MONOMIAL CONVERGENCE ON ℓ_r

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ABSTRACT. For $1 < r \leq 2$, we study the set of monomial convergence for spaces of holomorphic functions over ℓ_r. For $H_b(\ell_r)$, the space of entire functions of bounded type in ℓ_r, we prove that $\operatorname{mon} H_b(\ell_r)$ is exactly the Marcinkiewicz sequence space m_{Ψ_r} where the symbol Ψ_r is given by $\Psi_r(n) := \log(n+1)^{1-\frac{1}{r}}$ for $n \in \mathbb{N}_0$. For the space of m-homogeneous polynomials on ℓ_r, we prove that the set of monomial convergence $\operatorname{mon}\mathscr{P}(^m\ell_r)$ contains the sequence space ℓ_q where $q = (mr')'$. Moreover, we show that for any $q \leq s < \infty$, the Lorentz sequence space $\ell_{q,s}$ lies in $\operatorname{mon}\mathscr{P}(^m\ell_r)$, provided that m is large enough. We apply our results to make an advance in the description of the set of monomial convergence of $H_\infty(B_{\ell_r})$ (the space of bounded holomorphic on the unit ball of ℓ_r). As a byproduct we close the gap on certain estimates related with the *mixed* unconditionality constant for spaces of polynomials over classical sequence spaces.

1. INTRODUCTION AND MAIN RESULTS

A basic fact taught on every course of one complex variable is that every function that is differentiable at all points of a disc centred at 0 can be represented as a power series, and vice-versa. In other words, the derivative $f'(z)$ exists (i.e. f is differentiable at z) for every $|z| < r$ if and only if there is a sequence of coefficients $(c_n(f))_n \subseteq \mathbb{C}$ so that

$$f(z) = \sum_{n=0}^{\infty} c_n(f) z^n \tag{1}$$

for every $|z| < r$ (i.e. it is analytic). In this case the coefficients can be computed either by differentiation or by the Cauchy integral formula, and the convergence is absolute and uniform on each compact subset of the disc. It also rather elementary to see that in fact this extends also to functions on several complex variables: a function defined on a Reinhardt domain $\mathscr{R} \subset \mathbb{C}^n$ (all needed definitions in this introduction can be found in Section 2), is differentiable at every $z \in \mathscr{R}$ if and only it is analytic (and has a power series representation as in (1)). So, differentiability and analiticity are two equivalent ways to define holomorphy in one and several complex variables.

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The idea of developing a sort of function theory in infinitely many variables (or, to put in nowadays terms, on infinite dimensional spaces) started at the beginning of the 20th century with the work, among others, of Hilbert, Fréchet and Gâteaux. Here the problem becomes much more subtle. To begin with, while a notion such as differentiability can be considered for functions on any Banach space the idea of analiticiy, where one needs power expansions with monomials of the form $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$, is much more restrictive. A Schauder basis, where an idea of 'coordinate' makes sense, is at least needed. This shows that the approach to holomorphy through differentiability is much more far reaching than the one through analiticity. We say, then, that a function $f : U \to \mathbb{C}$ (where U is some open subset of a Banach space X) is holomorphic if it is Fréchet differentiable at every point of U (or, equivalently, continuous and holomorphic when restricted to any one-dimensional affine subspace, see [Muj86, Din99]).

It is also worthy to explore the analytic approach whenever it makes sense (as, for example Banach sequences spaces, the definition is given below). Let us succinctly explain how this works (a detailed account on this can be found in [DGMSP19, Chapter 15]). Let f be a holomorphic function on some Reinhardt domain $\mathscr{R}$ in a Banach sequence space X. For each fixed n, the restriction of f to $\mathscr{R}_n = \mathscr{R}\cap\mathbb{C}^n$ (which is a Reinhardt domain) is holomorphic and, therefore, has a monomial expansion with coefficients $(c_\alpha^{(n)}(f))_{\alpha\in\mathbb{N}_0^n}$. It is easy to check that $c_\alpha^{(n)} = c_\alpha^{(n+1)}$ for $\alpha\in\mathbb{N}_0^n\subset\mathbb{N}_0^{n+1}$. In other words, we have a a unique family $(c_\alpha(f))_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$, such that

$$f(z) = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha \tag{2}$$

for all $n\in\mathbb{N}$ and all $z\in\mathscr{R}_n$. The coefficients can be computed, for each $\alpha = (\alpha_1,\dots,\alpha_n,0,0,\dots)$, by

$$c_\alpha(f) = \frac{\partial^\alpha f(0)}{\alpha!} = \frac{1}{(2\pi i)^n}\int_{\{|z|=r\}}\frac{f(z)}{z^{\alpha+1}}dz, \tag{3}$$

where $r>0$ such that $\{|z|\le r\}\subset\mathscr{R}$. As usual, the power series $\sum_\alpha c_\alpha z^\alpha$ is called the monomial expansion of f.

One could expect that in the settings where these two approaches coexist they are equivalent, just as in the finite dimensional setting. But this is not the case. When dealing with a totally different problem, related to the convergence of Dirichlet series, Toeplitz gave in [Toe13] an example that, to what we are concerned here, provided a holomorphic function on c_0 and a point in c_0 for which the monomial expansion does not converge absolutely. This shows that there are holomorphic functions that are not analytic (the converse, however, holds true: every analytic function is holomorphic).

Then the question arises in a natural way: for which z's does the monomial expansion of every holomorphic function converge absolutely? (note that when this is the case when the series converges to $f(z)$). From (2) we have that this happens for every $z\in\mathscr{R}_n$ but, are there other ones? Ryan showed in [Rya80] that the monomial expansion of every holomorphic function on ℓ_1 converges at every $z\in\ell_1$. Later Lempert in [Lem99] proved that the monomial expansion of every holomorphic function on

ρB_{ℓ_1} (for $\rho > 0$) converges at every $z \in \rho B_{\ell_1}$. This is a somewhat extremal case, where the analytic and differential approaches coincide. What happens in other spaces? or if we consider smaller families of holomorphic functions? To tackle this questions the set of monomial convergence of a family $\mathscr{F}(\mathscr{R})$ of holomorphic functions on $\mathscr{R}$ was defined in [DMP09] as

$$\operatorname{mon}\mathscr{F}(\mathscr{R}) = \left\{ z \in \mathbb{C}^{\mathbb{N}} \ : \ \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \left| c_\alpha(f) z^\alpha \right| < \infty \ \text{ for all } f \in \mathscr{F}(\mathscr{R}) \right\},$$

and a systematic study was started. We are mostly interested in studying the set of monomial convergence of the following three families:

- $H_b(\ell_r)$ (the space of holomorphic functions of bounded type on ℓ_r)
- $H_\infty(B_{\ell_r})$ (the space of bounded holomorphic functions on the open unit ball of ℓ_r)
- $\mathscr{P}(^m\ell_r)$ (the space of m-homogeneous polynomials on ℓ_r).

The results of Ryan and Lempert mentioned before imply $\operatorname{mon} H_b(\ell_1) = \operatorname{mon}\mathscr{P}(^m\ell_1) = \ell_1$ for every m and $\operatorname{mon} H_\infty(B_{\ell_1}) = B_{\ell_1}$. On the other endpoint of the scale ($p = \infty$) [BDF$^+$17] gives a precise description of $\operatorname{mon}\mathscr{P}(^m\ell_\infty)$ as $\ell_{\frac{m-1}{2m},\infty}$ and lower and upper inclusions for $\operatorname{mon} H_\infty(B_{\ell_\infty})$ that, although not optimal, are pretty tight. The study for $1 < r < \infty$ was started in [DMP09] and continued in [BDS], where several interesting results in this direction for polynomials and bounded holomorphic functions were obtained. To our best knowledge, nothing has been done so far to describe the set of monomial convergence of the holomorphic functions of bounded type. In this note we make progress towards the description of these set of monomial convergence in the case $1 < r \le 2$.

In Theorem 4.1 we provide a complete characterization of the set of monomial convergence of the space of holomorphic functions of bounded type for $1 < r \le 2$ as

$$\operatorname{mon} H_b(\ell_r) = \left\{ z \in \mathbb{C}^{\mathbb{N}} : \sup_{n \ge 1} \frac{\sum_{l=1}^n z_l^*}{\log(n+1)^{1-\frac{1}{r}}} < \infty \right\}.$$

The proof is given in Section 4 and the main tool developed is a decomposition of the multi-indices (in an even and a pure tetrahedral part), which allows us to split the monomial expansion in different pieces, for which we are able to find proper bounds.

Regarding set of monomial convergence of bounded holomorphic functions on B_{ℓ_r} is considered, there are a number of deep results (see [DMP09, Example 4.9 (1)(a)]) that in the case we are dealing with here ($1 < r \le 2$) imply

$$B_{\ell_r} \cap \ell_1 \subsetneq \operatorname{mon} H_\infty(B_{\ell_r}) \subseteq B_{\ell_r} \cap \ell_{1+\varepsilon} \text{ for every } \varepsilon > 0. \tag{4}$$

We give here some upper and lower inclusions, in the spirit of the ones obtained for $H_\infty(B_{\ell_\infty})$. We show in Theorem 5.1 that

$$\left\{z \in \mathbb{C}^{\mathbb{N}} \colon 2eC_r \left(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}}\right)^r + \|z\|_{\ell_r}^r < 1\right\} \subset \operatorname{mon} H_\infty(B_{\ell_r}) \subset \left\{z \in B_{\ell_r} \colon \limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \leq 1\right\},$$

where $0 < C_r \leq \left(\sum_{k=1}^{\infty} \frac{\log(k+1)^{r-1}}{k^r}\right)^{1/r}$ and depends on the interplay between ℓ_r and the Marcinkiewicz sequence space m_{Ψ_r} (see Remark 4.4). Let us point out that this is connected with the question stated in [BDS, Remark 5.8]. We will see in Remark 5.5 that these lower and upper inclusions recover (4).

Regarding m-homogeneous polynomials we know from [BDS, Theorem 5.1] and [DMP09, Example 4.6] that $\ell_{q-\varepsilon} \subset \operatorname{mon}\mathscr{P}(^m\ell_r) \subset \ell_{q,\infty}$ for every $\varepsilon > 0$ (where $1 < r \leq 2$ and $q := (mr')'$). Using elementary methods we show in Theorem 6.3 that we can even take $\varepsilon = 0$ (this proves a conjecture made by Defant, Maestre and Prengel in [DMP09]). We go one step further, showing in Theorem 6.1 that

$$\ell_{q,\frac{m}{\log m}} \subset \operatorname{mon}\mathscr{P}(^m\ell_r)$$

for every $m \geq 5$ (we also give lower inclusions for $m \leq 4$). The proof is technically involved and uses interpolation of linear operators defined on cones. All this is presented in Section 6.

Finally, as a byproduct, in Section 7 we provide correct estimates of the asymptotic growth of the mixed-(p,q) unconditional constant (a notion by Defant, Maestre and Prengel in [DMP09, Section 5]) as n tends to infinity for every $1 \leq p, q \leq \infty$; closing the gap of the remaining cases of [GMMb].

2. Preliminaries

For every $x, y \in \mathbb{C}^{\mathbb{N}}$ we denote by $|x|$ the sequence $(|x_1|, |x_2|, \ldots, |x_n|, \ldots)$. If $|x_i| \leq |y_i|$ for every $i \in \mathbb{N}$ we write $|x| \leq |y|$. A Banach sequence space is a Banach space $(X, \|\cdot\|_X)$ such that $\ell_1 \subset X \subset \ell_\infty$ satisfying that, if $x \in \mathbb{C}^{\mathbb{N}}$ and $y \in X$ with $|x| \leq |y|$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$. A non-empty open set $\mathscr{R} \subset X$ is called a Reinhardt domain if given $x \in \mathbb{C}^{\mathbb{N}}$ and $y \in \mathscr{R}$ such that $|x| \leq |y|$ then $x \in \mathscr{R}$. Given a bounded sequence x its decreasing rearrangement x^* is the sequence defined as $x_n^* = \inf\{\sup_{j\in\mathbb{N}\setminus J} |x_j| : J \subset \mathbb{N}, \operatorname{card}(J) < n\}$. A Banach sequence space $(X, \|\cdot\|_X)$ is said to be symmetric if $x^* \in X$ whenever $x \in X$ and, moreover $\|x\|_X = \|x^*\|_X$. A set $A \subset X$ is symmetric if $x \in A$ if and only if $x^* \in A$. For every $x \in c_0$ there is some injective mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that $x_n^* = |x_{\sigma(n)}|$ for all $n \in \mathbb{N}$. We will say that a sequence $x \in \mathbb{C}^{\mathbb{N}}$ is decreasing whenever $|x|$ is decreasing.

We are going to deal basically with three classes of Banach sequence spaces: the classical Minkowski ℓ_r spaces, the Lorentz $\ell_{p,q}$ spaces and the Marcinkiewicz sequence spaces. Let us recall some definitions. For $1 \le p, q \le \infty$ the space $\ell_{p,q}$ consists of those sequences z for which (we use the convention $\frac{1}{\infty} = 0$)

$$\|z\|_{\ell_{p,q}} := \left\| \left(z_n^* n^{\frac{1}{p} - \frac{1}{q}} \right)_{n=1}^{\infty} \right\|_{\ell_q} < \infty.$$

Observe that in general this is a quasi-norm and only defines a norm for $1 \le q \le p \le \infty$. For $z \in \ell_{p,q}$ we define

$$\|z\|^*_{\ell_{p,q}} := \left(\sum_{n=1}^{\infty} n^{\frac{q}{p} - 1} \left(\frac{1}{n} \sum_{k=1}^{n} z_k^* \right)^q \right)^{1/q}.$$

It should be noted (see [BS88, Lemma 4.5]) that for $1 \le p, q \le \infty$ and $z \in \ell_{p,q}$, it holds

$$\|z\|_{\ell_{p,q}} \le \|z\|^*_{\ell_{p,q}} \le p' \|z\|_{\ell_{p,q}},$$

so we can always work with the quasi-norm $\|\cdot\|_{\ell_{p,q}}$ and treat $(\ell_{p,q}, \|\cdot\|_{\ell_{p,q}})$ as a Banach sequence space at the expense of p' (the conjugate exponent of p) as a price every time we do so . Let $\Psi = (\Psi(n))_{n=0}^{\infty}$ be an increasing sequence of nonnegative real numbers with $\Psi(0) = 0$ and $\Psi(n) > 0$ for every $n \in \mathbb{N}$. These functions are usually known as symbols. The Marcinkiewicz sequence space associated to the symbol Ψ, denoted by m_Ψ, is the vector space of all bounded sequences $(z_n)_n$ such that

$$\|z\|_{m_\Psi} := \sup_{n \ge 1} \frac{\sum_{k=1}^{n} z_k^*}{\Psi(n)} < \infty.$$

An m-homogeneous polynomial in n variables is a function P of the form

$$P(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_1 + \cdots + \alpha_n = m}} c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

Given $\alpha \in \mathbb{N}_0^n$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\Lambda(m,n) = \{\alpha \in \mathbb{N}_0^n \colon |\alpha| = m\}$. We also consider the set $\mathcal{J}(m,n) = \{\mathbf{j} = (j_1, \ldots, j_m) \in \mathbb{N}^m \colon 1 \le j_1 \le \cdots \le j_m \le n\}$. Each $\alpha \in \Lambda(m,n)$ defines $\mathbf{j}_\alpha = (1, \overset{\alpha_1}{\ldots}, 1, 2, \overset{\alpha_2}{\ldots}, 2, \ldots, n, \overset{\alpha_n}{\ldots}, n) \in \mathcal{J}(m,n)$. Conversely, each $\mathbf{j} \in \mathcal{J}(m,n)$ defines $\alpha \in \Lambda(m,n)$ by $\alpha_k = \operatorname{card}\{i \colon j_i = k\}$. In this way these two indexing sets are injective and, denoting $z_1^{\alpha_1} \cdots z_n^{\alpha_n} = z^\alpha$ and $z_{j_1} \cdots z_{j_m} = z_{\mathbf{j}}$ we can write each homogeneous polynomial in two alternative ways

$$P(z) = \sum_{\alpha \in \Lambda(m,n)} c_\alpha z^\alpha = \sum_{1 \le j_1 \le \cdots \le j_m \le n} c_{j_1, \ldots, j_m} z_{j_1} \cdots z_{j_m} = \sum_{\mathbf{j} \in \mathcal{J}(m,n)} c_{\mathbf{j}} z_{\mathbf{j}}. \tag{5}$$

We will freely change from the α to the $\mathbf{j}$ notation whenever it is more convenient (always assuming that α and $\mathbf{j}$ are related to each other). We write

$$|\mathbf{j}| = \operatorname{card}\{\mathbf{i} \in \mathbb{N}^m \colon \text{ there exists a permutation } \sigma \text{ of } 1, \ldots, m \text{ so that } i_{\sigma(k)} = j_k \text{ for all } k\}.$$

Note that if $\mathbf{j}$ and α are associated to each other, then

$$|\mathbf{j}| = \frac{m!}{\alpha_1! \cdots \alpha_n!} = \frac{m!}{\alpha!}. \tag{6}$$

We will sometimes denote this by $|[\alpha]|$. We write $\mathscr{P}(^m\mathbb{C}^n)$ for the space of all m-homogeneous polynomials in n variables. Each ℓ_r-norm on $\mathbb{C}^n$ induces a different (though all equivalent) norm $\|P\|_{\mathscr{P}(^m\ell_r^n)} = \sup_{\|z\|_r \le 1} |P(z)|$.

We follow the theory of holomorphic functions on arbitrary Banach spaces as presented in [Muj86, Din99]. If X is a (finite or infinite dimensional) Banach space, a function $P : X \to \mathbb{C}$ is a (continuous) m-homogeneous polynomial if there exists a (unique) continuous symmetric m-linear form (denoted by $\check{P}$) on X such that $P(x) = \check{P}(x, \dots, x)$ for every x. A function $f : U \to \mathbb{C}$ (where U is some open subset of a Banach space X) is holomorphic if it is Fréchet differentiable at every point of U. If U is balanced there are $P_m(f)$ for $m = 0, 1, 2, \dots$, each an m-homogeneous polynomial on X, such that $f = \sum_m P_m(f)$ uniformly on U. The space of all holomorphic functions on U is denoted by $H(U)$. The space of bounded holomorphic functions on B_X (the open unit ball of X) with the norm $\|f\| = \sup_{\|x\| \le 1} |f(x)|$ is denoted by $H_\infty(B_X)$. The space of m-homogeneous polynomials on X is denoted by $\mathscr{P}(^m X)$, and is endowed with the norm $\|P\| = \sup_{\|x\| \le 1} |P(x)|$. Every homogeneous polynomial is entire (holomorphic on X) and, then, its coefficients can be computed through (3). Let us note that $c_\alpha(P) \neq 0$ only if $|\alpha| = m$ and that, if $\mathbf{j} \in \mathscr{J}(m, n)$ is associated to α, then

$$c_\alpha(P) = \frac{m!}{\alpha!} \check{P}(e_{j_1}, \dots, e_{j_m}).$$

An entire function is said to be of bounded type if it is bounded on every bounded set of X. The space of entire functions of bounded type is denoted by $H_b(X)$. It is a Fréchet space with the family of seminorms defined by $p_n(f) = \sup_{\|x\| \le n} |f(x)|$.

