GEOMETRY AND ISOMETRIES OF THE MARCINKIEWICZ SEQUENCE SPACE

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Abstract. We characterise the real extreme points of the unit ball of m_Ψ^0, the complex extreme points of the unit ball of m_Ψ and the real extreme and exposed points of the unit ball of $(m_\Psi^0)'$. Using these characterisations we show that, depending on the length of the extreme points, the multipliers of m_Ψ^0 are either constant multiple of the identity or diagonal operators.

1. Introduction

Since the 1930s, the concepts of rearrangement invariant spaces in general and Marcinkiewicz spaces in particular have played an important role in many areas of analysis. Marcinkiewicz spaces are an intrinsic part of interpolation theory, (see [4, 14]) and the control of their norm given to them by the fundamental function has meant that they are a useful source of counterexamples. Recently, Marcinkiewicz sequence space have been used by Bayart et al., [2], in the description of sets of absolute monomial convergence and of ℓ_1-multipliers of Dirichlet series.

The goal of this paper is twofold. Firstly we want to understand the geometry of the unit ball of the Marcinkiewicz sequence spaces m_Ψ^0, its dual, $(m_\Psi^0)'$ and bidual m_Ψ. Secondly we wish to use this geometry to determine the isometries of m_Ψ. Previously, in [5], the authors had determined the complex extreme points of the unit ball of the (little) Marcinkiewicz space m_Ψ^0 where, it was observed, that this geometry was determined by the finite dimensional subspaces m_Ψ^n. As we will observe in this paper the geometry of the unit ball of m_Ψ depends heavily on the asymptotic values of Ψ. We also determine the geometry of the unit ball of $(m_\Psi^0)'$. It is the geometry of this space which allows us to characterise the isometries of m_Ψ.

Marcinkiewicz sequence spaces and their duals are rearrangement invariant sequence spaces. In order to understand their geometry it is essential to understand precisely what is meant by decreasing rearrangement of a complex bounded sequence. Given a bounded sequence $z = (z_k)_k$ we define the distribution of z, d_z, by $d_z(s) = \#\{j \in \mathbf{N} \colon |z_j| > s\}$ for $s \geq 0$. Bounded sequence x and y are said to be equimeasuarable, $x \sim y$, if $d_x = d_y$. A Banach sequence space $(E, \|\cdot\|_E)$ is said to be a rearrangement invariant space if for any $y \in E$ and x a bounded sequence such that $x \sim y$, then $x \in E$ and $\|x\|_E = \|y\|_E$. To each rearrangement invariant sequence space $(E, \|\cdot\|_E)$ we associate its fundamental sequence, ϕ_E, given by $\phi_E(n) = \|e_1 + \cdots + e_n\|_E$.

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Setting $z^*_k = \inf\{s > 0\colon d_z(s) < k\}$ we obtain the decreasing rearrangement, z^*, of z. For a bounded sequence z we denote the maximal sequence of z^* by

$$z^{**}_n := \frac{1}{n}\sum_{i=1}^{n} z^*_i.$$

By a symbol Ψ we understand an increasing sequence of non-negative real numbers, $\Psi = (\Psi(k))_{k=0}^{\infty}$, with $\Psi(0) = 0$ and $\Psi(k) > 0$ if $k \geq 1$. The Marcinkiewicz sequence space associated to the symbol Ψ, m_Ψ, is the Banach space of all bounded sequences $(z_k)_k$ such that

$$\|z\| := \sup_{k\geq 1} \frac{\sum_{j=1}^{k} z^*_j}{\Psi(k)} < \infty,$$

where $z^* = (z^*_k)_k$ is the decreasing rearrangement of $(z_k)_k$. We denote by m^0_Ψ the subspace of m_Ψ consisting of all z such that

$$\lim_{k\to\infty} \frac{\sum_{j=1}^{k} z^*_j}{\Psi(k)} = 0.$$

To avoid the case where $m^0_\Psi = \{0\}$ we shall assume that $\lim_{k\to\infty}\Psi(k) = \infty$.

We assume without loss of generality that $\Psi(1) = 1$. This condition is equivalent to the assumption that $\|e_k\| = 1$ for all k in $\mathbf{N}$. It follows from [12] that we can also assume that $(\Psi(k)/k)_k$ is decreasing. From this it follows that if $z \in m^0_\Psi$ then $\lim_k |z_k| = 0$ and $\|z\|_\infty \leq \|z\|$. Thus $m^0_\Psi \hookrightarrow c_0$ and the standard unit vectors $(e_k)_k$ form an unconditional basis for m^0_Ψ. This condition will also be important in our understanding of the geometry of $(m^0_\Psi)'$. If $z \in m_\Psi$ we let $\operatorname{supp}(z)$ denote $\#\{j : z_j \neq 0\}$.

Given a Banach space E, B_E denotes its closed unit ball. Also, Δ and $\overline{\Delta}$ denote, respectively, the open and closed complex unit disc. A point z in B_E is said to be a real extreme point of B_E if z is not the midpoint of any line segment which is contained in B_E. When E is a complex Banach space we shall say that z in B_E is a complex extreme point of B_E if $\|z + \lambda y\| \leq 1$ for all $\lambda \in \overline{\Delta}$ implies that $y = 0$. The real extreme points of B_E are denoted by $\mathrm{Ext}_{\mathbf{R}}(B_E)$ while the complex extreme points are denoted by $\mathrm{Ext}_{\mathbf{C}}(B_E)$.

Let E be a complex Banach space. We denote by $\mathcal{A}_b(B_E)$ the Banach algebra of all continuous bounded functions on B_E, which are holomorphic on the interior of B_E. A point z in B_E is said to be a peak point for $\mathcal{A}_b(B_E)$ if there is f in $\mathcal{A}_b(B_E)$ such that $|f(x)| < f(z)$ for all x in $B_E \setminus \{z\}$. A point z in E is said to be an exposed point of the unit ball of E if there is a norm one linear function, $\varphi \in E'$, such that $\varphi(z) = 1$ and $Re(\varphi(x)) < 1$ for all $x \in B_E$, $x \neq z$. When $E = F'$ is a dual space and φ is in F we shall say that z is weak*-exposed. Both, peak and exposed points are complex extreme points, see [11, Theorem 4] and [10, Exercise 3.5].

In [5] we undertook a detailed study of the geometry of the unit ball of m^0_Ψ giving a complete description of its complex extreme points. Necessary and sufficient conditions for the existence of complex extreme points in the unit ball of m^0_Ψ had previously been obtained by Kamińska and Lee [12]. In the same year Choi and Han obtained a characterisation of the extreme points of the unit ball of the finite dimensional Marcinkiewicz sequence space m^n_Ψ. An important class of Marcinkiewicz sequence spaces is formed by the duals of Lorentz sequence spaces. Their geometry and its interaction with boundaries of spaces of analytic functions has been investigated by Moraes and Romero-Grados [15] and by Acosta, Aron and Moraes [1]. In particular, the complex extreme points of the unit ball of a Lorentz sequence space are determined in [1] while the real extreme points of the unit ball of a Lorentz sequence space and its predual are determined by Kamińska, Lee and Lewicki in [13]. In this paper the results in [5] are generalised in two ways. First we determine the real extreme points of the unit ball of m^0_Ψ, then, in the second section, we characterise the complex extreme point of the unit ball of m_Ψ.

Our final section contains a description of the geometry of the unit ball of $(m^0_\Psi)'$ the dual of m^0_Ψ. In particular, we are able to characterise the weak* exposed points of the unit ball of the Lorentz spaces $d(w,1)$ and $\gamma_{1,w}$ extending result of Kamińska, Lee and Lewicki, [13, Theorem 2.6], and of Ciesielski and Lewicki, [8, Theorem 4.7], which characterised the extreme points of $d(w,1)$ and $\gamma_{1,w}$ respectively. We then use our results to give a characterisation of the multipliers of m^0_Ψ. Multiplier and more generally centralisers play a fundamental role in extending the idea of the centre of a unital Banach algebra to a general Banach space setting. See [3] for more details.

2. Real extreme points of m^0_Ψ

Given a symbol Ψ and a positive integer n we let $\mathcal{T}_n$ denote the set of all decreasing n-tuples of strictly positive real numbers $(z_j)_{j=1}^n$ in $\mathbf{C^n}$ with $\sum_{j=1}^k z_j \le \Psi(k)$, for all $k<n$ and $\sum_{j=1}^n z_j = \Psi(n)$.

