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# Pure Variable Inclusion Logics

**Abstract.** The aim of this article is to discuss pure variable inclusion logics, that is, logical systems where valid entailments require that the propositional variables occurring in the conclusion are included among those appearing in the premises, or vice versa. We study the subsystems of Classical Logic satisfying these requirements and assess the extent to which it is possible to characterise them by means of a single logical matrix. In addition, we semantically describe both of these companions to Classical Logic in terms of appropriate matrix bundles and as semilattice-based logics, showing that the notion of consequence in these logics can be interpreted in terms of truth (or non-falsity) and meaningfulness (or meaninglessness) preservation. Finally, we use Płonka sums of matrices to investigate the pure variable inclusion companions of an arbitrary finitary logic.

**Keywords**: logics of variable inclusion; significance logics; analytic entailment; weak Kleene logics; Płonka sums

## 1. Introduction

The aim of this article is to discuss refinements of the so-called variable inclusion logics, that is to say, of what are dubbed the right companion and the left companion of a given logical system in [7]. These target refinements will be referred to by us as *pure variable inclusion logics*, whence we will speak of the *pure right companion* and the *pure left companion* of a given logical system. For this purpose, here we will start our investigations focussing on these variable inclusion logics associated with Classical Logic. From these considerations, we will later extract technical and conceptual generalisations regarding the possibility of endowing the

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pure right and pure left companion of every Tarskian logic with matrix semantics.

The article is organised as follows. Section 2 presents both a historical and conceptual motivation for pure variable inclusion logics based on Classical Logic. In Section 3 we show that the pure right companion of Classical Logic has no single characteristic matrix, when this system is conceived within the Set-Fmla framework. However, we duly provide a matrix semantics for both logics with the help of a pair of logical matrices, also facilitating a philosophical reading of this binomial approach to logical consequence. Furthermore, we prove that the pure left companion of Classical Logic has a 5-valued characteristic matrix. Section 4 is devoted to a thorough discussion of the abstract semantics for the pure variable inclusion logics associated with arbitrary logics, with the tools of Płonka sums of matrices.

This being said, before delving into the proper contents of the article, let us briefly make explicit that we will be working with propositional languages or similarity types $\mathcal{L}$, $\mathcal{L}'$, $\ldots$ counting with a denumerable set $Var$ of propositional variables $x, y, z, x_1, x_2 \ldots$, and logical connectives. In particular, $\mathcal{L}_0$ will denote the language whose connectives are $\neg$ (negation, unary), $\wedge$ (conjunction, binary) and $\vee$ (disjunction, binary). $\mathbf{FOR}(\mathcal{L})$ will stand for the algebra of $\mathcal{L}$-formulae, standardly defined, whose universe is the set of $\mathcal{L}$-formulae $FOR(\mathcal{L})$. In this respect, lower case Greek letters $\varphi, \psi, \chi, \ldots$ will be considered as metavariables for formulae, whereas upper case Greek letters $\Gamma$ and $\Delta$ will be considered as metavariables for sets of formulae. Below, $Var(\varphi)$ denotes the set of propositional variables occurring in $\varphi$, while $Var(\Gamma)$ denotes the set $\bigcup\{ Var(\gamma) \mid \gamma \in \Gamma\}$.

A logic $L$ is a pair $\langle \mathbf{FOR}(\mathcal{L}), \vdash_L \rangle$ consisting of a formula-algebra and a substitution-invariant Tarskian consequence relation. For the purpose of the discussion below, we will interchangeably refer to logics and their consequences relations, hoping that no confusion arises — furthermore, taking "logic" to always mean such systems.

## 2. The target logics

The label "variable inclusion logic" and also "containment logic" has been applied to a variety of different systems that have certain similarities — among which it is possible to include those discussed in [7, 8, 9, 11, 12, 17, 18]. More recently, the literature distinguishes between two types of variable inclusion logics: a given logical system, indeed, can have a left and a right variable inclusion companion. For a logic $L = \langle \mathrm{FOR}(\mathcal{L}), \vdash_L \rangle$, these companions can be denoted as $L^{\mathrm{l}}$ and $L^{\mathrm{r}}$, respectively. Their consequence relations can be characterised as follows.

$$\begin{aligned} \varGamma \vdash_{L^{\mathrm{l}}} \varphi &\iff \exists \varDelta \subseteq \varGamma\colon \varDelta \vdash_L \varphi \text{ and } \mathit{Var}(\varDelta) \subseteq \mathit{Var}(\varphi), \\ \varGamma \vdash_{L^{\mathrm{r}}} \varphi &\iff \text{either } \varGamma \text{ contains an } L\text{-anti-theorem or} \\ &\qquad \varGamma \vdash_L \varphi \text{ and } \mathit{Var}(\varphi) \subseteq \mathit{Var}(\varGamma). \end{aligned}$$

Thanks to the results of [9, 17, 41] it is immediate to observe that the left and right companions of Classical Logic (CL, for short), formulated in the language $\mathcal{L}_0$, are the so-called paracomplete and paraconsistent weak Kleene logics $\mathrm{K}_3^{\mathrm{w}}$ and PWK — investigated also in [6, 24, 28]. It is worth observing that these logics share, respectively, all the theorems and all the anti-theorems of Classical Logic — and that, in general, if a logic has theorems its left companion will also have them, and similarly for anti-theorems and right companions. This serves the purpose of noticing that as variable inclusion logics, the left and right companion of CL, or indeed of any logic, are rather unsatisfactory precisely because they seem to allow for notable exemptions to the variable inclusion clauses. In other words, they are variable inclusion logics provided the premises or the conclusions, respectively, are not or do not include anti-theorems or theorems.

In this article, however, we will not be concerned with companions of the above sorts, but rather with certain refinements thereof. Since in these logics the variable inclusion pattern is unconstrained, we will refer to them as *pure variable inclusion logics*, or pure companions for a rather succinct denomination. For a logic $L$ with consequence relation $\vdash_L$, these pure companions can be denoted as $L^{\mathrm{pl}}$ and $L^{\mathrm{pr}}$, respectively, and their consequence relations can be characterised as follows.

$$\begin{aligned} \varGamma \vdash_{L^{\mathrm{pl}}} \varphi &\iff \exists \varDelta \subseteq \varGamma\colon \varDelta \neq \emptyset, \varDelta \vdash_L \varphi \text{ and } \mathit{Var}(\varDelta) \subseteq \mathit{Var}(\varphi), \\ \varGamma \vdash_{L^{\mathrm{pr}}} \varphi &\iff \varGamma \vdash_L \varphi \text{ and } \mathit{Var}(\varphi) \subseteq \mathit{Var}(\varGamma). \end{aligned}$$

As documented in [13, 17, 31], it has long been known that, when taking into account single premises and single conclusions, $\mathrm{CL}^{\mathrm{pr}}$ coincides with the first-degree entailment fragment of the famous logic of Analytic Implication of Parry in [34]. Analogously, when taking into account single premises and single conclusions, $\mathrm{CL}^{\mathrm{pl}}$ coincides with the first-degree entailment fragment of the perhaps less famous logic of Dual Dependence of Epstein in [16]; see, e.g., [27] for a discussion of these systems in the broader context of relating logics and relating semantics. It may be for these reasons that in [18] Ferguson refers to the pure right companion of a given logical system as its Parry fragment, and in [38] Szmuc refers to the pure left companion of a given logical system as its Dual Parry fragment. Notice that not only in the case of the pure variable inclusion companions of CL, but in the case of the systems of this sort based on any logic whatsoever, the pure left and pure right companions will have neither theorems nor anti-theorems.