We denote by $\mathbb{N}_0^{(\mathbb{N})}$ the set of eventually zero multi-indices. In other words, $\mathbb{N}_0^{(\mathbb{N})} = \bigcup_{n=1}^\infty \mathbb{N}_0^n \times \{0\}$. From now on we will identify $\mathbb{N}_0^n \times \{0\}$ with $\mathbb{N}_0^n$ without further notice.

3. Rearrangement families of holomorphic functions.

A very useful tool in the study of sets monomial convergence (see [BDF+17]) is that usually, a sequence belongs to the set of monomial convergence if and only if its decreasing rearrangement does (see also [DGMPG08]). We isolate this property, and say in this case that $\mathscr{F} \subset H(\mathscr{R})$ is a *rearrangement family* (where $\mathscr{R}$ is a Reinhardt domain in a Banach sequence space X). In [BDF+17] it was proved that $H_\infty(B_{c_0})$ and $\mathscr{P}(^m c_0)$ are rearrangement families. The fact that this is also the case for ℓ_r for $1 \le r < \infty$ is implicitly used in [BDS]. Our aim now is to find other rearrangement families of holomorphic functions (compare this with [Sch15, Chapter 7] where similar results appear).

To this purpose we introduce another concept. We say a family $\mathscr{F} \subset H(\mathscr{R})$ is *linearly balanced* if $f \circ T|_{\mathscr{R}} \in \mathscr{F}$ for every $f \in \mathscr{F}$ and $T : X \to X$ linear with $\|T\| = 1$ and $T(\mathscr{R}) \subset \mathscr{R}$.

Remark 3.1. Rather straightforward arguments show that $H_b(X)$, $\mathscr{A}_u(B_X)$ (all uniformly continuous and holomorphic functions on B_X), $H_\infty(B_X)$ and $\mathscr{P}(^m X)$ for every $m \geq 2$ are linearly balanced families.

Theorem 3.2. *Let $\mathscr{R}$ be a symmetric Reinhardt domain of a symmetric Banach sequence space X and $\mathscr{F} \subset H(\mathscr{R})$ a linearly balanced family such that* $\operatorname{mon}\mathscr{F} \subset c_0$, *then $\mathscr{F}$ is a rearrangement family.*

We give a series of preliminary results needed for the proof of Theorem 3.2. Given an injective mapping $\sigma : \mathbb{N} \to \mathbb{N}$ we define two mappings in the following way. First

$$\begin{aligned} T_\sigma : \mathbb{C}^{\mathbb{N}} &\to \mathbb{C}^{\mathbb{N}} \\ x &\mapsto (x_{\sigma(k)})_{k\in\mathbb{N}}. \end{aligned} \tag{7}$$

Second, $S_\sigma : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is defined for $x \in \mathbb{C}^{\mathbb{N}}$ by

$$(S_\sigma x)_k = \begin{cases} 0 & \text{if } k \notin \sigma(\mathbb{N}) \\ x_{\sigma^{-1}(k)} & \text{if } k \in \sigma(\mathbb{N}). \end{cases} \tag{8}$$

Both are clearly linear and $T_\sigma(S_\sigma x) = x$ for every x.

Remark 3.3. Let us see now how these two mappings behave with the decreasing rearrangement of a bounded sequence x. Fixed $n \in \mathbb{N}$ and $J \subset \mathbb{N}$ such that $\operatorname{card}(J) < n$ we have

$$\sup_{\sigma(j)\in\mathbb{N}\setminus J} |x_{\sigma(j)}| = \sup_{j\in(\mathbb{N}\setminus J)\cap\sigma(\mathbb{N})} |x_j| \leq \sup_{j\in\mathbb{N}\setminus J} |x_j|.$$

Thus

$$\big(T_\sigma(x)\big)^*_n = \inf\{\sup_{\sigma(j)\in\mathbb{N}\setminus J} |x_{\sigma(j)}| : J \subset \mathbb{N}, \operatorname{card}(J) < n\} \leq \inf\{\sup_{j\in\mathbb{N}\setminus J} |x_j| : J \subset \mathbb{N}, \operatorname{card}(J) < n\} = x^*_n.$$

That is, $T_\sigma(x)^* \leq x^*$. A similar argument shows that $(S_\sigma x)^* = x^*$.

The following lemma shows that the restrictions of S_σ and T_σ to symmetric Banach sequence spaces are endomorphisms of norm 1.

Lemma 3.4. *Let X be a symmetric Banach sequence space and $\sigma : \mathbb{N} \to \mathbb{N}$ an injective mapping. Then $T_\sigma, S_\sigma : X \to X$ defined by (7) and (8) respectively are well defined, $\|T_\sigma\| = 1$ and S_σ is an isometry.*

Proof. Remark 3.3 together with the symmetry of the space imply that both operators are well defined, that S_σ is an isometry and $\|T_\sigma\| \leq 1$. The fact that $\|T_\sigma\| = 1$ follows from the equality $T_\sigma(S_\sigma x_0) = x_0$. □

Now we are able to give the proof of Theorem 3.2.

Proof of Theorem 3.2. To begin with we take $z \in \operatorname{mon}\mathscr{F}$ and see that $z^* \in \operatorname{mon}\mathscr{F}$. As $\operatorname{mon}\mathscr{F} \subset c_0$ there is some injective mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that $z^*_k = |z_{\sigma(k)}|$ for every $k \in \mathbb{N}$. Observe that $|T_\sigma(z)| = z^*$. We take $f \in \mathscr{F}$, then $f \circ T_\sigma$ also belongs to $\mathscr{F}$ and what we want to see first is that, if $\alpha(\sigma) \in \mathbb{N}_0^{(\mathbb{N})}$ denotes the multi-index that fulfils $T_\sigma(z)^\alpha = z^{\alpha(\sigma)}$, then

$$c_\alpha(f) = c_{\alpha(\sigma)}(f \circ T_\sigma) \tag{9}$$

for every α. Take, then, some $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and set $N = \max\{k\colon \alpha_k \neq 0\}$. On one hand we have

$$(f \circ T_\sigma)(w) = \sum_{\beta \in \mathbb{N}_0^N} c_\beta(f \circ T_\sigma) w^\beta,$$

for all $w \in \mathbb{C}^N \cap \mathscr{R}$. Define $M = \max\{\sigma(k)\colon k = 1, \ldots, N\}$ and note that $T_\sigma(w) \in \mathbb{C}^M \cap \mathscr{R}$. Thus

$$(f \circ T_\sigma)(w) = f(T_\sigma(w)) = \sum_{\gamma \in \mathbb{N}_0^N} c_\gamma(f) T_\sigma(w)^\gamma = \sum_{\gamma \in \mathbb{N}_0^N} c_\gamma(f) w^{\gamma(\sigma)}.$$

The uniqueness of the Taylor coefficients gives (9). Once we have this we obtain (recall that $f \circ T_\sigma \in \mathscr{F}$ and $z \in \operatorname{mon}\mathscr{F}$)

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)(z^*)^\alpha| = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)||(T_\sigma(z))^\alpha| = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_{\alpha(\sigma)}(f \circ T_\sigma)||z^{\alpha(\sigma)}| \leq \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f \circ T_\sigma) z^\alpha| < \infty,$$

which proves our claim.

For the converse, suppose $z^* \in \operatorname{mon}\mathscr{F}$. Again, as $\operatorname{mon}\mathscr{F} \subset c_0$, there is some injective mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that $z^*_k = |z_{\sigma(k)}|$ for every $k \in \mathbb{N}$. Now it will be useful to notice $|z| = S_\sigma(z^*)$. Given $f \in \mathscr{F}$ we have

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f) z^\alpha| = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)||(S_\sigma(z^*))^\alpha|. \tag{10}$$

Besides,

$$\sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f \circ S_\sigma) w^\alpha = f(S_\sigma(w)) = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f) S_\sigma(w)^\alpha = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f) S_\sigma(w)^\alpha.$$

Observe that for $\alpha \in \mathbb{N}^{(\mathbb{N})}$, if there is $k \in \mathbb{N} \setminus \sigma(\mathbb{N})$ such that $\alpha_k \neq 0$ then $S_\sigma(w)^\alpha = 0$, otherwise we define $\alpha(\sigma^{-1}) \in \mathbb{N}^{(\mathbb{N})}$ as the only multi-index which fulfils $S_\sigma(w)^\alpha = w^{\alpha(\sigma^{-1})}$. By the uniqueness of the coefficients of the Taylor expansion for $f \circ S_\sigma : \mathbb{C}^N \to \mathbb{C}$ it follows

$$c_\alpha(f) S_\sigma(z^*)^\alpha = \begin{cases} 0 & \text{if there is } k \notin \sigma(\mathbb{N}) \text{ such that } \alpha_k \neq 0 \\ c_{\alpha(\sigma^{-1})}(f \circ S_\sigma)(z^*)^{\alpha(\sigma^{-1})} & \text{otherwise,} \end{cases}$$

then

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f) z^\alpha| &= \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)||(S_\sigma(z^*))^\alpha| \\ &= \sum_{\alpha \in (\sigma(\mathbb{N}) \cup \{0\})^{(\mathbb{N})}} \left| c_{\alpha(\sigma^{-1})}(f \circ S_\sigma)(z^*)^{\alpha(\sigma^{-1})} \right| \leq \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f \circ S_\sigma)||(z^*)^\alpha| < \infty, \end{aligned} \tag{11}$$

as we wanted. ☐

Remark 3.5. Let $\mathscr{R}$ be a symmetric Reinhardt domain in a Banach sequence space X and consider a family of homolorphic functions $\mathscr{F} \subset H(\mathscr{R})$ such that for some $m \geq 2$ the space $\mathscr{P}(^m X)$ lies inside $\mathscr{F}$. Then, as $X \subset \ell_\infty$ continuously we have $\mathscr{P}(^m \ell_\infty) \subset \mathscr{P}(^m X) \subset \mathscr{F}$. With this, [BDF+17, Theorem 2.1] yields

$$\operatorname{mon} \mathscr{F} \subset \operatorname{mon} \mathscr{P}(^m \ell_\infty) = \ell_{\frac{2m}{m-1}, \infty} \subset c_0.$$

Corollary 3.6. *For every symmetric Banach sequence space X the families of holomorphic functions $H_b(X), \mathscr{A}_u(B_X), H_\infty(B_X)$ and $\mathscr{P}(^m X)$ with $m \geq 2$ are rearrangement families.*

Proof. Each of these families satisfies the condition in Remark 3.5. Then Remark 3.1 and Theorem 3.2 give the conclusion. ☐

Remark 3.7. As we have already pointed out, we are mainly interested in $H_\infty(B_{\ell_r})$, $H_b(\ell_r)$ and $\mathscr{P}(^m \ell_r)$. The set of monomial convergence of each one of these spaces is, by Remark 3.5 contained in c_0. But, as matter of fact, we can say more. By [DGMSP19, Proposition 20.3] we have $\operatorname{mon} H_\infty(B_{\ell_r}) \subseteq B_{\ell_r}$. Noting that every functional $f \in \ell_r^*$ belongs to $H_b(\ell_r)$ and using the definition of the set of monomial convergence we have $\operatorname{mon} H_b(\ell_r) \subseteq \ell_r$. Finally, exactly the same argument as in [DGMSP19, Remark 10.7] shows that $\operatorname{mon} \mathscr{P}(^m \ell_r) \subseteq \mathscr{P}(^1 \ell_r) = \operatorname{mon} \ell_r^* = \ell_r$.

4. Monomial convergence for holomorphic functions of bounded type on ℓ_r

We can now describe the set of monomial convergence of $H_b(\ell_r)$ for $1 < r \leq 2$. It happens to be a Marcinkiewicz space m_{Ψ_r} where the symbol is given by

$$\Psi_r(n) := \log(n+1)^{1-\frac{1}{r}},$$

for $n \in \mathbb{N}_0$.

Theorem 4.1. *For* $1 < r \leq 2$,

$$\operatorname{mon} H_b(\ell_r) = m_{\Psi_r} := \left\{ z \in \mathbb{C}^{\mathbb{N}} : \sup_{n \geq 1} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-\frac{1}{r}}} < \infty \right\}.$$

We handle the upper and the lower inclusions separately in the following two sections.

4.1. **The upper inclusion** $\operatorname{mon} H_b(\ell_r) \subset m_{\Psi_r}$**.** Typically, the way to prove upper inclusions for a set of monomial convergence goes through providing polynomials satisfying certain convenient properties. Over the last years probabilistic techniques have shown to be extremely helpful to find such polynomials. This is, for instance, what is done in [BDF+17, Theorem 2.2], where the probabilistic device is the well known Kahane-Salem-Zygmund inequality. Here we follow essentially the same lines, replacing the polynomials provided by this inequality by other ones. Following techniques of Boas and Bayart (see [Boa00], [Bay12] and also [DGMSP19, Corollary 17.6]) for every $1 \leq r \leq 2$ there is a constant $C_r > 0$ such that for all n and $m \geq 2$ we can find a choice of signs $(\varepsilon_\alpha)_\alpha$ so that

$$\sup_{\|z\|_r < 1} \Big| \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \Big| \leq C_r (\log(m) m!)^{1-\frac{1}{r}} \, n^{1-\frac{1}{r}}. \tag{12}$$

These polynomials are the main tool for the proof of the upper inclusion. We also need the following result, an extension of [DMP09, Lemma 4.1] whose proof follows the same lines.

Lemma 4.2. *Let* $\mathscr{R}$ *be a Reinhardt domain in a Banach sequence space* X *and let* $(\mathscr{F}, (q_n)_n)$ *be a Fréchet space of holomorphic functions continuously included in* $H_b(\mathscr{R})$*. Then, for each* $z \in \operatorname{mon}(\mathscr{F})$*, there exist* $C > 0$ *and* n *such that*

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha z^\alpha| \leq C q_n(f).$$

for every $f \in \mathscr{F}$*. In particular, if* $z \in \operatorname{mon} H_b(X)$*, there exists* $C > 0$*, such that*

$$\sum_{\alpha \in \Lambda(m,n)} |c_\alpha(P) z^\alpha| \leq C^m \|P\|_{\mathscr{P}(^m X)},$$

for every $P \in \mathscr{P}(^m X)$*.*

We have now everything at hand to proceed with the proof of the upper inclusion.

Proof of the upper inclusion in Theorem 4.1. Fix $1 < r \leq 2$ and choose $z \in \operatorname{mon} H_b(\ell_r)$. Now fix n, m, choose signs as in (12) and define the polynomial $P(w) := \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} w^\alpha$. By Corollary 3.6 we know that $z^* \in \operatorname{mon} H_b(\ell_r)$. Using first the multinomial formula, then Lemma 4.2 and finally (12) we have

$$\left(\sum_{j=1}^{n}|z_j^*|\right)^m = \sum_{\alpha\in\Lambda(m,n)}\frac{m!}{\alpha!}|(z^*)^\alpha| = \sum_{\alpha\in\Lambda(m,n)}\left|\varepsilon_\alpha\frac{m!}{\alpha!}(z^*)^\alpha\right| \le C_{z^*}^m \sup_{u\in B_{\ell_r^n}}\left|\sum_{\alpha\in\Lambda(m,n)}\varepsilon_\alpha\frac{m!}{\alpha!}u^\alpha\right|_{\mathscr{P}(^m\ell_r^n)} \le C_{z^*,r}^m(\log(m)m!n)^{1-\frac{1}{r}}. \tag{13}$$

Taking the power $1/m$ and using Stirling's formula ($m!\le\sqrt{2\pi m}e^{\frac{1}{12m}}m^me^{-m}$) yield

$$\sum_{j=1}^{n}|z_j^*|\le C_{z^*,r}\left[\log(m)^{\frac{1}{m}}(2\pi m)^{\frac{1}{2m}}e^{\frac{1}{12m^2}}\frac{m}{e}n^{\frac{1}{m}}\right]^{1-\frac{1}{r}}. \tag{14}$$

Finally, choosing $m=\lfloor\log(n+1)\rfloor$ gives that the term $\frac{1}{\log(n+1)^{1-\frac{1}{r}}}\sum_{k=1}^{n}|z_n^*|$ (for every $n\ge2$) is bounded independently of n, so $z\in m_{\Psi_r}$. □

4.2. **The lower inclusion $m_{\Psi_r}\subset\operatorname{mon}H_b(\ell_r)$.** We face now the proof of the lower inclusion in Theorem 4.1. The main tool is the following result, the proof of which requires some work, that we perform all along this section.

Theorem 4.3. *Fix $1<r\le2$. For every $\varepsilon>0$ there is $C_r=C_r(\varepsilon)>0$ such that for every $m,n\in\mathbb{N}$, every m-homogeneous polynomial in n complex variables P and every $z\in\mathbb{C}^n$, we have*

$$\sum_{\mathbf{j}\in\mathscr{J}(m,n)}|c_{\mathbf{j}}(P)z_{\mathbf{j}}^*|\le C_r(\varepsilon)m^{2+\frac{1}{r}}((1+\varepsilon)2e)^{\frac{m}{r}}\|\operatorname{id}:m_{\Psi_r}\to\ell_r\|^m\|z\|_{m_{\Psi_r}}^m\|P\|_{\mathscr{P}(^m\ell_r^n)}.$$

Before we start with the proof of this result, let us see how, having it at hand, we can prove the lower inclusion we are aiming at.

Proof of the lower inclusion in Theorem 4.1. Choose $z\in m_{\Psi_r}$ and let us see that $z\in\operatorname{mon}H_b(\ell_r)$. By Corollary 3.6 we may assume without loss of generality $z=z^*$. Given $f\in H_b(\ell_r)$ (recall that we denote $P_m(f)$ for the m-homogeneous part of the Taylor expansion) and Theorem 4.3 (with $\varepsilon=1$) gives

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}|c_\alpha(f)z^\alpha| &= \sup_{n\in\mathbb{N}}\sum_{m=0}^{\infty}\sum_{\mathbf{j}\in\mathscr{J}(m,n)}|c_{\mathbf{j}}(f)z_{\mathbf{j}}|\\ &\le \sup_{n\in\mathbb{N}}\sum_{m=0}^{\infty}C_rm^{2+\frac{1}{r}}(4e)^{\frac{m}{r}}\|\operatorname{id}\|^m\|z\|_{m_{\Psi_r}}^m\sup_{u\in B_{\ell_r^n}}\left|\sum_{\mathbf{j}\in\mathscr{J}(m,n)}c_{\mathbf{j}}(f)u_{\mathbf{j}}\right|\\ &= C_r\sum_{m=0}^{\infty}(m^{(2+\frac{1}{r})\frac{1}{m}}(4e)^{\frac{1}{r}}\|\operatorname{id}\|\|z\|_{m_{\Psi_r}})^m\|P_m(f)\|_{\mathscr{P}(^m\ell_r)}.\end{aligned}$$

Let us see that this sum is finite. Take $R > \sup_m \left(m^{(2+\frac{1}{r})\frac{1}{m}}(4e)^{\frac{1}{r}}\|\operatorname{id}\|\|z\|_{m_{\Psi_r}}\right)$, then by the homogeneity of $P_m(f)$

$$\sum_{m=0}^{\infty}(m^{(2+\frac{1}{r})\frac{1}{m}}(4e)^{\frac{1}{r}}\|\operatorname{id}\|\|z\|_{m_{\Psi_r}})^m\|P_m(f)\|_{\mathscr{P}(^m\ell_r)}$$
$$= \sum_{m=0}^{\infty}\left(\frac{m^{(2+\frac{1}{r})\frac{1}{m}}(4e)^{\frac{1}{r}}\|\operatorname{id}\|\|z\|_{m_{\Psi_r}}}{R}\right)^m \sup_{w\in R\cdot B_{\ell_r}}|P_m(f)(w)|$$
$$\leq \sum_{m=0}^{\infty}\left(\frac{m^{(2+\frac{1}{r})\frac{1}{m}}(4e)^{\frac{1}{r}}\|\operatorname{id}\|\|z\|_{m_{\Psi_r}}}{R}\right)^m \sup_{w\in R\cdot B_{\ell_r}}|f(w)| < \infty,$$

where the last step is due to Cauchy's inequality. This completes the proof. □

We start now the way to the proof of Theorem 4.3. We begin with a simple remark.

