# AN OPTIMIZATION PROBLEM WITH VOLUME CONSTRAINT FOR AN INHOMOGENEOUS OPERATOR WITH NONSTANDARD GROWTH

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Abstract. We consider an optimization problem with volume constraint for an energy functional associated to an inhomogeneous operator with nonstandard growth. By studying an auxiliary penalized problem, we prove existence and regularity of solution to the original problem: every optimal configuration is a solution to a one phase free boundary problem—for an operator with nonstandard growth and non-zero right hand side—and the free boundary is a smooth surface.

## 1. Introduction

A classical problem asks for the properties of the following optimal configuration: given a body and a fixed amount of insulating material, what is the best way of insulating it?

In general, the problem of minimizing the flow of heat through the boundary of a region $\Omega$ by including in $\Omega$ a fixed amount of insulating material, can be reduced to the problem of minimizing an energy functional within $\Omega$ over functions satisfying a constraint on the measure of their support. This reduction can be done, under the assumption that the temperature is constant on the boundary of the region, by using that it satisfies a differential equation on its support. When there are external sources, the equation satisfied by the temperature is inhomogeneous.

This, as well as other applications, suggest the interest of analyzing the minimization of functionals associated to some differential equations with restrictions on the measure of the support of the admissible functions.

In the pioneering article [3], Aguilera, Alt and Caffarelli studied an optimal design problem with volume constraint of this type. The authors introduced a penalization term in the energy functional (the Dirichlet integral) and minimized without the volume constraint. For fixed values of the penalization parameter, the penalized functional was very similar to the one considered in [5] and regularity results for minimizers of the penalized problem followed once the authors proved that minimizers were weak solutions to the free boundary problem in [5].

The main result in [3]—that makes this method so useful—is that the right volume is already attained for small values of the penalization parameter. In this way, all the regularity results apply to the solution of the optimal design problem as well. Moreover, the minimizer is a solution of the associated Euler Lagrange equation on its support so that, when the boundary datum is constant, it is a solution to the problem of minimizing the boundary flux.

The regularity of the boundary of the support of the minimizers as well as the free boundary condition allow, in many cases, to characterize the optimal configurations.

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This method has been applied to other problems with similar success. In the first ones, the differential equation satisfied by the minimizers was uniformly elliptic and homogeneous, i.e., having zero right hand side (see [4, 15, 33]). Still in the homogeneous case, the method was applied for nonlinear degenerate equations in [16, 28, 30] and in [34] a related problem for a space dependent operator with $p$-Laplacian type growth with $p$ constant was analyzed. The case of an equation with non-zero right hand side was treated in the linear case in [23].

In this article we prove similar results for an inhomogeneous equation with nonstandard growth. In fact, we study the following problem which is a generalization of the one in [3]:

We take $\Omega$ a $C^1$ bounded domain in $\mathbb{R}^N$ and $\varphi_0 \in W^{1,p(\cdot)}(\Omega)$, a nonnegative Dirichlet datum, with $\varphi_0 \geq c_0 > 0$ in $\mathcal{A}$, where $\mathcal{A}$ is a nonempty relatively open subset of $\partial\Omega$ of class $C^2$. Let $f \in L^\infty(\Omega)$ and $0 < \omega_0 < |\Omega|$. Let

$$\mathcal{K}_{\omega_0} = \{v \in W^{1,p(\cdot)}(\Omega) \,/\, |\{v > 0\}| = \omega_0,\, v - \varphi_0 \in W_0^{1,p(\cdot)}(\Omega)\}.$$

Our purpose is to find nonnegative solutions of the problem:

$$(P) \qquad \text{Minimize} \quad \mathcal{J}(v) = \int_\Omega \Big(\frac{|\nabla v|^{p(x)}}{p(x)} + fv\Big)\, dx \quad \text{in} \quad \mathcal{K}_{\omega_0},$$

and study their properties.

In order to find nonnegative solutions to problem $(P)$ in a way that allows us to perform non volume preserving perturbations we consider instead the following penalized problem: We let, for $0 < \varepsilon < 1$,

$$\mathcal{K} = \{v \in W^{1,p(\cdot)}(\Omega) \,/\, v - \varphi_0 \in W_0^{1,p(\cdot)}(\Omega)\}$$

and

$$\mathcal{J}_\varepsilon(v) = \int_\Omega \Big(\frac{|\nabla v|^{p(x)}}{p(x)} + fv\Big)\, dx + F_\varepsilon(|\{v > 0\}|),$$

where

$$F_\varepsilon(s) = \begin{cases} \varepsilon(s - \omega_0) & \text{if } s < \omega_0 \\ \frac{1}{\varepsilon}(s - \omega_0) & \text{if } s \geq \omega_0. \end{cases}$$