One historical and conceptual reason why one might be interested in pure variable inclusion logics, and in pure right companions in particular, actually comes from reflecting upon some themes connected with Parry's logic. Thus, Dunn interpreted Parry's analytic implication as analytic in Kant's sense, e.g., in [14, p. 17]. In this respect, his suggestion was to understand analytic implications by analogy with analytic judgments in the context of Kantian philosophy. Just like in analytic judgments the predicate was implicit, occult, or included in the subject, in analytic implications, the consequent could be taken to be implicit, occult, or included in the antecedent. This, furthermore, helped to push forward the interpretation of analytic implication as a sort of content containment relation — as done by Anderson and Belnap in [2, p. 23].

Recently, T. Ferguson contested this reading of Parry's project, claiming that his motivations were mainly syntactic. Thus, while the containment interpretation requires meanings (which are paradigmatically semantic items) to be mereologically related so that one is included in the other, Ferguson argues that originally Parry had in mind syntactic concerns.[1] However, even allowing Ferguson's reservations about Parry's original motivations, it is worth noting that there might be some room to support the containment interpretation of analytic implication, and therefore of pure right companions.

One way in which this could be done is by looking at S. Yablo's own theory of content inclusion, which describes the inclusion of the content of $\psi$ in the content of $\varphi$ as the joint satisfaction of truth preservation

[1] This may also be reflected by the fact that Parry's reason to call this principle "proscriptive" was allegedly to oppose it to the otherwise "prescriptive" principles (i.e., transformation rules) described by Sheffer's [see, e.g., 35, p. 103].

and subject-matter preservation from $\varphi$ to $\psi$ [see, e.g., 44, p. 3]. Indeed, granting some standard assumptions in the contemporary subject-matter literature,[2] the subject-matter preservation from $\varphi$ to $\psi$ would amount to the corresponding entailment satisfying the Proscriptive Principle, whereas truth preservation from $\varphi$ to $\psi$ would amount to the CL-validity of the inference with premise $\varphi$ and conclusion $\psi$. In other words, the inclusion of the content of $\psi$ in the content of $\varphi$ could be characterised — if we were to follow Yablo's account — as the joint satisfaction of the requirements of CL-validity and variable inclusion. Yet, as discussed a few paragraphs above, the simultaneous satisfaction of these requirements is precisely what characterises the validity of the corresponding first-degree entailment in Parry's logic and, therefore, in $\mathrm{CL}^{\mathrm{pr}}$. Finally, this explains why content inclusion could be a sensible way of reading analytic implications and entailments according to right companions.

Indeed, accounts of logical consequence based on content inclusion tend to resurface from time to time in the history of logic. One of the founding fathers of relevance logics, Ackermann, actually referred to relevant entailment in his original piece as content inclusion of some form in [1]; perspectives on entailment in terms of meaning inclusion are explicitly endorsed by some early connexive logicians like, e.g., Nelson [30] or Baylis [4]; moreover, some interpreters argue that the fourth reading of the conditional discussed in ancient Greece and reviewed by Sextus Empiricus, also works along these lines [42].

A different philosophical reading of $\mathrm{CL}^{\mathrm{pl}}$ and $\mathrm{CL}^{\mathrm{pr}}$ will emerge in the next section, when examining the matrix semantics for these logics. It is well-known that their "impure" counterparts $\mathrm{K}_3^w$ and PWK were originally investigated as *significance logics*, namely, as logics that are apt for reasoning in the presence of possibly meaningless statements. The status of $\mathrm{CL}^{\mathrm{pl}}$ and $\mathrm{CL}^{\mathrm{pr}}$ as significance logics will be more thoroughly assessed against the backdrop of the semantic completeness results provided below.

Admittedly, this does not necessarily translate into motivation for all pure right companions, and certainly, this does not by itself constitute a reason to analogously pursue pure left companions. Nevertheless, it does

[2] In particular, granting that the subject-matter of a complex proposition is to be identified with the sum or collection of the subject matter of all the propositional variables appearing in it — an idealised but relatively standard assumption, as discussed, e.g., in [5, p. 503]

guarantee that these systems are not purely formal contraptions devoid of any philosophical and historical pedigree.

## 3. Semantics

The aim of this section is to discuss semantics for pure variable inclusion companions of CL, hoping to draw general conclusions from the study of these cases.

In this respect, one key piece of information will be to determine somehow if these subsystems can be captured via a single logical matrix. Roughly speaking, a matrix semantics is given by a set of truth-values and an associated set of truth-tables describing the operations these values partake on, and the results thereof. Logical consequence is then defined as necessary preservation from premises to conclusion of a certain subset of truth-values, called designated values. Whence, if the premises are designated, so must be the conclusion.

Were this to be the case concerning $\mathrm{CL}^{\mathrm{pl}}$ and $\mathrm{CL}^{\mathrm{pr}}$, this would constitute a major insight into these systems. The reason is that these logics could be ultimately thought of as "being about" the truth-values and the operations in the corresponding algebras, whereas its characteristic notion of logical consequence could be thought of as preservation of the designated truth-values — all these being more philosophically interesting than the mere imposition of a syntactic sieve on the CL-valid inferences. Next, in order to establish whether or not this is possible, we will appeal to well-known metalogical results due to Łoś, Suszko, Wójcicki, and others.

More formally, for a given propositional language $\mathcal{L}$, a logical $\mathcal{L}$-matrix $\mathcal{M}$ is a pair $\langle \boldsymbol{A}, D\rangle$, where $\boldsymbol{A}$ is an algebra of the same similarity type as $\mathcal{L}$, and $D$ is a subset of $A$. A logical $\mathcal{L}$-matrix $\mathcal{M}$ induces a consequence relation $\vdash_{\mathcal{M}}$ in the following, standard manner, where $\Gamma \cup \{\varphi\} \subseteq \mathrm{FOR}(\mathcal{L})$:

$$\Gamma \vdash_{\mathcal{M}} \varphi \iff \forall v \in \mathrm{Hom}(\mathrm{FOR}(\mathcal{L}), \boldsymbol{A})\colon v(\Gamma) \subseteq D \text{ implies } v(\varphi) \in D.$$

Now, to examine whether $\mathrm{CL}^{\mathrm{pl}}$ and $\mathrm{CL}^{\mathrm{pr}}$ are characterisable in terms of a single logical matrix we will discuss if they enjoy a metalogical property called the *cancellation property* [see 29, 37, 43]. It is well-known from the literature that if a logic doesn't have this property, then it cannot be characterised by a single logical matrix. In what follows we will show that when we are working with multiple premises and single conclusions, then $\mathrm{CL}^{\mathrm{pl}}$ has this property whereas $\mathrm{CL}^{\mathrm{pr}}$ doesn't have it.

Definition 3.1 (37). A logic $L = \langle \mathrm{FOR}(\mathcal{L}), \vdash_L \rangle$ has the *cancellation property* if and only if:

$$\Gamma \cup \{\Gamma_i \mid i \in I\} \vdash_L \varphi \quad \text{implies} \quad \Gamma \vdash_L \varphi$$

for all $\varphi \in \mathrm{FOR}(\mathcal{L})$ and $\Gamma, \Gamma_i \subseteq \mathrm{FOR}(\mathcal{L})$, $i \in I$, such that:

(i) $\mathit{Var}(\Gamma \cup \{\varphi\}) \cap \mathit{Var}(\bigcup\{\Gamma_i \mid i \in I\}) = \emptyset$,
(ii) $\mathit{Var}(\Gamma_i) \cap \mathit{Var}(\Gamma_j) = \emptyset$, for all $i \neq j$,
(iii) for any $i \in I$ there is $\psi \in \mathrm{FOR}(\mathcal{L})$ such that $\Gamma_i \nvdash_L \psi$.