Remark 4.4. If $z \in m_{\Psi_r}$, then

$$n|z_n^*| \leq \sum_{l=1}^{n} z_l^* \leq \|z\|_{m_{\Psi_r}} \log(n+1)^{\frac{1}{r'}}.$$

That is

$$|z_n^*| \leq \|z\|_{m_{\Psi_r}} \frac{\log(n+1)^{\frac{1}{r'}}}{n}$$

for every $n \in \mathbb{N}$. This gives

$$\sum_{j=1}^{n}|z_j|^r \leq \sum_{j=1}^{n}|z_j^*|^r \leq \|z\|_{m_{\Psi_r}}^r \sum_{j=1}^{n}\frac{\log(j+1)^{\frac{r}{r'}}}{j^r}.$$

This implies $\|\operatorname{id} : m_{\Psi_r} \to \ell_r\| \leq \left(\sum_{j=1}^{\infty}\frac{\log(j+1)^{\frac{r}{r'}}}{j^r}\right)^{1/r}$ (note that this series is convergent for $1 < r$).

Our first ingredient is the following lemma, that follows with a careful analysis of the proof of [BDS, Lemma 3.5], that relates the summability of certain coefficients of a polynomial and its uniform norm in ℓ_r^n. It has been very useful to provide a proof 'at an elementary level' (in the sense that it does not require tools from the local theory of Banach space) of the asymptotic growth of the unconditional constant of the space of m-homogeneous polynomials on ℓ_r^n as n goes to infinite with suitable care on the dependence of m (in fact this has been proved for general index sets, see [BDS, Theorem 3.2]). As a consequence the behaviour of the Bohr radii of holomorphic functions on ℓ_r for $1 \leq r \leq 2$ has been described in [BDS, Theorem 3.9]. It has recently been used also to study the asymptotic growth of the mixed Bohr radii in [GMMa]. In some sense, for $1 \leq r \leq 2$, it plays the role of the Bohnenblust–Hille inequality for the case $r = \infty$.

Lemma 4.5. *Let $1 \le r \le \infty$ and P be an m-homogeneous polynomial in n variables. Then for each $\mathbf{i} \in \mathcal{J}(m-1,n)$ with associated multi-index $\alpha(\mathbf{i}) \in \Lambda(m-1,n)$ we have*

$$\left(\sum_{k=j_{m-1}}^{n} |c_{(\mathbf{i},k)}(P)|^{r'}\right)^{\frac{1}{r'}} \le em\Big(\frac{(m-1)^{m-1}}{\alpha(\mathbf{i})^{\alpha(\mathbf{i})}}\Big)^{\frac{1}{r}} \|P\|_{\mathscr{P}(^m\ell_r^n)}. \tag{15}$$

Since $\frac{(m-1)^{m-1}}{\alpha(\mathbf{i})^{\alpha(\mathbf{i})}} \le e^{m-1}|\mathbf{i}|$ we immediately have

$$\left(\sum_{k=j_{m-1}}^{n} |c_{(\mathbf{i},k)}(P)|^{r'}\right)^{\frac{1}{r'}} \le me^{1+\frac{m-1}{r}}|\mathbf{i}|^{\frac{1}{r}} \|P\|_{\mathscr{P}(^m\ell_r^n)}. \tag{16}$$

This is in fact the statement of [BDS, Lemma 3.5.]. With it we can give the first step towards the proof of Theorem 4.3.

Lemma 4.6. *Let $1 < r \le 2$, there is $A_r > 0$ such that for every $m,n \in \mathbb{N}$, every $P \in \mathscr{P}(^m\mathbb{C}^n)$ and every decreasing $z \in \mathbb{C}^n$ we have*

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| \le A_r m^{1+\frac{1}{r}} e^{\frac{m}{r}} \|z\|^2_{m_{\Psi_r}} \left(\sum_{k=1}^{n} \frac{\log(k+1)^{\frac{2}{r'}}}{k^{1+\frac{1}{r'}}} \sum_{\mathbf{i}\in\mathcal{J}(m-2,k)} |z_{\mathbf{i}}||\mathbf{i}|^{\frac{1}{r}}\right) \|P\|_{\mathscr{P}(^m\ell_r^n)}.$$

Proof. Consider $P \in \mathscr{P}(^m\mathbb{C}^n)$ as in (5) and $z \in \mathbb{C}^n$ decreasing. Using first Hölder's inequality and then (16) we have

(17)
$$\begin{aligned}
\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| &= \sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} \sum_{j_m=j_{m-1}}^{n} |c_{(\mathbf{j},j_m)}(P)z_{\mathbf{j}}z_{j_m}| \\
&\le \sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |z_{\mathbf{j}}|\Big(\sum_{j_m=j_{m-1}}^{n} |c_{(\mathbf{j},j_m)}(P)|^{r'}\Big)^{\frac{1}{r'}}\Big(\sum_{j_m=j_{m-1}}^{n} |z_{j_m}|^r\Big)^{\frac{1}{r}} \\
&\le e^{1-\frac{1}{r}} me^{\frac{m}{r}} \|P\|_{\mathscr{P}(^m\ell_r)} \sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |z_{\mathbf{j}}||\mathbf{j}|^{\frac{1}{r}}\Big(\sum_{j_m=j_{m-1}}^{n} |z_{j_m}|^r\Big)^{\frac{1}{r}} \\
&= e^{1-\frac{1}{r}} me^{\frac{m}{r}} \|P\|_{\mathscr{P}(^m\ell_r)} \sum_{j_{m-1}=1}^{n} |z_{j_{m-1}}| \sum_{\mathbf{i}\in\mathcal{J}(m-2,j_{m-1})} |z_{\mathbf{i}}||(\mathbf{i},j_{m-1})|^{\frac{1}{r}}\Big(\sum_{j_m=j_{m-1}}^{n} |z_{j_m}|^r\Big)^{\frac{1}{r}} \\
&\le e^{1-\frac{1}{r}} me^{\frac{m}{r}} \|P\|_{\mathscr{P}(^m\ell_r)}(m-1)^{\frac{1}{r}} \sum_{j_{m-1}=1}^{n} |z_{j_{m-1}}|\Big(\sum_{j_m=j_{m-1}}^{n} |z_{j_m}|^r\Big)^{\frac{1}{r}} \sum_{\mathbf{i}\in\mathcal{J}(m-2,j_{m-1})} |z_{\mathbf{i}}||\mathbf{i}|^{\frac{1}{r}},
\end{aligned}$$

where the last inequality is due to the fact that $|(\mathbf{i},j_{m-1})| \le (m-1)|\mathbf{i}|$ for every $\mathbf{i} \in \mathcal{J}(m-2,j_{m-1})$.

We now bound the factor $|z_{j_{m-1}}|\Big(\sum_{j_m=j_{m-1}}^n |z_{j_m}|^r\Big)^{\frac{1}{r}}$. For each $1 \le j \le n$ we use Remark 4.4 to obtain (note that $\frac{r}{r'} - 1 = r - 2 \le 0$).

$$|z_j|\Big(\sum_{k=j}^n |z_k|^r\Big)^{\frac{1}{r}} \le \|z\|^2_{m\Psi_r} \frac{\log(j+1)^{\frac{1}{r'}}}{j}\Big(\sum_{k=j}^n \frac{\log(k+1)^{\frac{r}{r'}}}{k^r}\Big)^{\frac{1}{r}}$$

$$\le \|z\|^2_{m\Psi_r} \frac{\log(j+1)^{\frac{1}{r'}}}{j} \log(j+1)^{\frac{1}{r'}-\frac{1}{r}}\Big(\sum_{k=j}^n \frac{\log(k+1)}{k^r}\Big)^{\frac{1}{r}}.$$

We deal with the last sum

$$\sum_{k=j}^n \frac{\log(k+1)}{k^r} \le \Big(1+\frac{1}{j}\Big)^r \sum_{k=j}^n \frac{\log(k+1)}{(k+1)^r} \le 2^r \sum_{k=j+1}^{n+1} \frac{\log(k)}{k^r} \le 2^{r+2} \int_j^{n+1} \frac{\log(x)}{x^r}dx$$

$$\le 2^{r+2}\frac{(r-1)\log(j)+1}{(r-1)^2 j^{r-1}} \le 2^{r+2}\frac{2r}{(r-1)^2}\frac{\log(j+1)}{j^{r-1}},$$

and

$$|z_j|\Big(\sum_{k=j}^n |z_k|^r\Big)^{\frac{1}{r}} \le 2^{r+2}\frac{2r}{(r-1)^2}\|z\|^2_{m\Psi_r}\frac{\log(j+1)^{\frac{2}{r'}}}{j^{1+\frac{1}{r'}}}$$

This and (17) give the conclusion □

In view of Lemma 4.6, now we need to bound $\sum_{\mathbf{i}\in\mathscr{J}(m-2,k)} |z_{\mathbf{i}}||\mathbf{i}|^{\frac{1}{r}}$ in a suitable way (depending on k). To this purpose we switch to the α-notation of multi-indices (recall (5)), that is going to be more convenient. Then the sum reads

$$\sum_{\alpha\in\Lambda(m-2,k)} |z|^\alpha |[\alpha]| \tag{18}$$

and the strategy is to decompose this sum into two sums: a tetrahedral and an even part and, then, bound each one of these. This lies in the general philosophy of decomposing index sets into some smaller subset in which a certain problem results easier and, at the same time, are the bricks in which any general index can be recovered. This philosophy has alredy been used in [GMMa].

Let us be more precise and introduce some notation. A multi-index α is tetrahedral if all its entries are either 0 or 1. We consider the set of tetrahedral multi-indices

$$\Lambda_T(m,n) = \big\{\alpha \in \Lambda(m,n) : \alpha_i \in \{0,1\}\big\}.$$

A multi-index is called even if all its non-zero entries are even (note that this forces the multi-index to have even order). We consider then the set

$$\Lambda_E(m,n) = \big\{\alpha \in \Lambda(m,n) : \alpha_i \text{ is even for every } i = 1,\dots,n\big\}.$$

Observe that for every $\alpha \in \Lambda_E(m,n)$ there is a unique $\beta \in \Lambda(m/2,n)$ such that $\alpha = 2\beta$.

Remark 4.7. Given $\alpha \in \Lambda(M,N)$ define α_T (the tetrahedral part) and α_E (the even part) as

$$\left(\alpha_T\right)_i = \begin{cases} 1 & \text{if } \alpha_i \text{ is odd} \\ 0 & \text{if } \alpha_i \text{ is even} \end{cases} \quad \text{and} \quad \left(\alpha_E\right)_i = \begin{cases} \alpha_i - 1 & \text{if } \alpha_i \text{ is odd} \\ \alpha_i & \text{if } \alpha_i \text{ is even} \end{cases}.$$

If $0 \le k \le M$ is the number of odd entries in α, then clearly $\alpha_T \in \Lambda_T(k,N)$ and $\alpha_E \in \Lambda_E(M-k,N)$ and $\alpha = \alpha_T + \alpha_E$. As $(\alpha_E)_i \le \alpha_i$ for every i then $\alpha_E! \le \alpha!$. On the other hand, $\alpha_T! = 1$, then $\alpha_T!\alpha_E! \le \alpha!$, and

$$|[\alpha]| = \frac{M!}{\alpha!} \le \frac{M!}{\alpha_T!\alpha_E!} = \frac{M!}{(M-k)!k!}\frac{k!}{\alpha_T!}\frac{(M-k)!}{\alpha_E!} = \binom{M}{k}|[\alpha_T]||[\alpha_E]| \le 2^M |[\alpha_T]||[\alpha_E]|.$$

Our next step is to bound a sum as in (18) when we just consider even or tetrahedral indices. We start with the latter.

Lemma 4.8. *For every $1 < r \le 2$ and $M, N \in \mathbb{N}$, and every decreasing $z \in \mathbb{C}^N$ we have*

$$\sum_{\alpha \in \Lambda_T(M,N)} |z^\alpha||[\alpha]|^{\frac{1}{r}} \le 2(1+\varepsilon)^{\frac{M}{r'}} \|z\|_{m_{\Psi_r}}^M N^{\frac{1}{(1+\varepsilon)r'}},$$

for every $\varepsilon > 0$ and

$$\sum_{\alpha \in \Lambda_E(M,N)} |z^\alpha||[\alpha]|^{\frac{1}{r}} \le \|z\|_{\ell_r}^M \le \|\operatorname{id} : m_{\Psi_r} \to \ell_r\|^M \|z\|_{m_{\Psi_r}}^M.$$

Proof. We begin with the first inequality, observing that it is obvious if $N = 1$. We may, then, assume $N \ge 2$. Then, given $\alpha \in \Lambda_T(M,N)$, note that $\alpha! = 1$ and $|[\alpha]|$ is exactly $M!$. Then,

$$\sum_{\alpha \in \Lambda_T(M,N)} |z^\alpha||[\alpha]|^{\frac{1}{r}} = \sum_{\alpha \in \Lambda_T(M,N)} |z^\alpha||[\alpha]|\frac{1}{|[\alpha]|^{\frac{1}{r'}}} \le \Big(\sum_{k=1}^N |z_k|\Big)^M \frac{1}{M!^{\frac{1}{r'}}}$$
$$\le \|z\|_{m_{\Psi_r}}^M \log(N+1)^{\frac{M}{r'}} \frac{1}{M!^{\frac{1}{r'}}} \le 2\|z\|_{m_{\Psi_r}}^M \Big(\frac{\log(N)^M}{M!}\Big)^{\frac{1}{r'}}.$$

A simple calculus argument shows that the function $f : [1,\infty[\to \mathbb{R}$ given by $f(x) = \frac{\log(x)^M}{x^{1/(1+\varepsilon)}}$ is bounded by $\big(\frac{(1+\varepsilon)M}{e}\big)^M$, then $\log(N)^M \le N^{1/(1+\varepsilon)}\big(\frac{(1+\varepsilon)M}{e}\big)^M$. On the other hand $M! \ge \big(\frac{M}{e}\big)^M$.This gives the conclusion.

For the proof of the second inequality let us recall first that for each $\alpha \in \Lambda_E(M,N)$ there is a unique $\beta \in \Lambda(M/2,N)$ such that $\alpha = 2\beta$ and, moreover,

$$|[\alpha]| = \frac{M!}{\alpha_1!\cdots\alpha_N!} = \Big(\frac{(M/2)!}{\beta_1!\cdots\beta_N!}\Big)^2 \frac{M!}{(M/2)!(M/2)!}\prod_{i=1}^N \frac{\beta_i!\beta_i!}{(2\beta_i)!} \le |[\beta]|^2,$$

where last inequality holds because $2^k \le \frac{(2k)!}{k!^2} \le 2^{2k}$ and then

$$\frac{M!}{(M/2)!(M/2)!}\prod_{i=1}^N \frac{\beta_i!\beta_i!}{(2\beta_i)!} \le 2^M \prod_{i=1}^N \frac{1}{2^{\beta_i}} = 1.$$

Then (note that, since $2/r \geq 1$, the ℓ_1 norm bounds the $\ell_{2/r}$ norm)

$$\sum_{\alpha\in\Lambda_E(M,N)} |z^\alpha||\alpha|^{\frac{1}{r}} \leq \sum_{\beta\in\Lambda(M/2,N)} |(z^2)^\beta||\beta|^{2/r} = \sum_{\beta\in\Lambda(M/2,N)} \Big(|(z^r)^\beta||\beta|\Big)^{2/r}$$
$$\leq \Big(\sum_{\beta\in\Lambda(M/2,N)} |(z^r)^\beta||\beta|\Big)^{2/r} = \Big(\sum_{l=1}^N |z_l|^r\Big)^{M/r} \leq \| \mathrm{id}: m_{\Psi_r} \to \ell_r\|^M \|z\|^M_{m_{\Psi_r}}.$$

□

Lemma 4.9. *Given $1 < r \leq 2$ there is a constant $K_r \geq 1$ such that for every $M, N \in \mathbb{N}$, and every decreasing $z \in \mathbb{C}^N$ we have*

$$\sum_{\alpha\in\Lambda(M,N)} |z^\alpha||[\alpha]|^{\frac{1}{r}} \leq K_r(M+1)(1+\varepsilon)^{\frac{M}{r'}} 2^{\frac{M}{r}+1} N^{\frac{1}{(1+\varepsilon)r'}} \| \mathrm{id}: m_{\Psi_r} \to \ell_r\|^M \|z\|^M_{m_{\Psi_r}},$$

for every $\varepsilon > 0$.

Proof. Choose some decreasing z and use by Remark 4.7 and Lemma 4.8 to get

$$\sum_{\alpha\in\Lambda(M,N)} |z^\alpha||[\alpha]|^{\frac{1}{r}} = \sum_{k=0}^M \sum_{\alpha_T\in\Lambda_T(k,N)} \sum_{\alpha_E\in\Lambda_E(M-k,N)} |z^{(\alpha_T+\alpha_E)}||[\alpha_T+\alpha_E]|^{\frac{1}{r}}$$
$$\leq 2^{\frac{M}{r}} \sum_{k=0}^M \Big(\sum_{\alpha_T\in\Lambda_T(k,N)} |z_T^\alpha||[\alpha_T]|^{\frac{1}{r}}\Big)\Big(\sum_{\alpha_E\in\Lambda_E(M-k,N)} |z_E^\alpha||[\alpha_E]|^{\frac{1}{r}}\Big)$$
$$\leq 2^{\frac{M}{r}} \sum_{k=0}^M \Big((1+\varepsilon)^{\frac{k}{r'}} \|z\|^k_{m_{\Psi_r}} N^{\frac{1}{(1+\varepsilon)r'}}\Big)\Big(\| \mathrm{id}: m_{\Psi_r} \to \ell_r\|^{M-k} \|z\|^{M-k}_{m_{\Psi_r}}\Big)$$
$$\leq 2^{\frac{M}{r}+1}(1+\varepsilon)^M \| \mathrm{id}: m_{\Psi_r} \to \ell_r\|^M \|z\|^M_{m_{\Psi_r}} N^{\frac{1}{(1+\varepsilon)r'}} \sum_{k=0}^M 2^{k(1-\frac{2}{r})}.$$

For $r = 2$ the last sum is exactly $M+1$. If $1 < r < 2$ the series converges to $\frac{2^{2/r}}{2^{2/r}-2}$. This completes the proof □

We are finally in the position to give the proof of Theorem 4.3 from which (as we already saw) the lower inclusion in Theorem 4.1 follows.