Then, the penalized problem is

$$(P_\varepsilon) \qquad \text{Find } u_\varepsilon \in \mathcal{K} \quad \text{such that} \quad \mathcal{J}_\varepsilon(u_\varepsilon) = \inf_{v \in \mathcal{K}} \mathcal{J}_\varepsilon(v).$$

Existence of solutions to $(P_\varepsilon)$ follows by direct minimization. We obtain the regularity of nonnegative solutions to $(P_\varepsilon)$ and their free boundaries $\partial\{u_\varepsilon > 0\}$ by first proving that any nonnegative local minimizer $u_\varepsilon$ of $\mathcal{J}_\varepsilon$ is a weak solution of the free boundary problem: $u_\varepsilon \geq 0$ and

$$(P(f, p, \lambda^*_{u_\varepsilon})) \qquad \begin{cases} \Delta_{p(x)} u_\varepsilon := \operatorname{div}(|\nabla u_\varepsilon(x)|^{p(x)-2} \nabla u_\varepsilon) = f & \text{in } \{u_\varepsilon > 0\} \\ u_\varepsilon = 0,\ |\nabla u_\varepsilon| = \lambda^*_{u_\varepsilon}(x) & \text{on } \partial\{u_\varepsilon > 0\}, \end{cases}$$

with $\lambda^*_{u_\varepsilon}(x) = \Big(\frac{p(x)}{p(x)-1}\, \lambda_{u_\varepsilon}\Big)^{1/p(x)}$, where $\lambda_{u_\varepsilon} > 0$ is a constant.

Then, from [26] we obtain the regularity of $\partial\{u_\varepsilon > 0\}$. In fact, in [26] we developed a regularity theory for weak solutions of the free boundary problem $P(f, p, \lambda^*)$, with the notion of weak solution we employ here.

As in [3], the reason why this penalization method is so useful is that there is no need to pass to the limit in the penalization parameter $\varepsilon$ for which uniform, in $\varepsilon$, regularity estimates would be needed. In fact, we show that, under suitable assumptions, for small values of $\varepsilon$ the right volume is already attained. That is, $|\{u_\varepsilon > 0\}| = \omega_0$ for $\varepsilon$ small. Therefore, any nonnegative solution to $(P_\varepsilon)$ is a solution to our original problem $(P)$.

In particular, the fact that, for small $\varepsilon$, any nonnegative solution to $(P_\varepsilon)$ satisfies $|\{u_\varepsilon > 0\}| = \omega_0$ implies that *any nonnegative solution to our original optimization problem* $(P)$ *is also a nonnegative solution to* $(P_\varepsilon)$ so that it is locally Lipschitz continuous with smooth free boundary.

Let us remark that our study of the penalized problem $(P_\varepsilon)$ presents new features—it required delicate arguments due to the nonlinear degenerate/singular nature and the $x$-dependence of the operator associated to the original energy functional $\mathcal{J}$.

On the one hand, in order to prove basic properties of nonnegative local minimizers of $\mathcal{J}_\varepsilon$ (see Definition 3.1) such as Lipschitz continuity and nondegeneracy, we use a method introduced in [9] and then used in [8] for a minimization problem related to the $p$-Laplacian with a linear dependence on the volume of the positivity set. This method requires multiple rescalings. Due to the nonlinear and nonlocal nature of our penalization term it is not clear that these rescalings are minimizers of a similar functional so that the method cannot be directly applied. This difficulty is not due to the presence of a right-hand side $f$ nor to the fact that the exponent $p(x)$ is not constant.

In order to see that a somewhat similar approach is still possible, we introduce the concepts of local minimizers from above and from below of $J^{p,\lambda,f}$ (see Definition 3.2). This allows us to deal with the penalization term—which is *nonlinear and nonlocal*, depending on the positivity set of the function in the whole domain $\Omega$—in a *linear and local* way, that at the same time is *preserved under successive rescalings.* Once we change in this way our point of view, we are able to prove the desired basic properties (Corollary 3.1 and Theorems 3.3 to 3.6) with the aid of the arguments from our previous work [27].

On the other hand, the derivation of the free boundary condition—i.e., at points $x$ in the free boundary there holds that $\big(\frac{p(x)-1}{p(x)}\big)|\nabla u_\varepsilon(x)|^{p(x)} = \lambda_{u_\varepsilon}$ (in the weak sense of Definition 2.2), with $\lambda_{u_\varepsilon}$ a positive constant—required a subtle procedure not present in previous literature, that we develop in Lemmas 4.2 to 4.4 and Theorems 4.1 to 4.3. This subtlety comes from several facts.