If $L$ is finitary, we lose no generality in the next theorem if we assume that $I$ is a singleton.

Theorem 3.1 (29, 37, 43). *A logic $L = \langle \mathrm{FOR}(\mathcal{L}), \vdash_L \rangle$ has the cancellation property if and only if there is a single $\mathcal{L}$-matrix $\mathcal{M}$ such that $\vdash_L = \vdash_{\mathcal{M}}$.*

In this respect, two things can be shown. First, that $\mathrm{CL}^{\mathrm{pr}}$ doesn't have the cancellation property and is therefore not characterisable by a single logical matrix. Second, that $\mathrm{CL}^{\mathrm{pl}}$ is characterisable in terms of a single matrix and thus enjoys the cancellation property.[3] We detail the proof of the former fact here, while the latter one will descend as a corollary of the general theory deployed in Section 4.

Lemma 3.1. $\mathrm{CL}^{\mathrm{pr}}$ *doesn't have the cancellation property.*

Proof. Let $\Gamma = \{x\}$, $I = \{1\}$, $\Gamma_1 = \{y, \neg y\}$, $\varphi = \neg x$. Notice that $\Gamma, \Gamma_1 \vdash_{\mathrm{CL}} \varphi$ and $\mathit{Var}(\varphi) \subseteq \mathit{Var}(\Gamma \cup \Gamma_1)$, whence $\Gamma, \Gamma_1 \vdash_{\mathrm{CL}^{\mathrm{pr}}} \varphi$. Additionally, (i) $\mathit{Var}(\Gamma \cup \{\varphi\}) \cap \mathit{Var}(\Gamma_1) = \emptyset$, (ii) is vacuously satisfied, and (iii) for some $\psi$ we have $\Gamma_1 \nvdash_{\mathrm{CL}^{\mathrm{pr}}} \psi$. Yet, although all these things are true, $\Gamma \nvdash_{\mathrm{CL}^{\mathrm{pr}}} \varphi$. $\dashv$

By the above lemma and Theorem 3.1 we obtain:

Corollary 3.2. $\mathrm{CL}^{\mathrm{pr}}$ *has no single characteristic matrix.*

[3] Notice that this does not contradict what is proved in [39], i.e., that there are single matrix semantics for the single-premise single-conclusion versions of $\mathrm{CL}^{\mathrm{pr}}$ and $\mathrm{CL}^{\mathrm{pl}}$ — the difference is that here we are discussing the more general and more widely adopted Tarskian approach to logical systems, allowing for multiple premises and single conclusions.

For the proof of the following lemma see Theorem 4.4 below.

Lemma 3.2. $\mathrm{CL}^{\mathrm{pl}}$ *has a 5-element characteristic matrix, whose algebraic reduct is described by the tables below and whose set of designated elements is* $\{\mathbf{t}, \mathbf{u}\}$*:*

| | $\neg$ |
|---|---|
| **t** | **f** |
| **m** | **n** |
| **u** | **u** |
| **n** | **m** |
| **f** | **t** |

| $\wedge$ | **t** | **m** | **u** | **n** | **f** |
|---|---|---|---|---|---|
| **t** | **t** | **t** | **u** | **f** | **f** |
| **m** | **t** | **m** | **u** | **n** | **f** |
| **u** | **u** | **u** | **u** | **u** | **u** |
| **n** | **f** | **n** | **u** | **n** | **f** |
| **f** | **f** | **f** | **u** | **f** | **f** |

| $\vee$ | **t** | **m** | **u** | **n** | **f** |
|---|---|---|---|---|---|
| **t** | **t** | **t** | **u** | **t** | **t** |
| **m** | **t** | **m** | **u** | **m** | **t** |
| **u** | **u** | **u** | **u** | **u** | **u** |
| **n** | **t** | **m** | **u** | **n** | **f** |
| **f** | **t** | **t** | **u** | **f** | **f** |

It should be noted that even though $\mathrm{CL}^{\mathrm{pr}}$ does not have an interpretation in terms of a single logical matrix, it is still possible for it to have semantics in terms of a class of logical matrices. In fact, we will show that it is possible to provide symmetric understandings of both these pure variable inclusion companions thanks to pairs of logical matrices.

To this end, we need to introduce a number of matrices built on top of a 3-element algebra which is very dear to those working on logics of variable inclusion — the weak Kleene algebra, symbolised as **WK**, whose operations are depicted in Figure 1 below. It is useful to keep in mind that this structure is usually introduced when discussing the so-called logics of nonsense or significance logics, i.e., systems that allow for grammatical though meaningless sentences to be around otherwise true or false (hence, meaningful) sentences — for which see the previously referred works and, e.g., [19, 23, 40]. There is some consensus in the literature to the effect that the most appropriate description of logical operations between sentences of these categories results in the weak Kleene tables and thus, for logical purposes, all that is left is to discuss appropriate choices of designated values.

| | $\neg$ |
|---|---|
| **t** | **f** |
| **u** | **u** |
| **f** | **t** |

| $\wedge$ | **t** | **u** | **f** |
|---|---|---|---|
| **t** | **t** | **u** | **f** |
| **u** | **u** | **u** | **u** |
| **f** | **f** | **u** | **f** |

| $\vee$ | **t** | **u** | **f** |
|---|---|---|---|
| **t** | **t** | **u** | **t** |
| **u** | **u** | **u** | **u** |
| **f** | **t** | **u** | **f** |

Figure 1. The weak Kleene truth-tables

Homomorphisms from $\mathrm{FOR}(\mathcal{L}_0)$ to **WK** will be called *valuations*, for short, while *Boolean* valuations will be valuations with range $\{\mathbf{t}, \mathbf{f}\}$. Elements of this last set will be occasionally called Boolean values.

Observe that the weak Kleene tables obey a principle, variously called *contamination principle* [10], *principle of component homogeneity* [23] or *doctrine of the predominance of the atheoretical element* [3]: whenever a propositional variable occurring in a formula is assigned the nonsensical value $\mathbf{u}$, the whole formula assumes that value. This behaviour is meant to formally mirror the "infectiousness" of nonsensical sentences, which can never be a part of a context that is meaningful as a whole.

The two variable inclusion logics discussed in Section 2, $\mathrm{K}_3^w$ and PWK, constitute the two more debated options of this sort. These correspond to criteria of forward *truth preservation* and forward *non-falsity preservation*, respectively encoded by the matrices $\langle \mathbf{WK}, \{\mathbf{t}\}\rangle$ and $\langle \mathbf{WK}, \{\mathbf{t}, \mathbf{u}\}\rangle$. But there certainly are other options. If for example we selected the set $\{\mathbf{t}, \mathbf{f}\}$ as our set of designated truth-values we would be looking at forward *meaningfulness preservation*, whereas if we selected the set $\{\mathbf{u}\}$ of truth-values we would be looking at forward *meaninglessness preservation.*

Needless to say, these last two options may sound artificial. On the one hand, having $\{\mathbf{t}, \mathbf{f}\}$ as the set of designated values may seem odd because falsity is not something that usually we look forward to preserving from premises to conclusion, when reasoning validly. Similarly, having $\{\mathbf{u}\}$ as the set of designated values may seem odd not only because we may not want to preserve meaninglessness from premises to conclusion, but because we are not requiring that truth be preserved in this same direction — something that is usually an indispensable requirement. Nevertheless, when one is confronted with the question of whether meaningful premises may validly lead to meaningless conclusions, or whether meaningful conclusions may validly be inferred from meaningless premises, these options may end up not sounding as bizarre as they appear.

