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Integral Equations and Operator Theory

A Geometry for the Set of Split Operators

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Abstract. We study the set $\mathcal{X}$ of *split operators* acting in the Hilbert space $\mathcal{H}$:

$$\mathcal{X} = \{T \in \mathcal{B}(\mathcal{H}) : N(T) \cap R(T) = \{0\} \text{ and } N(T) + R(T) = \mathcal{H}\}.$$

Inside $\mathcal{X}$, we consider the set $\mathcal{Y}$:

$$\mathcal{Y} = \{T \in \mathcal{X} : N(T) \perp R(T)\}.$$

Several characterizations of these sets are given. For instance $T \in \mathcal{X}$ if and only if there exists an oblique projection Q whose range is $N(T)$ such that $T + Q$ is invertible, if and only if T posseses a commuting (necessarilly unique) pseudo-inverse S (i.e. $TS = ST, TST = T$ and $STS = S$). Analogous characterizations are given for $\mathcal{Y}$. Two natural maps are considered:

$$\mathbf{q} : \mathcal{X} \to \mathbb{Q} := \{\text{oblique projections in } \mathcal{H}\},\ \mathbf{q}(T) = P_{R(T)//N(T)}$$

and

$$\mathbf{p} : \mathcal{Y} \to \mathbb{P} := \{\text{orthogonal projections in } \mathcal{H}\},\ \mathbf{p}(T) = P_{R(T)},$$

where $P_{R(T)//N(T)}$ denotes the projection onto $R(T)$ with nullspace $N(T)$, and $P_{R(T)}$ denotes the orthogonal projection onto $R(T)$. These maps are in general non continuous, subsets of continuity are studied. For the map $\mathbf{q}$ these are: similarity orbits, and the subsets $\mathcal{X}_{c_k} \subset \mathcal{X}$ of operators with rank $k < \infty$, and $\mathcal{X}_{F_k} \subset \mathcal{X}$ of Fredholm operators with nullity $k < \infty$. For the map $\mathbf{p}$ there are analogous results. We show that the interior of $\mathcal{X}$ is $\mathcal{X}_{F_0} \cup \mathcal{X}_{F_1}$, and that $\mathcal{X}_{c_k}$ and $\mathcal{X}_{F_k}$ are arc-wise connected differentiable manifolds.

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1. Introduction

Given a Hilbert space $\mathcal{H}$ we consider the set $\mathcal{X}$ of all bounded linear operators T such that the image $R(T)$ and the nullspace $N(T)$ intersect at $\{0\}$ and

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$R(T) + N(T) = \mathcal{H}$; in symbols

$$\mathcal{X} = \{T \in \mathcal{B}(\mathcal{H}) : R(T) \dot{+} N(T) = \mathcal{H}\}.$$

A particular subset $\mathcal{Y}$ of $\mathcal{X}$ is the one corresponding to orthogonal decompositions:

$$\mathcal{Y} = \{T \in \mathcal{X} : R(T) \perp N(T)\} = \{T \in \mathcal{B}(\mathcal{H}) : R(T) \oplus N(T) = \mathcal{H}\}.$$

Elements in $\mathcal{X}$ are usually said to *admit a group inverse* [17], or to have Drazin index 1 [2,8,11]. Elements of $\mathcal{Y}$ are called *EP operators* [4,7], or *range-Hermitian* operators [3].

In this paper we study $\mathcal{X}$ and $\mathcal{Y}$ from a topological viewpoint, and continuity properties of the natural mapping which assigns to each $T \in \mathcal{X}$ the (non orthogonal) projection $P_{R(T)//N(T)}$ with image $R(T)$ and nullspace $N(T)$; if $T \in \mathcal{Y}$, then $N(T) = R(T)^{\perp}$ and we write $P_{R(T)}$. We denote by $\mathbb{Q}(\mathcal{H}) = \mathbb{Q}$ the set of all (bounded and linear) projections in $\mathcal{H}$, and by $\mathbb{P}(\mathcal{H}) = \mathbb{P} \subset \mathbb{Q}$ the subset of orthogonal projections.

The inclusion $\mathcal{Y} \hookrightarrow \mathcal{X}$ admits many right inverses $\mathcal{X} \to \mathcal{Y}$ which retract $\mathcal{X}$ onto $\mathcal{Y}$ (in general, without continuity). Together with corresponding right inverses of the inclusion $\mathbb{P} \hookrightarrow \mathbb{Q}$ there is a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\rho"] \arrow[d, "\mathbf{q}"'] & \mathcal{Y}, \arrow[d, "\mathbf{p}"] \\ \mathbb{Q} \arrow[r, "\theta"] & \mathbb{P} \end{tikzcd}$$

where

$$\mathbf{q} : \mathcal{X} \to \mathbb{Q}, \mathbf{q}(T) = P_{R(T)//N(T)},$$

and

$$\mathbf{p} : \mathcal{Y} \to \mathbb{P}, \mathbf{p}(T) = P_{R(T)}.$$

The horizontal maps ρ and θ are retractions related to the polar decomposition (see details below). The lack of continuity of $\mathbf{q}$ and $\mathbf{p}$ suggests considering different "sets of continuity" in $\mathcal{X}$ and $\mathcal{Y}$. Restricted to these sets, ρ is shown to have remarkable topological properties. These sets are also differentiable manifolds.

We describe the contents of the paper. Section 2 contains the definitions of the retractions mentioned above and several useful properties and characterizations of split operators. Section 3 studies continuity of the maps $\mathbf{q}$ and $\mathbf{p}$ at certain subsets of $\mathcal{X}$ and $\mathcal{Y}$. We treat at Section 4 the case of similarity orbits. The main result here is that $\mathbf{q}$ is a locally trivial fibre bundle from the similarity orbit of $T \in \mathcal{X}$ onto the connected component of $\mathbf{q}(T) = P_{R(T)//N(T)}$ in $\mathbb{Q}$. At Section 5 we consider those operators of $\mathcal{X}$ which are also Fredholm, or which are compact. This allows to identify the interior of $\mathcal{X}$. More precisely, if

$$\mathcal{X}_{F_k} = \{T \in \mathcal{X} : T \text{ is Fredholm with } \dim N(T) = k\},$$

we prove that $int\mathcal{X} = \mathcal{X}_{F_0} \cup \mathcal{X}_{F_1}$, to us a quite surprising result. Section 6 contains an introduction to the study of the differential structure of subsets of $\mathcal{X}$. The last Section 7 contains results about the polar decomposition of elements of $\mathcal{X}$.

2. Definitions and Some Properties of $\mathcal{X}$ and $\mathcal{Y}$

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators in $\mathcal{H}$. Denote by $\mathbb{Q} = \mathbb{Q}(\mathcal{H})$ the space of idempotent operators, or oblique projections, and $\mathbb{P} = \mathbb{P}(\mathcal{H}) \subset \mathbb{Q}$ the space of orthogonal projections. If $T \in \mathcal{B}(\mathcal{H})$, let $R(T)$ denote the range of T, and $N(T)$ the nullspace of T. Put $\alpha(T) = \dim N(T)$. If $\mathcal{S}, \mathcal{T}$ are closed linear subspaces of $\mathcal{H}$, the notation $\mathcal{S} \dot{+} \mathcal{T} = \mathcal{H}$ means that the direct sum is $\mathcal{H}$, and we shall write $\mathcal{S} \oplus \mathcal{T} = \mathcal{H}$ if the sum is orthogonal. We shall denote by $P_{\mathcal{S}}$ the orthogonal projection onto $\mathcal{S}$, and by $P_{\mathcal{S}//\mathcal{T}}$ the idempotent with range $\mathcal{S}$ and nullspace $\mathcal{T}$.

Note that if $T \in \mathcal{X}$, the fact that $R(T)$ has a closed complement, implies that it is also closed (see [18], Theorem 5.10). Selfadjoint operators with closed range are examples of EP operators.

These sets are related by the following commuting square of natural maps:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\rho"] \arrow[d, "\mathbf{q}"'] & \mathcal{Y}, \arrow[d, "\mathbf{p}"] \\ \mathbb{Q} \arrow[r, "\theta"] & \mathbb{P} \end{tikzcd}$$

where

$$\mathbf{q} : \mathcal{X} \to \mathbb{Q}, \mathbf{q}(T) = P_{R(T)//N(T)},$$

and

$$\mathbf{p} : \mathcal{Y} \to \mathbb{P}, \mathbf{p}(T) = P_{R(T)}.$$

The horizontal maps ρ and θ are orthogonalization maps, in fact retractions, and are defined in the remark below.

We shall use several facts concerning idempotents and projections, the proofs can be found in [6], or checked by direct computations.

Remark 2.1. Let $Q \in \mathbb{Q}$, and put $\epsilon_Q = 2Q - 1$, which verifies $\epsilon_Q^2 = 1$. Consider the polar decomposition

$$\epsilon_Q = |\epsilon_Q| U_Q,$$

where U_Q is a unitary operator. Then U_Q is a selfadjoint symmetry

$$U_Q^* = U_Q = U_Q^{-1},$$

and

$$|\epsilon_Q| U_Q = U_Q |\epsilon_Q|^{-1}.$$

In particular, $\epsilon_Q = |\epsilon_Q|^{1/2} U_Q |\epsilon_Q|^{-1/2}$.

