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Mario Castagnino & Sebastian Fortin

International Journal of Theoretical Physics

ISSN 0020-7748
Volume 52
Number 5

Int J Theor Phys (2013) 52:1379-1398
DOI 10.1007/s10773-012-1456-4

Springer

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Formal Features of a General Theoretical Framework for Decoherence in Open and Closed Systems

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Received: 11 June 2012 / Accepted: 18 December 2012 / Published online: 5 January 2013

Abstract Two complementary decoherence formalisms, *Environment Induced Decoherence* (EID) for open systems and *Self Induced Decoherence* (SID) for close systems are compared under a common *General Theoretical Formalism for Decoherence* (GTFD). The differences and similarities of EID and SID are studied, e.g. that the main difference is that EID only considers the relevant information of the proper system S and neglects the rest, while SID considers all possible information available from a certain class of measurement instruments and neglects the non-available information.

Keywords Decoherence · Preferred basis · Relaxation time · Decoherence time

1 Introduction

In papers [1] and [2] we have begun a unified study of decoherence in open and closed systems, with or without dissipation. In this paper we continue this study focused in some important details of a common formalism on this subject. As considered in [1, 3], and [4] decoherence is a particular case of one of the phenomenon of quantum mechanics: irreversibility. Decoherence is just an example of an irreversible process.

The problem of irreversibility is that when a quantum state $\rho(t)$ follows a unitary evolution given by the time-operator $\mathcal{U}(t) = e^{-i\frac{H}{\hbar}t}$, the unitary nature of this evolution prevents the state to reach equilibrium when $t \to \infty$. Therefore, if the non-unitary evolution towards equilibrium is to be accounted for, a further element must be added to the unitary evolution. From the most general viewpoint, this element consists in the splitting of the maximal information about the system into both a relevant part and an irrelevant part: whereas the

M. Castagnino
CONICET, IAFE (CONICET-UBA), IFIR and FCEN (University of Buenos Aires), Buenos Aires, Argentina

S. Fortin (✉)
CONICET, IAFE (CONICET-UBA) and FCEN (University of Buenos Aires), Buenos Aires, Argentina
e-mail: sfortin@gmx.net

irrelevant part is disregarded, the relevant part is retained and its evolution may reach a final equilibrium situation. It is important to observe that from the more general point of view, when we speak about the relevant part and the irrelevant one we are referring to part of the information of the system and it does not imply a separation into groups of particles that belong to the system. This last case would only be a particular way of separating the maximal system information. From our perspective since the split into relevant and irrelevant part can be performed in many ways, with no privileged decomposition, there is no need of an unequivocal criterion for deciding where to place the cut between "the" system and "the" environment. In paper [5] we argue that decoherence is a relative phenomenon, better understood from a closed-system perspective according to which the split of a closed quantum system into an open subsystem and its environment is just a way of selecting a particular space of relevant observables of the whole closed system.

In operators language: the maximal information about the system is given by the space of all potentially possible observables $\mathcal{O}$, i.e. self adjoint operators in a Hilbert space. The splitting of this maximal information into both a relevant part and an irrelevant one is done by choosing the observed part of the system and ignore the rest. Then, usually we select a particular subspace $\mathcal{O}_R$, of the space $\mathcal{O}$, as the set that gets the relevant information. Moreover, we want to emphasize that choosing to observe a part of the system does not necessarily imply a loss of dimensionality.

We have already mentioned in [1] that, to explain decoherence, in all its possible versions, it is necessary to choose a space of relevant observables. In this paper we will develop two examples:

1. In the *Self Induced Decoherence* (SID) (see [1, 2, 6–13], and [14, 15]) approach the choice of $O_R \in \mathcal{O}_R$ corresponds to the van Hove observables $O_{VH} \in \mathcal{O}_{VH}$ (as it is defined below in Eq. (42)). This choice of the relevant observables removes the non-relevant observables from the space $\mathcal{O}$. This choice does not imply that we necessarily ignore the information about some particles, or that we only observe one subsystem. But we can just ignore the information of some observables. In this particular example there is no reduction of the "size" of $\mathcal{O}_R = \mathcal{O}_{VH}$, precisely

$$\dim(\mathcal{O}) = \dim(\mathcal{O}_{VH}) \tag{1}$$

   because $\mathcal{O}_{VH}$ is a dense space [16, 17]. In this paper we consider the case of systems with continuous spectrum, for the discrete case (particles with spin) see [18].
2. Another choice of $O_R \in \mathcal{O}_R$ can be the *Environment Induced Decoherence* (EID) choice (see [1, 19–26]), where the space of observables is decomposed in $\mathcal{O} = \mathcal{O}_S \otimes \mathcal{O}_E$ and the relevant observables are:

$$O_R = O_S \otimes I_E \tag{2}$$

   where O_S only gets information from a subspace $\mathcal{O}_S$ and I_E is the unit operator of the correspondent space $\mathcal{O}_E$. In cases like this, many authors call S the factor space of the *system* and E the factor space of the *environment*. EID is a formalism with many choices since we can define different S and E.

The expectation values $\langle O_R \rangle_{\rho(t)} = \mathrm{Tr}(\rho(t) O_R)$ of the observables $O_R \in \mathcal{O}_R$ in the state $\rho(t)$ express the relevant information about the system. Of course, the decision about which observables are to be considered as relevant depends on the particular purposes of each situation; but without this decision irreversible evolutions cannot be described.

Based on these ideas the phenomenon of decoherence can be expressed in a general way leading to a *General Theoretical Framework for Decoherence* (GTFD) that was presented in

a previous paper (see [1] and [27]). According to this general framework, the phenomenon of decoherence can be explained in four general steps:

1. *First step*. The space $\mathcal{O}_R$ of relevant observables is defined.
2. *Second step*. The expectation value $\langle O_R\rangle_{\rho(t)}$, for any $O_R \in \mathcal{O}_R$, is obtained. This step can be formulated in two different but equivalent ways:
   - $\langle O_R\rangle_{\rho(t)}$ is computed as the expectation value of O_R in the unitarily evolving state $\rho(t)$.
   - A coarse-grained state $\rho_G(t)$ is defined with a non-unitary evolution. The quantum system state and the coarse-grained state are not equal and they evolve in a different way because we only consider the relevant observables (see [28] for details). The coarse-grained state is a state such that if we compute the mean value of a relevant observable using the quantum system state, then this value must be the same as the mean value obtained from the same observable using the coarse-grained state, i.e.

$$\langle O_R\rangle_{\rho(t)} = \langle O_R\rangle_{\rho_G(t)} \quad \text{for any } O_R \in \mathcal{O}_R \tag{3}$$

3. *Third step*. In many cases (see paper [29]) it is proved that $\langle O_R\rangle_{\rho(t)} = \langle O_R\rangle_{\rho_G(t)}$ reaches a final equilibrium value $\langle O_R\rangle_{\rho_*}$:

$$\lim_{t\to\infty}\langle O_R\rangle_{\rho(t)} = \lim_{t\to\infty}\langle O_R\rangle_{\rho_G(t)} = \langle O_R\rangle_{\rho_*} \tag{4}$$

