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The concept of Functions in Secondary School Textbooks: The Hybrid Approach

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ABSTRACT: This work is part of a broader investigation, in which we use the Theory of Conceptual Fields to analyse the meaning of function that emerges from the way high school books teach it. We analysed 24 high school books published between 1953 and 2010. In particular, we described the characteristics of the books that we classify into an approach we call a hybrid. This classification focuses on the identification of the representation systems used, the meaning and the situation. The results show that this approach, although it strives for a process of pragmatic knowledge elaboration, does not promote a true study of the problem, before its formalization. Rather, the problem works like a school task whose functionality is linked to the definitions.

Keywords – Books, Conceptual Field Theory, Function, Secondary School.

I. INTRODUCTION

Teaching the concept of function is central in all Argentine secondary schools [1]. This importance is sustained in the relevance that this concept has in the modelling of phenomena of different sciences. However, its conceptualization presents some difficulties related to representation systems, which have already been warned by teachers during the teaching process and documented by some research. For example, the works of [2], and [3] raise the need to teach functions through the coherent articulation of representation registers.

The teaching of functions, and the rest of the mathematical concepts taught in secondary school, is usually largely sustained in the mathematical discourse of secondary education books since it is usual for the teacher to select from the books the know-how to teach. The teaching practice is thus largely maintained by the mathematical discourse of the books that go on to regulate practically all their teaching actions, or at least having a great influence on them [4]. For this reason, the book is an essential device when it comes to analyzing the meanings to be reconstructed in secondary school.

In this paper, we present a part of the analysis carried out in 24 high school mathematics books published between 1953 and 2010, where we describe and examine how the books "teach" the concept of function [5]. In the analysis, we take into account, on the one hand, the theoretical construct of "Concept" of the Theory of Conceptual Fields [6], and on the other hand, the conjunct, graphic analytical and hybrid categories proposed by [7]. The Theory of Conceptual Fields (CBT) allows us to analyse the gap between the idea of the concept of function, formalized in textbooks and the operational invariants that students construct from the situation presented in the textbook. The dialectic between concepts in action and theorems in action is fundamental to achieve this [5].

II. THE THEORY OF CONCEPTUAL FIELDS (TCF)

The Theory of Conceptual Fields [6], [8] takes as a premise that knowledge is organized in conceptual fields whose domain, by the subject, occurs over an extensive period through experience, maturity and learning.[9]. A Conceptual Field, such as that of functions, is a heterogeneous set of problems, situations, concepts, relationships, structures, contents and operations of thoughts, interconnected and probably linked during the acquisition process.

In this context, the concept is defined as a triplet of three sets: C (S, I, Γ). **The reference [S]:** It is the set of situations that give meaning to the concept. **The meaning [I]:** It is the set of Operational Invariants (a) concepts-in-action, conformed by an object, a predicate or a category of thoughts considered pertinent, and b) theorems-in-action, formed by propositions, where the real can be either true or false), and upon which rests the

operation of the schemes. The schemes are the cognitive structure of the subject. **The signifier [Γ]:** It is the set of linguistic and non-linguistic forms that represent symbolically the concept, its properties, situations, and procedures.

Regarding this definition, it is important to emphasize that signifiers should not be confused with meanings: the words recover different meanings according to the situation. Besides, the meaning constructed by the subject can partially correspond to the conventional meaning of words and sentences, or to the one given by the teacher, since there is no direct homomorphism but partial, between the reality and the language, including the scientific language.

Finally, the concept of Operational Invariants [9], provides the elements to firstly determine the pieces of knowledge formalized in the textbooks to analyse the distance from the Operational Invariants that students will build in the classroom.

III. METHODOLOGY

We selected 24 books between 1953 and 2010 in which we prioritize the most popular publishers among secondary school teachers, that is, the ones they prefer when choosing a book to design their classes.