Using these facts, let

$$\rho : \mathcal{X} \to \mathcal{Y},\ \rho(T) = |\epsilon_{\mathbf{q}(T)}|^{-1/2} T |\epsilon_{\mathbf{q}(T)}|^{1/2},$$

and

$$\theta : \mathbb{Q} \to \mathbb{P}, \theta(Q) = \frac{1}{2}(1 + U_Q).$$

The map θ was considered in [6], and it is one of the many ways to orthogonalize an oblique projection. Note that given a selfadjoint symmetry U, $\frac{1}{2}(1+U)$ recovers the projection onto the the eigenspace of U corresponding to the eigenvalue $+1$. This map θ is indeed a retraction, and using the above formulas, one has

$$\begin{aligned}\theta(Q) &= \frac{1}{2}(1 + |\epsilon_Q|^{-1/2}\epsilon_Q|\epsilon_Q|^{1/2}) = |\epsilon_Q|^{-1/2}\frac{1}{2}(1+\epsilon_Q)|\epsilon_Q|^{1/2} \\ &= |\epsilon_Q|^{-1/2} Q |\epsilon_Q|^{1/2}.\end{aligned}$$

This implies that $|\epsilon_Q|^{-1/2}(R(Q)) \perp |\epsilon_Q|^{-1/2}(N(Q))$.

Note that ρ is well defined: this last formula implies that

$$R(\rho(T)) = R(|\epsilon_{\mathbf{q}(T)}|^{-1/2} T |\epsilon_{\mathbf{q}(T)}|^{1/2}) = |\epsilon_{\mathbf{q}(T)}|^{-1/2} R(\mathbf{q}(T))$$

and

$$N(\rho(T)) = |\epsilon_{\mathbf{q}(T)}|^{-1/2} N(T) = |\epsilon_{\mathbf{q}(T)}|^{-1/2} N(\mathbf{q}(T))$$

which are orthogonal subspaces which sum $\mathcal{H}$. Moreover,

$$|\epsilon_{\mathbf{q}(T)}|^{-1/2} R(\mathbf{q}(T)) = R(|\epsilon_{\mathbf{q}(T)}|^{-1/2}\mathbf{q}(T)|\epsilon_{\mathbf{q}(T)}|^{1/2}) = R(\theta(\mathbf{q}(T))),$$

and similarly for the nullspace, which imples that

$$\theta(\mathbf{q}(T)) = P_{R(\rho(T))} = \mathbf{p}(\rho(T)),$$

i.e. the square commutes. Finally, note that ρ is a retraction: if $T \in \mathcal{Y}$, $R(T) \perp N(T)$ and thus $\mathbf{q}(T) = P_{R(T)} = \mathbf{p}(T)$. Then $|\epsilon_{\mathbf{q}(T)}| = 1$.

These maps are defined in terms of continuous (even smooth) maps in the arguments $\mathbf{p}(T) = P_{R(T)}$ and $\mathbf{q}(T) = P_{R(T)//N(T)}$.

The following formula gives a characterization of the class $\mathcal{X}$.

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$, and $k(T)$ an oblique projection whose range is the nullspace of T (for instance, $k(T) = P_{N(T)}$). Then $T \in \mathcal{X}$ if and only*

$$T + k(T)$$

is invertible. In this case, one has that

$$\mathbf{q}(T) = P_{R(T)//N(T)} = T(T + k(T))^{-1}.$$

Proof. Suppose first that $T \in \mathcal{X}$. In particular, $R(T)$ is closed. Since the kernel of $k(T)$ is a supplement for $N(T)$, it follows that

$$T|_{N(k(T))} : N(k(T)) \to R(T)$$

is an isomorphism. Then, on the decomposition $\mathcal{H} = N(k(T)) \dot{+} N(T), T + k(T)$ acts as $T|_{N(k(T))} + id_{N(T)}$, onto $\mathcal{H} = R(T) \dot{+} N(T)$, and is clearly an isomorphism.

Conversely, suppose that $T + k(T)$ is invertible. Then

$$R(T) = T(N(k(T)) = (T + k(T))(N(k(T))$$

is closed. Let $\xi \in R(T) \cap N(T)$. Then there exists a vector η, which can be chosen in $N(k(T))$, such that $\xi = T(-\eta) = k(T)\xi$. Then

$$T(-\eta - \xi) = T(-\eta) = k(T)\xi = k(T)(\xi + \eta),$$

i.e. $\eta + \xi \in N(T + k(T)) = \{0\}$, or $\eta = -\xi$, which implies that they are both zero. Pick $\xi \in \mathcal{H}$, then there exists $\xi \in \mathcal{H}$ such that $\xi = (T + k(T))\eta$. Let $\eta = \eta_0 + \eta_1 \in N(T)\dot{+}N(k(T))$. Then $\xi = T\eta_1 + k(T)\eta_0 = T\eta_1 + \eta_0 \in R(T) + N(T)$.

Let us now prove the formula. As remarked above, $T+k(T)$ is an isomorphism mapping $N(k(T))$ onto $R(T)$, and acting as the identity on $N(T)$. Then $(T + k(T))^{-1}$ acts as the identity on $N(T)$, and as the inverse of $T|_{N(k(T))}$ in $R(T)$. Let $\eta \in R(T)$, then $\eta = T\xi$ for $\xi \in N(k(T))$,

$$T(T + k(T))^{-1}\eta = T\xi = \eta.$$

Clearly $T(T + k(T))^{-1}(N(T)) = 0$, which completes the proof. □

In particular, if $T \in \mathcal{X}$,

$$\mathbf{q}(T) = P_{R(T)//N(T)} = T(T + P_{N(T)})^{-1},$$

and this implies that continuity of the maps defined above depends on the continuity of the maps $T \mapsto P_{R(T)}$ and $T \mapsto P_{N(T)}$. It is well known that these maps are not continuous in $\mathcal{B}(\mathcal{H})$ (not even among closed range operators, as it can be shown with trivial finite dimensional examples). In the next section, we shall restrict our attention to subsets where they are continuous.

Let us show further characterizations of $\mathcal{X}$ and $\mathcal{Y}$. In particular the first property shows that elements of $\mathcal{X}$ have a commuting pseudo-inverse. The following result, in a different and broader context was proved in [13]. We include a proof.

Theorem 2.3. *The following are equivalent*

1. $T \in \mathcal{X}$
2. *There exists* $S \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST$ *and* $TST = T$, $STS = S$.
3. *There exists* $P \in \mathbb{Q}$ *such that* $TP = 0 = PT$ *and* $T + P$ *is invertible.*

Proof. Suppose that $T \in \mathcal{X}$. Then $R(T)$ is closed and $T_0 = T|_{R(T)} : R(T) \to R(T)$ is invertible. Put $S = T_0^{-1}\mathbf{q}(T)$ then $TS = ST = \mathbf{q}(T)$ and $TST = T, STS = S$. Hence T verifies 2.

Suppose that T verifies 2., and put $P = I - TS$. Then $TP = PT = 0 = SP = PS$,

$$(T + P)(S + P) = TS + P = TS + I - TS = I, \quad \text{and} \quad (T + P)^{-1} = S + P.$$

Hence, T verifies 3.

Suppose that T verifies 3. Since $(T + P)P = P = P(T + P)$ and $T + P$ is invertible, we have $P = (T + P)^{-1}P = P(T + P)^{-1}$. Let us show that $N(T) = R(P)$. If $\eta = P\eta \in R(P)$, then $T\eta = TP\eta = O$. Then $R(P) \subseteq N(T)$. For the other inclusion, if $T\xi = 0$ then $(T+P-P)\xi = 0$. Thus $(T+P)\xi = P\xi$

and $\xi = (T+P)^{-1}P\xi = P\xi \in R(P)$. In a similar fashion it is shown that $R(T) = N(P)$: if $\eta \in R(T)$ then $\eta = T\xi$ and $P\eta = PT\xi = 0$ i.e $\eta \in N(P)$. If $P\xi = 0$ then $(T+P-T)\xi = 0$. Thus $(T+P)\xi = T\xi$ and $\xi = (T+P)^{-1}T\xi = T(T+P)^{-1}\xi \in R(T)$.

Therefore $\mathcal{H} = R(P)\dot{+}N(P) = R(T)\dot{+}N(T)$, i.e $T \in \mathcal{X}$. □

There is an analogous characterization for $\mathcal{Y}$:

Theorem 2.4. *The following are equivalent*

1. $T \in \mathcal{Y}$
2. *There exists* $S \in \mathcal{B}(\mathcal{H})$ *such that* $TS = ST = (ST)^*$ *and* $TST = T$, $STS = S$.
3. *There exists* $P \in \mathbb{P}$ *such that* $TP = 0 = PT$ *and* $T+P$ *is invertible.*

Proof. The proof is essentially the same as above. Suppose that the first property holds, and pick $S = T^\dagger$. Then S is a pseudo-inverse for $T, TS = P_{R(T)}$ and $ST = P_{N(T)^\perp} = P_{R(T)}$.

If the second property holds, as in the proof above, pick $P = 1 - ST$, which now is an orthogonal projection.

If the third property holds then, by the above theorem, $T \in \mathcal{X}$, and as in the proof above, the orthogonal projection P verifies $N(P) = N(T)$ and $R(P) = R(T)$. Thus $T \in \mathcal{Y}$. □

Remark 2.5. 1. If $T \in \mathcal{X}$ then the commuting pseudo-inverse S is in fact unique.