First, the free boundary condition is not constant, as was the case in previous results on these kind of problems. But we prove that there is still something that is constant, namely, $\lambda_{u_\varepsilon}$. This fact is very important for some of the proofs leading to the main result in the following section. Next, in the derivation of the free boundary condition we can not follow the arguments in [3] because we are dealing with a different notion of weak solution more suitable for the nonlinear operator we are dealing with. Finally, neither can we argue as it was done in [8] for the case of the $p$-Laplacian because the derivation of the free boundary condition in [8] relies on their Theorem 2.1, which gives the free boundary condition in a very weak sense. The proof of that theorem strongly uses the linear dependence of the energy on the volume of the positivity set and does not make sense for a nonlinear and nonlocal penalization term as ours.

Then, in Section 5 we recover the original optimization problem $(P)$ and we prove our main result. We point out that the fact that we are dealing with an operator with nonstandard growth like the $p(x)$-Laplace operator, with a variable exponent $p(x)$ and a possibly non identically zero right hand side $f$, required the development of novel results such as Proposition 5.1, which is of independent interest. In fact, this proposition extends to the variable exponent setting the corresponding result proven in [5], Lemma 3.2, for the case $p(x) \equiv 2, f \equiv 0$, and it is new even when $p(x) \equiv p$. We remark that its proof is particularly delicate because of the form of the weak Harnack inequality when $p(x)$ is not constant and/or $f$ is not identically zero. Also at this stage it was necessary to construct new and nontrivial barriers (Lemma 5.3) on rings of arbitrarily small width needed for the proof of Lemma 5.4. In fact, the proof of this latter lemma differs deeply from the corresponding one for the case $p$ constant and $f \equiv 0$.

We want to emphasize that there was no need to impose a sign restriction on $f$ in the study of problem $(P_\varepsilon)$ performed in Sections 3 and 4.

On the other hand, given a nonnegative solution $u_\varepsilon$ to $(P_\varepsilon)$, in order to show that $|\{u_\varepsilon > 0\}| = \omega_0$ for $\varepsilon$ small, we proved that the constant $\lambda_{u_\varepsilon}$, appearing in the free boundary condition in $P(f, p, \lambda^*_{u_\varepsilon})$, stays away from zero and infinity, independently of $\varepsilon$.

We obtained the upper bound without a sign restriction on $f$. In order to obtain the lower bound for $\lambda_{u_\varepsilon}$, it was sufficient to have that nonnegative solutions $u_\varepsilon$ satisfy a nondegeneracy condition at some free boundary point, uniformly in $\varepsilon$. We called this condition $(H_\kappa)$ (see Definition 5.1). Such a condition is satisfied, for instance, if $||f^+||_{L^\infty}$ is small enough (Lemma 5.5) or if the prescribed volume $\omega_0$ is small enough (Lemma 5.6). In this situation we proved a partial existence and regularity result for problem $(P)$ (Theorem 5.1).

On the other hand, the assumption $f \le 0$ implies that any solution to problem $(P_\varepsilon)$ is nonnegative. The same holds for any solution to problem $(P)$.

The main result in the paper is:

**Theorem 1.1.** *Let $\Omega$, $p$, $f$ and $\varphi_0$ satisfying the assumptions in Subsection 1.3. Assume $f \le 0$.*

*Then there exists a nonnegative solution $u$ to problem $(P)$.*

*Moreover, any solution $u$ to $(P)$ is nonnegative and locally Lipschitz continuous.*

*Assume further that $f \in W^{1,q}(\Omega)$ and $p \in W^{2,q}(\Omega)$ with $q > \max\{1, N/2\}$. Then, any solution $u$ to $(P)$ satisfies that there is a subset $\mathcal{R}$ of $\Omega \cap \partial\{u > 0\}$ ($\mathcal{R} = \partial_{\rm red}\{u > 0\}$) which is locally a $C^{1,\alpha}$ surface, for some $0 < \alpha < 1$. Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u > 0\}$ and the remainder has $\mathcal{H}^{N-1}-$measure zero.*

*Assume moreover that $p \in C^2(\Omega)$ and $f \in C^1(\Omega)$, then $\mathcal{R} \in C^{2,\mu}$ for every $0 < \mu < 1$. If $p \in C^{m+1,\mu}(\Omega)$ and $f \in C^{m,\mu}(\Omega)$ for some $0 < \mu < 1$ and $m \ge 1$, then $\mathcal{R} \in C^{m+2,\mu}$. Finally, if $p$ and $f$ are analytic, then $\mathcal{R}$ is analytic.*

We remark that we did not use the regularity of the free boundary of the solutions to the penalized problem $(P_\varepsilon)$ in the existence proof for problem $(P)$, as was the case in previous articles (see Theorem 5.1).