   This also means that the coarse-grained state $\rho_G(t)$ evolves towards a final equilibrium state:

$$\lim_{t\to\infty}\langle O_R\rangle_{\rho_G(t)} = \langle O_R\rangle_{\rho_{G*}} \tag{5}$$

   This is possible because in an infinite dimensional Poincare system time is infinite. The final equilibrium state ρ_{G*} is obviously diagonal in its own eigenbasis, which turns out to be the final preferred basis. But, from Eqs. (4) or (5) we cannot say that $\lim_{t\to\infty}\rho(t) = \rho_*$ or $\lim_{t\to\infty}\rho_G(t) = \rho_{G*}$. But rigorously the unitarily evolving quantum state $\rho(t)$ of the whole system only has a *weak convergence* (see [28]) or *weak limit*, symbolized as:

$$W - \lim_{t\to\infty}\rho(t) = \rho_* \tag{6}$$

   This formula is simply another way to formulate Eq. (5). As a consequence, the coarse-grained state $\rho_G(t)$ also has a weak limit, as follows from Eq. (5):

$$W - \lim_{t\to\infty}\rho_G(t) = \rho_{G*} \tag{7}$$

   The meaning of Eqs. (6) and (7) is that although the off-diagonal terms of $\rho(t)$ never vanish through the unitary evolution, the system reaches equilibrium *from an observational point of view*, that is, from the viewpoint given by any relevant observable $O_R \in \mathcal{O}_R$.
4. *Fourth step*. Also a *moving preferred basis* $\{|j(t)\rangle_P\}$ must be defined as we will see in Sect. 1.2. This basis is the eigen basis of certain state $\rho_P(t)$ such that

$$\lim_{t\to\infty}\langle O_R\rangle_{(\rho_R(t)-\rho_P(t))} = 0, \quad \forall O_R \in \mathcal{O}_R \tag{8}$$

   The characteristic time for this limit is the t_D, the *decoherence time* (see [2] for details).

Then our *General Theoretical Framework for Decoherence* (GTFD) is introduced but we are unable to use it until we will specify the $\mathcal{O}_R$ for each particular case. Only then we will find the ρ_G evolution for EID and SID.

In this paper we show how the GTFD improves the understanding of approaches EID, SID and decoherence in general. In Sect. 2 we will present a general theoretical formalism for decoherence. Section 2 is devoted to EID and Sect. 3 to SID. Section 4 deals with the physical relevance of EID and SID and contains what would be the main conclusion of the paper. Section 5 deals with the characteristic times. We present our conclusion in Sect. 6.

1.1 Coarse-Grained Process

We will call *coarse-graining* to a process such that we can select a part of the information of the system under study and only consider the physical quantities that correspond to the selected information. Thus, considering only one part of the complete system, we can reduce, in some cases, the number of degrees of freedom that we use in the description. In quantum mechanics such a process leads, to the elimination of some components of the state. This process can be understood as the projection of the Hilbert space, associated with the complete system, on a smaller subsystem that contains relevant observables. If in the complete system the state operator is $\rho(t)$ and $\mathcal{O}$ is the space of all possible observable then, there is a subspace of relevant observable $\mathcal{O}_R$ observables that contains those that provide the physical information.

Thus, we have explained the first step of Sect. 1. Precisely, we define the *coarse-grained state* $\rho_G(t)$, associated with the relevant subsystem, to a state $\rho_G(t)$ such that it would satisfy Eq. (3). The space where the operators act are specified case by case (see e.g. Eqs. (26) and (43)). This $\rho_G(t)$[1] would contain the maximal possible information that can be obtained from the observables of $\mathcal{O}_R$. Is important to observe that the condition $\langle O_R\rangle_{\rho(t)} = \langle O_R\rangle_{\rho_G(t)}$ does not imply that $\rho(t) = \rho_G(t)$.

1.2 The *Coarse-Grained State* as a Projection in the Complete State

As a consequence of the definition (3), a coarse-graining usually implies a projection whose action is to eliminate some components of the state vector corresponding to the thinner description. If this idea is generalized, coarse-graining can be conceived as a projection that defines the properties of the relevant observables and also as a consequence the space of states. In this subsection we will prove that the coarse-grained state $\rho_G(t)$ can be conceived as the projection of the complete state $\rho(t)$ on the relevant observables subspace $\mathcal{O}_R$. Let us use the notation $\langle O\rangle_\rho = (\rho|O)$ inspired in the algebraic formalism which was initiated by the Brussels school in [30]. Let the basis of $\mathcal{O}_R$ be $\{|O_R^\alpha)\}$, where α is, e.g., a continuous index that identifies the basis elements and let us define a projector on $\mathcal{O}_R$ as

$$\pi = \int \left|O_R^\alpha\right)\left(\rho^\alpha\right| d\alpha \tag{9}$$

Let $\mathcal{O}_R'$ be a space of the linear functional on $\mathcal{O}_R$, the states are a linear combination of functionals $(\rho^\alpha|$, satisfying:[2]

$$\left(\rho^\alpha|O_R^\beta\right) = \delta(\alpha-\beta) \tag{10}$$

[1] See the mathematical definition in Eq. (35).

[2] If we were working in a finite dimensional space $\mathcal{O}$, we could choose $\alpha=(i,j)$, $\beta=(k,l)$, $|O_R^\alpha) = |i\rangle\langle j|$, $(\rho^\beta| = |k\rangle\langle l|$ so $(\rho^\beta|O_R^\alpha) = \mathrm{Tr}(|i\rangle\langle j|k\rangle\langle l|) = \delta_{jk}\delta_{il}$.

It is clear that π is a projector, because $\pi^2 = \pi$. Then we can define:

$$(\rho_G| = (\rho|\pi \tag{11}$$

Therefore,

$$\begin{aligned}\left(\rho_G|O_R^\beta\right) &= (\rho|\pi\left|O_R^\beta\right) = (\rho|\int \left|O_R^\alpha\right)\left(\rho^\alpha|O_R^\beta\right)d\alpha \\ &= (\rho|\int \left|O_R^\alpha\right)\delta(\alpha-\beta)d\alpha = \left(\rho|O_R^\beta\right)\end{aligned} \tag{12}$$

i.e. Eq. (3) since $\langle O\rangle_\rho = (\rho|O)$. Thus, making linear combinations of the $|O_R^\alpha)$, we obtain:

$$\langle O_R\rangle_\rho = (\rho|O_R) = (\rho|\pi|O_R) = (\rho_G|O_R) = \langle O_R\rangle_{\rho_G} \quad \forall O_R \in \mathcal{O}_R \tag{13}$$

i.e. Eq. (3). This demonstration is also valid for discrete spectra [18].

It is interesting to remark, that the lost of irrelevant information, just described, is somehow, similar to the one that appears in thermodynamics. In a classical mechanical system we know the position and velocity of all its particles. From a thermodynamical point of view this information is excessive and cannot be handled. Moreover we are just interested in some macroscopic magnitudes and their relation through thermodynamical equations. Then we must introduce a coarse graining to eliminate the excess of information i.e. the position and velocity of all the molecules.