Lemma 2.1. *Let Ψ be a symbol, $z=(z_j)_j$ be a point of B_{m_Ψ} with strictly positive real coefficients and k be a positive integer with $\sum_{j=1}^k z_j = \Psi(k)$. Let $y=(y_j)_j$ in m_Ψ be such that $\|z\pm y\|\le 1$, then for $1\le j\le k$, y_j is real and $\sum_{j=1}^k y_j = 0$.*

Proof. We now adapt the proof of [1, Lemma 2.7]. For each $j\in\mathbf{N}$, $1\le j\le k$, set $t_j = |z_j+y_j|+|z_j-y_j|-2|z_j|$ and $T_n=\sum_{j=1}^n t_j$. The Triangle Inequality implies that $t_j\ge 0$ for each $j\in\mathbf{N}$ and thus $(T_n)_n$ is an increasing sequence of positive real numbers. We have

$$\sum_{j=1}^k |z_j| = \frac{1}{2}\left(\sum_{j=1}^k |z_j+y_j|+|z_j-y_j|\right) - \frac{1}{2}T_k \le \Psi(k)-\frac{1}{2}T_k.$$

Thus

$$\frac{1}{2}T_k \le \Psi(k) - \sum_{j=1}^k |z_j| = 0$$

and therefore $t_j=0$ all j. Hence we have

$$|z_j+y_j+z_j-y_j| = 2|z_j| = |z_j+y_j|+|z_j-y_j| \qquad (*)$$

for all j in $1\le j\le k$. The argument of [1, Lemma 2.7] now implies that each y_j, $1\le j\le k$, is real. Replacing y with ry we may assume that $\max_{1\le j\le k}|y_j| <$

$\min_{1\le j\le k}|z_j|$. As $\|z\pm y\|\le 1$ we have $\sum_{j=1}^k z_j \pm \sum_{j=1}^k y_j \le \Psi(k) = \sum_{j=1}^k z_j$ and thus $\sum_{j=1}^k y_j = 0$. $\square$

Let Ψ be a symbol. It follows from [5, Theorem 2.10] that if z is a real extreme point of the unit ball of $B_{m_\Psi^0}$ then z has finite support. Baring this in mind, we now present the following characterisation of the unit ball of m_Ψ^0.

Theorem 2.2. *Let Ψ be a symbol and let $z=(z_j)_j$ in $B_{m_\Psi^0}$ such that z^* has n nonzero coordinates. Then, z is a real extreme point of $B_{m_\Psi^0}$ if and only if the following conditions hold.*

(a) *Either*
  (i) $z_1^* = \Psi(1)$*; or*
  (ii) $z_1^* < \Psi(1)$ *and, for some $p>2$, $z_1^*=z_2^*=\cdots=z_p^*>z_{p+1}^*$ then there exists l, $2\le l<p$ such that $\sum_{j=1}^l z_j^* = \Psi(l)$;*

(b) *If there is k, $1<k<n$, and $z_{k-1}^*>z_k^*>z_{k+1}^*>z_{k+2}^*$ then $\sum_{j=1}^k z_j^*=\Psi(k)$;*

(c) *If there are k and p with $k+1<p$ so that $1<k<n-1$ and $z_{k-1}^*>z_k^*>z_{k+1}^*=z_{k+2}^*=\cdots=z_p^*>z_{p+1}^*$ then there exists l, $k\le l<p$ such that $\sum_{j=1}^l z_j^*=\Psi(l)$;*

(d) *If there are k and p with $k<p$ so that $1<k<n-1$ and $z_{k-1}^*>z_k^*=z_{k+1}^*=\cdots=z_p^*>z_{p+1}^*$ then there exists l, $k\le l<p$ such that $\sum_{j=1}^l z_j^*=\Psi(l)$;*

(e) *If $z_{n-1}^*>z_n^*$ then $\sum_{j=1}^n z_j^*=\Psi(n)$.*

Proof. First suppose that there is $n\in\mathbf{N}$ such that (a), (b), (c), (d) and (e) hold. Let y in m_Ψ^0 be such that $\|z^*\pm y\|\le 1$. We claim that $y=0$. Our proof is by induction on k, the nonzero coordinates of y. Firstly, if $z_1=\Psi(1)$ then by (a) (i) it follows immediately that $y_1=0$. On the other hand, if $z_1^*<\Psi(1)$ and for some $p>2$ we have $z_1^*=z_2^*=\cdots=z_p^*>z_{p+1}^*$ then by (a) (ii) we there exists l, $2\le l<p$, with $\sum_{j=1}^l z_j^*=\Psi(l)$. By Lemma 2.1 we have $\sum_{i=1}^l y_{q_i}=0$ for every l-tuple $(q_1,\dots,q_l)$ in $(1,\dots,p)$. As $l<p$ this means that $y_i=0$ for $1\le i\le p$. In particular, $y_1=0$.

Suppose that we have shown that $y_1=\cdots=y_{k-1}=0$. If z^* is such that $z_{k-1}^*>z_k^*>z_{k+1}^*>z_{k+2}^*$ then by (b) we have $\sum_{j=1}^k z_j^*=\Psi(k)$. Again Lemma 2.1 implies that $\sum_{j=1}^k y_j=0$ and hence $y_k=0$. If we have k and p with $k+1<p$ so that $1<k<n-1$ and $z_{k-1}^*>z_k^*>z_{k+1}^*=z_{k+2}^*=\cdots=z_p^*>z_{p+1}^*$ then by (c) there exists l, $k\le l<p$ such that $\sum_{j=1}^l z_j^*=\Psi(l)$. By Lemma 2.1, we have $\sum_{j=k}^l y_j=0$. Hence, $y_k+\sum_{i=k+1}^l y_{q_i}=0$ for any $(l-k)-1$-tuple $(q_1,\dots,q_l)$ in $(1,\dots,p)$ and hence as $l<p$, $y_i=0$ for $k\le i\le p$. In particular, $y_k=0$. If we have k and p with $k<p$ so that $1<k<n-1$ and $z_{k-1}^*>z_k^*=z_{k+1}^*=\cdots=z_p^*>z_{p+1}^*$ then by (d) there exists l, $k\le l<p$ such that $\sum_{j=1}^l z_j^*=\Psi(l)$. Again, by Lemma 2.1, we obtain that $\sum_{i=1}^l y_{q_i}=0$ for every l-tuple $(q_1,\dots,q_l)$ in $(1,\dots,p)$ and hence, as $l<p$, $y_i=0$ (and therefore $y_k=0$) for $1\le i\le p$.

Finally, suppose we have $y_1=\cdots=y_{n-1}=0$. If $z_{n-1}^*=z_n^*$ it follows immediately that $y_n=0$. On the other hand, if $z_{n-1}^*<z_n^*$ then by (e) we have $\sum_{j=1}^n z_j^*=\Psi(n)$ and Lemma 2.1 gives that $y_n=0$. Thus, y is zero and z is a real extreme point of $B_{m_\Psi^0}$.

Conversely, let us first suppose that (a) does not occur. Then, suppose that $z_1^* = z_2^* > z_3^*$ and that $\sum_{j=1}^{l} z_j^* < \Psi(l)$ for $l = 1, 2$. We may choose $\epsilon > 0$ given by $\epsilon = \min\left\{z_2^* - z_3^*, \min_{l=1,2}\left\{\Psi(l) - \sum_{j=1}^{l} z_j^*\right\}\right\}$ and define y in m_Ψ^0 by $y_1 = \epsilon$, $y_2 = -\epsilon$ and all other y_j equal to 0. We get $\|z^* \pm y\| \leq 1$ which shows that z is not an extreme point of the unit ball of m_Ψ^0.

Next suppose that for some $p \in \mathbf{N}$, $p > 2$, $z_1^* = z_2^* = \cdots = z_p^* > z_{p+1}^*$ with $\sum_{j=1}^{l} z_j^* < \Psi(l)$ for all $1 \leq l < p$. Then we may consider the positive number $\epsilon = \min\left\{z_p^* - z_{p+1}^*, \min_{1\leq l<p}\left\{\Psi(l) - \sum_{j=1}^{l} z_j^*\right\}\right\}$. Let y in m_Ψ^0 be defined by $y_1 = \epsilon$, $y_p = -\epsilon$ and $y_j = 0$ otherwise. Then we get that $\|z^* \pm y\| \leq 1$ and thus z is not a real extreme point of the unit ball of m_Ψ^0.