Let us point out that in this article, for the sake of simplicity, we have chosen to work with the $p(x)$-Laplacian since it is a prototype operator with nonstandard growth. This operator has been used in the study of image processing ([1, 7]). The $p(x)$-Laplacian has also appeared as a model for a stationary non-newtonian fluid with properties depending on the point in the region where it moves. For example, such a situation corresponds to an electrorheological fluid. These are fluids such that their properties depend on the magnitude of the electric field applied to it ([32]).

The ideas and techniques in our work can be applied to any optimal design problem with volume constraint where the medium under consideration has properties possibly depending on the point, and where the corresponding energy functional is associated to an operator with nonstandard growth, with a possible non-zero right hand side.

Let us finally point out several problems similar to the one considered here that have appeared in shape optimization: for instance, in optimization of torsional rigidity [22], insulation of pipelines for hot liquids [18] and minimization of the current leakage from insulated wires and coaxial cables [2]. See also [20] and the references therein.

The paper is organized as follows: In Section 2 we define the notion of weak solution to the free boundary problem $P(f, p, \lambda^*)$ and include some related definitions and results.

In Section 3 we begin our analysis of problem $(P_\varepsilon)$ for fixed $\varepsilon$. First we prove the existence of a solution. Then, for nonnegative local minimizers of $\mathcal{J}_\varepsilon$, we prove local Lipschitz regularity and we study the behavior near the free boundary, such as nondegeneracy.

Then, in Section 4 we prove that any nonnegative local minimizer $u_\varepsilon$ of $\mathcal{J}_\varepsilon$ is a weak solution to the free boundary problem $P(f,p,\lambda^*_{u_\varepsilon})$—as defined in [26]. And, as a consequence we obtain that the free boundary is a $C^{1,\alpha}$ surface with the exception of a subset of $\mathcal{H}^{N-1}$-measure zero. We also get further regularity results on the free boundary, under further regularity assumptions on the data.

In Section 5 we prove that, under suitable assumptions, for small values of $\varepsilon$ we recover our original optimization problem $(P)$.

We also include a final section—Section 6—with some conclusions and remarks.

We end the paper with an Appendix where we collect some results on variable exponent Sobolev spaces as well as some other results that are used throughout the work.

We point out that we omit all the proofs that are very similar to the ones in other papers and we clearly refer to the corresponding results for the reader's convenience.

1.1. **Preliminaries on Lebesgue and Sobolev spaces with variable exponent.** Let $p:\Omega\to[1,\infty)$ be a measurable bounded function, called a variable exponent on $\Omega$ and denote $p_{\max}=\operatorname{esssup} p(x)$ and $p_{\min}=\operatorname{essinf} p(x)$. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u:\Omega\to\mathbb{R}$ for which the modular $\varrho_{p(\cdot)}(u)=\int_\Omega |u(x)|^{p(x)}\,dx$ is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)}=\|u\|_{p(\cdot)}=\inf\{\lambda>0:\varrho_{p(\cdot)}(u/\lambda)\le 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

There holds the following relation between $\varrho_{p(\cdot)}(u)$ and $\|u\|_{L^{p(\cdot)}}$:

$$\min\Big\{\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\min}},\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\max}}\Big\}\le\|u\|_{L^{p(\cdot)}(\Omega)}$$
$$\le\max\Big\{\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\min}},\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\max}}\Big\}.$$

Moreover, the dual of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ with $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$.

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions $u$ such that $u$ and the distributional derivative $\nabla u$ are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{1,p(\cdot)}:=\|u\|_{p(\cdot)}+\||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

For the sake of completeness we include in an Appendix at the end of the paper some additional results on these spaces that are used throughout the paper.

1.2. **Preliminaries on solutions to the $p(x)$-Laplacian.** Let $p(x)$ be as above and $g\in L^\infty(\Omega)$. We say that $u$ is a solution to

$$\Delta_{p(x)}u:=\operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u)=g(x)\ \text{ in }\ \Omega \tag{1.1}$$

if $u \in W^{1,p(\cdot)}(\Omega)$ and, for every $\varphi \in C_0^\infty(\Omega)$, there holds that

$$\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u \cdot \nabla\varphi \, dx = -\int_\Omega \varphi \, g(x)\, dx.$$

Under the assumptions of the present paper (see 1.3 below) it follows as in Remark 3.2 in [36] that $u \in L^\infty_{\rm loc}(\Omega)$.

Moreover, for any $x \in \Omega$, $\xi, \eta \in \mathbb{R}^N$ fixed we have the following inequalities

$$\begin{cases} |\eta-\xi|^{p(x)} \le C(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi)\cdot(\eta-\xi) & \text{if } p(x)\ge 2,\\ |\eta-\xi|^2\Big(|\eta|+|\xi|\Big)^{p(x)-2} \le C(|\eta|^{p(x)-2}\eta - |\xi|^{p(x)-2}\xi)\cdot(\eta-\xi) & \text{if } p(x)< 2, \end{cases} \tag{1.2}$$

with $C = C(N, p_{\min}, p_{\max})$. These inequalities imply that the function $A(x,\xi) = |\xi|^{p(x)-2}\xi$ is strictly monotone. Then, the comparison principle for equation (1.1) holds on bounded domains since it follows from the monotonicity of $A(x,\xi)$.