Indeed, if S' is another commuting pseudo-inverse for T, then by Theorem 2.3, one has $\mathbf{q}(T) = TS' = S'T$. Then

$$S' = S'TS' = S'\mathbf{q}(T) = S'TS = \mathbf{q}(T)S = STS = S.$$

Let us denote it by $S = s(T)$.

2. $T \in \mathcal{X}$ (resp. $T \in \mathcal{Y}$) if and only if $T^* \in \mathcal{X}$ (resp. $T^* \in \mathcal{Y}$). Moreover, apparently $s(T^*) = s(T)^*$.

   This is apparent using, for instance, property 2. in Theorem 2.3 (resp. 2.4). Analogously, if $T \in \mathcal{Y}$, then also $T^* \in \mathcal{Y}$.
3. Let T be a closed range operator. Then $T \in \mathcal{X}$ (resp. $T \in \mathcal{Y}$) if and only if $T^\dagger \in \mathcal{X}$ (resp. $T^\dagger \in \mathcal{Y}$).

   Indeed, since T, T^* have closed range, $R(T^\dagger) = N(T)^\perp = R(T^*)$ and $N(T^\dagger) = R(T)^\perp = N(T^*)$. These two subspaces are complementary because, for instance, $T^* \in \mathcal{X}$. For the converse statement, use that $(T^\dagger)^\dagger = T$.
4. Combining the first two statements, if $T \in \mathcal{X}$, then

$$T^\dagger = s(T^*T)T^* = T^*s(TT^*).$$

5. If $T \in \mathcal{X}$ and $P \in \mathbb{Q}$ such that $PT = 0 = TP$ and $T+P$ invertible, then

$$S = s(T) = (T+P)^{-1} - P$$

   Also $S = (T+P)^{-1}(I-P)$

6. As remarked above, a selfadjoint operator belongs to $\mathcal{Y}$ (or for that matter to $\mathcal{X}$) if and only if it has closed range. Indeed, if $A^* = A$, then $N(A)^\perp = \overline{R(A)}$.

   In particular if $T \in \mathcal{X}$, then $T^*T, TT^* \in \mathcal{Y}$, and for any $P \in \mathbb{Q}, P^*P, PP^* \in \mathcal{Y}$.
7. If $T \in \mathcal{X}$, then apparently $T^n \in \mathcal{X}$, for $n \geq 1$. The converse clearly does not hold.
8. If $T \in \mathcal{X}$ is non invertible, then $0 \in \sigma(T)$ is an isolated point. Define $f(z) = 0$ in a neighborhood of 0 and $f(z) = z^{-1}$ in a neighborhood of $\sigma(T)\backslash\{0\}$. Then f is a holomorphic function in a neighbourhood of $\sigma(T)$ and $S = f(T)$ (see for instance Theorem 10.10 in [18]). In particular S lies in the Banach algebra generated by T. Put $P = 1 - \mathbf{q}(T) = 1 - TS = 1 - ST$, where $S = s(T)$ as above is the unique commuting pseudo-inverse for T. Thus as remarked $S = (T+P)^{-1}(1-P)$. Using the resolvent formula for $0 < |z| < \gamma(T)$,

$$R(z,T) = \frac{1}{z}P + \sum_{n=0}^{+\infty} S^{n+1}z^n.$$

Indeed, for such z,

$$\begin{aligned}
R(z,T) &= (z-T)^{-1} = (z-T)^{-1}P + (z-T)^{-1}(1-P) \\
&= z^{-1}P + (z-(T+P)^{-1}(1-P) \\
&= z^{-1}P + \sum_{n\geq 0} z^n(t+P)^{-n-1}(1-P) \\
&= z^{-1}P + \sum_{n\geq 0} z^n(t+P)^{-n-1}(1-P)^{n+1} \\
&= z^{-1}P + \sum_{n\geq 0} z^n[(t+P)^{-1}(1-P)]^{n+1} = z^{-1}P + \sum_{n\geq 0} z^n S^{n+1}.
\end{aligned}$$

We prove now a type of perturbation result. More precisely, which conditions on an operator $T \in \mathcal{B}(\mathcal{H})$ which is close to $T_0 \in \mathcal{X}$ should be added, in order to get $T \in \mathcal{X}$.

Theorem 2.6. *Let $T_0 \in \mathcal{X}$ and $T \in \mathcal{B}(\mathcal{H})$ such that $\|T - T_0\| < \frac{1}{\|s(T_0)\|}$ and $\|T^2 - T_0^2\| < \frac{1}{\|s(T_0)\|^2}$.*

If either $R(T) \cap N(T_0) = \{0\}$, or $N(T) + R(T_0) = \mathcal{H}$, then $T \in \mathcal{X}$.

Proof. Suppose first that $R(T) \cap N(T_0) = \{0\}$. Note that $\|T - T_0\| < \frac{1}{\|S(T_0)\|}$ implies that $I + s(T_0)(T - T_0)$ and $I + (T - T_0)s(T_0)$ are invertible.

Let $P = I - s(T_0)T_0 = I - T_0 s(T_0)$. We have

$$I + s(T_0)(T - T_0) = I - s(T_0)T_0 + s(T_0)T = P + s(T_0)T.$$

Thus

$$P[I + s(T_0)(T - T_0)] = P[P + s(T_0)T] = P \quad \text{and} \quad P[I + s(T_0)(T - T_0)]^{-1} = P.$$

Therefore,

$$s(T_0)T[I + s(T_0)(T - T_0)]^{-1} = [P + s(T_0)T - P][I + s(T_0)(T - T_0)]^{-1} = I - P.$$

Thus, if $\xi \in \mathcal{H}$, then

$$T[I + s(T_0)(T - T_0)]^{-1} P\xi \in R(T) \cap N(s(T_0)) = R(T) \cap N(T_0) = \{0\}.$$

Thus

$$T[I + s(T_0)(T - T_0)]^{-1} P = 0.$$

Put

$$W = [I + s(T_0)(T - T_0)]^{-1} s(T_0).$$

Note that $TWT = T$:

$$\begin{aligned} TWT &= T[I + s(T_0)(T - T_0)]^{-1} s(T_0)T = T[P + s(T_0)T]^{-1} s(T_0)T \\ &= T[P + s(T_0)T]^{-1}[P + s(T_0)T - P] = T - T[P + s(T_0)T]^{-1} P = T. \end{aligned}$$

Also, we have

$$\begin{aligned} WTW &= [P + s(T_0)T]^{-1} s(T_0)TW = [P + s(T_0)T]^{-1}[P + s(T_0)T - P]W \\ &= [I - [P + s(T_0)T]^{-1}]P]W = W - [P + s(T_0)T]^{-1} PW \\ &= W - [P + s(T_0)T]^{-1} Ps(T_0) = W \end{aligned}$$

Thus $TWT = T$ and $WTW = W$.

Note that

$$\begin{aligned} I + s(T_0)(T^2 - T_0^2)s(T_0) &= I + s(T_0)T^2 s(T_0) - s(T_0)T_0^2 s(T_0) \\ &= I + s(T_0)T^2 s(T_0) - s(T_0)T_0 = P + s(T_0)T^2 s(T_0) \end{aligned}$$

and

$$\begin{aligned} [P - s(T_0)T][I - TW - WT][P + Ts(T_0)] &= P + s(T_0)T^2 s(T_0) \\ &= I + s(T_0)(T^2 - T_0^2)s(T_0). \end{aligned}$$

Since, $[P - s(T_0)T]$, $[P + Ts(T_0)]$ and $I + s(T_0)(T^2 - T_0^2)s(T_0)$ are invertible, $[I - TW - WT]$ is invertible.

Since WT and TW are idempotents, this implies that

$$\mathcal{H} = R(TW) \dot{+} N(WT) = R(T) \dot{+} N(T),$$

that is, $T \in \mathcal{X}$.

If $N(T) + R(T_0) = \mathcal{H}$, then $R(T^*) \cap N(T_0^*) = \{0\}$. Since $T_0^* \in \mathcal{X}$ and $S(T_0^*) = S(T_0)^*$ (see Remark 2.5), it follows that T^* satisfies the conditions of the first part of this proof. Thus $T^* \in \mathcal{X}$, and then $T \in \mathcal{X}$. $\square$

3. Continuity of the Maps

We shall study sets of continuity of the maps $\mathbf{q}$ and ρ. In general, they are not continuous. Easy examples can be constructed in finite dimension

Remark 3.1. Note that if $T_n \to T$, for $T_n, T \in \mathcal{X}$, and S_n, S are their commuting pseudo-inverses, then

$$\mathbf{q}(T_n) \to \mathbf{q}(T) \text{ if and only if } S_n \to S.$$

Indeed, if $S_n \to S$, clearly $\mathbf{q}(T_n) = T_n S_n \to \mathbf{q}(T) = ST$.

Conversely, suppose $\mathbf{q}(T_n) \to \mathbf{q}(T)$. If $P_n = 1 - \mathbf{q}(T_n)$ and $P = 1 - \mathbf{q}(T)$, clearly $T_n P_n = 0 = P_n T_n$, and P_n is an idempotent onto the nullspace of T_n, and therefore by Proposition 2.2, since $T_n \in \mathcal{X}, T_n + P_n$ is invertible. Then $S_n = (T_n + P_n)^{-1} + P_n$, and similarly for T. Then $S_n \to S$.