The concept of function is described and analysed with the theoretical tools of TCF. The description of the reference, the signifier and the meaning allow us to categorize the books according to the classification proposed by [7], as detailed below. The first is **the Set Approach**, which is based on the teaching of the concept of function through the Theory of Sets. All concepts are defined through their respective algebraic symbolic notation and the use of Ven diagrams. The construction of the concept of function is based on the concepts of ordered pair, relation, Cartesian product, equivalence relation, scope, rank, inverse relation, symmetric relation, reflexive relation, transitive relation, non-strict order relation, equivalence class, departure set, arrival set, equivalent elements, etc. This approach allows for great precision in the mathematical notation of the concept. The set approach exalts a static vision of the concept of the function relation, that is, it uses an assignment rule between ordered pairs that do not highlight the dependency between variables. On the other hand, in the **Analytic-Graphic Approach**, the concept of function is determined through the graphic representation in Cartesian axes of a problem or concrete situation framed in some science, in which a functional relationship is established between variables. It is from the graph that the analysis of the function and its properties is performed: domain, image, notable points, asymptotes, increasing or decreasing, etc. The analytical-graphic approach has a dynamic vision of the concept of the function relationship. In other words, it emphasizes the relationship between variables (e.g.: the amount of money based on hours worked, infected persons based on days, etc.). Thus, the graphical analytic approach defines the function by understanding. Lastly, the **Hybrid Approach** has characteristics of the two organizations mentioned: it establishes a relationship between variables, but also uses set concepts such as a starting and ending set; and allocation rules. The hybrid approach focuses on the graphical representation of the function from the formula that represents it.

On the other hand, we analyse which representation systems predominate in each approach. For this, we take the representation systems proposed by [7]. **Numeric [RSN]:** tables and numerical calculations, for example: $2.5 + 3 = 13$; **First Order Algebraic [RSA1]:** algebraic procedures in which the parameters are initialized: $f(x) = 2x + 3$; **Second Order Algebraic [RSA2]:** algebraic procedures in which the parameters are not initialized, for example: $f(x) = ax + b$; **Graphical Analytical [SRAG]:** graph on Cartesian axes; **Written Verbal [RSWV]:** they are written linguistic forms, whether they are situations, assertions, etc.; **Pictorial [RSP]:** Refers to the construction of schematic drawings;

With these elements, in this work, we describe the books that we have categorized within the hybrid approach.

III. ANALYSIS OF DATA AND RESULTS

Of the 24 books analysed, nine are framed in the Hybrid Approach: 1) Mathematics 2, Eds. Santillana (1989); 2) Mathematics 3, Eds. Santillana (1995); 3) Mathematics 8, Eds. Santillana (2002); 4) Pythagoras 8, Eds. SM (2004); 5) Mathematics, functions and Statistics, Eds. A-Z (2005); 6) Functions 1, Eds. Longseller (2002); 7) Mathematics I, Eds. Estrada (2004), 8) Elementary Functions to construct mathematical models, Eds. Inet (2010); 9) Mathematics 4, Eds. Estrada (2010). These textbooks were published between 1995 and 2010, yet there is one exception in 1989.

Due to lack of space, we cannot describe the nine books, but we describe and analyse the book "Matemática I" published in 2004 by Editorial Estrada to show the characteristics and relevant aspects of this approach from it. The book that is analysed corresponds to what is currently the third year of secondary school (15 years). According to the curriculum design of Argentina, in that year, Physics is studied in all the existing orientations of the secondary level.