### 1.3. **Assumptions.**

Throughout the paper we let $\Omega \subset \mathbb{R}^N$ a $C^1$ bounded domain with a nonempty relatively open subset $\mathcal{A}$ of $\partial\Omega$ of class $C^2$.

**Assumptions on $p(x)$.** We assume that the function $p(x)$ is measurable and verifies

$$1 < p_{\min} \le p(x) \le p_{\max} < \infty, \qquad x \in \Omega.$$

We also assume that $p(x)$ is Lipschitz continuous in $\Omega$ and we denote by $L$ the Lipschitz constant of $p(x)$, namely, $\|\nabla p\|_{L^\infty(\Omega)} \le L$.

**Assumptions on $f(x)$.** We assume that $f \in L^\infty(\Omega)$.

**Assumptions on $\varphi_0$.** We assume that $\varphi_0 \in W^{1,p(\cdot)}(\Omega)$, $\varphi_0 \ge 0$, with $\varphi_0 \ge c_0 > 0$ in $\mathcal{A}$.

### 1.4. **Notation.**

- $N$ spatial dimension
- $\Omega \cap \partial\{u > 0\}$ free boundary
- $|S|$ $N$-dimensional Lebesgue measure of the set $S$
- $\mathcal{H}^{N-1}$ $(N-1)$-dimensional Hausdorff measure
- $B_r(x_0)$ open ball of radius $r$ and center $x_0$
- $B_r$ open ball of radius $r$ and center 0
- $B_r^+ = B_r \cap \{x_N > 0\}$, $B_r^- = B_r \cap \{x_N < 0\}$
- $B'_r(x_0)$ open ball of radius $r$ and center $x_0$ in $\mathbb{R}^{N-1}$
- $B'_r$ open ball of radius $r$ and center 0 in $\mathbb{R}^{N-1}$
- $⨍_{B_r(x_0)} u = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u\,dx$
- $⨍_{\partial B_r(x_0)} u = \frac{1}{\mathcal{H}^{N-1}(\partial B_r(x_0))}\int_{\partial B_r(x_0)} u\,d\mathcal{H}^{N-1}$
- $\chi_S$ characteristic function of the set $S$
- $u^+ = \max(u,0)$, $u^- = \max(-u,0)$
- $\langle\, \xi \,,\, \eta\,\rangle$ and $\xi\cdot\eta$ both denote scalar product in $\mathbb{R}^N$

## 2. Weak solutions to the free boundary problem $P(f, p, \lambda^*)$

In this section, for the sake of completeness, we define the notion of weak solution to the free boundary problem $P(f, p, \lambda^*)$ and we give other related definitions and results that we are going to employ in the paper.

We point out that in [26] we derived some properties of the weak solutions to problem $P(f, p, \lambda^*)$ and we developed a theory for the regularity of the free boundary for weak solutions.

We first need

**Definition 2.1.** Let $u$ be a continuous and nonnegative function in a domain $\Omega \subset \mathbb{R}^N$. We say that $\nu$ is the exterior unit normal to the free boundary $\Omega \cap \partial\{u > 0\}$ at a point $x_0 \in \Omega \cap \partial\{u > 0\}$ in the measure theoretic sense, if $\nu \in \mathbb{R}^N$, $|\nu| = 1$ and

$$\lim_{r\to 0} \frac{1}{r^N} \int_{B_r(x_0)} |\chi_{\{u>0\}} - \chi_{\{x\,/\,\langle x-x_0,\nu\rangle<0\}}|\, dx = 0. \tag{2.1}$$

Then we have

**Definition 2.2.** Let $\Omega \subset \mathbb{R}^N$ be a domain. Let $p$ be a measurable function in $\Omega$ with $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$, $\lambda^*$ continuous in $\Omega$ with $0 < \lambda_{\min} \leq \lambda^*(x) \leq \lambda_{\max} < \infty$ and $f \in L^\infty(\Omega)$. We call $u$ a weak solution of $P(f, p, \lambda^*)$ in $\Omega$ if