Proof of Theorem 4.3. Fix $1 < r \leq 2$ and n, m. Pick then $P \in P \in \mathscr{P}(^m\mathbb{C}^n)$ and $z \in \mathbb{C}^n$. Since $\|z\|_{m_{\Psi_r}} = \|z^*\|_{m_{\Psi_r}}$, we may assume $z = z^*$. Applying Lemma 4.9 with $M = m-2$ and $N = k$ after Lemma 4.6

yields

$$\sum_{\mathbf{j}\in\mathscr{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}|$$

$$\leq 2A_r m^{1+\frac{1}{r}} e^{\frac{m}{r}} \|z\|^2_{m_{\Psi_r}} \left(\sum_{k=1}^{n} \frac{\log(k+1)^{\frac{2}{r'}}}{k^{1+\frac{1}{r'}}} K_r (m-1) 2^{\frac{(m-2)}{r}} (1+\varepsilon)^{\frac{m-2}{r'}} \|\operatorname{id}\|^{m-2} k^{\frac{1}{(1+\varepsilon)r'}} \|z\|^{m-2}_{m_{\Psi_r}} \right) \|P\|_{\mathscr{P}(^m\ell_r^n)}$$

$$\leq 2A_r K_r m^{2+\frac{1}{r}} ((1+\varepsilon)2e)^{\frac{m}{r}} \|\operatorname{id}\|^m \|z\|^m_{m_{\Psi_r}} \left(\sum_{k=1}^{n} \frac{\log(k+1)^{\frac{2}{r'}}}{k^{1+\frac{\varepsilon}{(1+\varepsilon)r'}}} \right) \|P\|_{\mathscr{P}(^m\ell_r^n)}.$$

Since $r > 1$ the series $\sum_{k=1}^{\infty} \frac{\log(k+1)^{\frac{2}{r'}}}{k^{1+\frac{\varepsilon}{(1+\varepsilon)r'}}}$ is convergent. This completes the proof. □

5. Monomial convergence for bounded holomorphic functions on B_{ℓ_r}

We change now our focus to the space $H_\infty(B_{\ell_r})$ of bounded holomorphic functions on B_{ℓ_r}. Our main contribution in this side is the following theorem, that provides with lower and upper inclusions for the set of monomial convergence of these spaces. It recovers (see Remark 5.5 and Corollary 5.6) some previously known results.

Theorem 5.1. *Let $1 < r \leq 2$ then,*

$$\left\{ z \in \mathbb{C}^{\mathbb{N}} \colon 2e \|\operatorname{id} \colon m_{\Psi_r} \to \ell_r\|^r \left(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \right)^r + \|z\|^r_{\ell_r} < 1 \right\} \subset$$

$$\operatorname{mon} H_\infty(B_{\ell_r}) \subset \left\{ z \in B_{\ell_r} \colon \limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \leq 1 \right\}.$$

The upper inclusion follows using probabilistic techniques, as in the case of $\operatorname{mon} H_b(\ell_r)$. The lower inclusion, on the other hand, relies on Theorem 4.3 and requires some preliminary work that we start with the following remark.

Remark 5.2. Given a Reinhardt domain $\mathscr{R}$ in a Banach sequence space X, a simple closed-graph argument (see [DMP09, Lemma 4.1] or [DGMSP19, Remark 20.1]) shows that $z \in \operatorname{mon} H_\infty(\mathscr{R})$ if and only if there is a constant $C_z > 0$ such that

$$\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f) z^\alpha| \leq C_z \|f\|_{\mathscr{R}}$$

for every $f \in H_\infty(\mathscr{R})$.

Lemma 5.3. *Let $1 < r \leq 2$ then, $\frac{1}{\|\operatorname{id}: m_{\Psi_r} \to \ell_r\| (2e)^{1/r}} B_{m_{\Psi_r}} \subset \operatorname{mon} H_\infty(B_{\ell_r})$.*

Proof. In order to keep thinsg readable we write $K = \|\operatorname{id} : m_{\Psi_r} \to \ell_r\| (2e)^{1/r}$. We first show that if $z \in \frac{1}{K} B_{m_{\Psi_r}}$ is non-decreasing, then $z \in \operatorname{mon} H_\infty(B_{\ell_r})$. The general result follows form the fact that

$B_{m\Psi_r}$ and $\operatorname{mon} H_\infty(B_{\ell_r})$ are both symmetric (Corollary 3.6). We choose now $f \in H_\infty(B_{\ell_r})$ and fix $\varepsilon > 0$ so that $(1+\varepsilon)^{1/r}\|z\|_{m\Psi_r} K < 1$. By Theorem 4.3 we can find $C_r(\varepsilon) > 0$ so that

$$\begin{aligned}
\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f) z^\alpha| &= \sup_{n\in\mathbb{N}} \sum_{m=0}^{\infty} \sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(f) z_{\mathbf{j}}| \\
&\leq \sup_{n\in\mathbb{N}} \sum_{m=0}^{\infty} C_r(\varepsilon) m^{2+\frac{1}{r}} (1+\varepsilon)^{\frac{m}{r}} K^m \|z\|_{m\Psi_r}^m \sup_{u\in B_{\ell_r^n}} \Big| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}}(f) u_{\mathbf{j}} \Big| \\
&\leq \sum_{m=0}^{\infty} C_r(\varepsilon) \left(m^{\frac{1}{m}(2+\frac{1}{r})} (1+\varepsilon)^{\frac{1}{r}} K \|z\|_{m\Psi_r} \right)^m \|P_m(f)\|_{\mathscr{P}(^m\ell_r)} \\
&\leq \|f\|_{B_{\ell_r}} C_r(\varepsilon) \sum_{m=0}^{\infty} \left(m^{\frac{1}{m}(2+\frac{1}{r})} (1+\varepsilon)^{\frac{1}{r}} K \|z\|_{m\Psi_r} \right)^m.
\end{aligned}$$

The choice of ε and fact that $m^{\frac{1}{m}(2+\frac{1}{r})} \to 1$ as $m \to \infty$ immediately give that the series converges and complete the proof. □

A useful tool when dealing with $\operatorname{mon} H_\infty(B_{c_0})$ is that, if a sequence belong to such a set of monomial convergence and we modify finitely many coordinates, then the resulting sequence remains in the set of monomial convergence (see [DGMPG08, Lemma 2] or [DGMSP19, Proposition 10.14]). It is unknown whether or not an analogous result result holds for ℓ_r (see the comments regarding this problem in [Sch15, Chapter 10]). We overcome this with the following proposition, a weaker version of this, but enough for our purposes.

Proposition 5.4. *Let $1 < r < \infty$ and $u, z \in B_{\ell_r}$ be such that $|u_n| \leq |z_n|$ for $1 \leq n \leq N$ and $|u_n| = |z_n|$ for $n > N$. Suppose that there exists $\rho > \sum_{n=1}^N |z_n|^r$ so that $u \in \operatorname{mon} H_\infty((1-\rho)^{1/r} B_{\ell_r})$. Then $z \in \operatorname{mon} H_\infty(B_{\ell_r})$.*

Proof. Let $a_1, \ldots, a_N$ be positive real numbers such that $|z_i| < a_i$ for every $1 \leq i \leq N$ and

$$a := \sum_{n=1}^{N} a_n^r < \rho.$$

Given for $f \in H_\infty(B_{\ell_r})$ and $k_1, \ldots, k_N \in \mathbb{N}$, we define (following the proof of [DGMPG08, Lemma 2])

$$f_{k_1,\ldots,k_N}(v) := \frac{1}{(2\pi i)^N} \int_{|w_1|=a_1} \cdots \int_{|w_N|=a_N} \frac{f(w_1,\ldots,w_N,v_{N+1},v_{N+2},\ldots)}{w_1^{k_1+1}\cdots w_N^{k_N+1}} dw_1 \cdots dw_N.$$

Note that $f_{k_1,\ldots,k_N}$ is well defined on the contracted ball $(1-a)^{1/r} B_{\ell_r}$ and, in fact, belongs to $H_\infty((1-a)^{1/r} B_{\ell_r})$ (because $f \in H_\infty(B_{\ell_r})$) and

$$\|f_{k_1,\ldots,k_N}\|_{(1-a)^{1/r}\cdot B_{\ell_r}} \leq \frac{\|f\|_{B_{\ell_r}}}{a_1^{k_1}\cdots a_N^{k_N}}. \tag{19}$$

Our next step is to understand the coefficients $c_\alpha(f_{k_1,\dots,k_N})$ in relation to those of f. For each multi-index $\alpha=(\alpha_1,\dots,\alpha_n,0,\dots)$ with $\alpha_n\neq 0$, an application of the Cauchy integral formula yields

$$c_\alpha(f_{k_1,\dots,k_n})=\begin{cases} c_{(k_1,\dots,k_N,\alpha_{N+1},\dots,\alpha_n)}(f) & \text{if } \alpha_1=\cdots=\alpha_N=0,\\ 0 & \text{otherwise.}\end{cases} \tag{20}$$

We have now everything we need to proceed. Note that, since $a<\rho$, we have $u\in \operatorname{mon} H_\infty((1-\rho)^{1/r}B_{\ell_r})\subset \operatorname{mon} H_\infty((1-a)^{1/r}B_{\ell_r})$. With Remark 5.2 and (19) we get

$$\sum_{\beta\in\mathbb{N}_0^{(\mathbb{N})}}|c_\beta(f_{k_1,\dots,k_N})||u_{N+1}^{\beta_1}\cdots u_{N+2}^{\beta_2}\cdots|\le C_u\|f_{k_1,\dots,k_N}\|_{(1-a)^{1/r}B_{\ell_r}}\le C_u\frac{\|f\|_{B_{\ell_r}}}{a_1^{k_1}\cdots a_N^{k_N}}. \tag{21}$$

Now using (20) and (21) (recall that $|u_n|=|z_n|$ for $n\ge N+1$) we have

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}|c_\alpha(f)||z^\alpha| &= \sum_{(k_1,\dots k_N)\in\mathbb{N}_0^N}|z_1^{k_1}\cdots z_N^{k_N}|\sum_{\beta\in\mathbb{N}_0^{(\mathbb{N})}}|c_{(k_1,\dots,k_N,\beta)}(f)||u_{N+1}^{\beta_1}\cdots u_{N+2}^{\beta_2}\cdots|\\ &= \sum_{(k_1,\dots k_N)\in\mathbb{N}_0^N}|z_1^{k_1}\cdots z_N^{k_N}|\sum_{\beta\in\mathbb{N}_0^{(\mathbb{N})}}|c_\beta(f_{k_1,\dots,k_N})||u_{N+1}^{\beta_1}\cdots u_{N+2}^{\beta_2}\cdots|\\ &\le \sum_{(k_1,\dots k_N)\in\mathbb{N}_0^N}|z_1^{k_1}\cdots z_N^{k_N}|C_u\frac{\|f\|_{B_{\ell_r}}}{a_1^{k_1}\cdots a_N^{k_N}}\\ &= C_u\|f\|_{B_{\ell_r}}\prod_{n=1}^N\sum_{k_n\ge 0}\left(\frac{|z_n|}{a_n}\right)^{k_n}<\infty,\end{aligned}$$

as we wanted. □

Let us make a last observation before we proceed with the proof of Theorem 5.1. Given a Banach sequence space X, for every $f\in H_\infty(tB_X)$ and $t>0$ the function f_t given by $f_t(x)=f(tx)$ for $x\in B_X$ belongs to $H_\infty(B_X)$ and $c_\alpha(f_t)=t^{|\alpha|}c_\alpha(f)$ for every α. Then, if $z\in \operatorname{mon} H_\infty(B_X)$ we have

$$\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}|c_\alpha(f)(tz)^\alpha|=\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}|c_\alpha(f)t^{|\alpha|}z^\alpha|=\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}|c_\alpha(f_t)z^\alpha|<\infty.$$

This implies $t\operatorname{mon} H_\infty(B_X)\subset \operatorname{mon} H_\infty(tB_x)$ for every Banach sequence space X and every $t>0$. Noting that tB_X is the open unit ball of the Banach sequence space $(X,t\|\cdot\|_X)$, the previous inclusion yields

$$t^{-1}\operatorname{mon} H_\infty(tB_X)\subset \operatorname{mon} H_\infty(t^{-1}tB_X)=\operatorname{mon} H_\infty(B_X).$$

This altogether shows

$$\operatorname{mon} H_\infty(tB_X)=t\operatorname{mon} H_\infty(B_X) \tag{22}$$

for every Banach sequence space X and every $t>0$. We are now in conditions of proving Theorem 5.1.

Proof of Theorem 5.1. Let us start with the upper inclusion

$$\operatorname{mon} H_\infty(B_{\ell_r}) \subset \Big\{ z \in B_{\ell_r} : \limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \le 1 \Big\}.$$

Fix $z \in \operatorname{mon} H_\infty(B_{\ell_r})$. Arguing as in the proof of the upper inclusion of Theorem 4.1, proceeding as in (13), replacing the role of Lemma 4.2 by Remark 5.2, and as in (14) we get

$$\sum_{j=1}^n |z_j^*| \le C_{z^*,r}^{\frac{1}{m}} \left[\log(m)^{\frac{1}{m}} (2\pi m)^{\frac{1}{2m}} e^{\frac{1}{12m^2}} \frac{m}{e} n^{\frac{1}{m}} \right]^{1-\frac{1}{r}}.$$

where $C_{z^*,r}$ is a positive constant that depends only on z^* and r. Choosing $m = \lfloor \log(n+1) \rfloor$ we get

$$\limsup_{n\to\infty} \frac{1}{\log(n+1)^{1-\frac{1}{r}}} \sum_{k=1}^n |z_n^*| \le 1,$$

which gives our claim.

We now face the proof of the lower inclusion

$$\Big\{ z \in \mathbb{C}^{\mathbb{N}} : 2e\| \operatorname{id} : m_{\Psi_r} \to \ell_r \|^r \Big(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \Big)^r + \|z\|_{\ell_r}^r < 1 \Big\} \subset \operatorname{mon} H_\infty(B_{\ell_r}).$$

In order to keep the notation as simple as possible, let $K = 2e\| \operatorname{id} : m_{\Psi_r} \to \ell_r \|^r$. Take $z \in \mathbb{C}^{\mathbb{N}}$ such that

$$K \Big(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \Big)^r + \|z\|_{\ell_r}^r < 1,$$

and note that this implies $z \in B_{\ell_r}$. Denote $L := \limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}}$, choose $\varepsilon > 0$ so that

$$K\big((1+\varepsilon)L\big)^r + \|z\|_{\ell_r}^r < 1, \tag{23}$$

and $N \in \mathbb{N}$ for which

$$\sup_{n\ge N} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} < (1+\varepsilon)L.$$

Let us observe that

$$z_N^* < \frac{\log(N+1)^{1-1/r}}{N}(1+\varepsilon)L, \tag{24}$$

(this follows essentially as in Remark 4.4) and define $u = (\underbrace{z_N^*, \ldots, z_N^*}_{N}, z_{N+1}^*, z_{N+2}^*, \ldots)$. On the one hand, for every $n < N$ we have, using (24),

$$\frac{\sum_{k=1}^n u_k^*}{\log(n+1)^{1-1/r}} < (1+\varepsilon)L.$$

On the other hand, for $n \ge N$,

$$\frac{\sum_{k=1}^n u_k^*}{\log(n+1)^{1-1/r}} = \frac{\sum_{k=1}^n z_n}{\log(n+1)^{1-1/r}} < (1+\varepsilon)L.$$

This altogether gives $\|u\|_{m_{\Psi_r}} < (1+\varepsilon)L$. We choose $\rho > \sum_{k=1}^N |z_k|^r$ such

$$\|\operatorname{id}: m_{\Psi_r} \to \ell_r\|^r (2e)(L(1+\varepsilon))^r + \rho < 1,$$

and, using (23) we get

$$\|u\|_{m_{\Psi_r}} < (1+\varepsilon)L < \frac{(1-\rho)^{1/r}}{\|\operatorname{id}: m_{\Psi_r} \to \ell_r\|(2e)^{1/r}}.$$

Lemma 5.3 and equation (22) imply $u \in \operatorname{mon} H_\infty((1-\rho)^{1/r} B_{\ell_r})$ and, then Proposition 5.4 gives $z^* \in \operatorname{mon} H_\infty(B_{\ell_r})$. Finally, Corollary 6.7 yields $z \in \operatorname{mon} H_\infty(B_{\ell_r})$ and completes the proof. □

Remark 5.5. Theorem 5.1 implies other known results which try to characterize the set of monomial convergence of $H_\infty(B_{\ell_r})$. Note first that, if $z \in \ell_1$, then

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} = 0.$$

Thus

$$B_{\ell_r} \cap \ell_1 \subset \left\{ z \in \mathbb{C}^{\mathbb{N}} : 2e\|\operatorname{id}: m_{\Psi_r} \to \ell_r\|^r \left(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \right)^r + \|z\|_{\ell_r}^r < 1 \right\}.$$

On the other hand, if $z \in B_{\ell_r}$ is such that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \le 1$$

then there is a constant $c > 0$ so that

$$z_n^* \le c \frac{\log(n+1)^{1-1/r}}{n}.$$

From this we easily get that $z \in \ell_{1+\varepsilon}$ for every $\varepsilon > 0$, and we recover (4) from Theorem 5.1.

The following corollary extends [BDS, Theorem 5.5(1a) and Corollary 5.7] for $1 < r \le 2$.