In what follows we will show that, irrespective of whether they admit a single characteristic matrix, $\mathrm{CL}^{\mathrm{pr}}$ and $\mathrm{CL}^{\mathrm{pl}}$ can be characterised in terms of matrix bundles (on which more below) for which interesting philosophical readings can be offered. In fact, the previous remarks will allow us to understand them as the logics, respectively, of truth and significance, and of non-falsity and non-significance preservation over the $\mathbf{WK}$ algebra. This implies that in $\mathrm{CL}^{\mathrm{pr}}$, unlike in $\mathrm{K}_3^w$, meaningful contradictions do not generally entail meaningless sentences, while in $\mathrm{CL}^{\mathrm{pl}}$, unlike in PWK, meaningful tautologies do not generally follow from meaningless premises. Let us now see how all this translates into a more technical terminology.

Definition 3.2 ([21], p. 187). A bundle of logical matrices is a set of logical matrices with the same algebraic reduct, that is to say, a set of the form $\{\langle \boldsymbol{A}, D\rangle \mid D \in \mathcal{F}\}$ for some family $\mathcal{F} \subseteq \wp(A)$.

Whenever we have a set of logical matrices $\mathbb{M}$, we take the intersection of the set of consequence relations induced by all of its members to be the consequence relation induced by such a set of matrices. That is, $\vdash_{\mathbb{M}} = \bigcap\{\vdash_{\mathcal{M}} \mid \mathcal{M} \in \mathbb{M}\}$. Thus, we say that a logic $L = \langle \mathrm{FOR}(\mathcal{L}), \vdash_L\rangle$ is characterised by a matrix bundle $\mathbb{M}$ if and only if $\vdash_L = \vdash_{\mathbb{M}}$ — for more on this [see [21], pp. 186–187].

Let $\mathcal{M}_1 := \langle \mathbf{WK}, \{\mathbf{t}\}\rangle$, $\mathcal{M}_2 := \langle \mathbf{WK}, \{\mathbf{t}, \mathbf{f}\}\rangle$ and $\mathbb{M} := \{\mathcal{M}_1, \mathcal{M}_2\}$.

Theorem 3.3. $\mathrm{CL}^{\mathrm{pr}}$ *is sound and complete w.r.t.* $\mathbb{M}$.

Proof. RTL: If $\mathit{\Gamma} \nvdash_{\mathrm{CL}^{\mathrm{pr}}} \varphi$, then either $\mathit{\Gamma} \nvdash_{\mathrm{CL}} \varphi$ or $\mathit{Var}(\varphi) \nsubseteq \mathit{Var}[\mathit{\Gamma}]$. Suppose first that $\mathit{\Gamma} \nvdash_{\mathrm{CL}} \varphi$. Then there exists a Boolean valuation $v$ s.t. $v[\mathit{\Gamma}] \subseteq \{\mathbf{t}\}, v(\varphi) = \mathbf{f}$. However, Boolean valuations are in particular valuations into $\mathbf{WK}$. Thus, $\mathit{\Gamma} \nvdash_{\mathcal{M}_1} \varphi$. If $\mathit{Var}(\varphi) \nsubseteq \mathit{Var}[\mathit{\Gamma}]$, there is $x \in \mathit{Var}(\varphi) \setminus \mathit{Var}[\mathit{\Gamma}]$. Consequently there is $v$ s.t. $v(x) = \mathbf{u}, v(y) = \mathbf{f}$ for $y \neq x$. Then $v[\mathit{\Gamma}] \subseteq \{\mathbf{t}, \mathbf{f}\}, v(\varphi) = \mathbf{u}$. Thus, $\mathit{\Gamma} \nvdash_{\mathcal{M}_2} \varphi$.

LTR: If $\mathit{\Gamma} \nvdash_{\mathbb{M}} \varphi$, we distinguish two cases. If $\mathit{\Gamma} \nvdash_{\mathcal{M}_1} \varphi$, then either there is a valuation $v$ s.t. $v[\mathit{\Gamma}] \subseteq \{\mathbf{t}\}, v(\varphi) = \mathbf{f}$ or there is a valuation $u$ s.t. $u[\mathit{\Gamma}] \subseteq \{\mathbf{t}\}, u(\varphi) = \mathbf{u}$. Given the way the operations behave in $\mathbf{WK}$, $v$ is Boolean, hence $\mathit{\Gamma} \nvdash_{\mathrm{CL}} \varphi$, while $u$ is such that for some $x$ we have that $u(x) = \mathbf{u}$ and $x \in \mathit{Var}(\varphi) \setminus \mathit{Var}[\mathit{\Gamma}]$. If $\mathit{\Gamma} \nvdash_{\mathcal{M}_2} \varphi$, there is a valuation $w$ s.t. $w[\mathit{\Gamma}] \subseteq \{\mathbf{t}, \mathbf{f}\}, w(\varphi) = \mathbf{u}$. Then some variable in $\varphi$ is assigned $\mathbf{u}$ by $w$, while none of the variables in $\mathit{\Gamma}$ is assigned $\mathbf{u}$ by $w$. It follows that $\mathit{Var}(\varphi) \nsubseteq \mathit{Var}[\mathit{\Gamma}]$. ⊣

Let $\mathcal{M}_3 := \langle \mathbf{WK}, \{\mathbf{t}, \mathbf{u}\}\rangle$, $\mathcal{M}_4 := \langle \mathbf{WK}, \{\mathbf{u}\}\rangle$ and $\mathbb{M}^* := \{\mathcal{M}_3, \mathcal{M}_4\}$.

Theorem 3.4. $\mathrm{CL}^{\mathrm{pl}}$ *is sound and complete w.r.t.* $\mathbb{M}^*$.

Proof. RTL: Suppose $\mathit{\Gamma} \vdash_{\mathbb{M}^*} \varphi$. If $\mathit{\Gamma} = \emptyset$, any valuation should assign $\mathbf{u}$ to $\varphi$, which is impossible. If $\mathit{\Gamma} \neq \emptyset$, let $\mathit{\Sigma} := \{\psi \in \mathit{\Gamma} : \mathit{Var}(\psi) \subseteq \mathit{Var}(\varphi)\}$. If $\mathit{\Sigma} = \emptyset$, any $\psi \in \mathit{\Gamma}$ contains $x_\psi$ s.t. $x_\psi \notin \mathit{Var}(\varphi)$. Then if we let $v(x_\psi) = \mathbf{u}$ for all $\psi \in \mathit{\Gamma}$ and $v(y) = \mathbf{f}$ for any other variable, $v[\mathit{\Gamma}] \subseteq \{\mathbf{u}\}, v(\varphi) \in \{\mathbf{t}, \mathbf{f}\}$, and $\mathit{\Gamma} \nvdash_{\mathcal{M}_4} \varphi$, a contradiction.