1. $u$ is continuous and nonnegative in $\Omega$, $u \in W^{1,p(\cdot)}_{\text{loc}}(\Omega)$ and $\Delta_{p(x)}u = f$ in $\Omega \cap \{u > 0\}$.
2. For $D \subset\subset \Omega$ there are constants $c_{\min} = c_{\min}(D)$, $C_{\max} = C_{\max}(D)$, $r_0 = r_0(D)$, $0 < c_{\min} \leq C_{\max}$, $r_0 > 0$, such that for balls $B_r(x) \subset D$ with $x \in \partial\{u > 0\}$ and $0 < r \leq r_0$
$$c_{\min} \leq \frac{1}{r} \sup_{B_r(x)} u \leq C_{\max}.$$
3. For $\mathcal{H}^{N-1}$ a.e. $x_0 \in \partial_{\text{red}}\{u > 0\}$ (that is, for $\mathcal{H}^{N-1}$-almost every point $x_0 \in \Omega \cap \partial\{u > 0\}$ such that $\Omega \cap \partial\{u > 0\}$ has an exterior unit normal $\nu(x_0)$ in the measure theoretic sense) $u$ has the asymptotic development
$$u(x) = \lambda^*(x_0)\langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|).$$
4. For every $x_0 \in \Omega \cap \partial\{u > 0\}$,
$$\limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)| \leq \lambda^*(x_0).$$
If there is a ball $B \subset \{u = 0\}$ touching $\Omega \cap \partial\{u > 0\}$ at $x_0$, then
$$\limsup_{\substack{x\to x_0\\ u(x)>0}} \frac{u(x)}{\operatorname{dist}(x, B)} \geq \lambda^*(x_0).$$

**Definition 2.3.** Let $v$ be a continuous nonnegative function in a domain $\Omega \subset \mathbb{R}^N$. We say that $v$ is nondegenerate at a point $x_0 \in \Omega \cap \{v = 0\}$ if there exist $c > 0$, $\bar{r}_0 > 0$ such that one of the following conditions holds:

$$⨍_{B_r(x_0)} v\, dx \geq cr \quad \text{for } 0 < r \leq \bar{r}_0, \tag{2.2}$$

$$⨍_{\partial B_r(x_0)} v\, dx \geq cr \quad \text{for } 0 < r \leq \bar{r}_0, \tag{2.3}$$

$$\sup_{B_r(x_0)} v \geq cr \quad \text{for } 0 < r \leq \bar{r}_0. \tag{2.4}$$

We say that $v$ is uniformly nondegenerate on a set $\Gamma \subset \Omega \cap \{v = 0\}$ in the sense of (2.2) (resp. (2.3), (2.4)) if the constants $c$ and $\bar{r}_0$ in (2.2) (resp. (2.3), (2.4)) can be taken independent of the point $x_0 \in \Gamma$.

**Remark 2.1.** Assume that $v \geq 0$ is locally Lipschitz continuous in a domain $\Omega \subset \mathbb{R}^N$, $v \in W^{1,p(\cdot)}(\Omega)$ with $\Delta_{p(x)} v \geq f\chi_{\{v>0\}}$, where $f \in L^\infty(\Omega)$, $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ and $p(x)$ is Lipschitz continuous. Then the three concepts of nondegeneracy in Definition 2.3 are equivalent (for the idea of the proof, see Remark 3.1 in [24], where the case $p(x) \equiv 2$ and $f \equiv 0$ is treated).

## 3. The penalized problem

In this section we begin by discussing the existence of solutions to problem $(P_\varepsilon)$ stated in Section 1. Then, for nonnegative local minimizers of the functional $\mathcal{J}_\varepsilon$ defined in Section 1, we prove local Lipschitz regularity and we study the behavior near the free boundary, such as nondegeneracy. Finally, we prove some results on the measure of the singular points of the boundary of the positivity set as well as a representation formula for the measure $\Delta_{p(x)} u_\varepsilon - f\chi_{\{u_\varepsilon>0\}}$.

We first prove

**Theorem 3.1.** *Assume that $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ with $\|\nabla p\|_{L^\infty} \leq L$ and $f \in L^\infty(\Omega)$. Then, there exists a solution $u_\varepsilon$ to $(P_\varepsilon)$.*

*Moreover, there exist positive constants $\bar{C}_1, \bar{C}_2$ and $\bar{C}_3$ such that, for any solution $u_\varepsilon$ to $(P_\varepsilon)$,*

1) $F_\varepsilon(|\{u_\varepsilon > 0\}|) \leq \bar{C}_1$,
2) $||u_\varepsilon||_{W^{1,p(\cdot)}(\Omega)} \leq \bar{C}_2$,
3) $\sup_{\Omega'} u_\varepsilon \leq \bar{C}_3$, *for every* $\Omega' \subset\subset \Omega$.

*The constants depend only on $N, \Omega, \|u_0\|_{1,p(\cdot)}, \|f\|_{L^\infty(\Omega)}, p_{\min}, p_{\max}, L$ and $\omega_0$, with the exception of $\bar{C}_3$, which depends also on $\Omega'$. Here $u_0$ is any function in $\mathcal{K}$ with $|\{u_0 > 0\}| \leq \omega_0$.*

*Proof.* The proofs of the existence of a minimizer and estimates 1) and 2) are straightforward.