Corollary 5.6. *Let $1 < r \le 2$. Then*

$$\left(\frac{1}{n^{1/r'} \log(n+2)^\theta} \right)_{n\ge1} \cdot B_{\ell_r} \subset \operatorname{mon} H_\infty(B_{\ell_r}) \tag{25}$$

for every $\theta > 0$. Also, denoting $K = \frac{1}{(2e\|\operatorname{id}: m_{\Psi_r} \to \ell_r\|+1)^{1/r}}$, we have

$$\left(\frac{1}{K n^{1/r'}} \right)_{n\ge1} \cdot B_{\ell_r} \subset \operatorname{mon} H_\infty(B_{\ell_r}). \tag{26}$$

Proof. Let us begin by proving (25). Fix $\theta > 0$ and choose $z \in \left(\frac{1}{n^{1/r'} \log(n+2)^\theta} \right)_{n\ge1} B_{\ell_r}$. We can find $w \in B_{\ell_r}$ so that $z_n = \frac{w_n}{n^{1/r'} \log(n+1)^\theta}$ for every $n \in \mathbb{N}$. Since $z \in c_0$, there is an injective $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$z_n^* = |z_{\sigma(n)}| = \frac{|w_{\sigma(n)}|}{\sigma(n)^{1/r'}\log(\sigma(n)+2)^{\theta}}$. Using Hölder's inequality we get

$$\begin{aligned}
\frac{1}{\log(n+1)^{1/r'}}\sum_{l=1}^{n} z_l^* &= \frac{1}{\log(n+1)^{1/r'}}\sum_{l=1}^{n}\frac{|w_{\sigma(l)}|}{\sigma(l)^{1/r'}\log(\sigma(l)+2)^{\theta}} \\
&\leq \frac{1}{\log(n+1)^{1/r'}}\left(\sum_{l=1}^{n}|w_{\sigma(l)}|^r\right)^{1/r}\left(\sum_{l=1}^{n}\frac{1}{\sigma(l)\log(\sigma(l)+2)^{r'\theta}}\right)^{1/r'} \\
&\leq \frac{1}{\log(n+1)^{1/r'}}\left(\sum_{l=1}^{n}\frac{1}{\sigma(l)\log(\sigma(l)+2)^{r'\theta}}\right)^{1/r'} \\
&\leq \frac{1}{\log(n+1)^{1/r'}}\left(\sum_{l=1}^{n}\frac{1}{l\log(l+2)^{r'\theta}}\right)^{1/r'},
\end{aligned}$$

where the last inequality holds because $x \mapsto \frac{1}{x\log(x+2)^{r'\theta}}$ defines a decreasing function for $x > 1$. The last term, $\frac{1}{\log(n+1)^{1/r'}}\left(\sum_{l=1}^{n}\frac{1}{l\log(l+2)^{r'\theta}}\right)^{1/r'}$, goes to 0 as $n \to \infty$, and therefore

$$\limsup_{n\to\infty}\frac{1}{\log(n+1)^{1/r'}}\sum_{l=1}^{n} z_l^* = 0.$$

Indeed, suppose that $\theta < \frac{1}{r'}$ (which me may always asume since $\frac{1}{l\log(l+2)^{r'\theta}}$ is decreasing on θ). Thus, there is some $C_{r',\theta} > 0$ such that

$$\left(\sum_{l=1}^{n}\frac{1}{l\log(l+2)^{r'\theta}}\right)^{1/r'} \leq C_{r',\theta}\left(\int_{l=2}^{n}\frac{1}{x\log(x)^{r'\theta}}dx\right)^{1/r'} = C_{r',\theta}\left(\int_{l=\log(2)}^{\log(n)}\frac{1}{y^{r'\theta}}dy\right)^{1/r'} \leq C_{r',\theta}\log(n)^{-\theta+\frac{1}{r'}},$$

Then, $\frac{1}{\log(n+1)^{1/r'}}\left(\sum_{l=1}^{n}\frac{1}{l\log(l+2)^{r'\theta}}\right)^{1/r'} \leq C_{r',\theta}\log(n)^{-\theta} \to 0$.

On the other hand, $z \in B_{\ell_r}$ (note that $|z_n| \leq |w_n|$ for every n and $w \in B_{\ell_r}$), then

$$2e\|\operatorname{id}: m_{\Psi_r} \to \ell_r\|^r\left(\limsup_{n\to\infty}\frac{\sum_{k=1}^{n} z_l^*}{\log(n+1)^{1-1/r}}\right)^r + \|z\|_{\ell_r}^r = \|z\|_{\ell_r}^r < 1,$$

and, by Theorem 5.1, $z \in \operatorname{mon} H_\infty(B_{\ell_r})$.

We give now the proof of (26). Take $z = \left(\frac{1}{Kn^{1/r'}}w_n\right)_{n\geq 1}$ with $w \in B_{\ell_r}$, and note that $\|z\|_{\ell_r}^r < \frac{1}{K^r}$. Proceeding as before we get

$$K\frac{1}{\log(n+1)^{1/r'}}\sum_{l=1}^{n} z_l^* \leq \frac{1}{\log(n+1)^{1/r'}}\left(\sum_{l=1}^{n}\frac{1}{l}\right)^{1/r'} \leq 1.$$

Since $K = (2e\|\operatorname{id}: m_{\Psi_r} \to \ell_r\|^r + 1)^{1/r}$,

$$2e\|\operatorname{id}: m_{\Psi_r} \to \ell_r\|^r\left(\limsup_{n\to\infty}\frac{\sum_{k=1}^{n} z_l^*}{\log(n+1)^{1-1/r}}\right)^r + \|z\|_{\ell_r}^r < (2e\|\operatorname{id}: m_{\Psi_r} \to \ell_r\|^r + 1)\frac{1}{K^r} = 1.$$

Again Theorem 5.1 gives the conclusion. □

6. Lower inclusions for the set of monomial convergence of $\mathscr{P}(^m\ell_r)$

We turn now our attention to the set of monomial convergence of homogeneous polynomials. We fix $1 < r \le 2$ and $m \ge 2$ and define $q = (mr')' = \frac{mr}{r(m-1)+1}$. As we already pointed out, we know from [BDS, Theorem 5.1] and [DMP09, Example 4.6] that

$$\ell_{q-\varepsilon} \subset \operatorname{mon}\mathscr{P}(^m\ell_r) \subset \ell_{q,\infty}$$

for every $\varepsilon > 0$. Our aim now is to tighten this lower bound. We find a lower inclusion that gets narrower when m gets bigger.

Theorem 6.1. *Fix $1 < r \le 2$ and, for each $m \ge 2$, define $q := (mr')'$. Then $\ell_q \subset \operatorname{mon}\mathscr{P}(^2\ell_r)$; $\ell_{q,2} \subset \operatorname{mon}\mathscr{P}(^3\ell_r)$; $\ell_{q,\frac{3+\sqrt{5}}{2}} \subset \operatorname{mon}\mathscr{P}(^4\ell_r)$ and*

$$\ell_{q,\frac{m}{\log(m)}} \subset \operatorname{mon}\mathscr{P}(^m\ell_r).$$

for $m \ge 5$.

We start with Theorem 6.3, which proves the case $m = 2$ in the previous theorem and also provides an elementary proof of the fact that ℓ_q is contained in $\operatorname{mon}\mathscr{P}(^m\ell_r)$. We even get a very good estimate for the sums. We will show later in Remark 6.14 (see also the comments after it) that for $m \ge 3$ something more can be achieved. We need first a lemma.

Lemma 6.2. *Let $r > 1$. There exists $C_r > 0$ such that, for every m,*

$$\sup\Big\{\frac{m^{m/r}}{m!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_k!}{n_k^{n_k/r}} : k \in \mathbb{N}, n_1, \dots, n_k \in \mathbb{N}\setminus\{0\}, n_1 + \cdots + n_k = m\Big\} \le C_r m^{\frac{e^{\frac{1}{r-1}}-1}{2}}.$$

Proof. We proceed by induction on m. The statement is trivially satisfied for $m = 2$ and we assume it holds for $m-1$. Fix then k and choose $n_1, \dots, n_k \in \mathbb{N}$, all non-zero, such that $n_1 + \cdots + n_k = m$. We may assume $n_1 \ge \cdots \ge n_k \ge 1$. We consider two possible cases. First, if $k < e^{\frac{1}{r-1}}$ Stirling formula and the fact that $n_j \le m$ for every j yield

$$\begin{aligned}
\frac{m^{m/r}}{m!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_k!}{n_k^{n_k/r}} &\le \frac{1}{\sqrt{2\pi m}}\frac{e^m}{m^{m/r'}}\prod_{j=1}^{k}\frac{\sqrt{2\pi n_j}\,n_j^{n_j/r'}e^{1/(12n_j)}}{e^{n_j}} \\
&\le (2\pi)^{\frac{k-1}{2}} e^{\sum_{j=1}^k \frac{1}{12n_j}}\Big(\frac{n_1^{n_1}\cdots n_k^{n_k}}{m^m}\Big)^{\frac{1}{r'}}\Big(\frac{n_1\cdots n_k}{m}\Big)^{\frac{1}{2}} \le (2\pi)^{\frac{k-1}{2}} e^{\sum_{j=1}^k \frac{1}{12j}} m^{\frac{k-1}{2}} \\
&\le (2\pi)^{\frac{e^{\frac{1}{r-1}}-1}{2}} e^{\frac{r}{12(r-1)}} m^{\frac{e^{\frac{1}{r-1}}-1}{2}}.
\end{aligned}$$

On the other hand, if $k \ge e^{\frac{1}{r-1}}$ we have

$$\frac{m^{m/r}}{m!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_k!}{n_k^{n_k/r}} = \Big(\frac{m}{m-1}\Big)^{\frac{m-1}{r}}\frac{1}{m^{1/r'}}\frac{(m-1)^{(m-1)/r}}{(m-1)!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_{k-1}!}{n_{k-1}^{n_{k-1}/r}}\frac{n_k!}{n_k^{n_k/r}}. \tag{27}$$

If $n_k = 1$ then $n_1 + \cdots + n_{k-1} = m-1$ and we may use the induction hypothesis and the fact that $k \le m$ to have

$$\frac{m^{m/r}}{m!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_k!}{n_k^{n_k/r}} \le \Big(\frac{m}{m-1}\Big)^{\frac{m-1}{r}}\frac{1}{k^{1/r'}}C_r(m-1)^{\frac{e^{\frac{1}{r-1}}-1}{2}} \le C_r e^{1/r}\frac{1}{e^{\frac{1}{(r-1)r'}}}(m-1)^{\frac{e^{\frac{1}{r-1}}-1}{2}} \le C_r m^{\frac{e^{\frac{1}{r-1}}-1}{2}}.$$

Finally, if $n_k > 1$ then

$$\frac{(n_k-1)^{\frac{n_k-1}{r'}}n_k}{n_k^{n_k/r}} = \Big(\frac{n_k-1}{n_k}\Big)^{\frac{n_k-1}{r'}}n_k^{\frac{1}{r'}} \le n_k^{\frac{1}{r'}}.$$

We may use again the induction hypothesis and the fact that $n_k \le m/k$ to obtain from (27)

$$\frac{m^{m/r}}{m!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_k!}{n_k^{n_k/r}} \le \Big(\frac{m}{m-1}\Big)^{\frac{m-1}{r}}\Big(\frac{n_k}{m}\Big)^{1/r'}C_r(m-1)^{\frac{e^{\frac{1}{r-1}}-1}{2}} \le \Big(\frac{m}{m-1}\Big)^{\frac{m-1}{r}}\frac{1}{k^{1/r'}}C_r(m-1)^{\frac{e^{\frac{1}{r-1}}-1}{2}}.$$

From here we conclude as in the previous case. □

Theorem 6.3. *For each $1 < r \le 2$, there exists $d_r > 1$ such that for each m and n, every $P \in \mathscr{P}(^m\mathbb{C}^n)$ and all $z \in \mathbb{C}^n$*

$$\sum_{1\le j_1\le\cdots\le j_m\le n}|c_{\mathbf{j}}(P)z_{j_1}\dots z_{j_m}| \le m^{d_r}\|P\|_{\mathscr{P}(^m\ell_r^n)}\|z\|_{\ell_q^n}^m, \tag{28}$$

where $q := (mr')'$. In particular

$$\ell_q \subset \operatorname{mon}\mathscr{P}(^m\ell_r).$$

Proof. Clearly it is enough to show (28) and, by (11) (see also [DGMSP19, Lemma 10.15]), we may assume without loss of generality $z = z^*$. First of all, by Hölder inequality we have

$$\begin{aligned}\sum_{1\le j_1\le\cdots\le j_m\le n}|c_{\mathbf{j}}(P)z_{j_1}\dots z_{j_{m-1}}z_{j_m}| &= \sum_{1\le j_1\le\cdots\le j_{m-1}\le n}|z_{j_1}\dots z_{j_{m-1}}|\sum_{j_m=j_{m-1}}^n|c_{\mathbf{j}}(P)z_{j_m}|\\ &\le \sum_{1\le j_1\le\cdots\le j_{m-1}\le n}|z_{j_1}\dots z_{j_{m-1}}|\Big(\sum_{j_m=j_{m-1}}^n|c_{\mathbf{j}}(P)|^{r'}\Big)^{\frac{1}{r'}}\Big(\sum_{j_m=j_{m-1}}^n|z_{j_m}^r|\Big)^{\frac{1}{r}}\end{aligned}$$

Using Lemma 4.5 together with the fact that for every $(\mathbf{i},k) \in \mathscr{J}(m-1,n)$ we have $\big(\frac{(m-1)^{m-1}}{\alpha(\mathbf{i},k)^{\alpha(\mathbf{i},k)}}\big) \le e(m-1)\big(\frac{(m-2)^{m-2}}{\alpha(\mathbf{i})^{\alpha(\mathbf{i})}}\big)$ we obtain

$$\begin{aligned}&\sum_{\mathbf{j}\in\mathscr{J}(m,n)}|c_{\mathbf{j}}(P)z_{\mathbf{j}}|\\ &\quad\le e^{1+\frac{1}{r}}(m-1)^{\frac{1}{r}}m\|P\|_{\mathscr{P}(^m\ell_r^n)}\sum_{j_{m-1}=1}^n|z_{j_{m-1}}|\sum_{\mathbf{i}\in\mathscr{J}(m-2,j_{m-1})}|z_{\mathbf{i}}|\Big(\frac{(m-2)^{m-2}}{\alpha(\mathbf{i})^{\alpha(\mathbf{i})}}\Big)^{\frac{1}{r}}\Big(\sum_{j_m=j_{m-1}}^n|z_{j_m}|^r\Big)^{\frac{1}{r}}\end{aligned}$$

For each fixed $1 \le k \le n$ we have, using (6) and Lemma 6.2 (we write $a_r = \frac{e^{\frac{1}{r-1}}-1}{2}$) and the fact that $q \le r$

$$|z_k| \sum_{\mathbf{i}\in\mathscr{J}(m-2,k)} |z_{\mathbf{i}}| \Big(\frac{(m-2)^{m-2}}{\alpha(\mathbf{i})^{\alpha(\mathbf{i})}}\Big)^{\frac{1}{r}} \Big(\sum_{j=k}^{n} |z_j|^r\Big)^{\frac{1}{r}} \le |z_k| \sum_{\mathbf{i}\in\mathscr{J}(m-2,k)} |z_{\mathbf{i}}||\mathbf{i}| \frac{(m-2)^{(m-2)/r}}{\alpha(\mathbf{i})^{\alpha(\mathbf{i})/r}|\mathbf{i}|} \Big(|z_k|^{r-q} \sum_{j=k}^{n} |z_j|^q\Big)^{\frac{1}{r}}$$

$$= C_r(m-2)^{a_r} |z_k|^{2-\frac{q}{r}} \sum_{i_1,\dots,i_{m-2}=1}^{k} |z_{i_1}\cdots z_{i_{m-2}}| \Big(\sum_{j=k}^{n} |z_j|^q\Big)^{\frac{1}{r}} = C_r(m-2)^{a_r} \|z\|_{\ell_q}^{\frac{q}{r}} |z_k|^{2-\frac{q}{r}} \Big(\sum_{i=1}^{k} |z_i|\Big)^{m-2}$$

$$\le C_r(m-2)^{a_r} \|z\|_{\ell_q^n}^{\frac{q}{r}+m-2} |z_k|^{2-\frac{q}{r}} k^{\frac{m-2}{q'}}.$$

Now

$$\sum_{k=1}^{n} |z_k|^{2-\frac{q}{r}} k^{\frac{m-2}{q'}} = \|z\|_{\ell^n_{q,2-q/r}}^{2-\frac{q}{r}} \le \|z\|_{\ell_q^n}^{2-\frac{q}{r}}$$

because $2-\frac{q}{r} \ge q$ for $m \ge 2$. This altogether gives

$$\sum_{\mathbf{j}\in\mathscr{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \le K_r m(m-1)^{\frac{1}{r}} (m-2)^{a_r} \|P\|_{\mathscr{P}(^m\ell_r^n)} \|z\|_{\ell_q^n}^m. \qquad \square$$

This gives the case $m=2$ in Theorem 6.1. We face now the problem of getting the result for other m's. The general philosophy is always to try to get a bound as that in (28), where in the right-hand-side we have some constants that depend on r and m (but not on n, the number of variables), the norm of the polynomial and the norm of z in some space X. This then implies $X \subset \mathrm{mon}\mathscr{P}(^m\ell_r)$. What we do is to take the sum as depending on m different variables; that is, for each polynomial P we consider

$$\sum_{1\le j_1\le\cdots\le j_m\le n} |c_{\mathbf{j}}(P) z^{(1)}_{j_1} \dots z^{(m)}_{j_m}| \tag{29}$$

with $z^{(1)},\dots,z^{(m)} \in \mathbb{C}^n$ and then try to get an estimate that involves the norms of the $z^{(j)}$ in (possibly) different spaces. This then gives that the smallest of these spaces is contained in the set of monomial convergence (see Remark 6.10). We do this (giving the proof of Theorem 6.1) in two stages (that we present in the following two subsections). First we give an estimate for the sum that involves both $\ell_{q,1}$ and $\ell_{q,\infty}$ norms (the precise statement is given in Proposition 6.4). Then we interpret this inequality as operators from $\ell_{q,\infty}\times\cdots\times\ell_{q,\infty}\times\ell_{q,1}\times\ell_{q,\infty}\times\cdots\times\ell_{q,\infty}$ to $\ell_1(\mathscr{J}(m,n))$ and use interpolation techniques to improve the $\ell_{q,1}$-norm (by weakening the $\ell_{q,\infty}$-norm). This is done in Theorem 6.9. What happens here is that, since in the estimate in Proposition 6.4 some of the variables have to be decreasing, we cannot use general multilinear interpolation, but interpolation in cones (a more detailed explanation is given in Section 6.2).

6.1. **First bound for the sum.** As we announced, our first step towards the proof of Theorem 6.1 is to get a bound for a sum like that in (29). This becomes the main result of this section.