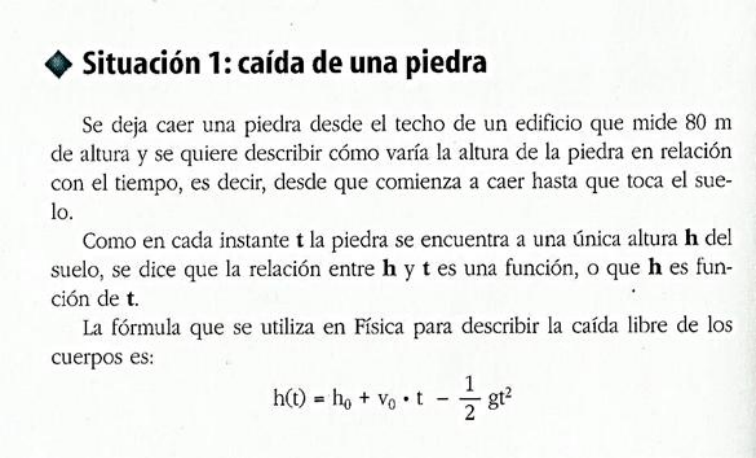
The reference [S]: The chapter on teaching functions (fig. 1a) begins with a free-fall problem*: "A stone is dropped from the roof of a building that is 80 meters high and we want to describe how the height of the stone in relation to time, that is, from when it begins to fall until it falls to the ground"*. Then, the author indicates that "as at each instant t the stone is at a single height h from the ground, it is said that the relationship between h and t is a function or that h is a function of t".

The problem establishes a relationship between magnitudes, using a problematic situation of free fall relative to the field of Physics. This link between a problem contextualized to a science and its algebraic formalization is a characteristic of the hybrid approach. In this situation, it is noted that, if the student does not know kinematics, he/she would not understand the problem. Also, the absence of concepts, which define a function, is observed, such as independent and dependent variable, and the relationship between variables.

In general, to perform the introduction of functions, less complex contextualized situations are used, such as price as a function of time, distance as a function of time, among others.

The signifier [Γ]: The textbook uses the RS Written Verbal to present the situation and the explanation of the content developed, and the RS Second Order Algebraic [RSA2] to formulate the height as a function of time.

The meaning [I]: The Theorems and Concepts in Act reconstructed from the concept in this approach are the following. RS Written Verbal [RSWV]: *"it is possible to establish a one-to-one relationship between height and time"*; *"The relationship between height h and time t is a function"*; *"For each instant of time t, a unique height h corresponds to it"*; *"The height h depends on the time t"*. RS Second-Order Algebraic [RSA2]: *"The formula that relates height and time is* $h(t) = h_0 + v_0.t + \frac{1}{2}.g.t^2$*"*.



Situación 1: caída de una piedra

Se deja caer una piedra desde el techo de un edificio que mide 80 m de altura y se quiere describir cómo varía la altura de la piedra en relación con el tiempo, es decir, desde que comienza a caer hasta que toca el suelo.

Como en cada instante **t** la piedra se encuentra a una única altura **h** del suelo, se dice que la relación entre **h** y **t** es una función, o que **h** es función de **t**.

La fórmula que se utiliza en Física para describir la caída libre de los cuerpos es:

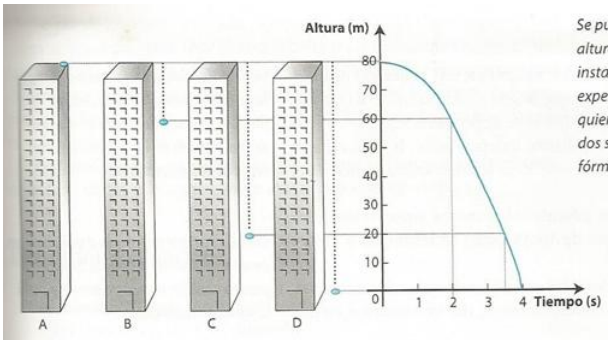
$$h(t) = h_0 + v_0 \cdot t - \frac{1}{2} gt^2$$

Figure 1a: Start-up problem

The author, then, continues developing the problem of free fall (see fig. 1b).