1.3 The Evolution of the *Coarse-Grained State* and Its Limit as a Projection of the Complete State

The just defined $\rho_G(t)$ is the result of the projection of the state $\rho(t)$ onto the space $\mathcal{O}_R$ of relevant observables (see Eq. (11)). Now we prove that the final state ρ_{G*} of $\rho_G(t)$ is the result of the projection of the final state ρ_* of $\rho(t)$ onto $\mathcal{O}_R$. In fact we have:

$$\left(\rho_G(t)\middle|O_R\right) = \left(\rho(t)\middle|\pi\middle|O_R\right) = \int \left(\rho(t)\middle|O_R^\alpha\right)\left(\rho^\alpha\middle|O_R\right)d\alpha \tag{14}$$

So, using Eq. (6) (if this limit exists, as in the case of SID and EID):

$$\begin{aligned}\lim_{t\to\infty}\left(\rho_G(t)|O_R\right) &= \lim_{t\to\infty}\left(\rho(t)\middle|\pi\middle|O_R\right) = \lim_{t\to\infty}\int \left(\rho(t)\middle|O_R^\alpha\right)\left(\rho^\alpha\middle|O_R\right)d\alpha \\ &= \int \left(\rho_*\middle|O_R^\alpha\right)\left(\rho^\alpha\middle|O_R\right)d\alpha = \left(\rho_*\middle|\pi\middle|O_R^\alpha\right) = (\rho_{G*}|O_R)\end{aligned} \tag{15}$$

where we have defined

$$(\rho_{G*}| = (\rho_*|\pi \tag{16}$$

This limit only exists for concrete examples, for example SID and EID cases. From Eq. (15) we obtain

$$W - \lim_{t\to\infty}\left(\rho_G(t)\right| = (\rho_{G*}| \tag{17}$$

1.4 The Master Equation as a Projection of Liouville Equation

As we said in Sect. 1, the second step of GTFD can be formulated computing the expectation value of O_R in the unitarily evolving state $\rho(t)$ e.g. $\langle O_R \rangle_{\rho(t)}$ or computing the expectation value of O_R in the non-unitarily evolving state $\rho_G(t)$ e.g. $\langle O_R \rangle_{\rho_G(t)}$, where $\rho_G(t)$ follows a non-unitary evolution governed by a master equation. In this subsection we show how the master equation can be written as a projected Liouville equation. In fact, let us consider the equation:

$$i\hbar \frac{d}{dt}|\rho) = [H, \rho] = L|\rho) \tag{18}$$

where L is the Liouville "superoperator". Let us project this equation as

$$i\hbar \frac{d}{dt}\pi|\rho) = \pi L|\rho) \tag{19}$$

where L is the Liouville "superoperator" (see definition in [31]). Now in general $[\pi, L] \neq 0$, in fact we define

$$[\pi, L] = N \tag{20}$$

So

$$i\hbar \frac{d}{dt}\pi|\rho) = L\pi|\rho) + N|\rho) \tag{21}$$

As $\pi|\rho) = |\rho_G)$, then we have

$$i\hbar \frac{d}{dt}|\rho_G) = L|\rho_G) + N|\rho) \tag{22}$$

This is the general form of a master equation. Clearly (22) is the Liouville equation with a extra term that in general, transforms the unitary evolution of the coarse-grained state $|\rho_G)$ in a non-unitary evolution.

For practical purposes the master equation will be presented in a more intuitive way. Precisely: if the projector π is known, we only need the operator $N = [\pi, L]$ and we can use it in Eq. (21). Of course in this case we can define $\pi = P$ and $Q = I - P$ and we can write the last equation as the system

$$i\hbar \frac{d}{dt}P|\rho) = PLP|\rho) + PLQ|\rho) \tag{23}$$

$$i\hbar \frac{d}{dt}Q|\rho) = QLP|\rho) + QLQ|\rho) \tag{24}$$

and solve this system by well-known methods (e.g. the Nakayima Zwanzig method [32, 33]) that yield a non-unitary evolution and finally they lead us to Eq. (17).

2 EID as Particular Case of the GTFD

In paper [1] we shown how the three first steps of the GTFD fit perfectly with EID. In EID a system S (usually a small system of macroscopic nature) and an environment E (usually a

big system of microscopic nature)[3] are defined (in more or less arbitrary way) and the closed system becomes $U = E \cup S$. Then we have the observable subspaces $\mathcal{O}_E$ and $\mathcal{O}_S$ and the operator

$$\mathcal{O}_U = \mathcal{O}_S \otimes \mathcal{O}_E \tag{25}$$

where the relevant observables O_R read

$$O_R = O_S \otimes I_E \tag{26}$$

As $U = E \cup S$ the corresponding spaces are $\mathcal{H}_U = \mathcal{H}_S \otimes \mathcal{H}_E$. Let $\{|i\rangle\}$ be the basis of $\mathcal{H}_S$, let $\{|\alpha\rangle\}$ be the basis of $\mathcal{H}_E$, therefore $\{|i,\alpha\rangle\}$ is the basis of $\mathcal{H}_U$, Under these conditions as we are only interested in the relevant information that the observable O_R sees, i.e. in the mean values

$$\langle O_R\rangle_\rho = \sum_{ij\alpha\beta} \rho_{i\alpha,j\beta} O_{S\,ij}\delta_{\alpha\beta} = \sum_{ij}\left(\sum_{\alpha}\rho_{i\alpha,j\alpha}\right) O_{ij} = \langle O_S\rangle_{\rho_S} \tag{27}$$

where

$$\rho_S = \mathrm{Tr}_E\,\rho \tag{28}$$

In many cases it can be proved that this $\rho_S(t)$ evolves in a non-unitary way and it reaches equilibrium [29].

2.1 The EID Projector

Let $\{|ij\alpha\beta) = |i,\alpha\rangle\langle j,\beta|\}$ the basis of $\mathcal{H}_U \otimes \mathcal{H}_U$, then the EID projector reads

$$P_S = \frac{1}{\sqrt{n}} \sum_{ij\alpha\beta} |ij\alpha\alpha)(ij\beta\beta| \tag{29}$$

In fact the generic state of $\mathcal{H}_U \otimes \mathcal{H}_U$ is

$$(\rho| = \sum_{ij\alpha\beta} \rho_{ij\alpha\beta}(ij\alpha\beta| \tag{30}$$

then

$$\begin{aligned}(\rho|P_S &= \frac{1}{\sqrt{n}} \sum_{ij\alpha\beta hk\gamma\delta} \rho_{hk\gamma\delta}(hk\gamma\delta|ij\alpha\alpha)(ij\beta\beta| \\ &= \frac{1}{\sqrt{n}} \sum_{ij\delta} \rho_{ij\delta\delta}(ij| \sum_{\beta}(\beta\beta| = \frac{1}{\sqrt{n}} \sum_{ij\delta} \rho_{ij\delta\delta}(ij|\end{aligned} \tag{31}$$

since $\sum_\beta(\beta\beta| = \sum_\beta |\beta\rangle\langle\beta| = 1$, and $\rho_{S,ij} = \sum_\delta \rho_{ij\delta\delta} = (\mathrm{Tr}_S\rho)_{ij}$ finally

$$(\rho|P_S = \frac{1}{\sqrt{n}} \sum_{ij\delta} \rho_{S,ij}(ij| \tag{32}$$

[3] In fact, decoherence is one of the steps of the classical limit for macroscopic systems.