Proposition 6.4. *Let $1 < r \le 2$ and $m \ge 2$. Define $q := (mr')'$. There exists $C_{m,r} > 1$ so that for every $n \in \mathbb{N}$, every $P \in \mathscr{P}(^m\mathbb{C}^n)$, every $z^{(1)}, \dots, z^{(m)} \in \mathbb{C}^n$ and $1 \le k \le m-1$ we have*

$$\sum_{1 \le j_1 \le \dots \le j_m \le n} \left| c_{\mathbf{j}}(P) z^{(1)}_{j_1} \cdots z^{(k)}_{j_k} z^{(k+1)*}_{j_{k+1}} \cdots z^{(m)*}_{j_m} \right| \le C_{m,r} \| z^{(k)} \|_{\ell_{q,1}} \prod_{i \ne k} \| z^{(i)} \|_{\ell_{q,\infty}} \| P \|_{\mathscr{P}(^m \ell_r^n)} .$$

The proof requires some work, that we prepare with a few lemmas. But before let us make a couple of elementary comments. First of all, by definition,

$$z^*_k \le \| z \|_{\ell_{q,\infty}} \frac{1}{k^{1/q}} \tag{30}$$

for every $z \in \mathbb{C}^n$ and, then

$$\sum_{k=N}^{M} z^*_k \le \| z \|_{\ell_{q,\infty}} \sum_{k=N}^{M} \frac{1}{k^{1/q}} . \tag{31}$$

Also, for $1 \ne \alpha < 0$,

$$\sum_{k=N}^{M} n^\alpha = N^\alpha + \sum_{k=N+1}^{M} n^\alpha \le N^\alpha + \int_N^M x^\alpha \, dx = N^\alpha + \frac{1}{\alpha+1} \big(M^{\alpha+1} - N^{\alpha+1} \big) . \tag{32}$$

Lemma 6.5. *Let $n, k \ge 1$ and $1 \le q < \infty$. Then for every $z^{(1)}, \dots, z^{(k)} \in \mathbb{C}^n$ and $1 \le j \le n$ we have*

$$\sum_{1 \le j_1 \le \dots \le j_k \le j} | z^{(1)}_{j_1} \dots z^{(k)}_{j_k} | \le (q')^k j^{\frac{k}{q'}} \prod_{1 \le i \le k} \| z^{(i)} \|_{\ell_{q,\infty}} .$$

Proof. We proceed by induction on k. For $k = 1$ the statement is a straightforward consequence of (31) and (32). Assume that the result holds for $k-1$. Then

$$\begin{aligned} \sum_{1 \le j_1 \le \dots \le j_k \le j} | z^{(1)}_{j_1} \cdots z^{(k)}_{j_l} | &= \sum_{j_k=1}^{j} | z^{(k)}_{j_k} | \Big(\sum_{1 \le j_1 \le \dots \le j_{k-1} \le j_k} | z^{(1)}_{j_1} \dots z^{(k-1)}_{j_{k-1}} | \Big) \\ &\le (q')^{k-1} \prod_{1 \le i \le k-1} \| z^{(i)} \|_{\ell_{q,\infty}} j_k^{\frac{k-1}{q'}} \sum_{j_k=1}^{j} | z^{(k)}_{j_k} | \le (q')^k j^{\frac{k-1}{q'}} j^{\frac{1}{q'}} \prod_{1 \le i \le k} \| z^{(i)} \|_{\ell_{q,\infty}} , \end{aligned}$$

which concludes the proof. ☐

Lemma 6.6. *Let $1 < r \le 2$, $m \ge 3$ and $n \in \mathbb{N}$. Fix $q := (mr')'$ and $1 \le k \le m-2$. For every $z^{(i_1)}, \dots, z^{(i_k)} \in \mathbb{C}^n$ and $1 \le t \le n$ we have*

$$\sum_{t \le j_1 \le \dots \le j_k \le n} | z^{(i_1)*}_{j_1} \dots z^{(i_k)*}_{j_k} | j_k^{\frac{1}{r} - \frac{1}{q}} \le \Big(\prod_{1 \le l \le k} \big(\frac{mr'}{m-l-1} + \frac{1}{t} \big) \Big) t^{\frac{k+1}{q'} - \frac{1}{r'}} \Big(\prod_{1 \le l \le k} \| z^{(i_l)} \|_{\ell_{q,\infty}} \Big).$$

Proof. First of all let us note that a simple computation shows that $\frac{s}{q'}-\frac{1}{r'}\le -\frac{1}{mr'}<0$ for every $1\le s\le m-1$. We now proceed by induction on k. For $k=1$ we use (31) and (32) to have

$$\sum_{j=t}^{n}|z_j^*|j^{\frac{1}{r}-\frac{1}{q}}\le \|z\|_{\ell_{q,\infty}}\sum_{j=t}^{n}j^{\frac{2}{q'}-\frac{1}{r'}-1}\le \|z\|_{\ell_{q,\infty}}\Big(t^{\frac{2}{q'}-\frac{1}{r'}}-(\frac{2}{q'}-\frac{1}{r'})^{-1}t^{\frac{2}{q'}-\frac{1}{r'}+1}\Big)=(\frac{r'm}{m-2}+\frac{1}{t})t^{\frac{2}{q'}-\frac{1}{r'}}\|z\|_{\ell_{q,\infty}}.$$

Let us suppose now that the statement holds for $k-1$ and prove it for k.

$$\begin{aligned}
&\sum_{t\le j_1\le\cdots\le j_k\le n}|z_{j_1}^{(i_1)*}\cdots z_{j_k}^{(i_k)*}|j_k^{\frac{1}{r}-\frac{1}{q}}\\
&=\sum_{j_1=t}^{n}|z_{j_1}^{(i_1)*}|\sum_{j_1\le j_2\le\cdots\le j_k\le n}|z_{j_2}^{(i_2)*}\ldots z_{j_k}^{(i_k)*}|j_k^{\frac{1}{r}-\frac{1}{q}}\\
&\le\sum_{j_1=t}^{n}|z_{j_1}^{(i_1)*}|\Big(\prod_{1\le l\le k-1}(\frac{mr'}{m-l-1}+\frac{1}{j_1})\Big)j_1^{\frac{k}{q'}-\frac{1}{r'}}\Big(\prod_{2\le l\le k}\|z^{(i_l)}\|_{\ell_{q,\infty}}\Big)\\
&\le\Big(\prod_{1\le l\le k-1}(\frac{mr'}{m-l-1}+\frac{1}{t})\Big)\Big(\prod_{2\le l\le k}\|z^{(i_l)}\|_{\ell_{q,\infty}}\Big)\sum_{j_1=t}^{n}|z_{j_1}^{(i_1)*}|j_1^{\frac{k}{q'}-\frac{1}{r'}}\\
&\le\Big(\prod_{1\le l\le k-1}(\frac{mr'}{m-l-1}+\frac{1}{t})\Big)\Big(\prod_{1\le l\le k}\|z^{(i_l)}\|_{\ell_{q,\infty}}\Big)\sum_{j_1=t}^{n}j_1^{\frac{k+1}{q'}-\frac{1}{r'}-1}\\
&\le\Big(\prod_{1\le l\le k-1}(\frac{mr'}{m-l-1}+\frac{1}{t})\Big)\Big(\prod_{1\le l\le k}\|z^{(i_l)}\|_{\ell_{q,\infty}}\Big)t^{\frac{k+1}{q'}-\frac{1}{r'}}\Big(\frac{1}{t}-(\frac{k+1}{q'}-\frac{1}{r'})^{-1}\Big)\\
&=\Big(\prod_{1\le l\le k-1}(\frac{mr'}{m-l-1}+\frac{1}{t})\Big)\Big(\prod_{1\le l\le k}\|z^{(i_l)}\|_{\ell_{q,\infty}}\Big)t^{\frac{k+1}{q'}-\frac{1}{r'}}\Big(\frac{1}{t}+\frac{mr'}{m-k-1}\Big).
\end{aligned}$$

∎

For the following next we need the following well known Hardy-Littlewood rearrangement inequality (see for example [HLP52, Section 10.2, Theorem 368]).

Lemma 6.7. *Let $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ two non-increasing sequences of non-negative real numbers. Then, for every $m\in\mathbb{N}$ and every injection $\sigma:\mathbb{N}\to\mathbb{N}$ we have*

$$\sum_{k=1}^{m}a_{\sigma(k)}b_k\le\sum_{k=1}^{m}a_kb_k.$$

Lemma 6.8. *Let $1<r\le 2$, $m\ge 3$. Fix $q:=(mr')'$ and $1\le k\le m-2$. For every $z^{(1)},\ldots,z^{(k)}\in\mathbb{C}^n$ we have*

$$\sum_{1\le j_1\le\cdots\le j_{m-1}\le n}|z_{j_1}^{(1)}\cdots z_{j_k}^{(k)}z_{j_{k+1}}^{(k+1)*}\cdots z_{j_{m-1}}^{(m-1)*}|j_{m-1}^{\frac{1}{r}-\frac{1}{q}}\le(q'+1)^{m-2}\|z^{(k)}\|_{\ell_{q,1}}\prod_{\substack{1\le i\le m-1\\ i\ne k}}\|z^{(i)}\|_{\ell_{q,\infty}}.$$

Proof. We begin by splitting the sum in a convenient way

$$\begin{aligned}
&\sum_{1\le j_1\le\cdots\le j_{m-1}\le n}|z_{j_1}^{(1)}\cdots z_{j_k}^{(k)}z_{j_{k+1}}^{(k+1)*}\cdots z_{j_{m-1}}^{(m-1)*}|j_{m-1}^{\frac{1}{r}-\frac{1}{q}}\\
&=\sum_{j_k=1}^{n}|z_{j_k}^{(k)}|\Big(\sum_{j_k\le j_{k+1}\le\cdots\le j_{m-1}\le n}|z_{j_{k+1}}^{(k+1)*}\ldots z_{j_{m-1}}^{(m-1)*}|j_{m-1}^{\frac{1}{r}-\frac{1}{q}}\Big)\Big(\sum_{1\le j_1\le\cdots\le j_{k-1}\le j_k}|z_{j_1}^{(1)}\ldots z_{j_{k-1}}^{(k-1)}|\Big).
\end{aligned}$$

We fix j_k and bound the first block using Lemma 6.6, taking into account that we have now $m-k-1$ z's and that $\frac{1}{j_k}+\frac{mr'}{m-l-1}\le q'+1$ for every $1\le l\le m-k-1$,

$$\begin{aligned}\sum_{j_k\le j_{k+1}\le\cdots\le j_{m-1}\le n}&|z^{(k+1)*}_{j_{k+1}}\dots z^{(m-1)*}_{j_{m-1}}|j_{m-1}^{\frac{1}{r}-\frac{1}{q}}\\ &\le j_k^{\frac{m-k}{q'}-\frac{1}{r'}}\Big(\prod_{1\le l\le m-k-1}\frac{1}{j_k}+\frac{mr'}{m-l-1}\Big)\Big(\prod_{k+1\le i\le m-1}\|z^{(i)}\|_{\ell_{q,\infty}}\Big)\\ &\le j_k^{\frac{m-k}{q'}-\frac{1}{r'}}(q'+1)^{m-k-1}\prod_{k+1\le i\le m-1}\|z^{(i)}\|_{\ell_{q,\infty}}.\end{aligned}$$

With this, and bounding the second block using Lemma 6.5 we get

$$\sum_{1\le j_1\le\cdots\le j_{m-1}\le n}|z^{(1)}_{j_1}\cdots z^{(k)}_{j_k}z^{(k+1)*}_{j_{k+1}}\cdots z^{(m-1)*}_{j_{m-1}}|j_{m-1}^{\frac{1}{r}-\frac{1}{q}}\le(q'+1)^{m-2}\prod_{i\ne k}\|z^{(i)}\|_{\ell_{q,\infty}}\sum_{j_k=1}^{n}|z^{(k)}_{j_k}|j_k^{\frac{k-1}{q'}+\frac{m-k}{q'}-\frac{1}{r'}}.$$

It easy to see that $\frac{k-1}{q'}+\frac{m-k}{q'}-\frac{1}{r'}=\frac{1}{q}-1$. Therefore, using Lemma 6.7 we have

$$\sum_{j_k=1}^{n}|z^{(k)}_{j_k}|j_k^{\frac{1}{q}-1}\le\sum_{j_k=1}^{n}|(z^{(k)})^*_{j_k}|j_k^{\frac{1}{q}-1}=\|z^{(k)}\|_{\ell_{q,1}}.\qquad\square$$

As it was the case for the study of holomorphic functions, Lemma 4.5 (in fact (16), which is [BDS, Lemma 3.5]) is a crucial tool for the proof of Proposition 6.4.

Proof of Proposition 6.4. We begin by using Hölder's inequality and (16) (noting that $|\mathbf{i}|\le(m-1)!$ for every $\mathbf{i}\in\mathcal{J}(m-1,n)$) and (30) to have

$$\begin{aligned}\sum_{1\le j_1\le\cdots\le j_m\le n}\big|c_{\mathbf{j}}(P)z^{(1)}_{j_1}\cdots z^{(k)}_{j_k}z^{(k+1)*}_{j_{k+1}}\cdots z^{(m)*}_{j_m}\big|&=\sum_{1\le j_1\le\cdots\le j_{m-1}\le n}|z^{(1)}_{j_1}\dots z^{(m-1)*}_{j_{m-1}}|\sum_{j_m=j_{m-1}}^{n}|c_{\mathbf{j}}(P)z^{(m)*}_{j_m}|\\ &\le\sum_{1\le j_1\le\cdots\le j_{m-1}\le n}|z^{(1)}_{j_1}\dots z^{(m-1)*}_{j_{m-1}}|\Big(\sum_{j_m=j_{m-1}}^{n}c_{\mathbf{j}}(P)^{r'}\Big)^{\frac{1}{r'}}\Big(\sum_{j_m=j_{m-1}}^{n}|z^{(m)*}_{j_m}|^r\Big)^{\frac{1}{r}}\\ &\le(m-1)!^{\frac{1}{r}}me^{1+\frac{m-1}{r}}\|P\|_{\mathcal{P}(^m\ell_r^n)}\|z^{(m)}\|_{\ell_{q,\infty}}\sum_{1\le j_1\le\cdots\le j_{m-1}\le n}|z^{(1)}_{j_1}\dots z^{(m-1)*}_{j_{m-1}}|\Big(\sum_{j_m=j_{m-1}}^{n}j_m^{-\frac{r}{q}}\Big)^{\frac{1}{r}}.\end{aligned}$$

Observe now that, for each $N\in\mathbb{N}$ we have $N^{-r/q}\le 2^{r/q}x^{-r/q}$ for every $N\le x<N+1$. Then

$$\sum_{j_m=j_{m-1}}^{n}j_m^{-\frac{r}{q}}\le 2^{\frac{r}{q}}\int_{j_{m-1}}^{n}x^{-\frac{r}{q}}dx\le 2^{\frac{r}{q}}\frac{q}{r-q}j_{m-1}^{1-\frac{r}{q}}.$$

The proof now finishes with a straightforward application of Lemma 6.8. $\square$

6.2. **Real interpolation on cones.** What we are going to do now is to look at the inequalities for sums like in (6) from the point of view of multilinear mappings. We fix a polynomial $P \in \mathscr{P}(^m\mathbb{C}^n)$ and consider the mapping $\mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \ell_1(\mathscr{J}(m,n))$, given by

$$(z^{(1)},\dots,z^{(m)}) \mapsto \big(c_{\mathbf{j}}(P) z^{(1)}_{j_1} \dots z^{(m)}_{j_m}\big)_{\mathbf{j}\in\mathscr{J}(m,n)}. \tag{33}$$

Note that, since everything here is finite dimensional, the mapping is well defined. The idea is, then, to consider norms on the domain spaces so that the norm of this mapping is bounded by a term involving the norm of the polynomial and some constant independent of n. Since the inequality that we get in Proposition 6.4 requires some variables to be decreasing we have to restrict ourselves to cones of decreasing sequences. To be more precise, if we denote $\ell^d_{q,s} := \{z \in \ell_{q,s} : |z| = z^*\}$ for $1 \le s \le \infty$, Proposition 6.4 tells us that there is a constant $C_{m,r} > 1$ (independent of P and n) such that, for every $1 \le k \le m-1$, the mapping

$$T_k : \underbrace{\ell^n_{q,\infty} \times \cdots \times \ell^n_{q,\infty}}_{k-1} \times \ell^n_{q,1} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-k} \to \ell_1(\mathscr{J}(m,n)), \tag{34}$$

given by (33) satisfies

$$\|T_k\| \le C_{m,r} \|P\|_{\mathscr{P}(^m\ell^n_r)}. \tag{35}$$

All these mappings have the same defining formula (which is m-linear), so it is tempting to apply multilinear interpolation. But, since we need to restrict ourselves to the cone of non-increasing sequences in the last $m-k$ variables, we are not able to directly apply the classical multilinear interpolation results, but interpolation in cones.

For the general theory of interpolation we follow (and refer the reader to) [BL76]. Since (as we have already explained) we have to consider linear operators on cones, we use the K-method of interpolation for operators on the cone of non-increasing sequences, as presented in [CM96]. Then the main result of this section, from which Theorem 6.1, follows is the following.

Theorem 6.9. *Let $1 < r \le 2$ and $m \ge 3$. Define $q := (mr')'$ and*

$$s = \begin{cases} 2 & \text{if } m = 3 \\ \frac{3+\sqrt{5}}{2} & \text{if } m = 4 \\ \frac{m}{\log(m)} & \text{if } m \ge 5 \end{cases}$$

There exists a constant $C_{m,r} \ge 1$ such that, for every $P \in \mathscr{P}(^m\mathbb{C}^n)$ the m-linear mapping

$$T : \underbrace{(\ell^n_{q,s})^d \times \cdots \times (\ell^n_{q,s})^d}_{m-1} \times (\ell^n_{q,\infty})^d \to \ell_1(\mathscr{J}(m,n))$$

given by

$$(z^{(1)},\dots,z^{(m)}) \mapsto \big(c_{\mathbf{j}}(P) z^{(1)}_{j_1} \dots z^{(m)}_{j_m}\big)_{\mathbf{j}\in\mathscr{J}(m,n)}$$

satisfies

$$\|T\| \leq C_{m,r} \|P\|_{\mathscr{P}(^m \ell_r^n)}.$$

Remark 6.10. If we take $z^{(1)} = \ldots = z^{(m)} = z$ and observe that $\|z\|_{\ell_{q,\infty}} \leq \|z\|_{\ell_{q,s}}$, Theorem 6.9 gives

$$\sum_{1 \leq j_1 \leq \cdots \leq j_m \leq n} |c_{\mathbf{j}}(P) z^*_{j_1} \cdots z^*_{j_m}| \leq C_{m,r} \|z\|^m_{\ell_{q,s}} \|P\|_{\mathscr{P}(^m \ell_r^n)}$$

for every $P \in \mathscr{P}(^m \mathbb{C}^n)$ and $z \in \mathbb{C}^n$. A standard argument shows that $z^* \in \mathrm{mon}\mathscr{P}(^m \ell_r)$ for every $z \in \ell_{q,s}$ and, then, Corollary 3.6 implies $\ell_{q,s} \subset \mathrm{mon}\mathscr{P}(^m \ell_r)$. This gives Theorem 6.1.

Before we proceed, let us fix some notation. Given a Banach function lattice X (in particular a sequence space or a finite dimensional Banach space, on which we are mainly interested), we write X^d for the cone of non-increasing functions in X. If Y is any Banach space and $S : X \to Y$ is a linear operator we can restrict it to the cone and denote

$$\|S : X^d \to Y\| = \inf\{\|S(x)\|_Y : x \in X^d, \|x\| < 1\}. \tag{36}$$

Clearly neither is X^d a vector space, nor is $\|S\|$ a norm. We will later use an analogous notation for m-linear mappings. We are now ready to state our main tool to interpolate in cones. It is a direct corollary of [CM96, Theorem 1–(b)] (recall that we are using the notation as introduced there).