The Reference [S]: In fig. 1b there is a graphic representation of the above-mentioned problematic situation. The graph indicates that: *"it can be seen that to calculate the height of the stone at each instant, it is not necessary to repeat the experience each time. For example, if you want to know the height reached after two seconds, just calculate the formula for the value of* $t = 2$ *seconds"*. Also, it specifies that the variables are represented with letters and that in this case, h represents the height and t the time. Then calculate h for the time $t = 2.5s$ by plugging it into the formula, and the time for a height $h = 20m$.

The presentation of the graph does not explain how it is constructed. It does not even mention that its understanding requires knowledge of kinematics, which is not compatible with students of that age. Finally, it is not explained why the value of gravity is rounded to $10\ m/s^2$either. In short, the graph responds to the proposed algebraic formula, but it is difficult to construct it from it.



Se puede observar que para averiguar la altura que alcanza la piedra en cada instante, no es necesario hacer la experiencia cada vez; por ejemplo, si se quiere conocer la altura alcanzada a los dos segundos, basta con calcular la fórmula para el valor t = 2 segundos.

b| ¿A qué altura se encuentra la piedra después de 2 segundos y medio?

Se calcula h(2,5), o sea $80 - 5 \cdot (2,5)^2 = 80 - 5 \cdot 6,25 = 48,75$ m.

Respuesta: La piedra se encuentra a una altura de 48,75 metros.

En los ejemplos que siguen, se calcula, en cada caso, el tiempo que tarda la piedra para alcanzar un valor h_1 de la altura. Para ello, se averigua para qué valor de **t** es $h(t) = h_1$ resolviendo la ecuación $80 - 5t^2 = h_1$.

*Para representar funciones y variables se utilizan letras. En este caso particular, se asignó la letra **t** para el tiempo y la letra **h** para la altura.*

c| ¿Cuántos segundos tarda la piedra en alcanzar una altura de 20 metros?

En este caso, se resuelve la ecuación $80 - 5t^2 = 20$ como sigue:

$$80 - 5t^2 = 20 \Rightarrow 80 = 20 + 5t^2 \Rightarrow 80 - 20 = 5t^2 \Rightarrow 60 = 5t^2 \Rightarrow t^2 = \frac{60}{5} = 12 \Rightarrow t = \pm\sqrt{12} \approx \pm 3,46 \text{ s}$$

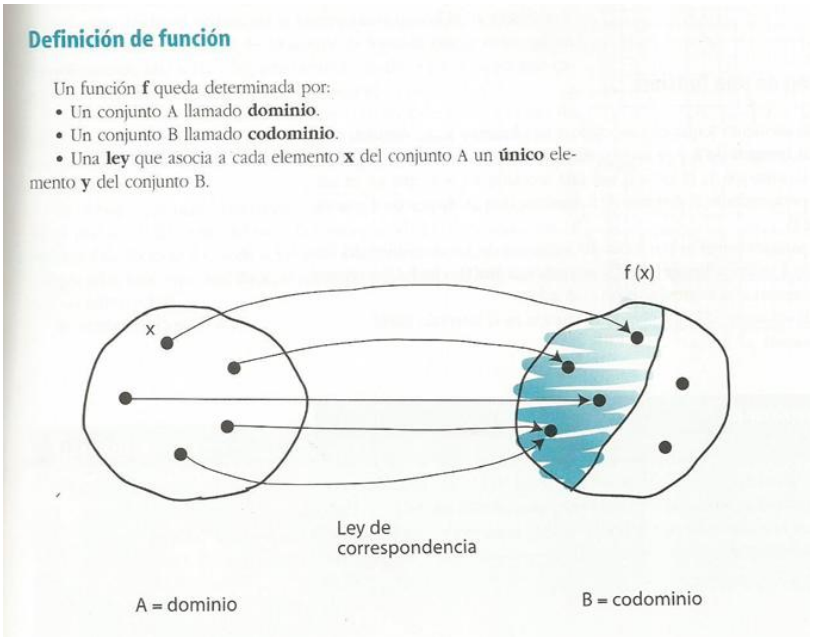
Respuesta: La piedra tarda 3,46 segundos.