Let us recall the definition of the reduced minimum modulus $\gamma(T)$ of an operator:

$$\gamma(T) = inf\{r \geq 0 : \|T\xi\| \geq r\|\xi\|, \text{ for all } \xi \in N(T)^{\perp}\}.$$

Clearly $\gamma(T) > 0$ if and only if $R(T)$ is closed. Let us recall as well the following facts and notations from [5], in the following remark.

Remark 3.2. Let $T \in \mathcal{B}(\mathcal{H})$ with closed range.

1. Let T' be a pseudo-inverse of T (i.e. $TT'T = T$). Then
$$\gamma(T) = \frac{1}{\|T^{\dagger}\|} \geq \frac{1}{\|T'\|}.$$
2. For $d > 0$, let $\mathcal{R}_d = \{B \in \mathcal{B}(\mathcal{H}) : \gamma(B) \geq \frac{1}{d}\}$. Then both maps
$$B \mapsto P_{R(B)} \quad \text{and} \quad B \mapsto P_{N(B)}$$
are continuous as maps from $\mathcal{R}_d$ to $\mathbb{P}$, for any fixed $d > 0$.

To study the continuity of ρ and $\mathbf{q}$, we shall need the following result by Markus [14] (see also [15], Theorem 17, page 102):

Let $B_n \to B$ be a convergent sequence of closed rank operators, the following are equivalent

1. $\gamma(B_n)$ is uniformly bounded from below.
2. $P_{N(B_n)} \to P_{N(B)}$.
3. $P_{R(B_n)} \to P_{R(B)}$.

From this result follow two lemmas (3.3 and 3.4 below), which will be useful.

This result is perhaps known, though we know no reference for it.

Lemma 3.3. *Let A_n be a sequence of positive operators which converges to A. Suppose that $\dim(R(A_n)) = \dim(R(A)) = k < \infty$. Then the reduced minimum moduli of A_n, A are uniformly bounded from below, i.e. there exists a constant $d > 0$ such that $\gamma(A_n) \geq d$.*

Proof. By hypothesis, the range projections $P_{R(A_n)}$ and $P_{R(A)}$ are unitarily equivalent. Let U_n be unitary operators such that

$$U_n P_{R(A)} U_n^* = P_{R(A_n)}.$$

Thus the operators $U_n^* A_n U_n, A$ have the same range $R(A)$. Let B_n and B be the restrictions of $U_n^* A_n U_n$ and A to $R(A) = N(A)^{\perp}$. Thus B_n, B are invertible operators acting in the finite dimensional space $R(A)$. Apparently

$$\gamma(A_n) = \gamma(U_n^* A_n U_n) = \gamma(B_n).$$

Suppose that $\gamma(B_n)$ is not bounded from below, then there exists a subsequence, which we will denote B_j, such that $\gamma(B_j) \to 0$. Note that $\|B_n\| = \|A_n\|$ is uniformly bounded. Since B_j act in a finite dimensional space, it follows that there exists a subsequence of B_j, which we will denote by B_l, such that $B_l \to C$. Since $\gamma(B_l) \to 0$ it follows that C, which is positive, is not invertible.

On the other hand, extending C trivially in $R(A)^{\perp}$, clearly $U_l^* A_l U_l \to C$ in $\mathcal{H}$. Note that $U_l^* A_l U_l$ are closed range operators, which converge to A, and such that $R(U_l^* A_l U_l) = R(A)$. Thus we can apply (a trivial version of) Markus Theorem cited above, to conclude that

$$\gamma(A_l) = \gamma(U_l^* A_l U_l) \to \gamma(A) > 0,$$

leading to a contradiction. $\square$

The following Lemma is again a consequence of Markus' Theorem:

Lemma 3.4. *Let A_n be a sequence of (closed range) positive operators which converges to A, such that $\dim(N(A_n)) = \dim(N(A)) = k < \infty$. Then there exists $d > 0$ such that $\gamma(A_n) \geq d$ for all $n \geq 1$.*

Proof. Note that $A \in \mathcal{X}$, and since it is positive, its commuting pseudo-inverse coincides with its Moore-Penrose pseudo-inverse $s(A) = A^{\dagger}$.

$$AA^{\dagger} = A^{\dagger}A = \mathbf{q}(A) = P_{R(A)}.$$

The sequence $B_n = A^{\dagger}A_n$ converges to $A^{\dagger}A = P_{R(A)}$, which is a Fredholm operator of index zero. It follows that there exists n_0 such that for $n \geq n_0, B_n$ is also a Fredholm operator of index zero. Moreover, since $N(A_n) \subset N(A^{\dagger}A_n)$, using Kato's theorem, one can choose n_0 in order that $n \geq n_0$ implies

$$k = \alpha(A_n) \leq \alpha(A^{\dagger}A_n) \leq \alpha(A^{\dagger}A) = \alpha(A) = k.$$

In particular, $N(A^{\dagger}A_n) = N(A_n)$. Note also that $R(B_n) \subset R(A^{\dagger}) = R(A)$. If $n \geq n_0, R(B_n)$ has co-dimension k. These facts together imply that for $n \geq n_0, R(B_n) = R(A)$.

Note that for $B_n = A^{\dagger}A_n$, a trivial application of Markus Theorem proves that the last statement holds trivially. Thus

$$P_{N(A_n)} = P_{N(B_n)} \to P_{N(A)}.$$

Again using the Markus result, this time for $A_n \to A$, it follows that $\gamma(A_n)$ is uniformly bounded from below. $\square$

4. Similarity Orbits

Let $Gl(\mathcal{H})$ be the group of invertible operators acting in $\mathcal{H}$. $Gl(\mathcal{H})$ acts in $\mathcal{X}$ by similarity: if $T \in \mathcal{X}$

$$GTG^{-1} \in \mathcal{X}.$$

Indeed, $R(GTG^{-1}) = G(R(T))$ and $N(GTG^{-1}) = G(N(T))$. Moreover, the map $\mathbf{q} : \mathcal{X} \to \mathbb{Q}$ is equivariant with respect to this action:

$$G(R(T) \dot{+} N(T)) = R(GTG^{-1}) \dot{+} N(GTG^{-1}),$$

and thus

$$\mathbf{q}(GTG^{-1}) = P_{G(R(T))//G(N(T))} = GP_{R(T)//N(T)}G^{-1} = G\mathbf{q}(T)G^{-1}.$$

Denote by $\mathcal{S}_{T_0}$ the similarity orbit of T_0,

$$\mathcal{S}_{T_0} = \{GT_0G^{-1} : G \in Gl(\mathcal{H})\}.$$

It is known that the similarity orbit of an oblique projection Q coincides with the connected component $\mathbb{Q}_Q$ of Q in $\mathbb{Q}$.

The following result will be useful, here and later. A set $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ is said to be similarity invariant if $T \in \mathcal{S}$ implies that $GTG^{-1} \in \mathcal{S}$, for any $G \in Gl(\mathcal{H})$. We refer the reader to [9] for the definitions and basic facts on fibre bundles.

Lemma 4.1. *Let $\mathcal{S}$ be a subset of $\mathcal{X}$ which is similarity invariant. Suppose also that*

$$\mathbf{q}|_{\mathcal{S}} : \mathcal{S} \to \mathbb{Q}, \quad \mathbf{q}(T) = P_{R(T)//N(T)}$$

is continuous. Then the image of $\mathbf{q}|_{\mathcal{S}}$ is a union of connected components of $\mathbb{Q}$, and $\mathbf{q}|_{\mathcal{S}}$ is a locally trivial fibre bundle onto its image.

Proof. If $Q = \mathbf{q}(T)$ for some $T \in \mathcal{S}$, then for any $G \in Gl(\mathcal{H})$,

$$GQG^{-1} = G\mathbf{q}(T)G^{-1} = \mathbf{q}(GTG^{-1}) \in \mathbf{q}(\mathcal{S}),$$

which proves the first assertion. Fix $Q_0 = \mathbf{q}(T_0)$ for $T_0 \in \mathcal{S}$.