Theorem 6.11. *Given a pair of quasi-Banach function lattices* (X_0, X_1)*, a pair of quasi-Banach spaces* (Y_0, Y_1) *and a linear operator* S *defined both* $X_0 \to Y_0$ *and* $X_1 \to Y_1$ *with*

$$\|S : X_0^d \longrightarrow Y_0\| \leq M_0 \quad \text{and} \quad \|S : X_1^d \longrightarrow Y_1\| \leq M_1.$$

Then for every $0 < \theta < 1$ *the operator* $S : (X_0^d, X_1^d)_{\theta,a} \longrightarrow (Y_0, Y_1)_{\theta,a}$ *is well defined and*

$$\|S : (X_0^d, X_1^d)_{\theta,a} \longrightarrow (Y_0, Y_1)_{\theta,a}\| \leq M_0^{1-\theta} M_1^{\theta}.$$

We are going to apply this to Lorentz sequence spaces. In this case, it was proved in [Sag72] (see also [CM96, Theorem 4]) that

$$(\ell^d_{q,p_0}, \ell^d_{q,p_1})_{\theta,a} = (\ell_{q,p_0}, \ell_{q,p_1})^d_{\theta,a}.$$

On the other hand, it is known (see for example [BL76, Theorem 5.3.1]) that whenever $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we have

$$(\ell_{q,p_0}, \ell_{q,p_1})_{\theta,p} = \ell_{q,p},$$

and therefore

$$(\ell^d_{q,p_0}, \ell^d_{q,p_1})_{\theta,p} = \ell^d_{q,p}. \tag{37}$$

Finally [BL76, Theorem 3.7.1] gives that (if p_0, p_1, p are related as before)

$$(\ell'_{q,p_0}, \ell'_{q,p_1})_{\theta,p} = (\ell_{q,p_0}, \ell_{q,p_1})'_{\theta,p} = \ell'_{q,p}. \tag{38}$$

The idea now is to use Theorem 6.11 to interpolate multilinear mappings. Let us explain how we are going to do this. Let $X_1,\dots,X_m$ be Banach function lattices (in our case they will always be finite dimensional Lorentz spaces), Y some Banach space ($\ell_1(\mathscr{J}(m,n)$ for us) and some continuous m-linear $T: X_1\times\cdots\times X_m\to Y$ (for us given by (33)). Now we fix $1\le j\neq k\le m$ and, for each $i\neq j,k$ pick $z^{(i)}\in X_i$ and $\varphi\in Y'$ and consider $v=(z^{(1)},\dots,z^{(m)},\varphi)$. Now we define

$$T_v : X_j \to X_k' \quad \text{by} \quad \big(T_v(z^{(j)})\big)(z^{(k)}) = \varphi(T(z^{(1)},\dots,z^{(m)})). \tag{39}$$

An easy computation shows that

$$\|T_v\| \le \|\varphi\|\,\|T\| \prod_{i\neq j,k} \|z^{(i)}\|\,. \tag{40}$$

Observe that in this procedure we may consider X_i^d for every i except for $i=k$, getting the same estimate for the norm (defining the "norm" for multilinear mappings on cones with the same idea as in (36)). We are now ready to present the main technical tool for the proof of Theorem 6.9.

Lemma 6.12. *Let $m\ge 3$, $1<r\le 2$, define $q:=(mr')'$ and let $C_{m,r}$ be the constant from Proposition 6.4. For each $0<\theta<1$, every $P\in\mathscr{P}({}^m\mathbb{C}^n)$ and all $1\le k\le m-2$ the m-linear mapping*

$$T^k(\theta) : \big(\ell^n_{q,(\frac{1}{1-\theta})^k}\big)^d \times \underbrace{\big(\ell^n_{q,\frac{1}{\theta}}\big)^d \times\cdots\times \big(\ell^n_{q,\frac{1}{\theta}}\big)^d}_{k} \times \underbrace{\big(\ell^n_{q,\infty}\big)^d\times\cdots\times\big(\ell^n_{q,\infty}\big)^d}_{m-k-1} \to \ell_1(\mathscr{J}(m,n))$$

given by (33) satisfies

$$\|T^k(\theta)\| \le C_{m,r}\|P\|_{\mathscr{P}({}^m\ell_r^n)}.$$

Proof. We proceed by induction on k and begin with the case $k=1$. We consider the mappings (see (34))

$$T_1 : \ell^n_{q,1} \times \underbrace{(\ell^n_{q,\infty})^d\times(\ell^n_{q,\infty})^d\times\cdots\times(\ell^n_{q,\infty})^d}_{m-1} \to \ell_1(\mathscr{J}(m,n))$$

$$T_2 : \ell^n_{q,\infty} \times \ell^n_{q,1} \times \underbrace{(\ell^n_{q,\infty})^d\times\cdots\times(\ell^n_{q,\infty})^d}_{m-2} \to \ell_1(\mathscr{J}(m,n)).$$

We fix $z^{(3)},\dots,z^{(m)}\in(\ell^n_\infty)^d$ and $\varphi\in\Big(\ell_1(\mathscr{J}(m,n))\Big)'$ and writing $v=(z^{(3)},\dots,z^{(m)},\varphi)$ define, following (39), two linear operators

$$(T_1)_v : \big(\ell^n_{q,\infty}\big)^d \to \big(\ell^n_{q,1}\big)' \quad \text{and} \quad (T_2)_v : \big(\ell^n_{q,1}\big)^d \to \big(\ell^n_{q,\infty}\big)'$$

that, by (35) and (40), satisfy (for $i=1,2$)

$$\|(T_i)_v\| \le C_{m,r}\|P\|_{\mathscr{P}({}^m\ell_r^n)}\|z^{(3)}\|_{\ell_{q,\infty}}\cdots\|z^{(m)}\|_{\ell_{q,\infty}}\|\varphi\|_{\ell_1(\mathscr{J}(m,n))'}\,.$$

Now we interpolate, using Theorem 6.11 and equations (37) and (38), to have

$$\Big\|\big(T^1(\theta)\big)_v : \big(\ell^n_{q,\frac{1}{\theta}}\big)^d \to \big(\ell^n_{q,\frac{1}{1-\theta}}\big)'\Big\| \le C_{m,r}\|P\|_{\mathscr{P}({}^m\ell_r^n)}\|z^{(3)}\|_{\ell_{q,\infty}}\cdots\|z^{(m)}\|_{\ell_{q,\infty}}\|\varphi\|_{\ell_1(\mathscr{J}(m,n))'}$$

for every $0<\theta<1$. This immediately gives (just taking supremums)

$$\left\| T^1(\theta) : \ell^n_{q,\frac{1}{1-\theta}} \times \left(\ell^n_{q,\frac{1}{\theta}}\right)^d \times \underbrace{\left(\ell^n_{q,\infty}\right)^d \times \cdots \times \left(\ell^n_{q,\infty}\right)^d}_{m-2} \to \ell_1(\mathscr{J}(m,n)) \right\| \leq C_{m,r} \|P\|_{\mathscr{P}(^m \ell^n_r)}.$$

Now let us assume that, for $1 \leq k \leq m-2$,

$$T^{k-1}(\theta) : \ell^n_{q,(\frac{1}{1-\theta})^{k-1}} \times \underbrace{(\ell^n_{q,\frac{1}{\theta}})^d \times \cdots \times (\ell^n_{q,\frac{1}{\theta}})^d}_{k-1} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-k} \to \ell_1(\mathscr{J}(m,n))$$

has norm $\leq C_{m,r}\|P\|_{\mathscr{P}(^m\ell^n_r)}$. On the other hand consider the mapping defined by Theorem 6.4 (see (34))

$$T_{k+1} : \underbrace{\ell^n_{q,\infty} \times \cdots \times \ell^n_{q,\infty}}_{k} \times \ell^n_{q,1} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-k-1} \to \ell_1(\mathscr{J}(m,n))$$

that (recall (40)) also has norm $\leq C_{m,r}\|P\|_{\mathscr{P}(^m\ell^n_r)}$. Since $\|\ell^n_{q,\frac{1}{\theta}} \hookrightarrow \ell^n_{q,\infty}\| = 1$ we have (recall (36))

$$T_{k+1} : \ell^n_{q,\infty} \times \underbrace{(\ell^n_{q,\frac{1}{\theta}})^d \times \cdots \times (\ell^n_{q,\frac{1}{\theta}})^d}_{k-1} \times \ell_{q,1} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-k-1} \to \ell_1(\mathscr{J}(m,n))$$

has again norm bounded by $C_{m,r}\|P\|_{\mathscr{P}(^m\ell^n_r)}$. We fix $\varphi \in \left(\ell_1(\mathscr{J}(m,n))\right)'$ and $z^{(i)} \in (\mathbb{C}^n)^d$ for $i \neq 1,k$ and, taking $v = (z^{(2)},\dots,z^{(k)},z^{(k+2)},\dots,z^{(m)},\varphi)$ we have, by (39) and (40)

$$\begin{aligned}
&\left\| (T^{k-1}(\theta))_v : (\ell^n_{q,\infty})^d \to \left(\ell^n_{q,(\frac{1}{1-\theta})^{k-1}}\right)' \right\| \\
&\qquad \leq C_{m,r}\|P\|_{\mathscr{P}(^m\ell^n_r)} \|\varphi\|_{\ell_1(\mathscr{J}(m,n))'} \|z^{(2)}\|_{\ell_{q,\frac{1}{\theta}}} \cdots \|z^{(k)}\|_{\ell_{q,\frac{1}{\theta}}} \|z^{(k+2)}\|_{\ell_{q,\infty}} \cdots \|z^{(m)}\|_{\ell_{q,\infty}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| (T_{k+1})_v : (\ell^n_{q,1})^d \to \left(\ell^n_{q,\infty}\right)' \right\| \\
&\qquad \leq C_{m,r}\|P\|_{\mathscr{P}(^m\ell^n_r)} \|\varphi\|_{\ell_1(\mathscr{J}(m,n))'} \|z^{(2)}\|_{\ell_{q,\frac{1}{\theta}}} \cdots \|z^{(k)}\|_{\ell_{q,\frac{1}{\theta}}} \|z^{(k+2)}\|_{\ell_{q,\infty}} \cdots \|z^{(m)}\|_{\ell_{q,\infty}}.
\end{aligned}$$

Once again, we may interpolate using Theorem 6.11, (37) and (38) to have

$$\begin{aligned}
&\left\| (T^{k}(\theta) v : (\ell^n_{q,\frac{1}{\theta}})^d \to \left(\ell^n_{q,\frac{1}{(1-\theta)^k}}\right)' \right\| \\
&\qquad \leq C_{m,r}\|P\|_{\mathscr{P}(^m\ell^n_r)} \|\varphi\|_{\ell_1(\mathscr{J}(m,n))'} \|z^{(2)}\|_{\ell_{q,\frac{1}{\theta}}} \cdots \|z^{(k)}\|_{\ell_{q,\frac{1}{\theta}}} \|z^{(k+2)}\|_{\ell_{q,\infty}} \cdots \|z^{(m)}\|_{\ell_{q,\infty}}
\end{aligned}$$

for every $0<\theta<1$. Taking supremum as before this gives

$$\left\| T^k(\theta) : \ell^n_{q,(\frac{1}{1-\theta})^k} \times \underbrace{(\ell^n_{q,\frac{1}{\theta}})^d \times \cdots \times (\ell^n_{q,\frac{1}{\theta}})^d}_{k} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-k-1} \to \ell_1(\mathscr{J}(m,n)) \right\| \leq C_{m,r}\|P\|_{\mathscr{P}(^m\ell^n_r)}.$$

$\square$

Proof of Theorem 6.9. For $m \geq 5$, we choose $\theta = \frac{\log(m+\frac{3}{2})}{m-1+\log(m+\frac{3}{2})}$. Then $\frac{1}{\theta} \geq \frac{m}{\log(m)}$ and

$$\left(\frac{1}{1-\theta}\right)^k = \left(1+\frac{\log(m+\frac{3}{2})}{m-1}\right)^{m-2} \geq \frac{m}{\log m}.$$

Therefore $\|\ell^n_{q,(\frac{1}{1-\theta})^k} \hookrightarrow \ell^n_{q,\frac{m}{\log(m)}}\| = \|\ell^n_{q,\frac{1}{\theta}} \hookrightarrow \ell^n_{q,\frac{m}{\log(m)}}\| = 1$. Using Lemma 6.12 with $k = m-2$ the result follows. For $m = 3$ and $m = 4$ just take $\theta = \frac{1}{2}$ and $\theta = \frac{3}{2} - \frac{\sqrt{5}}{2}$ in Lemma 6.12, respectively. □

We finish this section with some comments on the hypercontractivity of the inclusion of $\ell_{q,s}$ in $\mathrm{mon}\mathscr{P}(^m\ell_r)$. For the ℓ_∞ case it is known (see [BDF⁺17, Theorem 2.1]) that the inclusion $\ell_{\frac{2m}{m-1},\infty}$ in $\mathrm{mon}\mathscr{P}(^m\ell_\infty)$ is hypercontractive in the sense that there exists a constant $C > 0$ such for every $P \in \mathscr{P}(^m\ell_\infty)$,

$$\sum_{\mathbf{j}\in\mathscr{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| \leq C^m \|z\|^m_{\ell_{\frac{2m}{m-1},\infty}} \|P\|_{P(^m\ell_\infty)}.$$

For $1 < r \leq 2$, although we do not know if $\ell_{q,\infty}$ lies in the set $\mathrm{mon}\mathscr{P}(^m\ell_r)$ it is easy to see that we cannot expect to have a hypercontractive inequality as above.

Remark 6.13. Proceeding as in the proof of the upper inclusion in Theorem 4.1 (see (14)) with $m = \lfloor\log(n+1)\rfloor$ we would have that

$$\frac{1}{\|z\|_{\ell_{q,\log m}} \log(n+1)^{1-\frac{1}{r}}} \sum_{j=1}^n |z_j^*|$$

is bounded independently of n for every $z \in \ell_{q,\log m}$. Take now $z = (j^{-1/q}\log(j)^{-2/\log(m)})_j$. Then $\|z\|_{\ell_{q,\log m}} \leq \left(\sum_{j=1}^\infty \frac{1}{j\log^2(j)}\right)^{\frac{1}{\log m}}$. But,

$$\begin{aligned}\frac{1}{\|z\|_{\ell_{q,\log m}} \log(n+1)^{1-\frac{1}{r}}} \sum_{j=1}^n |z_j^*| &\gg \frac{1}{\log(n+1)^{1-\frac{1}{r}}} \sum_{j=1}^n \frac{1}{j^{1/q}\log(j)^{\frac{2}{\log m}}} \\ &\gg \frac{e^2}{c\log(n+1)^{1-\frac{1}{r}}} \sum_{j=1}^n \frac{1}{j^{1/q}} \geq \frac{e^2}{c\log(n+1)^{1-\frac{1}{r}}} n^{1/q'} q'.\end{aligned}$$

Since $q' = mr' = \lfloor\log(n+1)\rfloor r'$, the last expression is $\gg \log(n)^{\frac{1}{r}}$. This shows that there exists no constant $C > 0$ such that for every n and m and all $P \in \mathscr{P}(^m\mathbb{C}^n)$ we have

$$\sum_{\mathbf{j}\in\mathscr{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| \leq C^m \|z\|^m_{\ell_{q,\log m}} \|P\|_{\mathscr{P}(^m\ell_r^n)}.$$

On the other hand, applying carefully the ideas developed in this section, it is possible to obtain hypercontractive inequalities in some cases.

Remark 6.14. Given $\varepsilon > 0$, there exists a constant $C > 0$ such that for every $m \geq 3$, $n \in \mathbb{N}$ and every $P \in \mathscr{P}({}^m\mathbb{C}^n)$

$$\sum_{\mathbf{j} \in \mathscr{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leq C(1+\varepsilon)^m \|P\|_{\mathscr{P}({}^m\ell_r^n)} \|z\|^m_{\ell^n_{q,2}}.$$

To see this fix $1 < r \leq 2$, $m \geq 3$, and take z, $z^{(m-2)}$, $z^{(m-1)}$, $w \in \mathbb{C}^n$ such that $z^{(m-1)} = z^{(m-1)*}$ and $w = w^*$. Then we have, using Lemma 4.5 (see also (15)) and Lemma 6.2,

$$\begin{aligned}
&\sum_{\mathbf{j} \in \mathscr{J}(m,n)} |c_{\mathbf{j}}(P) z_{j_1} \dots z_{j_{m-3}} z^{(m-2)}_{j_{m-2}} z^{(m-1)}_{j_{m-1}} w_{j_m}| \\
&\quad \leq em \|P\|_{\mathscr{P}({}^m\ell_r^n)} \sum_{\mathbf{j} \in \mathscr{J}(m-1,n)} |z_{j_1} \dots z_{j_{m-3}} z^{(m-2)}_{j_{m-2}} z^{(m-1)}_{j_{m-1}}| \cdot \Big(\frac{(m-1)^{m-1}}{\alpha(\mathbf{j})^{\alpha(\mathbf{j})}} \Big)^{1/r} \Big(\sum_{j_m = j_{m-1}}^n w^r_{j_m} \Big)^{1/r} \\
&\quad \leq em^3 Cm^{e^{r'-1}} \|P\|_{\mathscr{P}({}^m\ell_r^n)} \sum_{j_{m-2}=1}^n |z^{(m-2)}_{j_{m-2}}| \Big(\sum_{\mathbf{j} \in \mathscr{J}(m-3, j_{m-2})} |\mathbf{j}||z_{\mathbf{j}}| \Big) \sum_{j_{m-1}=j_{m-2}}^n |z^{(m-1)}_{j_{m-1}}| \Big(\sum_{j_m = j_{m-1}}^n w^r_{j_m} \Big)^{1/r} \\
&\quad \leq Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|P\|_{\mathscr{P}({}^m\ell_r^n)} \sum_{j_{m-2}=1}^n |z^{(m-2)}_{j_{m-2}}| \Big(\sum_{l=1}^{j_{m-2}} |z_l| \Big)^{m-3} \sum_{j_{m-1}=j_{m-2}}^n |z^{(m-1)}_{j_{m-1}}| j_{m-1}^{\frac{1}{r}-\frac{1}{q}} \\
&\quad \leq Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|P\|_{\mathscr{P}({}^m\ell_r^n)} \sum_{j_{m-2}=1}^n |z^{(m-2)}_{j_{m-2}}| \Big((j_{m-2})^{1-\frac{1}{q}} \|z\|_{\ell_{q,\infty}} \Big)^{m-3} \|z^{(m-1)}\|_{\ell_{q,\infty}} (r'+1) j_{m-2}^{\frac{2}{q'}-\frac{1}{r'}} \\
&\quad \leq (r'+1) Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|z\|^{m-3}_{\ell_{q,\infty}} \|z^{(m-2)}\|_{\ell_{q,1}} \|z^{(m-1)}\|_{\ell_{q,\infty}} \|P\|_{\mathscr{P}({}^m\ell_r^n)},
\end{aligned}$$

where in the penultimate inequality we used the bound of the identity from ℓ_1^k to $\ell_{q,\infty}^k$ that may be found for example in [DM06, Lemma 22]. On the other hand, we also have,

$$\begin{aligned}
&\sum_{\mathbf{j} \in \mathscr{J}(m,n)} |c_{\mathbf{j}}(P) z_{j_1} \dots z_{j_{m-3}} z^{(m-2)}_{j_{m-2}} z^{(m-1)}_{j_{m-1}} w_{j_m}| \\
&\quad \leq em \|P\|_{\mathscr{P}({}^m\ell_r^n)} \sum_{\mathbf{j} \in \mathscr{J}(m-1,n)} |z_{j_1} \dots z_{j_{m-3}} z^{(m-2)}_{j_{m-2}} z^{(m-1)}_{j_{m-1}}| \cdot \Big(\frac{(m-1)^{m-1}}{\alpha(\mathbf{j})^{\alpha(\mathbf{j})}} \Big)^{1/r} \Big(\sum_{j_m = j_{m-1}}^n w^r_{j_m} \Big)^{1/r} \\
&\quad \leq em^3 Cm^{e^{r'-1}} \|P\|_{\mathscr{P}({}^m\ell_r^n)} \sum_{j_{m-1}=1}^n |z^{(m-1)}_{j_{m-1}}| \Big(\sum_{\mathbf{j} \in \mathscr{J}(m-3, j_{m-2})} |\mathbf{j}||z_{\mathbf{j}}| \Big) \sum_{j_{m-2}=1}^{j_{m-1}} |z^{(m-2)}_{j_{m-2}}| \Big(\sum_{j_m = j_{m-1}}^n w^r_{j_m} \Big)^{1/r} \\
&\quad \leq Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|P\|_{\mathscr{P}({}^m\ell_r^n)} \sum_{j_{m-1}=1}^n |z^{(m-1)}_{j_{m-1}}| \Big(\sum_{l=1}^{j_{m-2}} |z_l| \Big)^{m-3} \sum_{j_{m-2}=1}^{j_{m-1}} |z^{(m-2)}_{j_{m-2}}| j_{m-1}^{\frac{1}{r}-\frac{1}{q}} \\
&\quad \leq Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|P\|_{\mathscr{P}({}^m\ell_r^n)} \sum_{j_{m-1}=1}^n |z^{(m-1)}_{j_{m-1}}| \Big((j_{m-2})^{1-\frac{1}{q}} \|z\|_{\ell_{q,\infty}} \Big)^{m-3} j_{m-1}^{1-\frac{1}{q}} \|z^{(m-2)}\|_{\ell_{q,\infty}} j_{m-1}^{\frac{1}{r}-\frac{1}{q}} \\
&\quad = Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|z\|^{m-3}_{\ell_{q,\infty}} \|z^{(m-2)}\|_{\ell_{q,\infty}} \|z^{(m-1)}\|_{\ell_{q,1}} \|P\|_{\mathscr{P}({}^m\ell_r^n)}.
\end{aligned}$$