On the other hand

$$P_S^2 = \frac{1}{n}\sum_{ij\alpha\beta i'j'\alpha'\beta} |ij\alpha\alpha)(ij\beta\beta|i'j'\alpha'\alpha')(i'j'\beta'\beta'|' = P_S = \frac{1}{n}\sum_{ij\alpha\beta'} |ij\alpha\alpha)(ij\beta\beta'| \sum_{\alpha'\beta} \delta_{\alpha'\beta}$$
$$= P_S \tag{33}$$

so P_S is a projector.

2.2 The *Coarse-Grained State* in EID

To obtain the coarse-grained state in EID we must project the complete state on space $\mathcal{O}'_S$. So:

$$(\rho_G| = (\rho|P_S \tag{34}$$

Note that the dimension of the space that contains $(\rho_G|$ is equal that the dimension of the space that contains $(\rho|$ but this does not happen with $(\rho_G|$ and $(\rho_S|$ because $\rho_S = \mathrm{Tr}_E\rho$. If we want to recover ρ_G starting from ρ_S, we have: from the second step of GTFD that $\langle O_R\rangle_{\rho(t)} = \langle O_R\rangle_{\rho_G(t)}$, and frpm (27) that $\langle O_R\rangle_\rho = \langle O_S\rangle_{\rho_S}$ where $O_R = O_S \otimes I_E$. Then we can define:

$$\rho_G = \frac{\rho_S \otimes I_E}{\mathrm{Tr}(I_E)} \tag{35}$$

Then

$$\langle O_R\rangle_{\rho(t)} = \langle O_R\rangle_{\rho_G(t)} = \langle O_S \otimes I_E\rangle_{\rho_G(t)} = \mathrm{Tr}\left((O_S \otimes I_E)\left(\frac{\rho_S \otimes I_E}{\mathrm{Tr}(I_E)}\right)\right)$$
$$= \frac{\mathrm{Tr}(O_S\rho_S)\mathrm{Tr}(I_E)}{\mathrm{Tr}(I_E)} = \mathrm{Tr}(O_S\rho_S) = \langle O_S\rangle_{\rho_S(t)} \tag{36}$$

If we want to find the final coarse-grained state we can proceed proving, case by case, in each system or example that

$$\lim_{t\to\infty}\big(\rho(t)|O_R\big) = (\rho_*|O_R), \quad \forall O_R \in \mathcal{O}_R \text{ or } W - \lim_{t\to\infty}\big(\rho(t)\big| = (\rho_*| \tag{37}$$

then

$$\lim_{t\to\infty}\big(\rho(t)\big|\pi\big|O_R\big) = (\rho_*|\pi|O_R), \quad \text{thus} \quad \lim_{t\to\infty}\big(\rho_G(t)\big|O_R\big) = (\rho_{G*}|O_R), \tag{38}$$

We can use Eq. (38) and now we have enough equations to find $(\rho_{G*}|O_R)$ and therefore to find all the relevant coordinates of $(\rho_{G*}|$. So for any $O_R \in \mathcal{O}_R$ we have

$$W - \lim_{t\to\infty}\big(\rho_G(t)\big| = (\rho_{R*}| \tag{39}$$

The characteristic time of this evolution is t_R that can be computed using the poles technique. The decoherence time $t_D < t_R$ can also be computed with the same technique. The two times can be also computed case-by-case in several models [2, 34, 35].

2.3 Comments: EID Dissipative Environment

The intuitive explanation of EID is *dissipation*. EID would be, in principle, a dissipative formalism, since in many models the microscopic S gives its energy to the macroscopic E where this energy is stored. The kinetic energy of S becomes zero and S reaches equilibrium and classical motion stops in the macroscopic-collective variables of S. Decoherence is produced before equilibrium and it is proved that, for macroscopic systems, the decoherence time is a small fraction of the relaxation time (see [1] and [2])

I. *A trivial example*: This trivial example will become quite persuasive when we compare it with the SID analog.

Let us consider a (small) stone S and a (big) poll E. The stone (which initially has all the energy) falls into the motionless pool, creating big waves of big wave length and low frequency in the water. The evolution makes that waves would become smaller and smaller and their frequencies grow, ending in microscopic (thermal) waves, while the stone stops its motion and reaches equilibrium.[4] The stone has dissipated its energy into the pool. Essentially, in this example we see that *big-low-frequency-macroscopic* waves end in *small-high-frequency-microscopic* waves where the energy is dissipated.

Then, essentially we have two processes:

a. Macro to Micro dissipation. The energy of the macroscopic waves "dissipates" into those of microscopic word.
b. Evolution of the motion from low frequencies to high frequencies. The macroscopic wave has low frequency while the microscopic one has high frequency.

We do not say that classical dissipation leads to quantum dissipation, but this is a good analogy to understand the phenomena.

II. *For more general* (*non-trivial*) *example* (see [36], 3.2, p. 48). In Zwanzig's general formalism of the master equation we have relevant channels (corresponding to relevant observables) and irrelevant channels (corresponding to irrelevant observables) and the information goes to deeper and deeper spaces of irrelevant channels. So information is dissipated in this case.

3 SID as Particular Case of the GTFD

In SID approach the game is played in the complete set of commuting observables (CSCO) that contains the Hamiltonian H of the closed system U and the constants of motion C_i such that $[H, C_i] = 0$. The corresponding basis is $\{|\omega, c_i\rangle\}$, being these states stationary, and

$$H = \int \omega \sum_i |\omega, c_i\rangle\langle\omega, c_i| d\omega \tag{40}$$

We will see that in this case we can directly obtain a state equilibrium limit $\rho(t) \to \rho_*$. Then, as we will see, all the characters of the play: state, energy, etc. are constants of the motion, and therefore there is no energy transfer and no dissipation in the $\{|\omega, c_i\rangle\}$ context (SID is not a dissipative formalism). This is the main difference with EID.

Nevertheless point "b" of Sect. 2.3. I allows us to see a crucial resemblance with EID:

[4]Following the laws of the thermodynamic, the total energy is conserved, but the mechanical energy is "degraded" in heat.