If there is k, $1 < k \leq n$ with $z_{k-1}^* > z_k^* > z_{k+1}^* > z_{k+2}^*$ and (b) does not hold, then $\sum_{j=1}^{k} z_j^* < \Psi(k)$. We now consider the positive number

$$\epsilon = \min\left\{z_{k-1}^* - z_k^*, z_k^* - z_{k+1}^*, z_{k+1}^* - z_{k+2}^*, \Psi(k) - \sum_{j=1}^{k} z_j^*\right\}.$$

Let y be such that $y_k = \epsilon$, $y_{k+1} = -\epsilon$ and all other y_j equal to 0. Then we get that $\|z^* \pm y\| \leq 1$ so again z is not a real extreme point of the unit ball of m_Ψ^0.

Suppose there exist k and p with $k + 1 < p$ so that $1 < k < n - 1$ and $z_{k-1}^* > z_k^* > z_{k+1}^* = z_{k+2}^* = \cdots = z_p^* > z_{p+1}^*$. As (c) does not hold, we have $\sum_{j=1}^{l} z_j^* < \Psi(l)$ for $k \leq l < p$. Now, we may consider

$$\epsilon = \min\left\{z_{k-1}^* - z_k^*, z_p^* - z_{p+1}^*, \min_{k\leq l<p}\left\{\Psi(l) - \sum_{j=1}^{l} z_j^*\right\}\right\}.$$

Let y be defined by $y_k = \epsilon$, $y_p = -\epsilon$ and $y_j = 0$ otherwise. Then we have $\|z^* \pm y\| \leq 1$ and hence z is not a real extreme point of the unit ball of m_Ψ^0.

Suppose now that there are k and p with $k < p$ so that $1 < k < n - 1$ with $z_{k-1}^* > z_k^* = z_{k+1}^* = \cdots = z_p^* > z_{p+1}^*$ and (d) does not hold. Then $\sum_{j=1}^{l} z_j^* < \Psi(l)$ for each $k \leq l < p$ and we may take the positive number

$$\epsilon = \min\left\{z_{k-1}^* - z_k^*, z_p^* - z_{p+1}^*, \min_{k\leq l<p}\left\{\Psi(l) - \sum_{j=1}^{l} z_j^*\right\}\right\}.$$

Let y be defined by $y_k = \epsilon$, $y_p = -\epsilon$ and $y_j = 0$ otherwise. We have that $\|z^* \pm y\| \leq 1$ and thus z is not a real extreme point of the unit ball of m_Ψ^0.

Finally, suppose that $z_{n-1}^* > z_n^*$ and that (e) fails, that is $\sum_{j=1}^{n} z_j^* < \Psi(n)$. Let $y_j = 0$ for $1 \leq j \leq n-1$ and $y_n = \min\left\{z_{n-1}^* - z_n^*, \Psi(n) - \sum_{j=1}^{n} z_j^*\right\}$ then y satisfies $\|z^* \pm y\| \leq 1$ and so z^* is not and extreme point. This completes the proof. □

3. Complex extreme points of m_Ψ

We divide our classification of the complex extreme points of B_{m_Ψ} into two parts. In the first one we assume that there is a bijection σ of $\mathbf{N}$ such that $z_{\sigma(n)}^* = |z_n|$ for all $n \in \mathbf{N}$. In that case we say that z^* is a permutation of $(|z_n|)_n$. F We will deal with the general case later. By m_Ψ^n we understand $\mathbf{C}^n$ with the norm

$$\|z\| = \sup_{k\geq 1} \frac{1}{\Psi(k)} \sum_{j=1}^{k} z_j^*.$$

We use Π_n to denote the continuous linear projection of m_Ψ or m_Ψ^0 onto m_Ψ^n which sends $(z_j)_j$ to $(z_j)_{j=1}^n$.

Note that given $z = (z_j)_j$ in m_Ψ, whenever $z_n^* > z_{n+1}^*$ there are precisely n coordinates of z, $z_{j_1}, \ldots, z_{j_n}$ with $|z_{j_m}| \geq z_n^*$, $1 \leq m \leq n$. In this case $z_n^* = \min\{|z_{j_m}|\colon 1 \leq m \leq n\}$ and therefore $z_n^* = |z_j|$ for some j.

Theorem 3.1. *Let Ψ be a symbol and $z = (z_j)_j$ be a point in B_{m_Ψ}. Suppose that z^* is a permutation of $(|z_j|)_j$. Then z is a complex extreme point of B_{m_Ψ} if and only if it satisfies one of the following conditions.*

(a) $\liminf \left(\Psi(n) - \sum_{j=1}^n z_j^*\right) = 0,$

(b) *there is n in $\mathbf{N}$ with $(z_j^*)_{j=1}^n \in \mathcal{T}_n$, $\sum_{j=1}^k z_j^* < \Psi(k)$ and $z_k^* = z_n^*$, for all $k > n$.*

Proof. First, notice that as z^* is a permutation of $(|z_j|)_j$, z is a complex extreme point of B_{m_Ψ} if and only if z^* is a complex extreme point of B_{m_Ψ}.
Let us suppose that z satisfies (a). Replacing W_n in [1, Lemma 2.7] with $\Psi(n)$ we see that z^* is a complex extreme point of B_{m_Ψ}.
Now, let us suppose that z^* has the form (b) and that we have y in m_Ψ with $\|z^* + \lambda y\| \leq 1$ for all λ in $\overline{\Delta}$. For each l, $1 \leq l \leq n$ and each λ in $\overline{\Delta}$ we have

$$\sum_{j=1}^{l} |z_j^* + \lambda y_j| \leq \sum_{j=1}^{l} (z^* + \lambda y)_j^* \leq \Psi(l).$$

Since $(z_j^*)_{j=1}^n \in \mathcal{T}_n$, by [5, Proposition 2.2], it is a peak point for $\mathcal{A}_u(B_{m_\Psi^n})$ and hence a complex extreme point of $B_{m_\Psi^n}$. This implies that $y_j = 0$ for $1 \leq j \leq n$. As $z_k^* = z_n^*$ for all $k > n$, what happens for the n-th coordinate of y should happen for the k-th coordinate of y for any $k > n$. Then, $y_j = 0$ for $j \geq n$, and z^* is a complex extreme point.
For the converse, let us suppose that z^* is in B_{m_Ψ} and does not have the form (a) or (b). As (a) does not hold, we have

$$\liminf_{n\to\infty} \left(\Psi(n) - \sum_{j=1}^{n} z_j^*\right) > 0. \tag{1}$$

If for each n in $\mathbf{N}$ we have $\sum_{j=1}^n z_j^* < \Psi(n)$ let $\epsilon = \inf\{\Psi(n) - \sum_{j=1}^n z_j^*\} > 0$ and set $y = \epsilon e_1$. Then for all λ in $\overline{\Delta}$ we have $\|z^* + \lambda y\| \leq 1$ and therefore z^* is not a complex extreme point of B_{m_Ψ}. Thus we may suppose that there is a positive integer n, and in virtue of (1) there are only finite such integers, with $\Psi(n) = \sum_{j=1}^n z_j^*$. Let n be the largest positive integer with $\sum_{j=1}^n z_j^* = \Psi(n)$. As (b) does not hold and $\sum_{j=1}^r z_j^* < \Psi(r)$ for $r > n$ there is $k > n$ with $z_k^* < z_n^*$. We assume that k is the first such integer. Let $\epsilon = \frac{1}{2}\min\left\{z_n^* - z_k^*, \inf_{l>n}\left(\Psi(l) - \sum_{j=1}^{l} z_j^*\right)\right\}$ and let $y = \epsilon e_k$. Fix $|\lambda| \leq 1$. If $l \leq n$ we have

$$\sum_{j=1}^{l}(z^* + \lambda y)_j^* \leq \sum_{j=1}^{l} z_j^* \leq \Psi(l).$$

If $l > n$ we have

$$\sum_{j=1}^{l}(z^* + \lambda y)_j^* \leq \sum_{j=1}^{l} z_j^* + \epsilon \leq \Psi(l).$$

Thus $\|z^* + \lambda y\| \leq 1$ for all $|\lambda| \leq 1$ which shows that z^* is not a complex extreme point of B_{m_Ψ}. □

Let us now investigate the case when z^* is not a permutation of $(|z_j|)_j$. For a fixed z, we consider the (possibly empty) sets $\mathbf{N}_1 = \{j \in \mathbf{N} \colon z_j^* = |z_k| \text{ for some } k \text{ in } \mathbf{N}\}$ and $\mathbf{N}_2 = \{k \in \mathbf{N} \colon |z_k| = z_j^* \text{ for some } j \text{ in } \mathbf{N}\}$.