Figure 1b: Graphical representation of the Function and evaluation at a point

The signifier [Γ]: The book uses the RS Written Verbal [RSWV] to explain the graph and the resolution of the proposed tasks. Besides, the RS Graphical Analytical [RSAG] is observed for the representation of the function $h\,(t)$, and the RS First Order Algebraic [RSA1] and RS Numerical [RSN], for the height and time calculations in the proposed formula.

The meaning [I]: For the RS Written Verbal [RSWV]: *"the variables are represented with letters"*, *"to find a certain height or time, the values must be replaced in the function"*. For RS First Order and Numerical Algebraic [RSN] and [RSA1]: *"the letter must be replaced by the value to obtain the value of the other variable"*. For the RS Analytical - Graph [RSAG]: *"the function can be represented on the Cartesian axes", "the variable t is represented on the abscissa axis", "the variable h is represented on the coordinate axis"*.

The author continues to develop the concept of Function as seen in Fig. 1c.



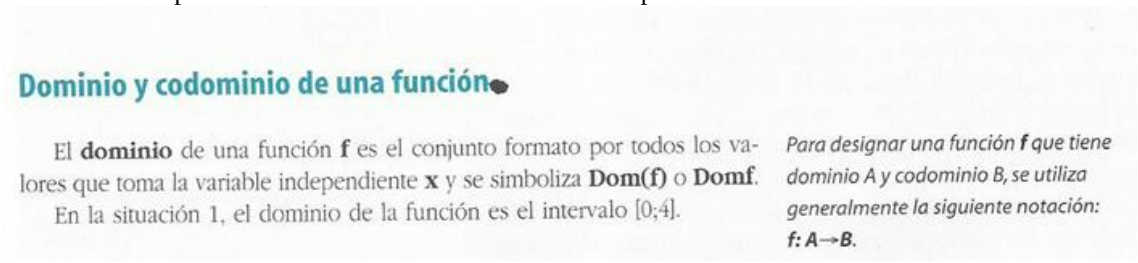
Definición de función

Un función **f** queda determinada por:

- Un conjunto A llamado **dominio**.
- Un conjunto B llamado **codominio**.
- Una **ley** que asocia a cada elemento **x** del conjunto A un **único** elemento **y** del conjunto B.

Figure 1c: Domain and Codomain of a Function

The Reference [S]: In a second instance, the author uses set theory to define the function from a relationship between two sets: the set A called the Domain and a set B called the Codomain. Then, it establishes that *"there is a law that associates to each element x of set A a single element and of set B"*. It continues with the proposal of a Venn diagram showing the set A (domain) and set B (codomain) and the association law between the sets that determine the function (fig. 1c). This definition, which is from set theory, does not previously carry out the definitions of ordered pair, relation, etc., characteristics of set theory, but simply use them. Other than that, it defines functions, not as a dependency between variables, but as a law that associates two elements. This is where the static characteristic of the function definition of this theory is framed.
In this context, the representation of the Venn diagram of the Domain and Codomain that is made does not justify the partition of the arrival set, where elements assigned to the function are observed and others are not, to which the student will have to make sense.
The signifier [Γ]: The RS Written Verbal [RSWV] is used to explain the function definition and the Domain and Codomain sets, and the RS Pictorial [RSP] to represent the sets A and B. While the second-order RS Algebraic [RSA2] is used to indicate the elements of the output set and the name of the arrival set.
The meaning [I]: For the RS Written Verbal [RSWV]: *"set A is the Domain of the function"*; *"Set B is the codomain of the function"*; *"There is a law that associates to each element x of set A a single element y of set B"*; *"A is the domain set of the Function"*; *"B is the image set of the function"*. For the RS Pictorial [RSP]: "the elements of set A are called x"; *"The set B is called* $f(x)$*"*; *"Each element x of set A is linked with a single element of set B"*; *"Set B is divided into two parts with elements that receive an arrow and others that do not"*. Note that the operational invariants that can be constructed in one representation system are different from those that can be constructed in another.
In what follows, the author defines the domain of a function, as shown in fig. 1d.
The Reference [S]: The author indicates that the domain of a function is "the set of all values determined by the independent variable x, and is symbolized $Domf$ or $Dom\,(f)$" (figure 4). In a column to its right indicates that: "To designate a function of a set A and a set B, the $notation\ is\ f: A \rightarrow B$". While the meaning of the dependent and independent variable of the function remains implicit.

