the non-stable states from one stabilizable set to the next one. Then, consider the following two-stage problem:

Problem 3a.

$$\min_{\Delta u_{a,k}, \delta_k^{un}} V_{a,k} = \delta_k^{un\ T} S^{un} \delta_k^{un}$$

subject to

$$\Delta u_a(k+j|k) \in U, \quad j = 0, 1, \dots, m-1$$

$$F^{um} x^{un}(k) + C^{un} \Delta u_{a,k} + F^{um} \delta_k^{un} = 0.$$

Problem 3b.

$$\min_{\Delta u_{b,k}, \delta_k^{un}, \delta_k^i} V_{b,k} = \sum_{j=0}^{\infty} (x(k+j|k) + \bar{\delta}(k,j))^T Q (x(k+j|k) + \bar{\delta}(k,j)) + \sum_{j=0}^{m-1} \Delta u_b(k+j|k)^T R \Delta u_b(k+j|k) + \delta_k^{iT} S^i \delta_k^i$$

subject to

$$\Delta u_b(k+j|k) \in U, \quad j = 0, 1, \dots, m-1$$

$$x^i(k) - y^{sp} + C^i \Delta u_{b,k} + \delta_k^i = 0$$

$$F^{um} x^{un}(k) + C^{un} \Delta u_{b,k} + F^{um} \delta_k^{un} = 0, \quad \delta_k^{un} = \delta_k^{un*}.$$

where δ_k^{un*} is the optimal slack of the unstable state obtained in Problem 3a,

$$C^{un} = \begin{bmatrix} F^{um^{m-1}} B^{un} & F^{um^{m-2}} B^{un} & \dots & B^{un} \end{bmatrix},$$

$$C^i = \begin{bmatrix} B^i & B^i & \dots & B^i \end{bmatrix}.$$

The sequential solution of Problems 3a and 3b produces a controller that drives the system output to the desired reference value, as shown in the following theorem:

Theorem 2. For systems with stable and non-stable modes in which each input affects at most only one unstable state x^{un} , the sequential solution of Problems 3a and 3b is always feasible with a domain of attraction given by $X^{at} = \{x \in X : x^{nst} \in \Theta\}$, where Θ is obtained as in Remark 1. Also, if the weight S^i is sufficiently large, then the control sequence obtained from the solution to Problems 3a and 3b at successive time steps drives the output of the closed loop system asymptotically to the reference value.

Proof. Since we are assuming that each input affects at most one unstable state x^{un} , we can develop the proof for the SISO case, without loss of generality. Consider that at time step k $x^{nst} \in St_m^{nst}(X^{nst}, \Omega)$. The orthogonal projection of this stabilizable set onto the x^{un} subspace, originates the stabilizable set $St_m^{un}(X^{un}, \{0\})$, which represents the m -step stabilizable set to the origin for the unstable states x^{un} . If each (scalar) unstable state, $x^{un}(k)$, is inside the set $St_m^{un}(X^{un}, \{0\})$ then the slack corresponding to this unstable state will be null and the sequence of problems 3a and 3b is equivalent to Problem 2 with $\delta_k^{un} = 0$. On the other hand, if the non-stable state $x^{nst}(k)$ is outside $St_m^{nst}(X^{nst}, \Omega)$, then, the slack will be different from zero. In this case, the corresponding optimal input increment will necessarily saturate one or more of the constraints defined by set U , which implies that $\Delta u_{b,k} = \Delta u_{a,k}$. If the initial non-stable state x^{nst} belongs to $St_j^{nst}(X^{nst}, \Omega)$, for a finite integer j such that $m < j \leq N$, then, the unstable state $x^{un}(k)$ belongs to $St_j^{un}(X^{un}, \{0\})$. This means that there exists a feasible sequence of j input increments $\Delta \tilde{u}(k+i|k) \in U$, for $i = 0, \dots, j$, such that $x^{un}(k+i|k) \in St_{j-i}^{un}(X^{un}, \{0\})$. Consider now the first m elements of this sequence, $[\Delta \tilde{u}(k|k) \dots \Delta \tilde{u}(k+m-1|k)]^T$, and the corresponding final unstable state $x^{un}(k+m|k) = F^{um} \delta_k^{un}$. Since $|x^{un}(k+m|k)|$ will be the smallest possible taking into account the input constraints (i.e. $x^{un}(k+m|k) \in St_{j-m}^{un}(X^{un}, \{0\}) \subset St_{j-m+1}^{un}(X^{un}, \{0\}) \subset \dots \subset St_j^{un}(X^{un}, \{0\})$), then $|\delta_k^{un}|$ will also be as small as possible, which implies that the optimal solution to Problem 3a will be