Let us exhibit a trivialization of $\mathbf{q}|_{\mathcal{S}}$ on a neighbourhood of T_0. Given Q_0, there exists $r_0 > 0$ such that if $Q \in \mathbb{Q}$ verifies $\|Q - Q_0\| < r_{Q_0}$, then

$$\sigma_Q = QQ_0 + (1 - Q)(1 - Q_0) \in Gl(\mathcal{H}).$$

Note that $\sigma_Q Q_0 = QQ_0 = Q\sigma_Q$, and thus $Q = \sigma_Q Q_0 \sigma_Q^{-1}$, if $\|Q - Q_0\| < r_{Q_0}$. Consider

$$\mathcal{B}_{T_0} = \{T \in \mathcal{S} : \|\mathbf{q}(T) - Q_0)\| < r_{Q_0}\},$$

the fiber space F_{Q_0} over Q_0, $F_{Q_0} = \{S \in \mathcal{S} : \mathbf{q}(S) = Q_0\}$. and

$$\Phi = \Phi_{T_0} : \mathcal{B}_{T_0} \to F_{Q_0} \times \{Q \in \mathbb{Q} : \|Q - Q_0\| < r_{Q_0}\}, \Phi(T) = (\sigma_{\mathbf{q}(T)}^{-1} T \sigma_{\mathbf{q}(T)}, \mathbf{q}(T)).$$

Apparently $T \in \mathcal{B}_{Q_0}$ means $\|\mathbf{q}(T) - Q_0\| < r_{Q_0}$ and thus $\mathbf{q}(\sigma_{\mathbf{q}(T)}^{-1} T \sigma_{\mathbf{q}(T)}) = \sigma_{\mathbf{q}(T)}^{-1} \mathbf{q}(T) \sigma_{\mathbf{q}(T)} = Q_0$. Consider

$$\Psi : F_{Q_0} \times \{Q \in \mathbb{Q} : \|Q - Q_0\| < r_{Q_0}\} \to \mathcal{S}, \Psi(S, Q) = \sigma_Q S \sigma_Q^{-1}.$$

Note that $\Psi(S, Q)$ is similar to T_0, because $S \in \mathcal{S}$. Also

$$\mathbf{q}(\Psi(S.Q)) = \sigma_Q \mathbf{q}(S) \sigma_Q^{-1} = \sigma_Q Q_0 \sigma_Q^{-1} = Q$$

Then

$$\begin{aligned}\Phi(\Psi(S, Q)) &= \Phi(\sigma_Q S \sigma_Q^{-1}) = (\sigma^{-1}_{\sigma_Q \mathbf{q}(S) \sigma_Q^{-1}} \sigma_Q S \sigma_Q^{-1} \sigma_{\sigma_Q \mathbf{q}(S) \sigma_Q^{-1}}, \mathbf{q}(\sigma_Q \mathbf{q}(S) \sigma_Q^{-1})) \\ &= (S, Q),\end{aligned}$$

and

$$\Psi(\Phi(T)) = \Psi(\sigma_{\mathbf{q}(T)}^{-1} T \sigma_{\mathbf{q}(T)}, \mathbf{q}(T)) = \sigma_{\mathbf{q}(T)} \sigma_{\mathbf{q}(T)}^{-1} T \sigma_{\mathbf{q}(T)} \sigma_{\mathbf{q}(T)}^{-1} = T. \qquad \square$$

Fix $T_0 \in \mathcal{X}$, and put $Q_0 = \mathbf{q}(T_0)$.

Proposition 4.2. *The mapping*

$$\mathbf{q} : \mathcal{S}_{T_0} \to \mathbb{Q}_{Q_0}, \ \ \mathbf{q}(T) = P_{R(T)//N(T)}$$

is continuous, onto and a locally trivial fibre bundle.

Proof. Let $G_n T_0 G_n^{-1}$ be a sequence in $\mathcal{S}_{T_0}$ converging to $T_1 = G_0 T_0 G_0^{-1}$. Let us show that

$$\mathbf{q}(G_n T_0 G_n^{-1}) \to \mathbf{q}(T_1).$$

By the continuity of the similarity action, and the equivariance of $\mathbf{q}$, it suffices to consider the case $T_1 = T_0$. We claim that there exists $d > 0$ such that $G_n T_0 G_n^{-1}$ and $T_0 \in \mathcal{R}_d$, for all $n \geq 1$, i.e. that the minimum modulus of $G_n T_0 G_n^{-1}$ is bounded from below. To prove this, note that if S_0 is a pseudo-inverse for T_0, then $G_n S_0 G_n^{-1}$ is a pseudo-inverse for $G_n T_0 G_n^{-1}$. Therefore, in view of the properties stated in Remark 3.2, it suffices to show that the norms $\|G_n S_0 G_n^{-1}\|$ are bounded from below. Suppose otherwise that there exists a subsequence such that $\|G_{n(k)} S_0 G_{n(k)}^{-1}\| \to 0$. Then in the expression

$$G_{n(k)} T_0 G_{n(k)}^{-1} G_{n(k)} S_0 G_{n(k)}^{-1} G_{n(k)} T_0 G_{n(k)}^{-1} = G_{n(k)} T_0 G_{n(k)}^{-1}.$$

The right hand terms tend to T_0, whereas the left hand terms tend to zero, leading to a contradiction. Therefore all terms $G_n T_0 G_n^{-1}$ belong to a set $\mathcal{R}_d$ for some $d > 0$, where $\mathbf{q}$ is continuous.

That $\mathbf{q}|_{\mathcal{S}_{T_0}}$ is a locally trivial fibre bundle, follows form Lemma 4.1 above. □

Remark 4.3. If $T_0 \in \mathcal{X}$, the fiber F_{T_0} consists of operators in the $T = G T_0 G^{-1}$ in the similarity orbit of T_0 such that $R(T) = R(T_0)$ and $N(T) = N(T_0)$. That is

$$G(N(T_0)) = N(T_0), G(R(T_0)) = R(T_0).$$

Denote by $\bar{T}_0$ the (invertible) operator $T_0|_{R(T_0)}$ as an operator in $\mathcal{B}(R(T_0))$. Then F_{T_0} identifies with the similarity orbit

$$\{K \bar{T}_0 K^{-1} : K \in Gl(R(T_0))\}.$$

Proposition 4.4. *Let $T_0 \in \mathcal{X}$. The map*

$$\rho : \mathcal{S}_{T_0} \to \mathcal{Y}, \rho(T) = |\epsilon_{\mathbf{q}(T)}|^{-1/2} T |\epsilon_{\mathbf{q}(T)}|^{1/2}$$

is continuous.

Proof. The map $\mathbb{Q} \to Gl(\mathcal{H}), Q \mapsto |\epsilon_Q|$ is apparently continuous. By the above result, $\mathcal{S}_{T_0} \to \mathbb{Q}, T \mapsto \mathbf{q}(T)$ is continuous. □

Let U be a unitary operator in $\mathcal{H}$. Then apparently, by uniqueness in the polar decompostion,

$$|\epsilon_{\mathbf{q}(U T_0 U^*)}| = |\epsilon_{U \mathbf{q}(T_0) U^*}| = U |\epsilon_{\mathbf{q}(T_0)}| U^*.$$

So that

$$\rho(U T_0 U^*) = U |\epsilon_{\mathbf{q}(T_0)}|^{-1/2} U^* U T_0 U^* U |\epsilon_{\mathbf{q}(T_0)}|^{1/2} U^* = U \rho(T_0) U^*.$$

In particular, this implies that the image $\rho(\mathcal{S}_{T_0})$ of the above retraction, contains the unitary orbit of $\rho(T_0)$.

5. Fredholm and Compact Split Operators

Split operators which are also Fredholm have finite dimensional nullspace, and therefore are special cases of zero index Fredholm operators. Split operators which have finite dimensional range, are compact operators. The converse of this latter statement is also apparent, split compact operators have finite rank, since split operators are invertible when restricted to their ranges.

Let us first treat the compact case, and examine the continuity of the maps ρ and $\mathbf{q}$ restricted to this class. Denote by

$$\mathcal{X}_c = \{T \in \mathcal{X} : T \text{ is compact}\} = \{T \in \mathcal{X} : T \text{ has finite rank}\},$$

and

$$\mathcal{X}_{c_k} = \{T \in \mathcal{X} : \dim(R(T)) = k\}.$$

Thus $\mathcal{X}_c = \cup_{k=0}^{\infty} \mathcal{X}_{c_k}$.

Proposition 5.1. *Restricted to $\mathcal{X}_{c_k}$ ($k < \infty$), the maps ρ and $\mathbf{q}$ are continuous.*

Proof. Let $T_n \to T$ in $\mathcal{X}_{c_k}$. Then $T_n T_n^*$ are positive operators of rank k, which converge to TT^* which is also of rank k. By Lemma 3.3, $\gamma(T_n) = \gamma(T_n T_n^*)$ is uniformly bounded from below. Thus, by the same argument as in Theorem 4.1 and Proposition 4.4, the maps ρ and $\mathbf{q}$ are continuous. □

Remark 5.2. These maps are non continuous in the whole class $\mathcal{X}_c$. Indeed, in the case of $\mathbf{q}$, one can easily find a sequence of finite (and fixed) rank operators which converges to an operator of different (lesser) rank. If $\mathbf{q}$ were continuous, one would have a sequence of idempotents of fixed rank, which converges to an idempotent of lesser rank, which cannot happen (close idempotents are similar).

As with the case of similarity orbits, continuity of the map $\mathbf{q}$ restricted to $\mathcal{X}_{c_k}$ implies that it is a fibre bundle.

Proposition 5.3. *The restriction*

$$\mathbf{q}_{c_k} = \mathbf{q}|_{\mathcal{X}_{c_k}} : \mathcal{X}_{c_k} \to \mathbb{Q}_k = \{Q \in \mathbb{Q} : \dim(R(Q)) = k\}$$

is a locally trivial fibre bundle.

Proof. By Lemma 4.1 and the above result, it suffices to prove that the set $\mathcal{S} = \mathcal{X}_{c_k}$ is similarity invariant, a fact which is apparent. □

Let us denote by

$$\mathcal{Y}_{c_k} = \mathcal{Y} \cap \mathcal{X}_{c_k} = \{T \in \mathcal{Y} : \dim(R(T)) = k\}.$$

Proposition 5.4. *The restriction*

$$\rho_{c_k} = \rho|_{\mathcal{X}_{c_k}} : \mathcal{X}_{c_k} \to \mathcal{Y}_{c_k}$$

is a strong deformation retraction.