So $\Sigma \neq \emptyset$. If ex absurdo $\Sigma \nvdash_{\mathrm{CL}} \varphi$, there is a Boolean $v$ s.t. $v[\Sigma] \subseteq \{\mathbf{t}\}, v(\varphi) = \mathbf{f}$. Let $u$ be such that:

$$u(x) = \begin{cases} v(x) & \text{if} \in \mathit{Var}(\varphi) \\ \mathbf{u} & \text{otherwise.} \end{cases}$$

Thus, $u[\Gamma] \subseteq \{\mathbf{t}, \mathbf{u}\}, u(\varphi) = \mathbf{f}$. So $\Gamma \nvdash_{\mathcal{M}_3} \varphi$, a contradiction.

LTR: Suppose there is a nonempty $\Delta \subseteq \Gamma$ s.t. $\mathit{Var}[\Delta] \subseteq \mathit{Var}(\varphi)$ and $\Delta \vdash_{\mathrm{CL}} \varphi$. If $v(\varphi) = \mathbf{f}$, then $v$ assigns Boolean values to all the variables in $\varphi$, hence to all the variables in $\Delta$. Since $\Delta \vdash_{\mathrm{CL}} \varphi$, there is $\psi \in \Delta$ s.t. $v(\psi) = \mathbf{f}$. We have shown that for every valuation $v$, either $v(\varphi) \in \{\mathbf{t}, \mathbf{u}\}$, or there exists $\psi \in \Delta$ s.t. $v(\psi) = \mathbf{f}$. So $\Gamma \vdash \varphi$ holds in $\mathcal{M}_3$.

If ex absurdo $v[\Gamma] \subseteq \{\mathbf{u}\}, v(\varphi) \in \{\mathbf{t}, \mathbf{f}\}$, since there is a *nonempty* $\Delta \subseteq \Gamma$ s.t. $\mathit{Var}[\Delta] \subseteq \mathit{Var}(\varphi)$, we would have $v(\varphi) = \mathbf{u}$, a contradiction. So $\Gamma \vdash \varphi$ holds in $\mathcal{M}_4$. ⊣

The last set of issues that we want to address in this section regarding semantics for pure variable inclusion logics pertains to logical consequence as related to ordered algebras. The literature has it that whenever an algebra $\boldsymbol{A}$ has some semilattice order $\leq$ on it, it is possible to study its associated *semilattice-based consequence relation* as defined below — for more [see 21, Def. 7.16, Def. 7.26].

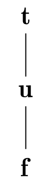
Definition 3.3. An *ordered structure* is a first-order structure where one of the relations is a partial order $\leq$. An ordered structure is said to be *semilattice-ordered* if $\leq$ is a semilattice order with induced meet $\wedge$.

Definition 3.4. Let $\boldsymbol{A}$ be a semilattice-ordered structure of algebraic type $\mathcal{L}$ and relational type $\langle 2 \rangle$. The *semilattice-based logic* of $\boldsymbol{A}$ is the finitary logic $\mathrm{A}^{\leq} = \langle \mathrm{FOR}(\mathcal{L}), \vdash^{\leq}_{\boldsymbol{A}} \rangle$ of type $\mathcal{L}$, where, for every $\gamma_1, \ldots, \gamma_n, \varphi \in \mathrm{FOR}(\mathcal{L})$:

- $\emptyset \vdash^{\leq}_{\boldsymbol{A}} \varphi \iff \forall a \in A, \forall v \in \mathrm{Hom}(\mathrm{FOR}(\mathcal{L}), \boldsymbol{A})\colon a \leq v(\varphi)$;
- $\gamma_1, \ldots, \gamma_n \vdash^{\leq}_{\boldsymbol{A}} \varphi \iff \forall v \in \mathrm{Hom}(\mathrm{FOR}(\mathcal{L}), \boldsymbol{A})\colon v(\gamma_1) \wedge \cdots \wedge v(\gamma_n) \leq v(\varphi)$.

This approach is slightly more general than the one followed, e.g., in [26]. In particular, we do not assume that $\mathrm{A}^{\leq}$ has a conjunction that is interpreted by the semilattice meet of $\boldsymbol{A}$ (in general, $\mathcal{L}$ might even contain no binary connectives).

In the literature, relations of logical consequence of this sort are sometimes construed as truth-degree-preserving consequence relations —



$\leq_{\mathbf{SK}}$

Figure 2. The **SK** algebra as a lattice

meaning, in the linearly ordered case, that the conclusion should have a degree of truth at least as great as that of the minimum of the premises [20]. This is an especially interesting point of view when working with infinitely-valued logics in the Łukasiewicz fashion and beyond [22], but ultimately it isn't impossible to see that it could also be applied to the case of 3-element algebras like the strong Kleene algebra **SK**, depicted as the lattice appearing in Figure 2.

In what concerns this algebra, it is interesting to observe that the semilattice-based consequence relation of type $\mathcal{L}_0$ defined on top of it renders the system sometimes called $\mathrm{S}_3$ but also referred to as $\mathrm{RM}_{\mathtt{fde}}$, because it codifies the first-degree entailments valid in Anderson and Belnap's logic R plus the "mingle" axiom [see 15]. This logic is interesting also because it is the intersection of LP and $\mathrm{K}_3$ [see 25, Sect. 2.1], whence it is weaker than the two main assertional logics derived from **SK**. It is important to notice that this system has no single characteristic matrix as it also fails to ensure the cancellation property, for which one can again consult [25, p. 214]. One may but wonder what sort of system would one get if the semilattice-based consequence relation were to be examined for **WK**.

In this respect, a crucial thing to highlight is that the operations of this algebra induce two different semilattice orders which cannot be identified. Indeed, **WK** is an instance of a *generalised involutive bisemilattice* as defined in [32, 33], whose meet-induced semilattice order $\leq_\wedge$ differs from the join-induced semilattice order $\leq_\vee$, as it can be seen in Figure 3.

Thus, it is natural to study which sorts of logical systems are obtained by taking into account these two different orders and examining the corresponding semilattice-based consequence relations $\vdash^{\leq_\wedge}_{\mathbf{WK}}$ and

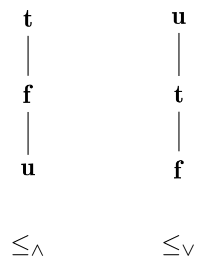



Figure 3. The **WK** algebra as a generalised involutive bisemilattice

$\vdash^{\leq_\vee}_{\mathbf{WK}}$. To this effect, it is useful to notice that in [31] it was proved that the logics $\mathrm{CL}^{\mathrm{pr}}$ and $\mathrm{WK}^{\leq_\wedge}$ coincide and also that the logics $\mathrm{CL}^{\mathrm{pl}}$ and $\mathrm{WK}^{\leq_\vee}$ coincide, when only single premises are considered. However, the result straightforwardly carries over to the multiple-premise case, as we highlight below.

THEOREM 3.5. $\mathrm{CL}^{\mathrm{pr}} = \mathrm{WK}^{\leq_\wedge}$, *and* $\mathrm{CL}^{\mathrm{pl}} = \mathrm{WK}^{\leq_\vee}$.

PROOF. We prove the latter identity; the proof of the former is analogous. Both halves of the proof rely on Theorem 3.4.