$\Delta u_a^*(k+i|k) = \Delta \tilde{u}(k+i|k)$ (the solution will be such that the successive states go from one stabilizable set to the next one). Following the receding horizon policy, the control action $\Delta u_b^*(k|k) = \Delta u_a^*(k|k)$ is applied to the system. At time step $k+1$, and provided that no mismatch exists between the nominal and the real system, we have $x^{un}(k+1|k+1) \in St_{j-1}^{un}(X^{un}, \{0\})$, and then $[\Delta \tilde{u}(k+1/k) \dots \Delta \tilde{u}(k+m/k)]^T$ constitutes a feasible solution to Problem 3a. Furthermore, a sequence with the same characteristics as the latter one will be the optimal one, since it produces the smallest slack satisfying $F^{um} \delta_{k+1}^{un} = x^{un}(k+m+1|k+1) \in St_{j-1-m}^{un}(X^{un}, \{0\})$. Then, by induction, we have that $x^{un}(k+j-m|k+j-m) \in St_m^{un}(X^{un}, \{0\})$, which implies that the slack variable corresponding to the unstable state goes to zero in $j-m$ time steps. Once the slack variables are zeroed, Problems 3a and 3b become equivalent to Problem 2 with $\delta_k^{un} = 0$, and the convergence is assured. \square

Remark 6. The reasoning adopted in the proof above can be followed whenever the successive non-stable stabilizable sets can be associated with a level curve of the norm of the slacks of the unstable state. In this case, it is not necessary to know in which set the initial state is located.

5. Domain of attraction of the proposed controller

As stated in Theorems 1 and 2, the domain of attraction of the proposed controller is given by $X^{at} = \{x \in X : x^{nst} \in \Theta\}$. This set, which represents the largest domain of attraction (for stable and non-stable states) that the system together with the state and input constraints permits, does not depend on the control horizon m . Furthermore, an *a priori* explicit characterization of the set of feasible initial conditions starting from where closed-loop stability is guaranteed is provided. On the other hand, the domain of attraction of the standard dual MPC (considering a velocity model) is given by the m -stabilizable set to the terminal set O_∞^K , that is, $St_m(X, O_\infty^K)$, where

$$O_\infty^K = \{x(0) \in X : (A - BK)^j x(0) \in X, K(A - BK)^j x(0) \in U$$

$$j = 0, 1, \dots\}$$

and K (terminal controller) is obtained from the solution to the algebraic Riccati equation. It can be shown that $St_\infty^{nst}(X^{nst}, \Omega) \supseteq St_m^{nst}(X^{nst}, O_\infty^K)$, where $St_m^{nst}(X^{nst}, O_\infty^K)$ is the orthogonal projection of $St_m(X, O_\infty^K)$ into the non-stable states space. Also, and opposite to the proposed controller, the standard dual MPC presents a limited domain of attraction for the stable modes x^{st} , even if X^{st} is unlimited.

In order to make a comparison between the proposed controller and the dual MPC–LQR, let us consider the following model:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.649 & 0 \\ 0 & 0 & 0.607 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.083 \\ -0.03 \end{bmatrix},$$

$$C = [10.204 \quad -101.534 \quad 101.534],$$

which corresponds to an inverted pendulum, with sampling period $T = 0.05$ s (Lundberg & Roberge, 2003). The output represents the angle position from the vertical (deg), and the input represents the cart position (cm). The input constraints are: $\Delta u_{\max} = 0.25$ (cm), $u_{\min} = -1$ (cm), $u_{\max} = 1$ (cm). The parameters corresponding to the dual MPC–LQR (which determine its domain of attraction) are: $Q = 0.1$, $R = 1$, $m = 2$. The LQR (terminal) controller is given by: $K = [-0.164 \quad 17.82 \quad 0]$. Fig. 1 shows the domain of attraction of the dual MPC–LQR corresponding to: $Q = 10$, $Q = 0.1$ and $Q = 0.0001$, and maintaining $R = 1$ and $m = 2$. It can be shown that, for the case $Q = 0.0001$, which shows the largest domain