Proof. First note that $\rho(\mathcal{X}_{c_k}) = \mathcal{Y}_{c_k}$. Since ρ is a retraction and $\mathcal{Y}_{c_k} \subset \mathcal{X}_{c_k}$, clearly $\mathcal{Y}_{c_k} \subset \rho(\mathcal{X}_{c_k})$. Note that by definition, $\rho(T)$ is similar to T. It follows that

$$\dim(R(\rho(T))) = \dim(R(T)),$$

and thus $T \in \mathcal{X}_{c_k}$ implies that $\rho(T) \in \mathcal{Y}_{c_k}$. Recall that

$$\rho(T) = |\epsilon_{\mathbf{q}(T)}|^{-1/2} T |\epsilon_{\mathbf{q}(T)}|^{1/2}.$$

Put

$$R_t(T) = |\epsilon_{\mathbf{q}(T)}|^{-t/2} T |\epsilon_{\mathbf{q}(T)}|^{t/2}.$$

Since $\mathcal{X}_{c_k}$ is similarity invariant, it follows that $R_t(\mathcal{X}_{c_k}) \subset \mathcal{X}_{c_k}$. If $T \in \mathcal{Y}, \epsilon_{\mathbf{q}(T)} = 1$, so that $R_t(T) = T$. Apparently, regarded as a two variable map

$$[0,1] \times \mathcal{X}_{c_k} \ni (t, T) \mapsto R_t(T) \in \mathcal{X}_{c_k},$$

it is continuous. Clearly, $R_0 = id_{\mathcal{X}_{c_k}}$ and $R_1 = \rho$. $\square$

Remark 5.5. In the case of $\mathbf{q}_{c_k}$, the fibre F_{Q_0} of this bundle over a given $Q_0 \in \mathbb{Q}_k$ is

$$F_{Q_0} = \{T \in \mathcal{B}(\mathcal{H}) : R(T) = R(Q_0),\ N(T) = N(Q_0)\} \simeq Gl(R(Q_0)) \simeq Gl(k).$$

One can use this bundle to prove that $\mathcal{X}_{c_k}$ is simply connected if $k \geq 2$. Indeed, in the next section we show that $\mathcal{X}_{c_k}$ is arc-wise connected. Let us write the tail of the homotopy exact sequence of the bundle $\mathbf{q}_{c_k}$ [9]:

$$0 = \pi_1(Gl(k)) \to \pi_1(\mathcal{X}_{c_k}) \to \pi_1(\mathbb{Q}_k) \to \pi_0(Gl(k)) = 0.$$

On the other hand, $\pi_1(\mathbb{Q}_k)$ is trivial for $k \geq 2$, using the homotopy exact sequence of the bundle

$$\mathcal{U}(\mathcal{H}) \to \mathbb{Q}_k, \quad U \mapsto UQ_0U^*$$

with fibre $\{U \in \mathcal{U}(\mathcal{H}) : UQ_0 = Q_0U\} \simeq \mathcal{U}(R(Q_0)) \times \mathcal{U}(N(Q_0)) \simeq \mathcal{U}(k) \times \mathcal{U}(N(Q_0))$. Note that in this case, since $\dim(R(Q_0)) = \infty$, $\mathcal{U}(N(Q_0))$ and $\mathcal{U}(\mathcal{H})$ are contractible by Kuiper's Theorem [12]. Thus $\pi_1(\mathbb{Q}_k) = \pi_1(\mathcal{U}(k)) = 0$.

We shall denote by

$$\mathcal{X}_F = \{T \in \mathcal{X} : T \text{ is a Fredholm operator}\}$$

the class of Fredholm split operators. Apparently, the Fredholm index of such operators is zero. For a non negative integer k, denote by

$$\mathcal{X}_{F_k} = \{T \in \mathcal{X}_F : \dim(N(T)) = k\}.$$

Clearly

$$\mathcal{X}_F = \mathcal{X} \cap \mathcal{F} = \mathcal{X} \cap \mathcal{F}_+ = \mathcal{X} \cap \mathcal{F}_- = \mathcal{X} \cap \mathcal{F}_0 = \cup_{k=0}^{\infty} \mathcal{X}_{F_k},$$

where $\mathcal{F}$ (resp.$\mathcal{F}_+$,$\mathcal{F}_-$, $\mathcal{F}_0$) denotes the class of Fredholm (resp. semi-Fredholm operators of non negative index, semi-Fredholm operators of non positive index, Fredholm operators of zero index). Note that

$$\mathcal{X}\backslash\mathcal{X}_F = \mathcal{X}\backslash\mathcal{F} = \{T \in \mathcal{X} : \dim(N(T)) = \infty\}.$$

If A is a Fredholm operator, denote

$$\alpha(A) = \dim(N(A)) \ \text{ and } \ \beta(A) = \dim(R(A)^{\perp}).$$

The following Lemma will be useful.

Lemma 5.6. *Let A be a Fredholm operator of index 0. The following are equivalent:*

1. *$A \in \mathcal{X}$.*
2. *$\alpha(A^2) = \alpha(A)$.*
3. *$\beta(A^2) = \beta(A)$.*

Proof. Note that $\alpha(A^2) = \alpha(A) < \infty$ means that $N(A^2) = N(A)$. Analogously, $\beta(A^2) = \beta(A)$ means that $R(A^2) = R(A)$. It is a general fact on the theory of the ascent and descent of an operator (see for instance [18]) that $N(A^2) = N(A)$ and $R(A^2) = R(A)$ together are equivalent to the condition $\mathcal{H} = N(A)\dot{+}R(A)$. Thus it suffices to show that (under the zero index assumption) $\alpha(A^2) = \alpha(A)$ if and only if $\beta(A^2) = \beta(A)$. Apparently, in general $\alpha(A^2) \geq \alpha(A)$ and $\beta(A^2) \geq \beta(A)$.

If $\alpha(A^2) = \alpha(A)$,

$$\begin{aligned} 0 &= index(A) = \alpha(A) - \beta(A) \\ &= \alpha(A^2) - \beta(A) \leq \alpha(A^2) - \beta(A^2) = index(A^2) = 0. \end{aligned}$$

The other statement is similar. □

Proposition 5.7. *The set*

$$\mathcal{X}_{F_0} \cup \mathcal{X}_{F_1}$$

is open in $\mathcal{B}(\mathcal{H})$. Moreover, fix $T_0 \in \mathcal{X}_{F_0} \cup \mathcal{X}_{F_1}$. If

$$\|T - T_0\| < \gamma(T_0) \quad \textit{and} \quad \|T^2 - T_0^2\| < \gamma(T_0^2),$$

then $T \in \mathcal{X}_{F_0} \cup \mathcal{X}_{F_1}$.

Proof. Clearly, the second statement implies that $\mathcal{X}_{F_0} \cup \mathcal{X}_{F_1}$ is open in $\mathcal{B}(\mathcal{H})$.

T. Kato (see for instance [10]) proved that if A_0 is a Fredholm operator and $\|A - A_0\| < \gamma(A_0)$, then A is also a Fredholm operator, $index(A) = index(A_0)$ and

$$\alpha(A) \leq \alpha(A_0), \beta(A) \leq \beta(A_0).$$

Applying this result to T and T^2, one has that T is a Fredholm operator of index 0, and

$$\alpha(T) \leq \alpha(T^2) \leq \alpha(T_0^2) = \alpha(T_0).$$

Note that $\alpha(T_0)$ equals 0 or 1. If $\alpha(T_0) = 0, T_0$ is invertible, and $\|T - T_0\| < \gamma(T_0) = \|T_0^{-1}\|^{-1}$ implies that T is also invertible, i.e.,

$$T \in Gl(\mathcal{H}) = \mathcal{X}_{F_0} \subset \mathcal{X}_{F_0} \cup \mathcal{X}_{F_1}.$$

If $\alpha(T_0) = 1$, by the above inequalities

$$0 \leq \alpha(T) \leq \alpha(T^2) \leq 1.$$

If $\alpha(T) = 0, T$ is invertible. If $\alpha(T) = 1$, then $\alpha(T) = \alpha(T^2) = 1$, and by the previous Lemma, $T \in \mathcal{X}$. Therefore $T \in \mathcal{X}_{F_1}$. □

Let us denote by $int(\mathcal{X})$ the interior of $\mathcal{X}$.

Theorem 5.8. $int(\mathcal{X}) = \mathcal{X}_{F_0} \cup \mathcal{X}_{F_1}$.

Proof. By the above proposition, $\mathcal{X}_{F_0} \cup \mathcal{X}_{F_1} \subset int(\mathcal{X})$. Let T in $\mathcal{X}$ such that $T \notin \mathcal{X}_{F_0} \cup \mathcal{X}_{F_1}$. Then $\alpha(T) \geq 2$ (and eventually equals $+\infty$). There exist $\xi, \nu \in N(T)$, with $\xi \perp \nu$ and $\|\xi\| = \|\nu\| = 1$. Consider $T_\delta = T + \delta\, \xi \otimes \nu$. Note that $T_\delta \xi = 0$ and that $T_\delta \nu = \delta \xi$, i.e. $\xi \in N(T_\delta) \cap R(T_\delta)$ and thus $T_\delta \notin \mathcal{X}$. Clearly $\|T_\delta - T\| = \delta$, and therefore $T \notin int(\mathcal{X})$. □

Proposition 5.9. *The maps* $\mathbf{q}$ *and* ρ*, when restricted to* $\mathcal{X}_{F_k}$ $(k < \infty)$*, are continuous.*

Proof. The proof follows as in the previous case, this time using Lemma 3.4. □

Proposition 5.10. *The map*

$$\mathbf{q}_{F_k} = \mathbf{q}|_{\mathcal{X}_{F_k}} : \mathcal{X}_{F_k} \to \mathbb{Q}_{\infty,k} = \{Q \in \mathbb{Q} : \dim(R(Q)^\perp) = k\}$$

is a locally trivial fibre bundle.