LTR: Suppose that $\Gamma \nvdash^{\leq_\vee}_{\mathbf{WK}} \varphi$, with $\Gamma$ finite or empty. If $\Gamma = \emptyset$, then the conclusion is trivial because $\mathrm{CL}^{\mathrm{pl}}$ is theoremless. If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, then there exist a valuation $v : \mathrm{FOR}(\mathcal{L}_0) arrow \mathbf{WK}$ such that $v(\gamma_1) \wedge^{\leq_\vee} \cdots \wedge^{\leq_\vee} v(\gamma_n) >_\vee v(\varphi)$ and thus, since $\mathbf{f} <_\vee \mathbf{t} <_\vee \mathbf{u}$, there are two possibilities: (i) $v(\gamma_1) = \cdots = v(\gamma_n) = \mathbf{u}, v(\varphi) \in \{\mathbf{t}, \mathbf{f}\}$; (ii) $\{v(\gamma_1), \ldots, v(\gamma_n)\} \subseteq \{\mathbf{t}, \mathbf{u}\}, v(\varphi) = \mathbf{f}$. In case (i), we have a counterexample in the matrix $\mathcal{M}_4$; in case (ii), we have a counterexample in the matrix $\mathcal{M}_3$.

RTL: Suppose $\Gamma \vdash^{\leq_\vee}_{\mathbf{WK}} \varphi$, with $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, and let $\{v(\gamma_1), \ldots, v(\gamma_n)\} \subseteq \{\mathbf{t}, \mathbf{u}\}$. Then $v(\varphi) \geqslant_\vee \mathbf{t}$, hence $v(\varphi) \in \{\mathbf{t}, \mathbf{u}\}$. If $v(\gamma_1) = \cdots = v(\gamma_n) = \mathbf{u}$, then $v(\varphi) \geqslant_\vee \mathbf{u}$ and thus $v(\varphi) = \mathbf{u}$. So $\Gamma \vdash \varphi$ holds in both $\mathcal{M}_3$ and $\mathcal{M}_4$. If $\vdash^{\leq_\vee}_{\mathbf{WK}} \varphi$, by definition for all $v$ we have that $v(\varphi) = \mathbf{u}$, which is impossible if we choose $v(x) \subseteq \{\mathbf{t}, \mathbf{f}\}$ for all $x \in \mathit{Var}(\varphi)$. Thus the implication is vacuously true. ⊣

This points out to a fortunate match between two ways in which this pure variable inclusion companions of CL are defined, connected to the question of how to interpret logical validity for the semilattice-based consequence relations. We mentioned that in the infinitely-valued case

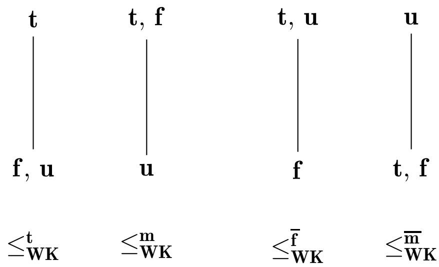



Figure 4. Multiple orders on the elements of the **WK** algebra

these could be seen as systems preserving degree of truth from premises to conclusions, but does something similar happen here?

It is our opinion that for a viable interpretation along some of these lines to be forthcoming, the $\leq_\wedge$ and $\leq_\vee$ orders need to be construed in a different way. Here's an option. Consider a truer-than order $\leq^{\mathbf{t}}_{\mathbf{WK}}$ and a more-meaningful-than order $\leq^{\mathbf{m}}_{\mathbf{WK}}$ which can be depicted as follows in Figure 4. Similarly, consider a less-false-than order $\leq^{\overline{\mathbf{f}}}_{\mathbf{WK}}$ and a less-meaningful-than order $\leq^{\overline{\mathbf{m}}}_{\mathbf{WK}}$. The idea is that being greater in the $\leq_\wedge$ ordering amounts to being greater or equal both in the $\leq^{\mathbf{t}}_{\mathbf{WK}}$ and in the $\leq^{\mathbf{m}}_{\mathbf{WK}}$ orderings. Similarly, being greater in the $\leq_\vee$ ordering amounts to being greater or equal both in $\leq^{\overline{\mathbf{f}}}_{\mathbf{WK}}$ and in $\leq^{\overline{\mathbf{m}}}_{\mathbf{WK}}$.

In this perspective, it is immediate to conclude that the semilattice-based consequence relation $\mathrm{WK}^{\leq_\wedge}$ can be read as preservation of degree of truth and meaningfulness, whereas $\mathrm{WK}^{\leq_\vee}$ can be read as preservation of degree of non-falsity and meaninglessness, so as to match the characterisation of these pure variable inclusion companions in terms of the matrix bundles presented above.

## 4. General completeness and related properties

In this section, we move back from the companions of CL to the general case and prove some completeness theorems for (finitary) logics of pure variable inclusion. This requires a small adaptation of the construction employed in [7] and [8] for logics of variable inclusion. Since the content of this section is more technical than the rest of the paper, we will have to resort to unexplained but standard concepts in abstract algebraic logic and in the theory of Płonka sums over semilattice direct systems of algebras, for which the reader can consult, respectively, [21] and [36].

We start by recalling the two matrix constructions arising in the theory of logics of variable inclusion.

Definition 4.1. An *l-direct system* of $\mathcal{L}$-matrices is an ordered pair $\mathbb{M} = \langle \mathbb{A}, \{F_i\}_{i\in I}\rangle$ such that:

1. $\mathbb{A} = \langle \{\boldsymbol{A}_i\}_{i\in I}, \boldsymbol{I}, \{p_{ij} : i \leq_{\boldsymbol{I}} j\}\rangle$ is a semilattice direct system of $\mathcal{L}$-algebras;
2. for every $i \in I$, $F_i \subseteq A_i$;
3. for every $i, j \in I$ such that $i \leq j$, $p_{ij}[F_i] \subseteq F_j$.

The dual notion, which applies to the right variable inclusion setting, is recalled below.

Definition 4.2. An *r-direct system* of $\mathcal{L}$-matrices is an ordered pair $\mathbb{M} = \langle \mathbb{A}, \{F_i\}_{i\in I}\rangle$ such that:

1. $\mathbb{A} = \langle \{\boldsymbol{A}_i\}_{i\in I}, \boldsymbol{I}, \{p_{ij} : i \leq_{\boldsymbol{I}} j\}\rangle$ is a semilattice direct system of $\mathcal{L}$-algebras;
2. for every $i \in I$, $F_i \subseteq A_i$;
3. $I^+ := \{i \in I : F_i \neq \emptyset\}$ is the universe of a subsemilattice of $\boldsymbol{I}$;
4. for every $i, j \in I$ such that $i \leq j$, if $F_j \neq \emptyset$, then $p_{ij}^{-1}[F_j] = F_i$.

Let $\mathbb{K}$ be a class of matrices. We denote by $\mathcal{P}_{\text{ł}}^{\text{l}}(\mathbb{K})$ the class of Płonka sums over the class of all l-direct systems of matrices in $\mathbb{K}$. Similarly for $\mathcal{P}_{\text{ł}}^{\text{r}}(\mathbb{K})$ w.r.t. r-direct systems of matrices.

In order to obtain a complete class of models for $L^{\text{pr}}$ starting from a complete class of models $\mathbb{K}$ for $L$ (containing a trivial matrix), we only need to apply the matrix operator $\mathcal{P}_{\text{ł}}^{\text{r}}(\cdot)$ to $\mathbb{K} \cup \{\langle \mathbf{1}, \emptyset\rangle\}$. It is easy to check that the only difference with the case of right variable inclusion logics is the demand that there be a trivial matrix in $\mathbb{K}$.