Lemma 3.2. *Let Ψ be a symbol and $z = (z_j)_j$ be a point in B_{m_Ψ}.*

(a) *We have* $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\left\{z_j^* \colon j \in \mathbf{N}\right\}$ *if and only if* $\mathbf{N}_1$ *is non-empty and* $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\left\{z_j^* \colon j \in \mathbf{N}_1\right\}$.

(b) *If* $\inf\{|z_j| \colon j \in \mathbf{N}\} < \inf\left\{z_j^* \colon j \in \mathbf{N}\right\}$ *then z is not a complex extreme point of* B_{m_Ψ}.

Proof. We prove the if part of (a) by the contra positive. That is, if $\mathbf{N}_1$ is empty or $\inf\{|z_j| \colon j \in \mathbf{N}\} < \inf\left\{z_j^* \colon j \in \mathbf{N}_1\right\}$ then the strict inequality

$$\inf\{|z_j| \colon \ j \in \mathbf{N}\} < \inf\left\{z_j^* \colon j \in \mathbf{N}\right\} \tag{2}$$

must hold. Assume first that $\mathbf{N}_1$ is non-empty and that $\inf\{|z_j| \colon j \in \mathbf{N}\} < \inf\left\{z_j^* \colon j \in \mathbf{N}_1\right\}$. In addition, suppose first that there are infinitely many n with $z_n^* > z_{n+1}^*$. By the comment above Theorem 3.1, each of these n belongs to $\mathbf{N_1}$ and therefore $\inf\left\{z_j^* \colon j \in \mathbf{N}_1\right\} = \inf\left\{z_j^* \colon j \in \mathbf{N}\right\}$. Then (2) is satisfied.

Now suppose that there are only finitely many n with $z_n^* > z_{n+1}^*$. Let n_0 be the biggest positive integer such that $z_{n_0-1}^* > z_{n_0}^*$. Then $z_{n_0}^* = \inf\left\{z_j^* \colon j \in \mathbf{N}\right\}$. If $n_0 \notin \mathbf{N}_1$, we have (infinitely) many $j \in \mathbf{N}$ with $|z_j| < z_{n_0}^*$ and so $\inf\{|z_j| \colon j \in \mathbf{N}\} < \inf\left\{z_j^* \colon j \in \mathbf{N}\right\}$ and (2) is satisfied.

On the other hand, if $n_0 \in \mathbf{N}_1$ then $\inf\left\{z_j^* \colon j \in \mathbf{N}_1\right\} = \inf\left\{z_j^* \colon j \in \mathbf{N}\right\}$ and we also obtain the strict inequality (2) holds.

Finally, assume that $\mathbf{N}_1$ is empty. Then $(z_j^*)_j$ is a constant sequence. Otherwise, $z_1^* > \inf\left\{z_j^* \colon j \in \mathbf{N}\right\}$ and as $(z_j^*)_j$ is non-increasing there are only finitely many terms equal to z_1^*. Then $z_n^* > z_{n+1}^*$ for some n, implying that $\mathbf{N_1}$ is non-empty. Now we claim that $\inf\{|z_j| \colon \ j \in \mathbf{N}\} < \inf\left\{z_j^* \colon j \in \mathbf{N}\right\} = z_1^*$ as otherwise, $(|z_j|)_j$ is a constant sequence. Indeed, suppose that $\inf\{|z_j| \colon \ j \in \mathbf{N}\} = \inf\left\{z_j^* \colon j \in \mathbf{N}\right\} = z_1^*$. As, $|z_j| \leq z_1^*$ for all j, we have

$$\inf\{|z_j| \colon \ j \in \mathbf{N}\} \leq |z_j| \leq z_1^* = \inf\left\{z_j^* \colon j \in \mathbf{N}\right\} = \inf\{|z_j| \colon \ j \in \mathbf{N}\},$$

and $|z_j| = z_1^*$ for all j. Hence, as $(z_j^*)_j$ is constant, $\mathbf{N}_1 = \mathbf{N}$ which is a contradiction. Then (2) holds and one side of the proof of (a) is complete.

For the converse note that since $\mathbf{N}_1$ is non-empty and $\left\{z_j^* \colon j \in \mathbf{N}_1\right\} \subseteq \left\{z_j^* \colon j \in \mathbf{N}\right\}$, then $\inf\left\{z_j^* \colon j \in \mathbf{N}\right\} \leq \inf\left\{z_j^* \colon j \in \mathbf{N}_1\right\}$. As $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\left\{z_j^* \colon j \in \mathbf{N}_1\right\}$ we have $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\left\{z_j^* \colon j \in \mathbf{N}\right\}$.

Now, let us prove (b). If $\inf\{|z_j| \colon j \in \mathbf{N}\} < \inf\left\{z_j^* \colon j \in \mathbf{N}\right\}$ then there is j_0 in $\mathbf{N}$ with $|z_{j_0}| < \inf\left\{z_j^* \colon j \in \mathbf{N}\right\}$. Letting $\epsilon = \inf\left\{z_j^* \colon j \in \mathbf{N}\right\} - |z_{j_0}|$ and setting $y = \epsilon e_{j_0}$ we see that $(z + \lambda y)^* = z^*$ for all $|\lambda| \leq 1$, and therefore z is not a complex extreme point of B_{m_Ψ}. □

Remark 3.3. *For $z = (z_j)_j \in m_\Psi$ such that $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\{z_j^* \colon j \in \mathbf{N}\}$ we have that $\mathbf{N}_1$ and $\mathbf{N}_2$ are both infinite. Indeed, we have a bijection of $\mathbf{N}_1$ onto $\mathbf{N}_2$. Observe that as $(z_j^*)_j$ is decreasing we have that either (i) there are infinitely many n with $z_n^* > z_{n+1}^*$ or (ii) there is a natural number n_0 with $z_j^* = z_{n_0}^*$ for all $n \geq n_0$. If (i) occurs then each n will belong to $\mathbf{N}_1$ and the corresponding k_n so that $|z_{k_n}| = z_n^*$ will belong to $\mathbf{N}_2$ and we have a bijection of $\mathbf{N}_1$ onto $\mathbf{N}_2$. If (ii) occurs we have that the sequence $(|z_j|)_j$ will eventually be equal to $\ell = \inf\{z_j^* \colon j \in \mathbf{N}\}$ and therefore each n will belong to $\mathbf{N}_1$ and the corresponding k_n will belong to $\mathbf{N}_2$ again giving a bijection between $\mathbf{N}_1$ and $\mathbf{N}_2$.*

Theorem 3.4. *Let Ψ be a symbol and $z = (z_j)_j$ be a point in B_{m_Ψ}. Suppose that z^* is not a permutation of $(|z_j|)_j$. Then z is a complex extreme point of B_{m_Ψ} if and only if $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\{z_j^* \colon j \in \mathbf{N}\}$ and $\liminf_{n\to\infty}\left(\Psi(n) - \sum_{j=1}^n z_j^*\right) = 0$.*