Thus, proceeding as in Lemma 6.12 we may construct an operator which is bounded from $\ell^d_{q,\infty}$ to $(\ell_{q,1})'$ and also from $\ell^d_{q,1}$ to $(\ell_{q,\infty})'$. Applying the K-interpolation method restricted to the cone of non-increasing sequences to this operator we can conclude that for any $z = z^*$,

$$\sum_{\mathbf{j}\in\mathscr{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| \le \sqrt{(1+r')}Cm^{e^{r'}}\|z\|_{\ell_{q,\infty}}^{m-2}\|z\|_{\ell_{q,2}}^{2}\|P\|_{\mathscr{P}(^m\ell_r^n)} \le C(1+\varepsilon)^m\|P\|_{\mathscr{P}(^m\ell_r^n)}\|z\|_{\ell_q^n}^m.$$

Therefore, by (11), we have proved our claim.

With some extra work it can proved, in a similar way, that given any $s \ge 1$ and $\varepsilon > 0$, there exist some m_0 and some $C > 0$ such that for every $n \in \mathbb{N}$, all $m \ge m_0$ and every polynomial $P \in \mathscr{P}(^m\mathbb{C}^n)$ we have

$$\sum_{\mathbf{j}\in\mathscr{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| \le C(1+\varepsilon)^m\|P\|_{\mathscr{P}(^m\ell_r^n)}\|z\|_{\ell_{q,s}^n}^m.$$

7. Some consequences

We now provide several consequences of the results obtained in the previous sections.

7.1. **Mixed unconditionality for spaces of m-homogeneous polynomials.** Let us recall that, if $(P_i)_{i\in\Lambda}$ is a Schauder basis of $\mathscr{P}(^m\mathbb{C}^n)$, for $1 \le r, s \le \infty$ and $n, m \in \mathbb{N}$, the mixed unconditional basis constant $\chi_{r,s}((P_i)_{i\in\Lambda})$ is defined as the best constant $C > 0$ such that

$$\|\sum_{i\in\Lambda}\theta_i c_i P_i\|_{\mathscr{P}(^m\ell_s^n)} \le C\|\sum_{i\in\Lambda} c_i P_i\|_{\mathscr{P}(^m\ell_r^n)},$$

for every $P = \sum_{i\in\Lambda} c_i P_i \in \mathscr{P}(^m\mathbb{C}^n)$ and every choice of complex numbers $(\theta_i)_{i\in\Lambda}$ of modulus one. Once we have this, the (r,s)-mixed unconditional constant of $\mathscr{P}(^m\mathbb{C}^n)$ is defined as

$$\chi_{r,s}(\mathscr{P}(^m\mathbb{C}^n)) := \inf\{\chi_{p,q}((P_i)_{i\in\Lambda}) : (P_i)_{i\in\Lambda} \text{ basis for } \mathscr{P}(^m\mathbb{C}^n)\}.$$

This notion was introduced by Defant, Maestre and Prengel in [DMP09, Section 5].

In [GMMb] the exact asymptotic asymptotic growth of the mixed-(r,s) unconditional constant as n tends to infinity was computed for many values of p and q's. To achieve this the authors proved that

$$\chi_{r,s}(\mathscr{P}(^m\mathbb{C}^n)) \sim \chi_{r,s}\big((z_{\mathbf{j}})_{\mathbf{j}\in\mathscr{J}(m,n)}\big).$$

We complete the result given in [GMMb, Theorem 3.4] by providing the exact asymptotic asymptotic growth for the remaining cases. In this way we have the behaviour of $\chi_{p,q}(\mathscr{P}(^m\mathbb{C}^n))$ for *every* $1 \le p, q \le \infty$.

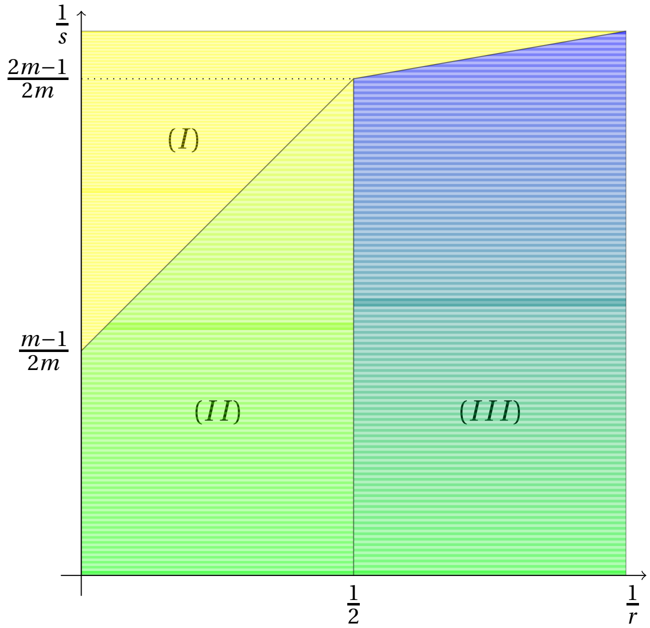

FIGURE 1. Graphical overview of the mixed unconditional constant described in Theorem 7.1.

Theorem 7.1. *For each $m \in \mathbb{N}$ we have*

$$\begin{cases} \chi_{r,s}(\mathscr{P}(^m\mathbb{C}^n)) \sim 1 & \text{for } (I): [\frac{1}{r} + \frac{m-1}{2m} \leq \frac{1}{s} \wedge \frac{1}{r} \leq \frac{1}{2}] \text{ or } [\frac{m-1}{m} + \frac{1}{mr} < \frac{1}{s} \wedge \frac{1}{2} \leq \frac{1}{r}], \\ \chi_{r,s}(\mathscr{P}(^m\mathbb{C}^n)) \sim n^{m(\frac{1}{r}-\frac{1}{s}+\frac{1}{2})-\frac{1}{2}} & \text{for } (II)\ [\frac{1}{r} + \frac{m-1}{2m} \geq \frac{1}{s} \wedge \frac{1}{r} \leq \frac{1}{2}], \\ \chi_{r,s}(\mathscr{P}(^m\mathbb{C}^n)) \sim n^{(m-1)(1-\frac{1}{s})+\frac{1}{r}-\frac{1}{s}} & \text{for } (III): [1-\frac{1}{m}+\frac{1}{mr} \geq \frac{1}{s} \wedge \frac{1}{2} < \frac{1}{r} < 1]. \end{cases}$$

Proof. The behaviour of $\chi_{r,s}(\mathscr{P}(^m\mathbb{C}^n))$ in regions (I) and (II) was already given in [GMMb, Theorem 3.4]. We now deal with (III). By Theorem 6.3 we know that $\ell_q \subset \operatorname{mon}\mathscr{P}(^m\ell_r)$. Thus, for every polynomial $P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha \in \mathscr{P}(^m\mathbb{C}^n)$ we have

$$\sum_{|\alpha|=m} |c_\alpha z^\alpha| \leq C^m \|z\|_{\ell_q}^m \|P\|_{\mathscr{P}(^m\ell_r)}, \tag{41}$$

where $q := (mr')'$. Since

$$\|z\|_{\ell_q} \leq n^{\frac{1}{q}-\frac{1}{s}} \|z\|_{\ell_s}, \tag{42}$$

combining (41) and (42) yields

$$\chi_{r,s}(\mathscr{P}(^m\mathbb{C}^n)) \leq n^{(\frac{1}{q}-\frac{1}{s})m} = n^{(1-\frac{1}{mr'}-\frac{1}{s})m} = n^{m(1-\frac{1}{s})-\frac{1}{r'}} = n^{(m-1)(1-\frac{1}{s})+\frac{1}{r}-\frac{1}{s}}. \tag{43}$$

□

7.2. **Mixed Bohr radius.** Let $K(B_{\ell_p^n}, B_{\ell_q^n})$ be the n-dimensional (p,q)-Bohr radius for holomorphic functions on $\mathbb{C}^n$. That is, $K(B_{\ell_p^n}, B_{\ell_q^n})$ denotes the greatest number $r \geq 0$ such that for every entire function $f(z) = \sum_\alpha a_\alpha z^\alpha$ in n-complex variables, we have the following (mixed) Bohr-type inequality

$$\sup_{z \in r \cdot B_{\ell_q^n}} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z \in B_{\ell_p^n}} |f(z)|.$$

The exact asymptotic growth of $K(B_{\ell_p^n}, B_{\ell_q^n})$ with n was given in [GMMa, Theorem 1.2]. More precisely, $K(B_{\ell_p^n}, B_{\ell_1^n}) \sim 1$ for every $1 \leq p \leq \infty$, and for $1 \leq p, q \leq \infty$, with $q \neq 1$,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \sim \begin{cases} 1 & \text{if (I): } 2 \leq p \leq \infty \wedge \frac{1}{2} + \frac{1}{p} \leq \frac{1}{q}, \\ \frac{\sqrt{\log(n)}}{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}} & \text{if (II): } 2 \leq p \leq \infty \wedge \frac{1}{2} + \frac{1}{p} > \frac{1}{q}, \\ \frac{\log(n)^{1-\frac{1}{p}}}{n^{1-\frac{1}{q}}} & \text{if (III): } 1 \leq p \leq 2. \end{cases}$$

As a consequence of our result we can give an alternative proof of the lower bounds for $K(B_{\ell_p^n}, B_{\ell_q^n})$ for the case $1 \leq p \leq 2$ (and every $1 \leq q \leq \infty$). It should be noted that this is the most complicated part of [GMMa, Theorem 1.2]. Let us see how.

By [DMP09, Theorem 5.1] and Lemma 5.3, there is a constant $C := C(p) > 0$ such that for every polynomial P in n complex variables we have

$$\sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leq C^m \|z\|^m_{(m_{\Psi_p})_n} \|P\|_{\mathcal{P}(^m\ell_p^n)}, \tag{44}$$

where $(m_{\Psi_p})_n$ is defined as the quotient space induced by the mapping

$$\begin{aligned} \pi_n : m_{\Psi_p} &\to \mathbb{C}^n \\ x &\mapsto (x_1, \ldots, x_n). \end{aligned}$$

Note that there is a constant $D = D(p,q) > 0$ such that $\|z\|_{(m_{\Psi_p})_n} \leq D \frac{n^{1-\frac{1}{q}}}{\log(n)^{1-\frac{1}{p}}} \|z\|_{\ell_q^n}$. Therefore, by (44) we have

$$\sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leq (CD)^m \left(\frac{n^{1-\frac{1}{q}}}{\log(n)^{1-\frac{1}{p}}} \right)^m \|z\|^m_{\ell_q^n} \|P\|_{\mathcal{P}(^m\ell_p^n)},$$

This implies that $\chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n))^{1/m} \ll \frac{n^{1-\frac{1}{q}}}{\log(n)^{1-\frac{1}{p}}}$. It should be noted that here is important to have control of the growth of the (p,q)-mixed unconditional constant also in terms of m (the homogeneity degree), contrary to problem treated in the previous subsection. The result now follows using that (see [GMMa, Lemma 2.2.]) for every $n \in \mathbb{N}$ and $1 \leq p, q \leq \infty$ we have

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \sim \frac{1}{\sup_{m \geq 1} \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n))^{1/m}}.$$

7.3. **Multipliers.** A sequence $(a_n)_{n\in\mathbb{N}}$ is a multiplier for mon$\mathscr{P}(^m\ell_r)$ if

$$(a_n)_{n\in\mathbb{N}}\cdot\ell_r \subset \mathrm{mon}\mathscr{P}(^m\ell_r),$$

where the product $(a_n)_{n\in\mathbb{N}}\cdot\ell_r$ is just the coordinate-wise multiplication. Let $p=(p_1,p_2,\ldots)$ be the sequence of the prime numbers. It is well-known that for $r\geq 2$, the sequence $\frac{1}{p^{\frac{m-1}{2m}}}$ is a multiplier for mon$\mathscr{P}(^m\ell_r)$ (this can be as an immediate consequence of [BDS, Theorem 5.1 (3)]).

For $1<r<2$ in [BDS, Theorem 5.3.] prove this up to an ε, showing that for each m and every $\varepsilon>\frac{1}{r}$

$$\frac{1}{p^{\sigma_m}\big(\log(p)\big)^{\varepsilon}}\cdot\ell_r \subset \mathrm{mon}\mathscr{P}(^m\ell_r), \tag{45}$$

where $\sigma_m=\frac{m-1}{m}\left(1-\frac{1}{r}\right)$. As a consequence of our results, we can improve this, showing that, for $1<r\leq 2$, even the sequence $(\frac{1}{n^{\sigma_m}})_{n\in\mathbb{N}}$ is a multiplier for mon$\mathscr{P}(^m\ell_r)$.

Theorem 7.2. *For $1<r<2$ and $m\geq 3$ put $\sigma_m=\frac{m-1}{m}\left(1-\frac{1}{r}\right)$. Then,*

$$\Big(\frac{1}{n^{\sigma_m}}\Big)_n\cdot\ell_r \subset \mathrm{mon}\mathscr{P}(^m\ell_r),$$

and σ_m is best possible.

Proof. As a consequence of Theorem 6.1 we know that $\ell_{q,r}\subset \mathrm{mon}\mathscr{P}(^m\ell_r)$, thus to prove the result it is sufficient to see that if $z\in\ell_r$ then, $\left(\frac{1}{n^{\sigma_m}}\right)_n\cdot z\in\ell_{q,r}$. Suppose that $z\in\ell_r$ is an arbitrary element (not necessarily equal to z^*). Since $r>q$ we know that the norm $\|\cdot\|_{\ell_{q,r}}$ is equivalent to the following maximal norm (see [BS88, Lemma 4.5])

$$\|w\|^*_{\ell_{q,r}}=\left(\sum_{n=1}^{\infty} n^{\frac{r}{q}-1}\left(\frac{1}{n}\sum_{k=1}^{n} w_k^*\right)^r\right)^{1/r}.$$

Then, if $w=\left(\frac{z_n}{n^{\sigma_m}}\right)_n$, by the Hardy-Littlewood rearrangement inequality (Lemma 6.7) it is easy to see that

$$\sum_{k=1}^{n} w_k^* \leq \sum_{k=1}^{n} z_k^*\frac{1}{k^{\sigma}}$$

for every $n\in\mathbb{N}$. Then

$$\left\|\left(\frac{z_n}{n^{\sigma_m}}\right)_n\right\|_{\ell_{q,r}} \sim \left\|\left(\frac{z_n}{n^{\sigma_m}}\right)_n\right\|^*_{\ell_{q,r}} \leq \left(\sum_{n=1}^{\infty} n^{\frac{r}{q}-1}\left(\frac{1}{n}\sum_{k=1}^{n} z_k^*\frac{1}{k^{\sigma}}\right)^r\right)^{1/r}$$
$$=\left\|\left(\frac{z_n^*}{n^{\sigma_m}}\right)_n\right\|^*_{\ell_{q,r}} \sim \left\|\left(\frac{z_n^*}{n^{\sigma_m}}\right)_n\right\|_{\ell_{q,r}} = \left(\sum_{n=1}^{\infty}\left((\frac{z_n^*}{n^{\sigma_m}})^* n^{\frac{1}{q}-\frac{1}{r}}\right)^r\right)^{1/r} = \|z\|_{\ell_r}<\infty,$$

where, in the last equality, we have used the fact that $\sigma_m=\frac{1}{q}-\frac{1}{r}$.
To see that the exponent is optimal take, as always, $q=(mr')'$. Now, if $(z_n)_n=\left(\frac{1}{n^{1/r}\log(n+1)^{2/r}}\right)_n\in\ell_r$ for every $\varepsilon>0$ it is easy to check that the sequence $\left(\frac{z_n}{n^{\sigma_m-\varepsilon}}\right)_n\notin\ell_{q,\infty}\supset \mathrm{mon}\mathscr{P}(^m\ell_r)$. □

For $m=2$ we cannot show that the sequence is $\left(\frac{1}{n^{\sigma_2}}\right)_n$ is a multiplier for $\mathrm{mon}\mathscr{P}(^2\ell_r)$ but using the fact that $\ell_q \subset \mathrm{mon}\mathscr{P}(^2\ell_r)$, Theorem 6.1, it is easy to see that we have the inclusion

$$\frac{1}{p^{\sigma_2}\big(\log(p)\big)^{\varepsilon}}\cdot \ell_r \subset \mathrm{mon}\mathscr{P}\left(^2\ell_r\right),$$

for *every* $\varepsilon > 0$ extending [BDS, Theorem 5.3.] (see also (45)). We leave the details for the reader.

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