The observables are (for simplicity we forget the c_i indices)

$$O = \int \widetilde{O}(\omega,\omega')|\omega,\omega')d\omega d\omega' \tag{41}$$

where $|\omega,\omega') = |\omega\rangle\langle\omega'|$ and $\widetilde{O}(\omega,\omega')$ is any kernel or distribution. The relevant observables are those obtained by the van Hove choice [16, 17]:[5]

$$\widetilde{O}(\omega,\omega') = O(\omega)\delta(\omega-\omega') + O(\omega,\omega') \tag{42}$$

where $O(\omega,\omega')$ is a regular function, precisely $O(\omega,\omega') \in \mathbb{L}^2(\omega-\omega')$. Then we define a

$$O_R = \int O(\omega)|\omega)d\omega + \int O(\omega,\omega')|\omega,\omega')d\omega d\omega' \tag{43}$$

where $|\omega) = |\omega\rangle\langle\omega|$, $|\omega,\omega') = |\omega\rangle\langle\omega'|$ and the states read

$$\rho_R = \int \rho(\omega)(\omega|d\omega + \int \rho(\omega,\omega')(\omega,\omega'|d\omega d\omega' \tag{44}$$

where $(\omega|$, $(\omega,\omega'|$ is the cobasis of $|\omega)$, $|\omega,\omega')$, where $\rho(\omega,\omega')$ is also a regular function, i.e. $\rho(\omega,\omega') \in \mathbb{L}^1(\omega-\omega')$, and

$$\rho_R(\omega) = \rho_R^*(\omega), \quad \rho_R(\omega) \geq 0, \quad \int \rho_R(\omega)d\omega = 1 \tag{45}$$

Then:

$$\begin{aligned}\langle O_R\rangle_{\rho_R(t)} &= (\rho_R|O_R) \\ &= \int \rho(\omega)O(\omega)d\omega + \iint \rho(\omega',\omega)O(\omega,\omega')\exp\left[-i\frac{(\omega-\omega')}{\hbar}t\right]d\omega d\omega'\end{aligned} \tag{46}$$

and

$$\lim_{t\to\infty}\langle O_R\rangle_{\rho_R(t)} = \lim_{t\to\infty}(\rho_R|O_R) = \int \rho(\omega)O(\omega)d\omega \tag{47}$$

since $\rho(\omega',\omega)O(\omega,\omega') \in \mathbb{L}^1(\omega-\omega')$.

In the particular case $O = H$ (a particular van Hove observables) equation (43) reads:

$$H = \int \omega|\omega)d\omega \tag{48}$$

and

$$\langle H\rangle_{\rho(t)} = \int \rho(\omega)\omega d\omega \tag{49}$$

Therefore the energy of the system remains constant in time and it is only concentrated in the diagonal terms $\rho(\omega)$. Thus, there is no energy transfer. Anyhow the van Hove observables

[5]The non-rigorous $\delta(\omega-\omega')$ will soon disappear from this text. In fact the formalism below is precisely a way to eliminate this $\delta(\omega-\omega')$. We will use this heuristic object "$\delta(\omega-\omega')$" just to give some examples below.

see the motion in the states $\rho(t)$ and therefore there is no quantum equilibrium at the initial stage.

We can follow the "more general example" at the end of the last section but now in the SID case. But upon a time there was a myth that said that dissipation was necessary for the quantum states to reach equilibrium and decoherence. Then as the states of a closed system cannot dissipate, because they have no environment, they can neither decohere nor reach equilibrium. The origin of this myth was a confusion between classical objects and quantum states. In fact, to reach equilibrium a classical object, e.g., needs friction to dissipate its kinetic energy in an environment. But a quantum state is not a classical object. So today this myth is dissipated (see [29], p. 93) but somehow the prejudice about closed systems subsists. To be didactic let us consider a closed system. The mean energy of a quantum state $\rho(t)$ in an arbitrary basis is:

$$\big(\rho(t)\big|H\big) = \sum_{ij} \rho_{ij}(t) H_{ji} = \sum_{i} \rho_{ii}(t) H_{ii} + \sum_{i \neq j} \rho_{ij}(t) H_{ji}$$

where the first term of the r.h.s. would be the mean energy of the diagonal terms $\rho_{ii}(t)$ and the second term the non-diagonal ones $\rho_{ij}(t)$. But in the energy eigenbasis this equation simply reads

$$\big(\rho(t)|H\big) = \sum_{i} \rho_{ii} \omega_i$$

where ω_i are the eigenvalues of H. Namely in the Hamiltonian basis the energy is concentrated in the constant diagonal terms and the variable non-diagonal terms do not contribute to the mean energy and therefore their vanishing (according to SID) is irrelevant for the energy balance.

For all these reasons decoherence is clearly unrelated with dissipation, at least in closed systems.

c. SID decoherence is originated in the physical phenomenon of destructive interference among the off diagonal terms of $\rho(t)$ or its mathematical version: the Riemann-Lebesgue theorem (also illustrated by [37]). Therefore SID is both physically and mathematically motivated. But nowadays SID has not a direct experimental verification but it has indirect proves as we will see. Also there is computational experiments as the Casati and Prosen model [38, 39]. Nevertheless there is a very long list of physical theories that were introduced, adopted, and even popularized before their experimental verification took place (e.g. Superstrings theory). Then the essential requirement for a (provisional) theoretical physical formalism is just that they would be soundly physically motivated.

3.1 The Algebraic Formalism

We can repeat this explanation in algebraic language [14]: The characteristic algebra $\mathcal{A}$ of the operators (see the complete version in [40]) contains the space of the self-adjoints observables $\mathcal{O}$ which in turn contains the minimal subalgebra $\widetilde{\mathcal{A}}$ of the operators that commute with the Hamiltonian H (that we can consider as the typical "diagonal" operators algebra). Then we have:

$$\widetilde{\mathcal{A}} \subset \mathcal{O} \subset \mathcal{A} \tag{50}$$

Now we can make the quotient

$$\mathcal{A}/\tilde{\mathcal{A}} = \mathcal{V}_{nd} \tag{51}$$

where $\mathcal{V}_{nd}$ would represent the set of equivalence classes of operators that do not commute with H (the "non-diagonal operators"). These equivalence classes read

$$[a] = a + \widetilde{\mathcal{A}}, \quad a \in \mathcal{A} \tag{52}$$

So we can decompose $\mathcal{A}$ as:

$$\mathcal{A} = \widetilde{\mathcal{A}} + \mathcal{V}_{nd} \tag{53}$$

But Eq. (52) is not a direct sum, since we can add an arbitrary $a \in \widetilde{\mathcal{A}}$ from the first term of the r.h.s. of the last equation and substrate a from the second term.