Proof. Let us first suppose that z satisfies $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\{z_j^* \colon j \in \mathbf{N}\}$ and also $\liminf_{n\to\infty}\left(\Psi(n) - \sum_{j=1}^n z_j^*\right) = 0$. In order to show that z is a complex extreme point, suppose that there is $y = (y_j)_j \in m_\Psi$ be such that $\|z + \lambda y\| \leq 1$ for all λ in $\overline{\Delta}$ and let us show that $y = 0$. Notice that, by Lemma 3.2, $\mathbf{N}_1$ (and therefore $\mathbf{N}_2$) is non-empty and $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\{z_j^* \colon j \in \mathbf{N}_1\}$. Fix $j_0 \in \mathbf{N}$. If $j_0 \in \mathbf{N}_2$ then, replacing W_n with $\Psi(n)$ in [1, Lemma 2.7] we see that $y_{j_0} = 0$. Next suppose that $j_0 \in \mathbf{N} \setminus \mathbf{N}_2$ and $y_{j_0} \neq 0$. For an appropriate choice of λ with $|\lambda| = 1$ we will have $|z_{j_0} + \lambda y_{j_0}| = |z_{j_0}| + |y_{j_0}| > |z_{j_0}|$. As $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\{z_j^* \colon j \in \mathbf{N}_1\}$, by the above remark, we can find n_0 in $\mathbf{N}_2$ with $|z_{j_0} + \lambda y_{j_0}| - z_{n_0}^* > \epsilon := |y_{j_0}|/2$. Since $\liminf_{n\to\infty}\left(\Psi(n) - \sum_{j=1}^n z_j^*\right) = 0$ we can choose $m \in \mathbf{N}$ with $m > n_0$ and $\Psi(m) - \sum_{j=1}^m z_j^* < \epsilon/2$. By the above remark, we have a bijection between $\mathbf{N}_1$ and $\mathbf{N}_2$ then, the fact that $y_k = 0$ for all k in $\mathbf{N}_2$ implies that $(z + \lambda y)_j^* = z_j^*$ for all $j \in \mathbf{N}$. Therefore we have

$$\begin{aligned} \Psi(m)\|z + \lambda y\| &\geq \sum_{j=1, j\neq n_0}^m z_j^* + |z_{j_0} + \lambda y_{j_0}| \\ &= \sum_{j=1}^m z_j^* + (|z_{j_0} + \lambda y_{j_0}| - z_{n_0}^*) \\ &> \Psi(m) - \epsilon/2 + \epsilon \\ &= \Psi(m) + \epsilon/2, \end{aligned}$$

contradicting the fact that $\|z+\lambda y\| \leq 1$ for all λ in $\overline{\Delta}$. Thus $y = 0$ and z is a complex extreme point of B_{m_Ψ}.

For the converse, first suppose that $\inf\{|z_j| \colon j \in \mathbf{N}\} < \inf\{z_j^* \colon j \in \mathbf{N}\}$. Then, by Lemma 3.2, z is not a complex extreme point of m_Ψ. Now, suppose that $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\{z_j^* \colon j \in \mathbf{N}\}$ and $\liminf \left(\Psi(n) - \sum_{j=1}^n z_j^*\right) > 0$. By Lemma 3.2, $\mathbf{N}_1$ is non-empty. Since z^* is not a permutation of $(|z_j|)_j$, we claim that the sequence $(z_j^*)_j$ cannot satisfy condition (b) of Theorem 3.1. To see this suppose that there is n in $\mathbf{N}$ with $(z_j^*)_{j=1}^n \in \mathcal{T}_n$ and, for all for all $k > n$, $\sum_{j=1}^k z_j^* < \Psi(k)$ and $z_k^* = z_n^*$. Then there is at most $n-1$ indices j with $z_j^* > z_n^*$ and hence at most $n-1$ indices j with $|z_j| > z_n^*$. As we are assuming that $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\{z_j^* \colon j \in \mathbf{N}\} = z_n^*$, all other z_j have modulus equal to z_n^* and thus $(z_j^*)_j$ is a permutation of $(|z_n|)_n$.

Now suppose that for each n in $\mathbf{N}$ we have $\sum_{j=1}^n z_j^* < \Psi(n)$. Let $\epsilon = \inf\{\Psi(n) - \sum_{j=1}^n z_j^*\}$ and set $y = \epsilon e_1$. Then for all λ in $\overline{\Delta}$ we have $\|z + \lambda y\| \leq 1$ proving that z is not a complex extreme point of B_{m_Ψ}.

On the other hand, if n is the largest positive integer with $\sum_{j=1}^n z_j^* = \Psi(n)$. Then $(z_j^*)_{j=1}^n \in \mathcal{T}_n$ and for $k > n$, $\sum_{j=1}^k z_j^* < \Psi(k)$. We claim that there is $k > n$ so that $z_k^* > z_n^*$. Otherwise as $\inf\{|z_j| : j \in \mathbf{N}\} = \inf\{z_j^* : j \in \mathbf{N}\}$ $(z_j^*)_j$ would be a permutation of $(z_n)_n$ contrary to our assumption.

Let $\epsilon = \frac{1}{2}\min\left\{z_n^* - z_k^*, \inf_{l>n}\left(\Psi(l) - \sum_{j=1}^l z_j^*\right)\right\}$ and let $y = \epsilon e_k$. Fix $|\lambda| \leq 1$. For $l \leq n$ we have

$$\sum_{j=1}^{l}(z^* + \lambda y)_j^* \leq \sum_{j=1}^{l} z_j^* \leq \Psi(l).$$

For $l > n$ we have

$$\sum_{j=1}^{l}(z^* + \lambda y)_j^* \leq \sum_{j=1}^{l} z_j^* + \epsilon \leq \Psi(l).$$

Thus $\|z^* + \lambda y\| \leq 1$ for all $|\lambda| \leq 1$ which shows that z^* is not a complex extreme point of B_{m_Ψ}. As k belongs to $\mathbf{N}_1$, by definition, we can find p in $\mathbf{N}_2$ so that $|z_p| = z_k^*$. If we now set $\tilde{y} = \epsilon e_p$ we see that $\|z + \lambda\tilde{y}\| = \|z^* + \lambda y\| \leq 1$ for all $|\lambda| \leq 1$ and we see that z is not an extreme point of B_{m_Ψ}. □

Note that the existence of an extreme point $z = (z_j)_j$ of B_{m_Ψ}, such that z^* is not a permutation of $(|z_j|)_j$ with $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\{z_j^* \colon j \in \mathbf{N}\} > 0$ will imply that that m_Ψ is isomorphic to ℓ_∞. An example of such a space and extreme point can be obtained by taking $\Psi(n) = \frac{n}{2} + \sum_{j=1}^n \frac{1}{2j}$, $z_{2j} = \frac{1}{2}(1 + \frac{1}{2j})$ and $z_{2j-1} = 1/2$, giving $z^* = (\frac{1}{2}(1 + \frac{1}{j}))_j$. If m_Ψ is not isomorphic to ℓ_∞ and z is an extreme point of B_{m_Ψ} we claim that for all z_j that do not contribute to z^* we must have $z_j = 0$. Indeed, If there is an index j with $z_j \neq 0$ such that z_j does not contribute to the norm of z then we have $\lim_{j\to\infty} z_j^* = c > 0$. Then, $nc \leq \sum_{j=1}^n z_j^* \leq \Psi(n)$ and therefore $\lim_{n\to\infty} \frac{\Psi(n)}{n} > c$. By [12, Theorem 3.2], m_Ψ is equivalent to ℓ_∞. Now, combining Theorem 3.1 and Theorem 3.4 we have the following.

Theorem 3.5. *Let Ψ be a symbol and $z = (z_j)_j$ be a point in B_{m_Ψ}. Then z is a complex extreme point of B_{m_Ψ} if and only if it satisfies one of the following conditions.*

(a) $\inf\{|z_j| \colon j \in \mathbf{N}\} = \inf\{z_j^* \colon j \in \mathbf{N}\}$ *and* $\liminf_{n\to\infty}\Big(\Psi(n) - \sum_{j=1}^n z_j^*\Big) = 0$,

(b) *there is n in $\mathbf{N}$ with $(z_j^*)_{j=1}^n \in \mathcal{T}_n$, $\sum_{j=1}^k z_j^* < \Psi(k)$ and $z_k^* = z_n^*$, for all $k > n$.*

Note that the extreme points described in (a) include the extreme points of the Lorentz sequence space, $d'(w,1)$, which are characterised by Acosta, Aron and Moraes in [1, Theorem 2.8]. The extreme points described in (b) contain the points satisfying that there is n in $\mathbf{N}$ with $(z_j^*)_{j=1}^n \in \mathcal{T}_n$, $\Psi(n) = \Psi(n+1)$ and $z_k^* = 0$ for all $k > n$. These are precisely the extreme points of the unit ball of m_Ψ^0, see [5, Theorem 2.5]. Finally, the extreme points described in (b), may have $z_k^* > 0$ for all $k > n$ which only occurs when m_Ψ is a renorming of ℓ_∞. To see this note that if l $k > n$ we have

$$\sum_{j=1}^k z_j^* = \Psi(n) + (k-n)z_n^* \leq \Psi(k).$$

Hence, $\lim_{k\to\infty} \frac{\Psi(k)}{k} \geq z_n^* > 0$. An application of [12, Theorem 3.2] shows that m_Ψ is isomorphic to ℓ_∞.