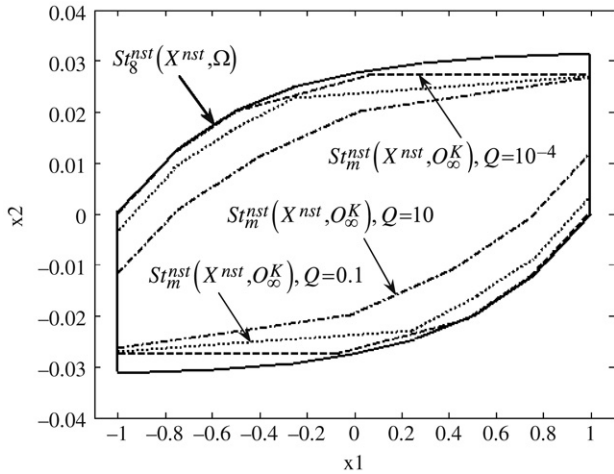


Fig. 1. Domain of attraction of the proposed MPC (solid line), and of the dual MPC with $Q = 0.0001$, $Q = 0.1$ and $Q = 10$.

of attraction, the closed-loop performance is extremely slow, and then unacceptable. On the other hand, the largest possible domain of attraction for the non-stable states is given, in this case, by: $\Theta = S_{N}^{nst}(X^{nst}, \Omega) = S_{\infty}^{nst}(X^{nst}, \Omega) \subseteq X^{nst}$, with $N = 8$ (that is, $S_{8+1}^{nst}(X^{nst}, \Omega) \approx S_{\infty}^{nst}(X^{nst}, \Omega)$), and is also shown in Fig. 1 (solid line). This set represents the domain of attraction of the proposed controller.

6. Simulation results

The performance of the controller proposed here is tested in a styrene reactor. The system is operated in an unstable steady state (Hidalgo & Brosilow, 1990). The inputs are the flow rate of initiator (u_1) and the flow rate of the cooling fluid (u_2). The controlled outputs are the viscosity (y_1) and the temperature (y_2). The linear discrete time model is the following:

$$x(k + 1) = Ax(k) + B\Delta u(k), \quad y(k) = Cx(k)$$

where A , B and C are given in Box I.

Matrix A has two integrating modes produced by the velocity form of the input, and one unstable mode. The tuning parameters of the proposed controller are: $T = 1$, $Q = 1$, $R = 1$, $m = 2$, $S^i = 10^2$ and $S^{un} = 10^2$. Parameters m , Q and R play the same role as in the conventional MPC. S^i should be selected to be larger than the lower limit given in González et al. (2007), (usually, one or two orders of magnitude larger than Q). Finally, S^{un} can be any positive weight matrix. The tuning parameters of the dual MPC are: $T = 1$, $Q = 1$, $R = 1$, $m = 2$. The input constraints are $u_{1,max} = 34.4$ ($L h^{-1}$), $u_{1,min} = -34.4$ ($L h^{-1}$), $\Delta u_{1,max} = 5$ ($L h^{-1}$), $u_{2,max} = 141.48$ ($L h^{-1}$), $u_{2,min} = -141.48$ ($L h^{-1}$), $\Delta u_{2,max} = 10$ ($L h^{-1}$).

For the dual MPC, each time a new output set point is introduced into the system, the terminal set must be re-computed in order to update the input constraints. Then, the optimization problem of the dual MPC may become infeasible, even for a feasible steady state. The sequence of changes in the set points considered in this example is as follows: at $t = 10$ h, $y_1^{sp} = [-0.08 \ -1]^T$; at $t = 50$ h, $y_2^{sp} = [0 \ 0]^T$; at $t = 100$ h, $y_3^{sp} = [0.02 \ 2.14]^T$ and at $t = 150$ h, $y_4^{sp} = [-0.18 \ -0.39]^T$. These set-points correspond to the following input values: $u_1^{ss} = [30 \ 20]^T$, $u_2^{ss} = [0 \ 0]^T$, $u_3^{ss} = [-34.4 \ 120]^T$ and $u_4^{ss} = [40 \ 160]^T$. Note that the third change saturates the first input, while the last one forces both inputs outside the feasible range. This simulation shows

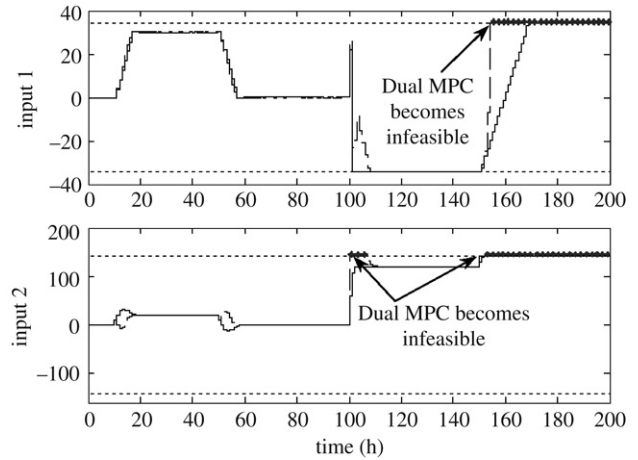


Fig. 2. Input responses. Proposed MPC (solid line), dual MPC (dashed line) and periods of infeasibility of dual MPC (asterisks).