In fact, in order to bound the functional $\mathcal{J}_\varepsilon$ from below we use Theorems A.3 and A.4 after subtracting any function $u_0$ in $\mathcal{K}$ with $|\{u_0 > 0\}| \leq \omega_0$.

In order to bound a minimizing sequence in $\varphi_0 + W_0^{1,p(\cdot)}$ we use Proposition A.1 and Theorem A.1. These estimates allow to pass to the limit and they also give estimates 1) and 2) for the minimizer. We use the convexity of the functional $\displaystyle\int_\Omega \Big(\frac{|\nabla v|^{p(x)}}{p(x)} + fv\Big)\,dx$ in order to prove that the weak limit of the minimizing sequence is a minimizer of $\mathcal{J}_\varepsilon$.

Finally, estimate 3) is a consequence of the application of Proposition 2.1 in [36], since, by Lemma 3.1 below, $\Delta_{p(x)} u_\varepsilon \geq f \geq -\|f\|_{L^\infty(\Omega)}$ in $\Omega$. □

We will next consider local minimizers of $\mathcal{J}_\varepsilon$. We have

**Definition 3.1.** Let $p$ and $f$ be as in Theorem 3.1. We say that $u_\varepsilon \in W^{1,p(\cdot)}(\Omega)$ is a local minimizer of $\mathcal{J}_\varepsilon$ if for every $\Omega' \subset\subset \Omega$ and for every $v \in W^{1,p(\cdot)}(\Omega)$ such that $v = u_\varepsilon$ in $\Omega \setminus \Omega'$ there holds that $\mathcal{J}_\varepsilon(v) \geq \mathcal{J}_\varepsilon(u_\varepsilon)$.

**Remark 3.1.** If $u_\varepsilon$ is a solution to $(P_\varepsilon)$, then $u_\varepsilon$ is a local minimizer of $\mathcal{J}_\varepsilon$.

From now on we denote by $u$ instead of $u_\varepsilon$ a solution to $(P_\varepsilon)$. The same consideration applies to local minimizers of $\mathcal{J}_\varepsilon$.

We first have

**Lemma 3.1.** *Let* $p$ *and* $f$ *be as in Theorem 3.1. Let* $u \in W^{1,p(\cdot)}(\Omega)$ *be a local minimizer of* $\mathcal{J}_\varepsilon$. *Then*

$$\Delta_{p(x)} u \geq f \qquad \text{in } \Omega.$$

*Proof.* See Lemma 3.1 in [27]. ☐

**Remark 3.2.** We are interested in studying the behavior of nonnegative local minimizers of the energy functional $\mathcal{J}_\varepsilon$.

If $u = u_\varepsilon$ is as in Theorem 3.1 and $f \leq 0$ in $\Omega$, since we have assumed that $\varphi_0 \geq 0$ in $\Omega$, then we have $u \geq 0$ in $\Omega$. In fact, the result follows by observing that $\xi = \min(u, 0) \in W_0^{1,p(\cdot)}(\Omega)$ so, for every $0 < t < 1$, $u - t\xi \in \varphi_0 + W_0^{1,p(\cdot)}(\Omega)$, with $|\{u - t\xi > 0\}| = |\{u > 0\}|$. Then, proceeding in a similar way as in Lemma 3.1 and using that $f \leq 0$, we obtain $\int_\Omega |\nabla \xi|^{p(x)}\, dx = 0$, which implies $u \geq 0$ in $\Omega$.

On the other hand, if $u$ is any local minimizer of $\mathcal{J}_\varepsilon$, the same argument employed at the end of Theorem 3.1 gives $\sup_{\Omega'} u \leq C^{\varepsilon}_{\Omega'}$, for any $\Omega' \subset\subset \Omega$. Therefore, if $u$ is any nonnegative local minimizer of $\mathcal{J}_\varepsilon$, then $u \in L^\infty_{\text{loc}}(\Omega)$.

Before continuing with the study of the behavior of nonnegative local minimizers of the energy functional $\mathcal{J}_\varepsilon$, we need to introduce the following concepts

**Definition 3.2.** Let $p$ and $f$ be as in Theorem 3.1, let $\lambda(x)$ measurable, $\lambda(x) > 0$ and let $a \in L^\infty(\Omega)$, $a(x) > 0$. For an open set $D \subset \Omega$, let

$$J_D^{a,p,\lambda,f}(v) = \int_D \Big( a(x) \frac{|\nabla v|^{p(x)}}{p(x)} + \lambda(x) \chi_{\{v>0\}} + f v \Big)\, dx.$$