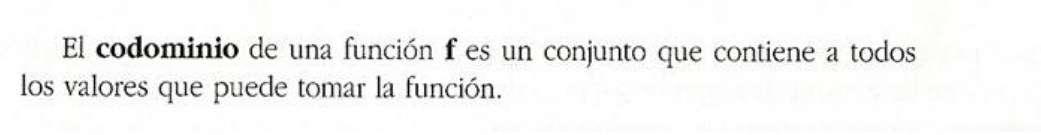


Dominio y codominio de una función

El **dominio** de una función **f** es el conjunto formato por todos los valores que toma la variable independiente **x** y se simboliza **Dom(f)** o **Domf**.
En la situación 1, el dominio de la función es el intervalo [0;4].

*Para designar una función **f** que tiene dominio A y codominio B, se utiliza generalmente la siguiente notación:* ***f: A→B.***

Figure 1d: Definition of Domain and Codomain of a Function
The signifier [Γ]: The RS Written Verbal [RSWV] and the RS Second-Order Algebraic [RSA2] are used for the explanation and notation of the domain of a function.
The meaning [I]: The Theorems in Act would be, for the RS Written Verbal [RSWV]: *"the domain is the set formed by all the values that the independent variable x takes"*; *"The domain is symbolized* $Domf\ o\ Dom\,(f)$*"*. RS Second-Order Algebraic [RSA2]: *"a function f that has domain A and codomain B, the notation is:* $f: A \rightarrow B$*"*.
Then, in Fig. 1e, define the codomain.



El **codominio** de una función **f** es un conjunto que contiene a todos los valores que puede tomar la función.

Figure 1e: Codomain definition of a function
The Reference [S]: The author defines the codomain *as "the set that contains all the values that a function can take"*, there is no notation or further explanation of said set, nor does the book define what a set is before. One of the characteristics of the hybrid approach consists precisely in presenting ambiguous definitions, which it takes from the set approach but without all its previous algebraic construction.
The signifier [Γ]: The RS Written Verbal is used.
The meaning [I]: In the RS written verbal [RSWV]: *"the codomain of a function is the set that contains all the values that a function can take"*.
Next, in fig. 1f the definition of the image of a function is presented.
The Reference [S]: The author explains that *"each element of y is associated to an element x of the domain of the function. He indicates that it is called the image of x and it is written* $f(x)$*"* clarifying that it is read *"ef of* x*"* (fig. 1f). It takes up the proposed example of free fall (figure 1a) and shows that for the height $h = 0$, the

time $t = 4$ is associated, which is written $h(4)$. Then, he states that *"the set formed by all the images of the domain of the function is called the image of f and is symbolized $Im(f)$ or Imf"*. And that the image is contained in the codomain. Finally, he takes up the problem of free fall indicating that the image is the interval [0; 80].