At this point we can ask ourselves which the observables are that really matter in the case of SID under an evolution e^{-iHt}. Certainly the observables that commute with H which are contained in $\widetilde{\mathcal{A}}$ (and correspond to diagonal matrices $\sim$"$\delta(\omega - \omega')$" of Eq. (42)). The observables that do not commute with H correspond to the off-diagonal terms contained in $\mathcal{V}_{nd}$. These terms, must vanish when $t \to \infty$, so they must be endowed with mathematical properties adequated to produce this limit. Riemann-Lebesgue theorem tells us that this fact takes place if functions $O(\omega, \omega')$, and therefore $\rho(\omega, \omega')$, are $\mathbb{L}_1$, in such a way that, via the Schwartz inequality the Riemann-Lebesgue theorem could be used as explained above. Then we add this property to $\mathcal{V}_{nd}$. So we define a sub algebra of $\mathcal{A}$ (that can be called a van Hove algebra [16, 17] since it is inspired in the works of this author) as:

$$\mathcal{A}_{vh} = \widetilde{\mathcal{A}} \oplus \mathcal{V}_r \subset \mathcal{A} \tag{54}$$

where the vector space $\mathcal{V}_r$ is the space of operators of Eq. (43) with $O(\omega) = 0$ and $O(\omega, \omega') \in \mathbb{L}_2(\omega - \omega')$ as required under Eq. (42). Moreover $\mathcal{O}_R = \mathcal{V}_{vhS}$, the space of self-adjoint operators of $\mathcal{A}_{vh}$, which can be constructed in such a way to be dense in $\mathcal{V}_S$ (because any distribution can be approximated by regular functions). Therefore, essentially the introduced restriction is the minimal possible coarse-graining. Now the $\oplus$ of Eq. (54) is a direct sum because $\widetilde{\mathcal{A}}$ contains the factor "$\delta(\omega - \omega')$" and $\mathcal{V}_r$ contains just regular functions and a kernel cannot be both a distribution δ and a regular function. Moreover, as our observables must be self-adjoint the space of observables must be the just defined

$$\mathcal{O}_R = \mathcal{V}_{vhS} = \widetilde{\mathcal{A}} \oplus \mathcal{V}_{rS} \subset \mathcal{V}_S \tag{55}$$

where $\mathcal{V}_{rS}$ is the space of the self-adjoint operators of $\mathcal{V}_r$. This decomposition corresponds to the one in Eq. (43) where $\mathcal{V}_{rS}$ only contains regular self-adjoint operators (namely $O(\omega', \omega)^* = O(\omega, \omega')$). Restriction (55) is just the choice (coarse-graining) of the relevant measurement apparatuses for our problem, those that measure the diagonal terms in $\widetilde{\mathcal{A}}$ and those that measure the non-diagonal terms that vanish when $t \to \infty$ in $\mathcal{V}_{rS}$.[6] Under Eq. (43) we have called $|\omega) = |\omega\rangle\langle\omega|$ the vectors of the basis of $\widetilde{\mathcal{A}}$ and $|\omega, \omega') = |\omega\rangle\langle\omega'|$ those of $\mathcal{V}_{rS}$. Then a generic observable of $\mathcal{O}_R$ reads as in Eq. (43).

The states must be considered as linear functionals over the space $\mathcal{O}$ ($\mathcal{O}'$ the dual of space $\mathcal{O}$):

$$\mathcal{O}'_R = \mathcal{V}'_{vhS} = \widetilde{\mathcal{A}}' \oplus \mathcal{V}'_{rS} \subset \mathcal{O}' \tag{56}$$

Therefore the state reads as in Eq. (44). The space of these generalized states (satisfying Eq. (45)) is the convex space $\mathcal{S}_R \subset \mathcal{O}'_R$. Now the mean value is given by Eq. (46) and we

[6] See [43] Sect. 8.2 (p. 210) for the definition of these observables.

can obtain the limits (38) or (39). *This is the simple trick that allows us to deal with the singularities* (i.e. the "$\delta(\omega-\omega')$") *in a rigorous mathematical way and to obtain correct physical results*. Essentially we have defined a new observable space $\mathcal{O}_R$ (that contains the observables O_R of Eq. (43)) and a space of states $\mathcal{S}_R$ that are adapted to solve our problem.

The algebraic approach has several applications in many chapters of physics. The most important are ARQFT [41] and Statistical Mechanics [42]. The approach presented here could be useful for defining decoherence into these fields.

3.2 The Projector into the Space of Regular Functions

Let us consider the rigged Hilbert space or Gel'fand triplet

$$\Phi \subset \mathcal{H} \subset \Phi' \tag{57}$$

where Φ is the test function space, $\mathcal{H}$ is a Hilbert space, and Φ' is the dual space of Φ.

Let

$$F \in \Phi', \quad F : \Phi \to \mathbb{R}, \quad F[\varphi] = x \in \mathbb{R} \tag{58}$$

be a functional or distribution on a space of test function Φ so $\varphi \in \Phi$ [1].[7] A regular function $f(x) \in \mathcal{H}$ can be used to define a generalized function (or distribution) as a functional

$$F_f[\varphi] = \int f(x)\varphi(x)dx \tag{59}$$

where $f(x) \in \mathcal{H}$, and $\varphi(x) \in \Phi$. Then, if $\{e_i(x)\}$ is a basis of $\mathcal{H}$ we can decompose $f(x)$ and $\varphi(x)$ as

$$f(x) = \sum_i f_i e_i(x), \varphi(x) = \sum_i \varphi_i e_i(x) \tag{60}$$

Then we can also define a projector acting in a generalized function on the space of regular functions $\mathcal{H}$ as

$$\pi F = \tilde{f}_F(x) \tag{61}$$

where

$$\tilde{f}_F(x) = \sum_i F\big[e_i(x)\big]e_i(x) \tag{62}$$

Then $\tilde{f}_F(x)$ is a $\mathcal{H}$ function if

$$\sum_i \big|F\big[e_i(x)\big]\big|^2 < \infty \tag{63}$$

But in general it will not be the case and it may happen that

$$\sum_i \big|F\big[e_i(x)\big]\big|^2 \sim \infty \tag{64}$$

We will assume that we can approximate a distribution F with a Hilbert space function $f(x)$ (which can be written in a distribution form as $F_f[\varphi]$ as close as we wish). Then we

[7] More precisely $\Phi \subset \mathcal{H} \subset \Phi^\times$, and $F : \Phi \to \mathbb{C}$ in the complex case, where $\Phi^\times$ is the anti-dual space (see [48]).

can assume that the space of functions of $\mathcal{H}$ is dense in the space of distributions Φ' in an adequate topology [44]. This mathematical idea will be enough for our physical purposes. Of course this fact must be demonstrated case by case choosing mathematical structure with adequate properties. Moreover we can study the problem using an algebra $\mathcal{A}$ and obtaining the space $\mathcal{H}$ using the GNS theorem and its generalization (see [45]). E.g. in paper [46] a detailed example can be found based in the algebra $\mathcal{L}(\mathcal{S}(\mathbb{R}^+))$ (also see a detailed example in [47]).

So $F[e_i(x)]$ can be approximated by a f_i satisfying

$$\sum_i |f_i|^2 < \infty \tag{65}$$

as close as we can and define a function

$$f(x) = \sum_i f_i e_i(x) \tag{66}$$

(this choice can be called a *smoothing* process) and defines an operator π such that

$$\pi F = f \tag{67}$$

Now, from Eq. (59) we have

$$F_f[\varphi] = \int f(x)\varphi(x)dx \tag{68}$$

Thus

$$F_f\big[e_i(x)\big] = \int f(x)e_i(x)dx = f_i \tag{69}$$

and from Eq. (62)

$$\pi F_f[\varphi] = \sum_i f_i e_i(x) = f(x) \tag{70}$$

and we have that

$$\sum_i |f_i|^2 < \infty \tag{71}$$

Then the projection of a $\mathcal{H}$ function is a $\mathcal{H}$ function and $\pi^2 = \pi$, so π is a projector and we have defined the projection

$$\pi : \Phi' \to \mathcal{H} \tag{72}$$

In a bra-ket language $\{e_i(x)\}$ becomes the basis $\{|e_i\rangle\}$ with cobasis is $\{\langle e_i|\}$ and the functional $F[\varphi]$ is a bra $\langle F|$. Then

$$\langle e_i|e_j\rangle = \delta_{ij}, \quad \pi = \sum_i |e_i\rangle\langle e_i| \quad \text{and therefore } \pi^2 = \pi \tag{73}$$

and

$$\langle F|\pi = \langle f|, \quad \pi = \sum_i \langle f|e_i\rangle\langle e_i| \tag{74}$$

namely Eq. (62) where we have smoothed the $\langle F|e_i\rangle$ to become the $\langle f|e_i\rangle$.