Theorem 4.1. *Let $L$ be a finitary logic and let $\mathbb{K}$ be a complete class of matrices for $L$ containing a trivial matrix. Then $L^{\text{pr}}$ is complete w.r.t. $\mathcal{P}_{\text{ł}}^{\text{r}}(\mathbb{K} \cup \{\langle \mathbf{1}, \emptyset\rangle\})$.*

Proof. ($L^{\text{pr}} \leq \vdash_{\mathcal{P}_{\text{ł}}^{\text{r}}(\mathbb{K}\cup\{\langle \mathbf{1},\emptyset\rangle\})}$). This direction can be proved as in [7], without considering the presence of anti-theorems.

($L^{\text{pr}} \geqslant \vdash_{\mathcal{P}_{\text{ł}}^{\text{r}}(\mathbb{K}\cup\{\langle \mathbf{1},\emptyset\rangle\})}$). Suppose $\varGamma \nvdash_{L^{\text{pr}}} \varphi$. If $\varGamma \nvdash_{L} \varphi$, the fact that $\mathbb{K}$ is complete for $L$ entails that there exists a matrix in $\mathbb{K}$ that falsifies the inference. If $Var(\varphi) \nsubseteq Var(\varGamma)$, fix a variable $x$ such that $x \in$

$Var(\varphi) \setminus Var(\Gamma)$ and let $\langle \boldsymbol{B}, B\rangle$ be the trivial matrix belonging to $\mathbb{K}$. Consider the matrix $\langle \boldsymbol{B} \oplus \mathbf{1}, B\rangle$, where $\boldsymbol{B} \oplus \mathbf{1}$ is the unique Płonka sum over the 2-element chain with $\boldsymbol{B}$ as the algebra with the lower index and $\mathbf{1}$ as the algebra with the upper index. Clearly $\langle \boldsymbol{B} \oplus \mathbf{1}, B\rangle$ can be obtained as a Płonka sum over a r-direct system of matrices in $\mathbb{K}$. Define an arbitrary homomorphism $h\colon \mathrm{FOR}(\mathcal{L}) \to \boldsymbol{B} \oplus \mathbf{1}$ mapping all the variables occurring in $\Gamma$ to elements of $B$, and $x$ to 1. Given the way operations are computed in a Płonka sum of algebras, we obtain $h[\Gamma] \subseteq F, h(\varphi) \notin F$, as desired. $\dashv$

We will use special notation for Płonka sums with just two fibres $\boldsymbol{A}_i$ and $\boldsymbol{A}_j$, where $i <_{\boldsymbol{I}} j$ in the underlying semilattice on $\{i, j\}$. Such algebras will be denoted as $\boldsymbol{A}_i \oplus \boldsymbol{A}_j$; when $\boldsymbol{A}_i$ is isomorphic to $\boldsymbol{A}_j$, the same symbol will be possibly used for both summands with no danger of confusion.

Observe that Theorem 3.3 above can be viewed as a corollary to Theorem 4.1. To see this, set $\mathbb{K} = \{\langle \boldsymbol{B}_2, \{1\}\rangle, \langle \boldsymbol{B}_2, B_2\rangle, \langle \mathbf{1}, \emptyset\rangle\}$, where $\boldsymbol{B}_2$ is the 2-element Boolean algebra. Clearly $\mathbb{K}$ is of the form $\mathbb{K}' \cup \{\langle \mathbf{1}, \emptyset\rangle\}$, and $\mathbb{K}'$ is a complete class of matrices for CL. Since the only Płonka sums of r-direct systems over $\mathbb{K}$ are matrices in $\mathbb{M}$, the theorem applies and we are done. It may also be worth noticing that although $\mathbb{M}$ is complete for $L^{\mathrm{pr}}$, $\mathcal{M}_2$ is not Leibniz-reduced. If we aim at a reduced, complete class of matrices for the same logic we must replace $\mathcal{M}_2$ with $\langle \mathbf{n} \oplus \mathbf{m}, \{m\}\rangle\}$, where $\mathbf{n}, \mathbf{m}$ are trivial algebras. However, there is a flip side to it, since this is not a matrix bundle.

The above theorem also has the following corollary, which determines how to turn some representative complete classes of models for $L$ into complete classes of models of $L^{\mathrm{pr}}$.

Corollary 4.2. *The following are complete for* $L^{\mathrm{pr}}$*:*

- $\mathcal{P}^{\mathrm{r}}_{\mathrm{I}}(\mathrm{Mod}(L) \cup \{\langle \mathbf{1}, \emptyset\rangle\})$
- $\mathcal{P}^{\mathrm{r}}_{\mathrm{I}}(\mathrm{Mod}^{*}(L) \cup \{\langle \mathbf{1}, \emptyset\rangle\})$
- $\mathcal{P}^{\mathrm{r}}_{\mathrm{I}}(\mathrm{Mod}^{Su}(L) \cup \{\langle \mathbf{1}, \emptyset\rangle\})$.

*If, moreover,* $L$ *has no theorems, all the occurrences of "*$\cup\{\langle \mathbf{1}, \emptyset\rangle\}$*" can be removed.*

As regards logics of the form $L^{\mathrm{l}}$, the unique difference with $L^{\mathrm{r}}$ is that we need to enlarge the initial class of matrices $\mathbb{K}$ with a matrix with empty filter.

Theorem 4.3. *Let $L$ be a logic of language $\mathcal{L}$ and let $\mathbb{K}$ be a sound and complete class of $\mathcal{L}$-matrices for $L$ containing $\langle \mathbf{1}, \{1\}\rangle$. Then $L^{\mathrm{pl}}$ is sound and complete w.r.t. $\mathcal{P}^{\mathrm{l}}_{\mathfrak{l}}(\mathbb{K} \cup \{\langle \boldsymbol{A}, \emptyset\rangle\})$, where $\boldsymbol{A}$ is an arbitrary $\mathcal{L}$-algebra.*

Proof. ($L^{\mathrm{pl}} \leq \vdash_{\mathcal{P}^{\mathrm{l}}_{\mathfrak{l}}(\mathbb{K}\cup\{\langle \boldsymbol{A},\emptyset\rangle\})}$). Suppose $\Gamma \vdash_{L^{\mathrm{pl}}} \varphi$, so there is $\emptyset \neq \Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$ and $Var(\Delta) \subseteq Var(\varphi)$. Let $\langle \boldsymbol{B}, F\rangle \in \mathcal{P}^{\mathrm{l}}_{\mathfrak{l}}(\mathbb{K} \cup \{\langle \boldsymbol{A}, \emptyset\rangle\})$ and consider a homomorphism mapping $\delta$ to $F$, for every $\delta \in \Delta$. For a formula $\alpha$, we denote by $i_h(\alpha)$ the index of the Płonka fibre the element $h(\alpha)$ belongs to. Clearly, $i_h(\delta) \leq i_h(\varphi)$, for each $\delta \in \Delta$, so by definition of l-direct system we have that $p_{i_h(\delta), i_h(\varphi)}(h(\delta))) \in F_{i_h(\varphi)}$, for each $\delta \in \Delta$.

The fact that $\langle \boldsymbol{B}_{i_h(\varphi)}, F_{i_h(\varphi)}\rangle$ is a model of $L$ ensures $h(\varphi) \in F_{i_h(\varphi)}$, so $\Delta \vdash_{\mathcal{P}^{\mathrm{r}}_{\mathfrak{l}}(\mathbb{K}\cup\{\langle \boldsymbol{A},\emptyset\rangle\})} \varphi$ and, by monotonicity, $\Gamma \vdash_{\mathcal{P}^{\mathrm{r}}_{\mathfrak{l}}(\mathbb{K}\cup\{\langle \boldsymbol{A},\emptyset\rangle\})} \varphi$.