Proof. Using the above proposition, and Lemma 4.1, it suffices to show that the set $\mathcal{S} = \mathcal{X}_{F_k}$ is similarity invariant, a fact which is apparent. Also it is apparent that $\mathbf{q}(\mathcal{X}_{F_k})$ is the connected component of $\mathbb{Q}$ consisting of idempotents with co-rank k, i.e. $\mathbb{Q}_{\infty,k}$. □

Denote by

$$\mathcal{Y}_{F_k} = \mathcal{Y} \cap \mathcal{X}_{F_k} = \{T \in \mathcal{Y} : \alpha(T) = k\}.$$

The proof of the following result is similar as the proof of Proposition 5.4.

Proposition 5.11. *The restriction*

$$\rho_{F_k} = \rho|_{\mathcal{X}_{F_k}} : \mathcal{X}_{F_k} \to \mathcal{Y}_{F_k}$$

is a strong deformation retraction.

Remark 5.12. The fibre of the bundle $\mathbf{q}_{F_k}$ over a given $Q_0 \in \mathbb{Q}_{\infty,k}$ is

$$F_{Q_0} = \{T \in \mathcal{X} : R(T) = R(Q_0),\ N(T) = N(Q_0)\} \simeq Gl(R(Q_0)).$$

If $\dim(\mathcal{H}) = \infty$, the fibre F_{Q_0} is contractible in the norm topology, by Kuiper's theorem.

In the next section, it will be shown that $\mathcal{X}_{F_k}$ and $\mathcal{Y}_{F_k}$ are arcwise connected.

Therefore, using the exact sequence of homotopy groups induced by the bundle $\mathbf{q}_{F_k}$, and the above proposition, one obtains that

$$\pi_n(\mathcal{Y}_{Fk}) \simeq \pi_n(\mathcal{X}_{F_k}) \simeq \pi_n(\mathbb{Q}_{\infty,k}).$$

On the other hand, the homotopy groups of $\mathbb{Q}_{\infty,k}$ can be examined using the fibre bundle

$$t_{Q_0} : Gl(\mathcal{H}) \to \mathbb{Q}_{\infty,k}, \pi_{Q_0}(G) = GQ_0G^{-1},$$

whose fibre is

$$G_{Q_0} = \{G \in Gl(\mathcal{H}) : GQ_0 = Q_0G\} \simeq Gl(R(Q_0)) \times Gl(N(Q_0)).$$

Since $\dim(R(Q_0)) = \infty$ and $\dim(N(Q_0)) = k$, again by Kuiper's theorem

$$\pi_n(\mathbb{Q}_{\infty,k}) \simeq \pi_{n-1}(Gl(k)).$$

6. Local Structure

The sets $\mathcal{X}$ and $\mathcal{Y}$ are connected. If $T \in \mathcal{X}$, then tT is a continuous path in the parameter t, which stays inside $\mathcal{X}$, and connects T and 0. Similarly for $\mathcal{Y}$.

Other special subsets of $\mathcal{X}$ and $\mathcal{Y}$ considered here are also connected. For instance (since $Gl(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ are connected), the unitary and similarity orbits of elements in $\mathcal{X}$ or $\mathcal{Y}$ are connected.

We have also considered $\mathcal{Y}_{c_k} \subset \mathcal{X}_{c_k}$ and $\mathcal{Y}_{F_k} \subset \mathcal{X}_{F_k}$ of finite rank and Fredholm split and EP operators. Let us add to this list

$$\mathcal{X}_\infty = \{T \in \mathcal{X} : \dim(N(T)) = \infty,\ \dim(R(T)) = \infty\}$$

and

$$\mathcal{Y}_\infty = \{T \in \mathcal{Y} : \dim(N(T)) = \infty,\ \dim(R(T)) = \infty\} = \mathcal{Y} \cap \mathcal{X}_\infty.$$

Apparently, $\mathcal{X} = (\cup_{k=0}^{\infty} \mathcal{X}_{c_k}) \cup (\cup_{k=0}^{\infty} \mathcal{X}_{F_k}) \cup \mathcal{X}_\infty$, and similarly for $\mathcal{Y}$.

Proposition 6.1. *The sets $\mathcal{X}_{c_k}, \mathcal{X}_{F_k}, \mathcal{X}_\infty, \mathcal{Y}_{c_k}, \mathcal{Y}_{F_k}$ and $\mathcal{Y}_\infty$ are arcwise connected.*

Proof. The proof is similar for all these sets, and is based in the following fact. Let $\mathcal{X}_*$ (resp. $\mathcal{Y}_*$) denote any of the sets related to $\mathcal{X}$ (resp. $\mathcal{Y}$). If T_1, T_2 lie in $\mathcal{X}_*$ (resp. $\mathcal{Y}_*$), then $\mathbf{q}(T_1)$ is similar to $\mathbf{q}(T_2)$ (resp. $\mathbf{p}(T_1)$ is unitarily equivalent to $\mathbf{p}(T_2)$). This is clear, if T_1, T_2 lie in the same set, then $\dim(N(T_1)) = \dim(N(T_2))$ and $\dim(R(T_1)) = \dim(R(T_2))$. Let us reason with $\mathcal{X}_*$ (the other case is analogous). Let $G \in Gl(\mathcal{H})$ such that $G\mathbf{q}(T_1)G^{-1} = \mathbf{q}(T_2)$. Then T_1 and $G^{-1}T_2G$ have the same nullspace and range. Then $T_1|_{R(T_1)}$ and $G^{-1}T_2G|_{R(T_1)}$ are invertible operators in $\mathcal{B}(R(T_1)$. Since $Gl(R(T_1))$ is connected, there exists a continuous path $K(t) \in Gl(R(T_1))$ such that $K(0) = T_1|_{R(T_1)}$ and $K(1) = G^{-1}T_2G|_{R(T_1)}$. Put

$$L(t) = K(t) \oplus 0 \text{ in } R(T_1) \oplus N(T_1) = \mathcal{H}.$$

Then clearly $L(t)$ is a continuous path in $\mathcal{X}_\infty$, with $L(0) = T_1$ and $L(1) = G^{-1}T_2G$. Thus it suffices to find a path connecting $G^{-1}T_2G$ and T_2 in $\mathcal{X}_*$. This can be done taking $G^{-1}(t)T_2G(t)$, where $G(t)$ is a continuous path in $Gl(\mathcal{H})$ with $G(0) = 1$ and $G(1) = G$. □

To establish the local regularity of the these subsets, let us refine our recollection of the geometric structure of $\mathbb{P}$ and $\mathbb{Q}$ (see [6,16])

Remark 6.2. 1. The manifold $\mathbb{Q}$ has local charts induced by its reductive structure. Explicitly, given $Q_0 \in \mathbb{Q}$, there exists an open set $\mathcal{V}$ in the complemented subspace of Q_0-co-diagonal operators

$$\{X \in \mathcal{B}(\mathcal{H}) : Q_0 X Q_0 = (1 - Q_0)X(1 - Q_0) = 0\}$$

and $r_{Q_0} < 0$, such that

$$exp : \mathcal{V} \to \{Q \in \mathbb{Q} : \|Q - Q_0\| < r_{Q_0}\}, \quad exp(X) = e^X Q_0 e^{-X}$$

is a diffeomorphism [16].

2. In the case of orthogonal projections, one can be more precise. If $P_0 \in \mathbb{P}$

$$exp : \{X \in \mathcal{B}(\mathcal{H}) : X^* = -X,\ P_0 X P_0 = (1 - P_0)X(1 - P_0) = 0, |X| < \pi/2\}$$
$$\to \{P \in \mathbb{P} : \|P - P_0\| < 1\}$$

is a diffeomorphism.