3.3 The SID Projector

We can define the projector π of SID such that

$$\pi|O) = |O_R) \quad \text{and} \quad (\rho_R| = (\rho|\pi \tag{75}$$

To begin with, we just stress that, intuitively, functions that oscillate with infinite frequency can be associated with some kind of distributions. Then these functions or distributions never reach equilibrium because they do not suffer the destructive interference that would produce the factor $\exp[-i\frac{(\omega-\omega')}{\hbar}t]$ in an infinite time. Precisely these distributions are the ones that are not taken into account by the van Hove observables.

From what we have explained in Sect. 3.2, in this case the projector π reads

$$\pi : \mathcal{O} \to \mathcal{O}_R = \mathcal{V}_{vhS} = \widetilde{\mathcal{A}} \oplus \mathcal{V}_{rS} \subset \mathcal{O}, \qquad \pi : \mathcal{O}' \to \mathcal{O}'_R = \mathcal{V}'_{vhS} = \widetilde{\mathcal{A}}' \oplus \mathcal{V}'_{rS} \subset \mathcal{O}' \tag{76}$$

Moreover at the end of calculation we have seen that the decohered states (namely the states that are candidates to become classical states when $\hbar \to 0$) only belong to space $\widetilde{\mathcal{A}}'$ (with basis $\{(\omega|\}$, see Eq. (44)). So the formalism yields the definition of an important projector π that projects the states over the sub space $\mathcal{O}'_R = \mathcal{V}'_{vhS}$. We can call π the *classical projector* because when $t \to \infty$ and $\hbar \to 0$, then $\mathcal{O}'_R = \mathcal{V}'_{vhS} \to \widetilde{\mathcal{A}}'$ so π projects on the "classical world" (see [49]).

Then according to the formalism of Sect. 3.2. we can define the projector π as

$$\pi|O) = |O_R) = \int |\omega)(\omega|d\omega + \iint_{\omega\neq\omega'} |\omega,\omega')(\omega,\omega'|d\omega d\omega' \tag{77}$$

and we can say that if $|O)$ and $(\rho|$ are generic operators or states the relevant ones will be

$$\pi|O) = |O_R) = \int O(\omega)|\omega)d\omega + \iint_{\omega\neq\omega'} O(\omega,\omega')|\omega,\omega')d\omega d\omega' \tag{78}$$

and

$$(\rho_R| = (\rho|\pi = \int \rho(\omega)(\omega|d\omega + \iint_{\omega\neq\omega'} \rho(\omega,\omega')e^{-i(\omega-\omega')t}(\omega,\omega'|d\omega d\omega' \tag{79}$$

and since $(\omega|$, $(\omega,\omega'|$ is the cobasis of $|\omega)$, $|\omega,\omega')$, the product results

$$(\rho|O_R) = (\rho|\pi|O) = (\rho_R|O) = \int \rho(\omega)O(\omega)d\omega + \iint \rho(\omega',\omega)O(\omega,\omega')d\omega d\omega' \tag{80}$$

Where we require that $O(\omega,\omega') \in \mathbb{L}_2(\omega-\omega')$, then $\rho(\omega',\omega) \in \mathbb{L}_2(\omega-\omega')$, and from the Schwarz inequality $\rho(\omega',\omega)O(\omega,\omega') \in \mathbb{L}_1(\omega-\omega')$, and precisely this is the condition to use Riemann-Lebesgue theorem.

3.4 The *Coarse-Grained State* in SID

To obtain the coarse-grained state of SID we must project the complete state on a van Hove space $\mathcal{O}'_{VH}$. So with a similar notation as the one of Sect. 2.2, we have:

$$(\rho_G| = (\rho|P_{VH} \tag{81}$$

Note that, in this case, $\dim(\rho_G| = \dim(\rho|$ but, unlike EID, $\dim(\rho_G| = \dim(\rho_{VH}|$ because $\rho_{VH} \neq \mathrm{Tr}_I \rho$ where I is the unit operator. If we want to recover ρ_G starting from ρ_{VH}, we need to do nothing because in this case $\rho_G = \rho_{VH}$. From the second step of GTFD we have that $\langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_G(t)}$, then

$$\begin{aligned} \langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_G(t)} = \langle O_{VH} \rangle_{\rho_G(t)} = \mathrm{Tr}(O_{VH}\rho_G) \\ = \mathrm{Tr}(O_{VH}\rho_{VH}) = \langle O_{VH} \rangle_{\rho_{VH}(t)} \end{aligned} \tag{82}$$

Let us now find the final coarse-grained state. In SID, using the Riemann-Lebesgue theorem, it is proved that

$$\lim_{t\to\infty} \big(\rho(t)|O_R\big) = (\rho_*|O_R), \quad \forall O_R \in \mathcal{O}_R \quad \text{or} \quad W - \lim_{t\to\infty} \big(\rho(t)\big| = (\rho_*| \tag{83}$$

Then

$$\lim_{t\to\infty} \big(\rho(t)\big|\pi|O\big) = (\rho_*|\pi|O), \quad \text{thus} \quad \lim_{t\to\infty} \big(\rho(t)\big|O_R\big) = (\rho_*|O_R), \tag{84}$$

From $(\rho_*|O_R)$ of Eq. (84) we can deduce that, according to the Riezs theorem, all the coordinates of $(\rho_*|$ e.g. in the finite space we have $\dim \mathcal{O}_R = \dim \mathcal{O} = n$ we could take n independent $|O_R^i)$ $i = 1, 2, \ldots, n$ and since we have n equations $(\rho_*|O_R) =$ *certain known mean value* we could obtain all the coordinates of $(\rho_*|$ in space $\mathcal{O}$. So for all $O_R \in \mathcal{O}_R$ we have

$$W - \lim_{t\to\infty} \big(\rho(t)\big| = (\rho_*| \tag{85}$$

4 Physical Relevance of EID and SID Observables

In the previous sections we have shown how the EID formalism fits perfectly in the GTFD. The main concept in this framework is the coarse graining, as explained in Sect. 1.1. But a question remains: if there is a loss of information with physical relevance in a coarse graining evolution. We have explained that the coarse graining is produced if we choose a space of relevant observables O_S of EID. All this is well known.