4. Geometry of the dual of Marcinkiewicz sequence spaces

In this section we consider the geometry of the unit ball of $(m_\Psi^0)'$ the dual of m_Ψ^0. We assume without loss of generality that $\frac{\Psi(n)}{n} \leq \frac{\Psi(k)}{k}$ for all $k \leq n$.

Theorem 4.1. *Let Ψ be a symbol and $v = (v_j)_j$ be a point in $B_{(m_\Psi^0)'}$. Then v is a weak*-exposed point of $B_{(m_\Psi^0)'}$ if and only if there is a positive integer n_0 with*

(a) $\Psi(n_0) < \Psi(n_0+1)$ *and* $\frac{\Psi(n_0)}{n_0} < \frac{\Psi(n_0-1)}{n_0-1}$ *f* $n_0 > 1$, *or* $\Psi(1) < \Psi(2)$ *if* $n_0 = 1$,

(b) $v_j^* = \frac{1}{\Psi(n_0)}$ *for* $1 \leq j \leq n_0$ *and* $v_j^* = 0$ *for* $j > n_0$.

Proof. Let $v = (v_j)_j$ be a weak*-exposed point of $B_{(m_\Psi^0)'}$ exposed by a norm one element $z = (z_j)_j$ in m_Ψ^0. Since z belongs to m_Ψ^0, z^* is a permutation of $(|z_j|)_j$, we may assume without loss of generality that $z_j^* = |z_j|$ for all j. Also, there is a positive integer n_0 so that $1 = \|z\| = \frac{1}{\Psi(n_0)}\sum_{j=1}^{n_0} z_j^* = \frac{1}{\Psi(n_0)}\sum_{j=1}^{n_0} z_j e^{-i\theta_j}$ where $\theta_j = \mathrm{Arg}(z_j)$. If we consider the finite support element u such that $u_j = \frac{e^{-i\theta_j}}{\Psi(n_0)}$ for $j = 1, \ldots, n_0$ and zero elsewhere, we have

$$|\langle x, u\rangle| = \frac{1}{\Psi(n_0)}\left|\sum_{j=1}^{n_0} e^{-i\theta_j} x_j\right| \leq \frac{1}{\Psi(n_0)}\sum_{j=1}^{n_0}|x_j| \leq \frac{1}{\Psi(n_0)}\sum_{j=1}^{n_0}|x_j^*| \leq 1, \tag{3}$$

for all $x \in B_{m_\Psi^0}$. Then, $\|u\| = 1$ and as $\langle z, u\rangle = 1$, by the definition of weak*-exposed point, it follows that $v = u$. Therefore, $v_j^* = \frac{1}{\Psi(n_0)}$ for $1 \leq j \leq n_0$ and $v_j^* = 0$ for $j > n_0$. Thus, (b) holds.

Next suppose that (a) fails. If $\Psi(n_0) = \Psi(n_0+1)$, consider $s = \frac{1}{\Psi(n_0)}\sum_{j=1}^{n_0+1} e_j$, and $t = \frac{1}{\Psi(n_0)}\sum_{j=1}^{n_0} e_j - \frac{1}{\Psi(n_0)} e_{n_0+1}$. Then, with a calculation similar to (3), we see that s and t belong to $B_{(m^0_\Psi)'}$. Since $v^* = \frac{1}{2}(s+t)$, v is not an extreme point and hence not a weak*-exposed point of the unit ball of $(m^0_\Psi)'$.

Now suppose that $n_0 > 1$ and $\frac{\Psi(n_0)}{n_0} = \frac{\Psi(n_0-1)}{n_0-1}$. Notice that by the choice of n_0, $(z_j)_{j=1}^{n_0}$ belongs to $\mathcal{T}_{n_0}$. Then, we claim that $z_j = \frac{\Psi(n_0)}{n_0}e^{i\theta_j}$ for $1 \leq j \leq n_0$. Suppose this is not the case. Then we have

$$z^*_{n_0} < \tfrac{1}{n_0-1}\textstyle\sum_{j=1}^{n_0-1} z^*_j \leq \tfrac{\Psi(n_0-1)}{n_0-1}$$

and hence

$$\tfrac{1}{n_0}\textstyle\sum_{j=1}^{n_0} z^*_j < \tfrac{1}{n_0-1}\textstyle\sum_{j=1}^{n_0-1} z^*_j \leq \tfrac{\Psi(n_0-1)}{n_0-1} = \tfrac{\Psi(n_0)}{n_0},$$

contradicting the fact that $(z_j)_{j=1}^{n_0}$ belongs to $\mathcal{T}_{n_0}$. In particular, the only weak*-exposing points of m^0_Ψ are those of the form $z_j = \frac{\Psi(n_0)}{n_0}e^{i\theta_j}$ for $1 \leq j \leq n_0$. Now set $u = (u_j)_j$ such that $u_j = \frac{1}{\Psi(n_0-1)}e^{-i\theta_j}$ for $1 \leq j \leq n_0-1$ and $u_j = 0$ for $j \geq n_0$. Notice that $\|u\| = 1$. As $z_j = \frac{\Psi(n_0)}{n_0}e^{i\theta_j}$ for $1 \leq j \leq n_0$, we see that $\langle z, u\rangle = 1$ and so v cannot be weak*-exposed.

Conversely, let $n_0 > 1$ be a positive integer with $\Psi(n_0) < \Psi(n_0+1)$ and $\frac{\Psi(n_0)}{n_0} < \frac{\Psi(n_0-1)}{n_0-1}$. Let $v_j = \frac{1}{\Psi(n_0)}$ for $1 \leq j \leq n_0$ and $v_j = 0$ for $j > n_0$. Set $z = (z_j)_j$ such that $z_j = \frac{\Psi(n_0)}{n_0}$ for $1 \leq j \leq n_0$ and $z_j = 0$ for $j > n_0$. Since $\frac{\Psi(n_0)}{n_0} \leq \frac{\Psi(k)}{k}$ for $1 \leq k \leq n_0$ we see that z belongs to the unit ball of m^0_Ψ. Moreover $\langle z, v\rangle = \sum_{j=1}^{n_0} z_j v_j = n_0\frac{\Psi(n_0)}{n_0}\frac{1}{\Psi(n_0)} = 1$.

Suppose that there exists $u = (u_j)_j$ in $(m^0_\Psi)'$ such that $\|u\| \leq 1$ and $\langle z, u\rangle = 1$. First observe that since $z_1u_1 + z_2u_2 + \cdots + z_{n_0}u_{n_0} = 1$ and u is in the unit ball of $(m^0_\Psi)'$, we see that each u_j is real and positive, $1 \leq j \leq n_0$. If $u_j \neq \frac{1}{\Psi(n_0)}$ for some $1 \leq j \leq n_0$ then we must have $u_k > u_{k+1}$ for some $1 \leq k < n_0$. Since $\frac{\Psi(n_0)}{n_0} < \frac{\Psi(n_0-1)}{n_0-1}$ we have that $\sum_{j=1}^{l} z_j < \Psi(l)$ for $1 \leq l < n_0 - 1$ and therefore we may choose $\epsilon > 0$ so that $\tilde{z} = (z_1, z_2, \ldots, z_k + \epsilon, z_{k+1} - \epsilon, \ldots, z_{n_0}, 0, \ldots)$ belongs to the unit ball of m^0_Ψ. Since $\langle \tilde{z}, u\rangle > 1$ we have a contradiction. Hence, $u_j = \frac{1}{\Psi(n_0)}$ for any $1 \leq j \leq n_0$. Finally, if $u^*_{n_0+1} \neq 0$ for $\theta \in \mathbf{R}$ take $\hat{z} = (\hat{z}_j)_j$ such that $\hat{z}_j = \frac{\Psi(n_0)}{n_0}$ for $1 \leq j \leq n_0$, $\hat{z}_{n_0+1} = e^{i\theta}\min\left\{\Psi(n_0+1) - \Psi(n_0), \frac{\Psi(n_0)}{n_0}\right\}$ and $\hat{z}_j = 0$ for $j > n_0 + 1$. Then $\hat{z}$ belongs to the unit ball of m^0_Ψ. As $\langle \hat{z}, u\rangle = 1 + e^{i\theta}u_{n_0+1} > 1$ for an appropriate choice of θ we see that $u^*_{n_0+1} = 0$ and the result is proven.