Theorem 6.3. *The sets $\mathcal{X}_{c_k}, \mathcal{X}_{F_k}, \mathcal{Y}_{c_k}$ and $\mathcal{Y}_{F_k}$ are differentiable manifolds. The subsets $\mathcal{X}_{c_k}, \mathcal{X}_{F_k}$ are analytic manifolds, the subsets $\mathcal{Y}_{c_k}, \mathcal{Y}_{F_k}$ are C^∞ manifolds.*

Proof. As above, denote by $\mathcal{X}_*, \mathcal{Y}_*$ any of the sets in the statement of this theorem. Note that in this case, the sets $\mathcal{X}_\infty$ and $\mathcal{Y}_\infty$ are excluded. Pick $T_0 \in \mathcal{X}_*$, and consider the set

$$\mathcal{W} = \{T \in \mathcal{X}_* : \|\mathbf{q}(T) - \mathbf{q}(T_0)\| < r_{Q_0}\}.$$

From the results of the previous section, it is clear that $\mathcal{W}$ is an open subset of $\mathcal{X}_*$. This set will serve as a local chart for $\mathcal{X}_*$ around T_0. Consider the map

$$\varphi : \mathcal{W} \to \mathcal{V} \times Gl(R(T_0)),\ \varphi(T) = (X_T, e^{-X_T} T e^{X_T}|_{R(T_0)}),$$

where $X_T = exp^{-1}(\mathbf{q}(T))$, and exp is the homeomorphism of the above remark. We claim that φ is a homeomorphism. First note that it is well defined. If $T \in \mathcal{W}$, then $\|\mathbf{q}(T) - \mathbf{q}(T_0)\| < 1$, and so $X_T = exp^{-1}(\mathbf{q}(T))$ belongs to $\mathcal{V}$. Also,

$$e^{X_T} \mathbf{q}(T_0) e^{X_T} = exp(X_T) = \mathbf{q}(T)$$

so that $e^{-X_T} T e^{X_T}$ has the same range and nullspace as T_0, and thus $e^{-X_T} T e^{X_T}|_{R(T_0)}$ is an invertible operator in $R(T_0)$. Apparently, φ is continuous, because $\mathbf{q}$ is continuous in $\mathcal{X}_*$. Its inverse is the mapping

$$\psi : \mathcal{V} \times Gl(R(T_0)) \to \mathcal{W},\ \psi(X, G) = e^X (G \oplus 0) e^{-X},$$

where $G \oplus 0$ is the extension of G acting in $R(T_0)$, defined as zero in the complement $N(T_0)$. Clearly ψ is continuous. Straightforward verifications show that φ and ψ are inverse maps. Thus we have constructed local charts around any point T_0 in $\mathcal{X}_*$. Apparently the transition maps between two charts are analytic.

The proof for $\mathcal{Y}_*$ is similar, replacing $\mathbf{q}$ by $\mathbf{p}$, and using the corresponding neighbourhoods of exp in $\mathbb{P}$ □

Remark 6.4. Note that the local charts of $\mathcal{X}_*$ and $\mathcal{Y}_*$ (excluding $\mathcal{X}_\infty$ and $\mathcal{Y}_\infty$) are open sets in the relative topology of $\mathcal{X}_*$ and $\mathcal{Y}_*$ induced by the norm topology. Thus these sets give a manifold structure, but not a submanifold structure (of $\mathcal{B}(\mathcal{H})$).

7. Split Partial Isometries

In [1] we studied the set $\mathcal{J}_0$ of split partial isometries, which consists of partial isometries V of $\mathcal{H}$ such that $N(V)\dot{+}R(V) = \mathcal{H}$. It is an open subset of the set $\mathcal{J}$ of all partial isometries, which is a differentiable submanifold of $\mathcal{B}(\mathcal{H})$, and therefore itself a differentiable submanifold. We also considered the set $\mathcal{J}_N$ of normal partial isometries, i.e., partial isometries with the same initial and final space.

If $T \in \mathcal{B}(\mathcal{H})$ let us denote by V_T the partial isometry in the polar decomposition of T,

$$T = V_T|T|,$$

with initial space $N(T)^\perp$ and final space $\overline{R(T)}$. Denote by ν the map

$$\nu : \mathcal{B}(\mathcal{H}) \to \mathcal{J},\ \nu(T) = V_T.$$

Proposition 7.1. *Suppose that $R(T)$ is closed.*

1. *$T \in \mathcal{X}$ if and only if $V_T \in \mathcal{J}_0$.*
2. *$T \in \mathcal{Y}$ if and only if $V_T \in \mathcal{J}_N$.*

Proof. The first assertion is apparent, because $N(V_T) = N(T)$ and $R(V_T) = R(T)$. For the second assertion, note that

$$V_T V_T^* = P_{R(V_T)} = P_{R(T)} \quad \text{and} \quad P_{N(T)^\perp} = P_{N(V_T)^\perp} = V_T^* V_T.$$

Thus both projections coincide if and only if $R(T) = N(T)^\perp$. □

Remark 7.2. Denote by $\mathcal{B}^+_{cr}(\mathcal{H})$ the set of positive operators with closed range. Note that if $V \in \mathcal{J}_0$, the fibre of ν over V is

$$\nu^{-1}(V) = \{T \in \mathcal{X} : V_T = V\} = \{VA : A \in \mathcal{B}^+_{cr}(\mathcal{H}), N(A) = N(V)\}.$$

With respect to the other factor in the polar decomposition, it is straightforward to verify that if $A \in \mathcal{B}^+_{cr}(\mathcal{H})$

$$\{T \in \mathcal{X} : |T| = A\} = \{VA : V \in \mathcal{J}_0, N(V) = N(A)\}.$$

Thus a retraction is defined:

$$\nu|_{\mathcal{X}} : \mathcal{X} \to \mathcal{J}_0 \subset \mathcal{X}, \nu(T) = V_T.$$

As with the nullspace and range projections, this map is in general not continuous. However it is continuous when restricted to several parts of $\mathcal{X}$.

Proposition 7.3. *If $T_0 \in \mathcal{X}$, the map $\nu(T) = V_T$ is continuous restricted to the following subsets of $\mathcal{X}$:*

- *The similarity orbit $\mathcal{S}_{T_0}$ of a fixed $T_0 \in \mathcal{X}$.*
- *The set $\mathcal{X}_{C_k}$ of elements in $T \in \mathcal{X}$ such that* $\dim R(T) = k, k < \infty$.
- *The set $\mathcal{X}_{F_k}$ of elements in $T \in \mathcal{X}$ such that* $\dim N(T) = k, k < \infty$.

Proof. Let T_n, T' in one of the above classes, such that $T_n \to T'$. As is well known,

$$\gamma(T) = \gamma(|T|).$$

It was shown in Sects. 3 and 4, that in any of these classes, there exists $d > 0$ such that $|T_n|, |T'| \in \mathcal{R}_d$. In [5] it was shown that the Moore-Penrose pseudo-inverse is continuous in $\mathcal{R}_d$ (in fact, it is a Lipschitz map with Lipschitz constant $\frac{3}{d^2}$). Then $|T_n|^\dagger \to |T'|^\dagger$, and thus

$$V_{T_n} = V_{T_n} P_{N(T_n)^\perp} = V_{T_n} |T_n||T_n|^\dagger = T_n |T_n|^\dagger \to T'|T'|^\dagger = V_{T'}. \qquad \square$$

References

[1] Andruchow, E., Corach, G., Mbekhta, M.: Split partial isometries. Complex Anal. Oper. Th. **7**(4), 813–829 (2013)

[2] Ben-Israel, A.: On matrices of index zero or one. SIAM J. Appl. Math. **17**, 1118–1121 (1969)

[3] Ben-Israel, A., Greville T.N.E.: Generalized inverses. Theory and Applications, 2nd edn. CMS Books in Mathematics/Ouvrages de Mathmatiques de la SMC, vol. 15. Springer, New York (2003)

[4] Campbell, S.L., Meyer, C.D.: *EP* operators and generalized inverses. Canad. Math. Bull. **18**(3), 327–333 (1975)

[5] Corach, G., Maestripieri, A., Mbekhta, M.: Metric and homogeneous structure of closed range operators. J. Oper. Theory **61**(1), 171–190 (2009)

[6] Corach, G., Porta, H., Recht, L.: The geometry of spaces of projections in C^*-algebras. Adv. Math. **101**(1), 59–77 (1993)

[7] Drivaliaris, D., Karanasios, S., Pappas, D.: Factorizations of EP operators. Linear Algebra Appl. **429**(7), 1555–1567 (2008)

[8] Erdelyi, I.: On the matrix equation $Ax = \lambda Bx$. J. Math. Anal. Appl. **17**, 119–132 (1967)

[9] Husemoller, D.: Fibre bundles, 3rd edn. Graduate Texts in Mathematics, vol. 20. Springer, New York (1994)

[10] Kato, T.: Perturbation theory for linear operators. Reprint of the 1980 edition. Classics in Mathematics. Springer, Berlin (1995)

[11] Koliha, J.J.: The Drazin and Moore-Penrose inverse in C^*-algebras. Math. Proc. R. Ir. Acad. **99**(1), 17–27 (1999)

[12] Kuiper, N.H.: The homotopy type of the unitary group of Hilbert space. Topology **3**, 19–30 (1965)

[13] Laursen, K.B., Mbekhta, M.: Closed range multipliers and generalized inverses. Studia Math. **107**(2), 127–135 (1993)

[14] Markus, A.S.: On some properties of linear operators connected with the notion of gap. Kishinev Gos. Uchen. Zap. **39**, 265–272 (Russian) (1959)

[15] Müller, V.: Spectral theory of linear operators and spectral systems in Banach algebras, 2nd edn. Operator Theory: Advances and Applications, vol. 139. Birkhuser Verlag, Basel (2007)

[16] Porta, H., Recht, L.: Minimality of geodesics in Grassmann manifolds. Proc. Am. Math. Soc. **100**(3), 464–466 (1987)

[17] Robert, P.: On the group-inverse of a linear transformation. J. Math. Anal. Appl. **22**, 658–669 (1968)

[18] Taylor, A.E., Lay, D.C.: Introduction to Functional Analysis, 2nd edn. Wiley, New York (1980)

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