We will now consider the case of SID where the relevant observables are the van Hove observables, of Eq. (43), that belong to a space $\mathcal{O}_R$. Then the corresponding states, of Eq. (44), belongs to a space $\mathcal{O}'_R$. Equations (43) and (44) show that in SID a particular choice and their consequences are introduced

(i) $O(\omega, \omega')$ is a regular function (i.e. $O(\omega, \omega') \in \mathbb{L}^2$) and not a generic distribution. This makes $\mathcal{O}_R = \mathcal{O}_{VH}$. This is the restriction.

(ii) $\rho(\omega, \omega')$ is also a regular function since it belongs to a $\mathcal{O}'_R = \mathcal{O}'_{VH}$ (and therefore also $\rho(\omega, \omega') \in \mathbb{L}^2$), this is the consequence. Then, we must ask ourselves if the obtained spaces $\mathcal{O}_{VH}$ and $\mathcal{O}'_{VH}$ are generic enough to take into account all physical reality. Below we give an argument to prove that it is so.

Let us consider the Hamiltonian of the system:

$$|H) = \int_0^\infty \omega|\omega)\, d\omega \tag{86}$$

and a particular observable $|Z) = |z\rangle\langle z|$ where $\langle z|z\rangle = 1$, i.e. $|Z)$ a is projector. As it is a usual observable we have that:

$$|Z) = \int_0^\infty \int_0^\infty \tilde{Z}(\omega,\omega')\big|\omega,\omega'\big)\, d\omega d\omega' \tag{87}$$

without any loss of generality we can write this equation as:

$$|Z) = \int_0^\infty Z(\omega)|\omega)\, d\omega + \int_0^\infty \int_0^\infty Z(\omega,\omega')\big|\omega,\omega'\big)\, d\omega d\omega' \tag{88}$$

where $Z(\omega)$ is a regular function and $Z(\omega,\omega')$ is a distribution (a "$\delta(\omega-\omega')$" is hidden in $|\omega)$); $Z(\omega)$ and $Z(\omega,\omega')$ represent the diagonal and non-diagonal components of the observable $|Z)$ which, in principle, it is not a van Hove observable. The non-diagonal components can be written as:

$$\begin{aligned}\left\langle Z^{ND}\right\rangle_{\rho_{\omega'\omega}} &= \langle z|\rho_{\omega'\omega}|z\rangle = \big\langle z\big|\omega'\big\rangle\langle\omega|z\rangle \\ &= \langle\omega|z\rangle\big\langle z\big|\omega'\big\rangle = Z\big(\omega,\omega'\big)\end{aligned} \tag{89}$$

The usual procedure to measure $Z(\omega,\omega')$ is to divide the plane (ω,ω') in squares of area $\Delta\omega\Delta\omega'$. For each one of these squares, i.e. for the square of the center (ω_k,ω_l) a state $(\rho_{\omega_k\omega_l}| = |\omega_k\rangle\langle\omega_l|$ can be prepared, and then, after the repetitions of many measurements the mean value $Z(\omega_k,\omega_l) = \langle z|\omega_k\rangle\langle\omega_l|z\rangle$ is computed. Once $Z(\omega_k,\omega_l)$ is chosen for each pair (ω_k,ω_l) a regular function $f(\omega,\omega')$ is defined such that it interpolates all the measured values. With this function we define:

$$|Z_{VH}) = \int_0^\infty Z(\omega)|\omega)\, d\omega + \int_0^\infty \int_0^\infty f\big(\omega,\omega'\big)\big|\omega,\omega'\big)\, d\omega d\omega' \tag{90}$$

which is a van Hove function since $Z(\omega)$ and $f(\omega,\omega')$ are regular functions. Of course $|Z_{VH})$ is not exactly $|Z)$, but the central point is that $\Delta\omega$ is maximal accuracy of the energy measurement instruments then $|Z_{VH})$ is indistinguishable of $|Z)$ from the experimental point of view. Then combining projectors, we can conclude that for any observable (according to the decomposition spectral theorem) there is a van Hove observable that is observationally indistinguishable from the former. Thus the van Hove observables can give an account of reality. A similar argument can be used in the case of states. As a consequence the observables and states that do not belong to the van Hove spaces cannot be characterized experimentally since they are beyond the measurement precision. Then SID is able to describe the physical reality with the measurement precision of nowadays.

Of course in EID, the criterion to neglect information is completely different. All the information that is irrelevant for the proper system S is neglected.

5 Characteristic Times

After this consideration we must complete the subject defining the characteristic times:

1. In EID there is a moving preferred basis[8] for the relevant subsystem and the off diagonal terms vanish in this basis in a characteristic time, known as the decoherence time t_D,

[8] In each example of EID this preferred basis is defined unambiguously, a general definition can be found in [2].

that we will call the proper system decoherence time in the moving pointer basis t_{DS}.[9] This is of course a quantum reasoning related with a quantum state, $\rho_S(t)$ of EID.

We also have a time where the relevant proper subsystem stops its motion, at a time t_R, that we will call the relaxation time of the proper subsystem t_{RS}, In this case we are in the usual grounds.

We know that for macroscopic bodies

$$t_{DS} \ll t_{RS}$$

2. In SID there is a final pointer basis (the eigenbasis of H) for the closed system and the off diagonal terms vanish in this basis in a characteristic time known as the relaxation time that we will call the proper system decoherence time of the "universe" t_{RU}. This time is studied in [6–14] and [50]. In SID there is also a moving preferred basis introduced in [50] and the corresponding decoherence time t_{DU}.

Of course we can prove that

$$t_{DS} < t_{RS}, \quad t_{US} < t_{DR}$$

and as proved in [2] and [50]

$$t_{DS} < t_{DU}, \quad t_{DS} < t_{RU}$$

This is only a general description about the characteristic times, a complete study about this issue can be found in [2, 34, 35] and [51].

6 Conclusion

1. Comparing EID and SID we can discuss the application of the these two formalisms. In the case of EID its experimental consequence and its general success to explain many physical phenomena is well known so it is useless to list all its applications. This is not the case of the new arrived SID. Besides we can list some important facts.
   (a) It explains the classical limit in the case of closed system as cosmological systems (see paper [52–55]) and other interesting closed systems, like the Casati-Prosen model [38, 39]. It also gives a closed-system solution to the Mott problem [52].
   (b) It may help to understand some formal aspects of quantum chaos [56].
   (c) It explains the classical limit in the case of the Modal Hamiltonian Interpretation of quantum mechanic [57], a new member of the group of modal interpretations that began with the work of van Frassen. This interpretation satisfies all the Mermin desiderata [58].
   (d) That closed systems reach an equilibrium at Khalfin time was experimentally proved in [59].
2. With this paper we have completed papers [1, 2] and we define the main common features of all decoherence formalisms, the choice of relevant observables, for the case of EID and SID.
3. We have also shown that EID and SID are perfectly compatible with the GTFD.
4. We have proved that coarse graining appears both in EID and in SID because some information has been neglected. In EID the neglected information is the one that does not come from the proper system S. In SID it is the information that cannot be obtained by any available physical device.

[9] See the discussion about t_{DS} in paper [2].

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