In the case where $n_0 = 1$, a close examination of the proof given above shows that a necessary and sufficient condition for each e_j to be an extreme point of the unit ball is that $\Psi(1) < \Psi(2)$. □

The following result extends [13, Theorem 2.6] which characterises the extreme points of the Lorentz sequence space $d(w, 1)$.

Corollary 4.2. *Let Ψ be a symbol and $v = (v_j)_j$ be a point in $B_{(m^0_\Psi)'}$. Then v is a real extreme point of $B_{(m^0_\Psi)'}$ if and only if there is a positive integer n_0 with*

(a) $\Psi(n_0) < \Psi(n_0+1)$ *and* $\frac{\Psi(n_0)}{n_0} < \frac{\Psi(n_0-1)}{n_0-1}$ *if* $n_0 > 1$*, or* $\Psi(1) < \Psi(2)$ *if* $n_0 = 1$*,*

(b) $v^*_j = \frac{1}{\Psi(n_0)}$ *for* $1 \leq j \leq n_0$ *and* $v^*_j = 0$ *for* $j > n_0$*.*

Proof. By Theorem 4.1, each point satisfying (a) and (b) of the statement is weak*-exposed and, therefore, it is also an extreme point of $B_{(m_\Psi^0)'}$.

For the converse, we recall that a Banach space E is weakly compactly generated if it contains a weakly compact set K whose span is dense in E. As it is readily shown that $(m_\Psi^0)'$ is separable it follows from [9, p.357] that $(m_\Psi^0)'$ is weakly compactly generated. Now, [16, Corollary 11] implies that $B_{(m_\Psi^0)'}$ is the closed unit ball is the weak*-closed convex hull of its weak*-exposed points. A result of Milman (see [9, Theorem 3.41]) now tells that each extreme point of the unit ball of $(m_\Psi^0)'$ is a weak*-limit of a sequence of weak*-exposed points. Therefore if we consider v an extreme point of $B_{(m_\Psi^0)'}$ then v is in the weak*-sequential closure of the set of weak*-exposed points of $B_{(m_\Psi^0)'}$. Let $(v^n)_n$ be a sequence of weak*-exposed points of $B_{(m_\Psi^0)'}$ which converges weak* to v. Choose $j_0 \in \mathbf{N}$ with $v_{j_0} \neq 0$. Then we can find $\epsilon > 0$ and $n_0 \in \mathbf{N}$ so that $|v_{j_0}^n| > \epsilon/2$ for all $n > n_0$. By Theorem 4.1, each v^n has finite support and each nonzero coordinate has the form $\frac{1}{\Psi(k)}$ for some k. Since there are only finitely many k with $\frac{1}{\Psi(k)} > \epsilon/2$, we can find a subsequence $(v^{n_k})_k$ of $(v^n)_n$ and $p \in \mathbf{N}$ such that v^{n_k} has length p, $\Psi(p) < \Psi(p+1)$ and $\frac{\Psi(p)}{p} < \frac{\Psi(p-1)}{p-1}$, and $|v_{j_0}^{n_k}| = \frac{1}{\Psi(p)}$ for all $k \in \mathbf{N}$. For every other index l we have that either $|v_l^{n_k}| = \frac{1}{\Psi(p)}$ or $|v_l^{n_k}| = 0$. Hence either $|v_l| = \frac{1}{\Psi(p)}$ or $|v_l| = 0$.

Let q be the number of non-zero indexes which v possess. If q was infinite then $\|v\|$ would also be infinite. If q is finite with $q > p$ then v^{n_k} will also have q non-zero indexes j with $|v_j^{n_k}|$ equal to $\frac{1}{\Psi(p)}$ for n sufficiently large which is a contradiction. Now suppose that $q < p$. If $\Psi(q) = \Psi(p)$ then the proof of the characterisation of the weak*-exposed points of $B_{(m_\Psi^0)'}$ show that v cannot be an extreme point. On the other hand if $\Psi(q) < \Psi(p)$ then v has norm strictly less than 1 and so cannot be an extreme point. Hence $p = q$ and v has length p with $v_j^* = \frac{1}{\Psi(p)}$ for $j = 1, \ldots, p$ and $v_j^* = 0$ for$j > p$, $\Psi(p) < \Psi(p+1)$ and $\frac{\Psi(p)}{p} < \frac{\Psi(p-1)}{p-1}$ all $p \in \mathbf{N}$. □

Since each weak*-exposed point of $B_{(m_\Psi^0)'}$ is exposed and every exposed point of $B_{(m_\Psi^0)'}$ is extreme we also have the following corollary.

Corollary 4.3. *Let Ψ be a symbol and $v = (v_j)_j$ be a point in $B_{(m_\Psi^0)'}$. Then v is an exposed point of $B_{(m_\Psi^0)'}$ if and only if there is a positive integer n_0 with*

(a) $\Psi(n_0) < \Psi(n_0+1)$ *and* $\frac{\Psi(n_0)}{n_0} < \frac{\Psi(n_0-1)}{n_0-1}$ *if* $n_0 > 1$, *or* $\Psi(1) < \Psi(2)$ *if* $n_0 = 1$,
(b) $v_j^* = \frac{1}{\Psi(n_0)}$ *for* $1 \leq j \leq n_0$ *and* $v_j^* = 0$ *for* $j > n_0$.

Let n be a positive integer so that $(m_\Psi^0)'$ has an extreme point, v, of length n. Note that the distance from v to any extreme points of $B_{(m_\Psi^0)'}$ of length different to n is at least $\frac{1}{\Psi(n)} - \frac{1}{\Psi(n+1)}$. Thus we see that for $i_1, i_2, \ldots, i_n$ in $\mathbf{N}$, the connected component of $\frac{1}{\Psi(n)}(e_{i_1} + e_{i_2} + \cdots + e_{i_n})$ in $\mathrm{Ext}_{\mathbf{R}}(B_{(m_\Psi^0)'})$ is $\{\lambda_{i_1}e_{i_1} + \lambda_{i_2}e_{i_2} + \cdots + \lambda_{i_n}e_{i_n} : |\lambda_{i_j}| = \frac{1}{\Psi(n)}\}$.

Example 4.4. Let $(w_n)_n$ be a decreasing sequence of positive real numbers which converge to 0. The Lorentz space $d(w,1)$ is defined by

$$d(w,1) = \left\{ (z_n)_n : \sum_{n=1}^{\infty} z_n^* w_n < \infty \right\}$$

endowed with the norm $\|z\|_w = \sum_{n=1}^{\infty} z_n^* w_n$. The space $d(w,1)$ is the dual of the Marcinkiewicz sequence space $d_*(w,1)$ whose fundamental sequence Ψ is given by $\Psi(n) = \sum_{k=1}^{n} w_k$. Since $w_n > 0$ for all n the first condition of Theorem 4.1, $\Psi(n_0) < \Psi(n_0+1)$, always holds. The violation of second condition of Theorem 4.1, $\frac{\Psi(n_0)}{n_0} = \frac{\Psi(n_0-1)}{n_0-1}$ holding, implies that $w_{n_0} = \frac{1}{n_0-1}\sum_{k=1}^{n_0-1} w_k$. As $(w_n)_n$ is decreasing the only way that this can be true is that $w_1 = w_2 = \cdots = w_{n_0}$. Hence, we see from Theorem 4.1 that the set of weak*-exposed points of the unit ball of $d(w,1)$ is

$$\left\{ (z_n)_n : \text{ there is } n_0 \text{ with } w_1 > w_{n_0}, z_k^* = \frac{1}{\Psi(n_0)} \text{ for } 1 \le k \le n_0 \text{ and } z_k^* = 0 \text{ for } k > n_0 \right\}.$$

We may write the set as:

$$\left\{ (z_n)_n : \text{ there is } n_0 > 1 \text{ with } w_1 > w_{n_0}, \text{ and } z^* = \frac{1}{\Psi(n_0)} \sum_{k=1}^{n_0} e_k \right\}.$$

Corollary 4.2 tells us that this set is also the of extreme points of the unit ball of $d(w,1)$, implying [13, Theorem 2.6].