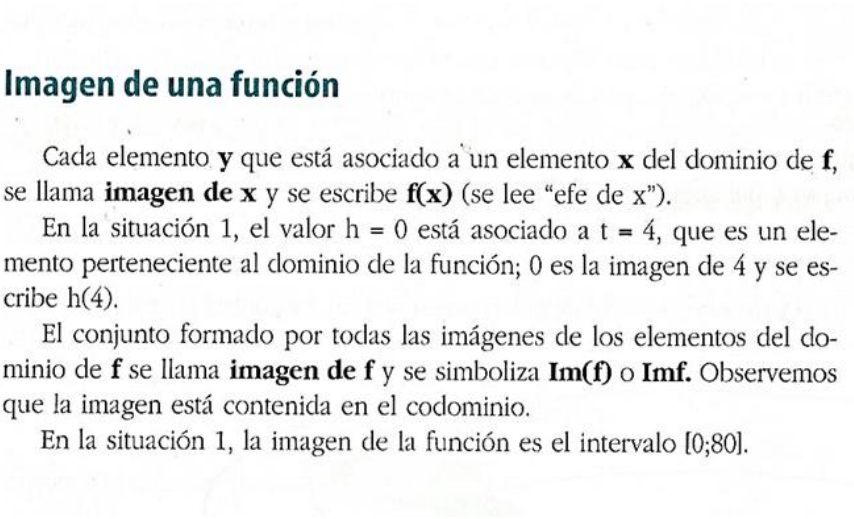


Imagen de una función

Cada elemento **y** que está asociado a un elemento **x** del dominio de **f**, se llama **imagen de x** y se escribe **f(x)** (se lee "efe de x").

En la situación 1, el valor h = 0 está asociado a t = 4, que es un elemento perteneciente al dominio de la función; 0 es la imagen de 4 y se escribe h(4).

El conjunto formado por todas las imágenes de los elementos del dominio de **f** se llama **imagen de f** y se simboliza **Im(f)** o **Imf.** Observemos que la imagen está contenida en el codominio.

En la situación 1, la imagen de la función es el intervalo [0;80].

Figure 1f: Image of a Function

The signifier [Γ]: The Representation Systems used are the RS Written Verbal [RSWV] to explain the definition of image. Also, the RS Numerical [RSN] is observed to find the height of the problem in figure 1a.
The meaning [I]: The Theorems in Act present in the aforementioned Representation Systems would be, in Written Verbal [RSWV]: *"to each element x of the domain corresponds a value y of the image", "the set formed by all the images of the elements of the domain of the function f, it is called the image of f ", "the image is contained in the codomain"*. RS Numerical [RSN]: *"to find the height, the time must be replaced by the assigned value; $t = 4$"*.

IV. RESULTS AND DISCUSSION

From the 24 books that propose the teaching of functions in secondary school, nine falls under the hybrid approach. This approach can be seen in the books published between 1999 and the present, which coincides with the transition from Federal Law on Education No. 24,195 to National Law on Education No. 26.206.

Books with characteristics of this approach begin the study of functions by formulating and solving a mathematical problem linked to real life, or some science such as physics, economics, biology, etc. As they propose the resolution of the problem, they also define the concepts of function, independent variable, dependent variable, domain, codomain, image. Thus, the problem works as an excuse to define mathematical concepts in terms of set theory that can hardly be made sense without all the previous theoretical construction. Well, even when this approach strives for a process of pragmatic knowledge elaboration, it does not promote a true study of the problem, before its formalization. Rather, the problem works as a school task whose functionality is linked to the definitions.

The resolution of the problems that arise is done utilizing an algebraic formula or expression as if the concept could be reduced to it. This way of privileging the algebraic expression of a function is a characteristic of the hybrid approach. In particular, this book sets out a contextualized problem in physics, a subject that is also developed in the third year, but that, if at the time of studying mathematics, the student does not have the basic knowledge of kinematics, the context will complicate the conceptualization of the concept of function.

V. CONCLUSION

On the one hand, the problem allows the pragmatic elaboration of the concept, which is essential for these to acquire meaning for the student [6]. On the other hand, these books try to recover the notation that is typical of mathematics. However, the study of functions in this approach would improve if the study of the initial problem worked as a real problem to be solved and not as an excuse. Thinking about this type of problem is the task of the didacts.

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