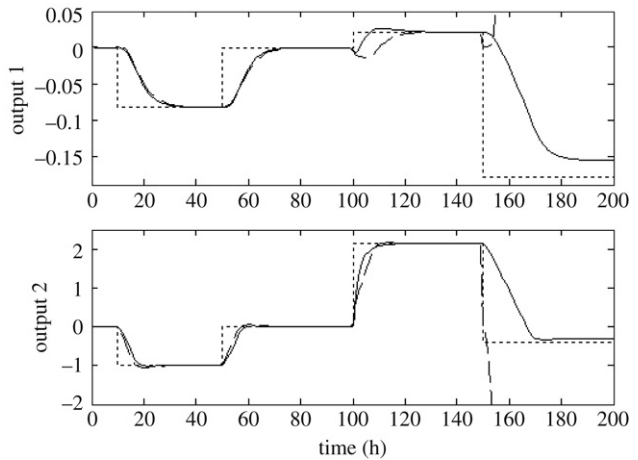


Fig. 3. Output responses. Proposed MPC (solid line), dual MPC (dashed line).

how the two controllers perform when the desired input steady state is close to the bounds. Fig. 2 shows the input responses and constraints, and Fig. 3 shows the output responses and the output set point. For the set point changes introduced at $t = 10$ h and $t = 50$ h, both controllers preserve their feasibility and show similar performances (because of the local optimality, the dual MPC has a slightly better performance). However, for the set-point change introduced at $t = 100$ h, the corresponding terminal set makes the optimization problem of the standard dual MPC infeasible. This can be seen in Fig. 2 where input u_2 tends to surpass the upper bound at time near $t = 100$ h, for about 5 h. In the real system, the maximum value is implemented.

The input increment constraints were not included as they would turn the dual MPC infeasible and would make the comparison impossible. Since the desired steady state is feasible, both controllers are able to stabilize the system at the set point. Finally, for the set point change introduced at time $t = 150$ h, the dual MPC becomes infeasible throughout the subsequent time period (since the unstable modes cannot be canceled), while the proposed MPC remains feasible and steers both inputs to their bounds. As expected, for the proposed MPC, the outputs show offset at steady state, because the inputs corresponding to desired steady state lie outside their feasible ranges. However, the loop remains stable throughout the whole time period. This property of the proposed controller is obtained through the use of the slack variables, which can assume non-null values (from time $t = 150$ until $t = 200$ h, in this case). When this happens, the control cost converges to a non-null value.

$$A = \text{diag}([0.73 \quad 0.73 \quad 3.83 \quad 0.59 \quad 0.77 + 0.02i \quad 0.77 - 0.02i \quad 1 \quad 1])$$

$$B = \begin{bmatrix} 0 & 0.32 & 0.08 & 0.03 & -0.07 + 0.14i & -0.07 - 0.14i & 1.03 & 0 \\ -0.0001 & 0 & -0.0081 & 0 & -0.03 + 0.23i & -0.03 - 0.23i & 0 & 1.003 \end{bmatrix}^T$$

$$C = \begin{bmatrix} -68.2546 & 0.0046 & -0.0027 & -0.0466 & -0.0281 + 0.0141i & -0.0281 - 0.0141i & -0.0023 & -0.0005 \\ 0 & 0 & 0.3656 & 0.7458 & 0.0397 + 0.0059i & 0.0397 - 0.0059i & -0.0367 & 0.0070 \end{bmatrix}$$

Box 1.**7. Conclusion**

In this paper, a different formulation of the stable MPC is presented, which includes an appropriate set of slack variables. The main benefits of the proposed approach become more effective in the application stage: a larger domain of attraction in comparison with a standard dual MPC, guarantee of recursive feasibility when the system is guided to a point in which the input saturates, or even, the desired operating point surpasses the bounds, and offset free (whenever the inputs do not saturate) without the necessity of a target calculation stage. In addition, despite the proposed controller not having local optimality, it shows a relatively good performance, similar in many cases to the standard dual MPC that uses a LQR as a local controller.

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