Example 4.5. Now let $(w_n)_n$ be a sequence of nonnegative real numbers (not necessarily decreasing). We assume that $w_1 \neq 0$. Recall that in [8] the sequence Lorentz space, $\gamma_{1,w}$, is defined as all sequences of complex numbers $(z_n)_n$ such that

$$\|z\|_{\gamma_{1,w}} := \sum_{n=1}^{\infty} z_n^{**} w_n < \infty.$$

The space $(\gamma_{1,w}, \|\cdot\|_{\gamma_{1,w}})$ is a rearrangement invariant sequence space. For $n \in \mathbf{N}$ let

$$W(n) = \sum_{k=1}^{n} w_k \qquad \text{and} \qquad W_1(n) = n \sum_{k=n+1}^{\infty} \frac{w_k}{k}.$$

The fundamental function of $\gamma_{1,w}$ is given by

$$\phi_{\gamma_{1,w}}(n) = W(n) + W_1(n).$$

It is shown in [8, Theorem 5.4] that if $\sum_{k=1}^{\infty} w_k$ diverges then $\gamma_{1,w}$ is the dual of the Marcinkiewicz sequence space m_Ψ^0 where Ψ is the symbol given by $\Psi(n) = \phi_{\gamma_{1,w}}(n)$ for all n.

Let us see that the conditions of Theorem 4.1 are satisfied for $\gamma_{1,w}$. Suppose we have a positive n_0 so that $\Psi(n_0) = \Psi(n_0+1)$. This implies that

$$w_{n_0+1} = -W_1(n_0+1),$$

which is impossible as $(w_k)_k$ is a sequence of nonnegative real numbers. If we now suppose that $\dfrac{\Psi(n_0)}{n_0} = \dfrac{\Psi(n_0-1)}{n_0-1}$ then we get that

$$\sum_{k=1}^{n_0-1} w_k = 0$$

which also impossible. Hence, we get that the set of weak*-exposed points of the unit ball of $\gamma_{1,w}$ is precisely

$$\left\{(z_n)_n : \; z^* = \frac{1}{\phi_{\gamma_{1,w}(n_0)}} \sum_{k=1}^{n_0} e_k, n_0 \in \mathbf{N}\right\}.$$

Applying Corollary 4.2 we see that the above set is also the 'set of the unit ball of $\gamma_{1,w}$. This provides us with an alternative proof of [8, Theorem 4.7].

Now, we describe the set of weak*-exposed (and extreme points) of the unit ball of $(m^0_\Psi)'$ when $(m^0_\Psi)' = \ell_1$, for two renormings of this space. In the first, we show that for each natural number k it is possible to obtain a renorming of ℓ_1 with extreme points $\{e_{i_1} + \cdots + e_{i_k} : i_1 < \cdots < i_k\}$. In the second, we show that for each natural number k it is possible to obtain a renorming of ℓ_1 with extreme points $\{\lambda_{r,k}(e_{i_1} + \cdots + e_{i_r}) : i_1 < \cdots < i_r, 1 \le r \le k,\}$, for normalizing sclars $\lambda_{r,k}$, $1 \le r \le k$.

Example 4.6. Let us consider our first renorming of ℓ_1. Fix $k > 2$ in $\mathbb{N}$ and define a symbol Ψ by $\Psi(n) = 1$ for $n < k$, $\Psi(k) = 2$ and $\Psi(n) = \frac{2}{k}n$ for $n > k$. Then Ψ is strictly increasing for $n \ge k-1$. We have that $\frac{\Psi(n)}{n} = \frac{1}{n}$ for $n < k$ and $\frac{\Psi(n)}{n} = \frac{2}{k}$ for $n \ge k$. As $\lim_{n\to\infty} \frac{\Psi(n)}{n} > 0$, by [12, Theorem 3.2], we know that $(m^0_\Psi)'$ is isomorphic to ℓ_1. In addition, Theorem 4.1 tells that the set of weak*-exposed (and extreme points) of the unit ball of $(m^0_\Psi)'$ is

$$\left\{(z_n)_n : z^* = \sum_{j=1}^{k-1} e_j\right\}.$$

Note that taking $k = 2$ in the above example gives us ℓ_1 isometrically.

For the second renorming we fix again $k > 2$ and define a symbol Ψ by $\Psi(1) = 1$, $\Psi(n) = 1 + \frac{n-1}{k-1}$ for $2 \le n \le k$ and $\Psi(n) = \frac{2n}{k}$ for $n > k$. The symbol Ψ is strictly increasing. For $2 \le n \le k$ we have that

$$\frac{\Psi(n)}{n} = \frac{1}{n} + \left(\frac{n-1}{n}\right)\frac{1}{k-1} = \frac{1}{k+1} + \frac{1}{n}\left(1 - \frac{1}{k-1}\right).$$

Considering the function $f\colon [1,\infty) \to \mathbb{R}^+$ given by $f(x) = \frac{1}{k+1} + \frac{1}{x}\left(1 - \frac{1}{k-1}\right)$ we observe that $\frac{\psi(n)}{n}$ is strictly decreasing for n between 1 and k.

Finally, for $n > k$ we have that $\frac{\Psi(n)}{n} = \frac{2}{k}$. As $\lim_{n\to\infty} \frac{\Psi(n)}{n} > 0$, applying again [12, Theorem 3.2], we see that $(m^0_\Psi)'$ is also isomorphic to ℓ_1. Now applying Theorem 4.1 we have that that the set of weak*-exposed (and extreme points) of the unit ball of $(m^0_\Psi)'$ is

$$\left\{(z_n)_n: \ z^* = \frac{k-1}{k+r-2}\left(e_1 + \cdots + e_r\right), 1 < r \le k\right\}.$$

Finally, as we have a description of the real extreme points of a dual of a Marcinkiewicz sequence space m^0_Ψ, we are able to characterise its multipliers. Recall that given a Banach space E, a linear operator $T\colon E \to E$ is said to be a multiplier of E if every extreme point of the unit ball of E' is an eigenvector of T', the adjoint of T. This means that for every extreme point e of $B_{E'}$ there is a_e in $\mathbf{C}$ so that

$$T'(e) = a_e e.$$

Proposition 4.7. *Suppose that Ψ is a symbol such that $(m^0_\Psi)'$ has extreme points.*

(a) *If $(m^0_\Psi)'$ only has an extreme points of length 1, then every multiplier of m^0_Ψ is diagonal.*
(b) *If $(m^0_\Psi)'$ has an extreme point of support at least 2, then every multiplier of m^0_Ψ is a constant multiple of the identity.*

Proof. Statement (a) holds since by Corollary 4.2, every unit vector e_j is an extreme point of $(m^0_\Psi)'$. To prove (b), let us suppose that $(m^0_\Psi)'$ has only extreme points with n non-zero coordinates, $n \ge 2$. Consider the subspace, V_1, of $(m^0_\Psi)'$ spanned by $e_1, e_2, \dots, e_n$ and the subspace, V_2, of $(m^0_\Psi)'$ spanned by $e_1, e_{j_2}, \dots, e_{j_n}$, so that $1, 2, \dots, n, j_2, \dots, j_n$ are distinct. Notice that every vector in V_1 can be written as linear combination of vectors of the form $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$ with $|\lambda_j| = \frac{1}{\Psi(n)}$. As, by Corollary 4.2, each of these elements is an extreme point, thus for any of them we have

$$T'(\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n) = \mu(\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n)$$

for some $\mu \in \mathbf{C}$. Then we get that T' maps V_1 into V_1. Similarly, T' maps V_2 into V_2. Hence we have that e_1, which is contained in the intersection of V_1 and V_2, is mapped to a multiple of e_1. With an analogous argument, we see that each e_j is mapped to a multiple of e_j for $j \in \mathbf{N}$. Let us suppose that $T'(e_j) = \mu_j e_j$ for some $\mu_j \in \mathbf{C}$ and for each $j \in \mathbf{N}$. Fix $j \in \mathbf{N}, \mathbf{j} \ge \mathbf{2}$ and consider the vector $v = e_1 + e_j + e_{j+1} + \cdots + e_{j+n-2}$ which is a multiple of an extreme point. Then we have

$$T'(v) = \mu_1 e_1 + \mu_j e_j + \mu_{j+1} e_{j+1} + \cdots + \mu_{j+n-2} e_{j+n-2}.$$

Also, as T is a multiplier, we have

$$T'(v) = a_v v.$$

Therefore, we obtain that $\mu_j = a_v = \mu_1$. As $j \ge 2$ was arbitrary, we conclude that T' and hence T is a multiple of the identity. □

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