($L^{\mathrm{pl}} \geqslant \vdash_{\mathcal{P}^{\mathrm{r}}_{\mathfrak{l}}(\mathbb{K}\cup\{\langle \boldsymbol{A},\emptyset\rangle\})}$). Suppose $\Gamma \nvdash_{L^{\mathrm{pl}}} \varphi$. If $\Gamma \nvdash_L \varphi$, the completeness of $\mathbb{K}$ w.r.t. $L$ ensures that some matrix in $\mathbb{K}$ falsifies the considered inference. So assume $\Gamma \vdash_L \varphi$, which entails that $Var(\Delta) \nsubseteq Var(\varphi)$ for every nonempty $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$. If $\vdash_L \varphi$, consider the matrix $\langle \boldsymbol{A} \oplus \mathbf{1}, \{1\}\rangle$, which is a Płonka sum over a l-direct system of matrices in $\mathbb{K} \cup \{\langle \boldsymbol{A}, \emptyset\rangle\}$. Any homomorphism mapping each $y \in Var(\varphi)$ to an arbitrary $a \in A$ falsifies $\varphi$.

So, the only case left is $\nvdash_L \varphi$. The conclusion follows by Theorem 14 of [8]. ⊣

*Remark* 4.1. By looking at the above proof it is possible to notice that, in general, we do not need to consider *all* the matrices belonging to $\mathcal{P}^{\mathrm{l}}_{\mathfrak{l}}(\mathbb{K} \cup \{\langle \boldsymbol{A}, \emptyset\rangle\})$ in order to obtain a complete class of models for $L^{\mathrm{pl}}$. This is particularly relevant when $L$ is complete with respect to a single matrix $\langle \boldsymbol{B}, F\rangle$. Under this circumstance, the only matrices in $\mathcal{P}^{\mathrm{l}}_{\mathfrak{l}}(\langle \boldsymbol{B}, F\rangle, \langle \mathbf{1}, \{1\}\rangle, \langle \boldsymbol{A}, \emptyset\rangle)$ the proof really uses are $\langle \boldsymbol{B} \oplus \mathbf{1}, F \cup \{1\}\rangle$ and $\langle \boldsymbol{A} \oplus \mathbf{1}, \{1\}\rangle$. If, for example, $L$ is CL, then by choosing $\boldsymbol{A} = \boldsymbol{B}_2$ and $\langle \boldsymbol{B}, F\rangle = \langle \boldsymbol{B}_2, \{1\}\rangle$, such matrices coincide with our old acquaintances $\mathcal{M}_3$ and $\mathcal{M}_4$. Thus, Theorem 3.4 can be subsumed as a corollary under Theorem 4.3.

More importantly, thanks to the acquired knowledge on pure left companions, we can prove the following theorem, which is particularly useful in applications.

Theorem 4.4. *Let $L$ be a logic which is complete with respect to the finite matrix $\langle \boldsymbol{A}, F\rangle$ with $|A| = n$. Then, $L^{\mathrm{pl}}$ is complete with respect to a finite matrix $\langle \boldsymbol{B}, F \cup \{1\}\rangle \in \mathcal{P}^{\mathrm{l}}_{\mathfrak{l}}(\langle \boldsymbol{A}, F\rangle, \langle \mathbf{1}, \{1\}\rangle, \langle \boldsymbol{A}, \emptyset\rangle)$ with $|B| = 2n+1$.*

Proof. The class $\{\langle \boldsymbol{A}, F\rangle, \langle \mathbf{1}, \{1\}\rangle, \langle \boldsymbol{A}, \emptyset\rangle\}$ can be organised as a Płonka sum over a l-direct system as follows. Its underlying semilattice is the 3-element chain $i < j < k$; the algebra indexed by $k$ is $\mathbf{1}$, so the algebras indexed by $i, j$ are the two isomorphic copies of $\boldsymbol{A}$. The filter indexed by $i$ is the empty filter; the filter indexed by $j$ is $F$ and the one indexed by $k$ is $\{1\}$. Concerning homomorphisms, $p_{ij}$ is an isomorphism, while $p_{jk}$ is the unique surjection from $\boldsymbol{A}$ to $\mathbf{1}$. So, let us call the resulting matrix $\langle \boldsymbol{B}, F \cup \{1\}\rangle$. Now, since $\{\langle \boldsymbol{A}, F\rangle \cup \langle \mathbf{1}, \{1\}\rangle\}$ is a complete class of matrices for $L$, Theorem 4.3 applies. This, and Remark 4.1, entail that $\mathbb{K} = \{\langle \boldsymbol{A} \oplus \mathbf{1}, F \cup \{1\}\rangle, \langle \boldsymbol{A} \oplus \mathbf{1}, \{1\}\rangle\}$ is complete for $L^{\mathrm{pl}}$. In order to conclude the proof, it remains to verify that $\vdash_{\mathbb{K}} = \vdash_{\langle \boldsymbol{B}, F \cup \{1\}\rangle}$. The $(\geqslant)$ inequality is justified by the fact that each matrix in $\mathbb{K}$ is a submatrix of $\langle \boldsymbol{B}, F \cup \{1\}\rangle$. The $(\leq)$ inequality follows by Theorem 4.3 since $\langle \boldsymbol{B}, F \cup \{1\}\rangle \in \mathcal{P}^{\mathrm{l}}_{\mathrm{ł}}(\langle \boldsymbol{A}, F\rangle, \langle \mathbf{1}, \{1\}\rangle, \langle \boldsymbol{A}, \emptyset\rangle)$. The fact that $|B| = 2n + 1$ is trivial. ⊣

Lemma 3.2 above instantiates this theorem, by letting $\langle \boldsymbol{A}, F\rangle$ be the 2-element Boolean algebra with universe $\{\mathbf{t}, \mathbf{f}\}$ and designated element $\mathbf{t}$, $\langle \boldsymbol{A}, \emptyset\rangle$ be the 2-element Boolean algebra with universe $\{\mathbf{n}, \mathbf{m}\}$ and no designated element, and $\langle \mathbf{1}, \{1\}\rangle$ be the trivial algebra with universe $\{\mathbf{u}\}$ and designated element $\mathbf{u}$. The reader can check that the resulting truth-tables are exactly as detailed in Lemma 3.2.

Corollary 4.5. *Let $L$ be a logic of language $\mathcal{L}$. The following classes of $\mathcal{L}$-matrices are complete for $L^{\mathrm{pl}}$, where $\boldsymbol{A}$ is an arbitrary $\mathcal{L}$-algebra:*

- $\mathcal{P}^{\mathrm{l}}_{\mathrm{ł}}(\mathrm{Mod}(L) \cup \langle \boldsymbol{A}, \emptyset\rangle)$
- $\mathcal{P}^{\mathrm{l}}_{\mathrm{ł}}(\mathrm{Mod}^{*}(L) \cup \langle \boldsymbol{A}, \emptyset\rangle)$
- $\mathcal{P}^{\mathrm{l}}_{\mathrm{ł}}(\mathrm{Mod}^{\mathrm{Su}}(L) \cup \langle \boldsymbol{A}, \emptyset\rangle)$.

*If, moreover, $L$ has no theorems, all the occurrences of "$\cup\langle \boldsymbol{A}, \emptyset\rangle$" can be removed.*

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