We say that $u \in W^{1,p(\cdot)}(\Omega)$ is a local minimizer from below of $J^{a,p,\lambda,f}$ in $\Omega$ if for every $B_r(x_0) \subset\subset \Omega$ and $v \in W^{1,p(\cdot)}(B_r(x_0))$ with $v - u \in W_0^{1,p(\cdot)}(B_r(x_0))$ and $v \geq u$ in $B_r(x_0)$, we have

$$J^{a,p,\lambda,f}_{B_r(x_0)}(u) \leq J^{a,p,\lambda,f}_{B_r(x_0)}(v).$$

Analogously, we say that $u \in W^{1,p(\cdot)}(\Omega)$ is a local minimizer from above of $J^{a,p,\lambda,f}$ in $\Omega$ if for every $B_r(x_0) \subset\subset \Omega$ and $v \in W^{1,p(\cdot)}(B_r(x_0))$ with $v - u \in W_0^{1,p(\cdot)}(B_r(x_0))$ and $v \leq u$ in $B_r(x_0)$, we have

$$J^{a,p,\lambda,f}_{B_r(x_0)}(u) \leq J^{a,p,\lambda,f}_{B_r(x_0)}(v).$$

When $a(x) \equiv 1$ we will denote $J^{p,\lambda,f} = J^{a,p,\lambda,f}$.

There holds

**Lemma 3.2.** *Let* $p$, $f$ *and* $u$ *be as in Lemma 3.1. Then* $u$ *is a local minimizer from below of* $J^{p,\frac{1}{\varepsilon},f}$ *and a local minimizer from above of* $J^{p,\varepsilon,f}$ *in* $\Omega$.

*Proof.* We first observe that

$$\varepsilon(s_1 - s_2) \leq F_\varepsilon(s_1) - F_\varepsilon(s_2) \leq \frac{1}{\varepsilon}(s_1 - s_2), \quad \text{if } s_1 \geq s_2. \tag{3.1}$$

Now let $B_r(x_0) \subset\subset \Omega$ and $v \in W^{1,p(\cdot)}(B_r(x_0))$ with $v - u \in W_0^{1,p(\cdot)}(B_r(x_0))$ and $v \geq u$ in $B_r(x_0)$ and define

$$w = \begin{cases} v & \text{in } B_r(x_0) \\ u & \text{elsewhere,} \end{cases}$$

then $w \in W^{1,p(\cdot)}(\Omega)$ and, since $u$ is a local minimizer of $\mathcal{J}_\varepsilon$, the second inequality in (3.1) gives

$$\begin{aligned}
0 \leq& \mathcal{J}_\varepsilon(w) - \mathcal{J}_\varepsilon(u)\\
=& \int_\Omega \Big(\frac{|\nabla w|^{p(x)}}{p(x)} + fw\Big)\,dx + F_\varepsilon(|\{w>0\}|) - \int_\Omega \Big(\frac{|\nabla u|^{p(x)}}{p(x)} + fu\Big)\,dx - F_\varepsilon(|\{u>0\}|)\\
=& \int_{B_r(x_0)} \Big(\frac{|\nabla v|^{p(x)}}{p(x)} + fv\Big)\,dx - \int_{B_r(x_0)} \Big(\frac{|\nabla u|^{p(x)}}{p(x)} + fu\Big)\,dx + F_\varepsilon(|\{w>0\}|) - F_\varepsilon(|\{u>0\}|)\\
\leq& \int_{B_r(x_0)} \Big(\frac{|\nabla v|^{p(x)}}{p(x)} + fv\Big)\,dx - \int_{B_r(x_0)} \Big(\frac{|\nabla u|^{p(x)}}{p(x)} + fu\Big)\,dx + \frac{1}{\varepsilon}(|\{w>0\}| - |\{u>0\}|)\\
=& J^{p,\frac{1}{\varepsilon},f}_{B_r(x_0)}(v) - J^{p,\frac{1}{\varepsilon},f}_{B_r(x_0)}(u).
\end{aligned}$$

Therefore $u$ is a local minimizer from below of $J^{p,\frac{1}{\varepsilon},f}$ in $\Omega$.

Similarly, we can prove that $u$ is a local minimizer from above of $J^{p,\varepsilon,f}$ in $\Omega$. ☐

Next, we prove that nonnegative local minimizers of functional $\mathcal{J}_\varepsilon$ are locally Hölder continuous.

**Theorem 3.2.** *Let $p$ and $f$ be as in Theorem 3.1. Let $u \in W^{1,p(\cdot)}(\Omega)$ be a nonnegative local minimizer of $\mathcal{J}_\varepsilon$. Then $u \in C^\gamma(\Omega)$ for some $0 < \gamma < 1$, $\gamma = \gamma(N, p_{\min})$. Moreover, if $\Omega' \subset\subset \Omega$, then $\|u\|_{C^\gamma(\overline{\Omega'})} \leq C$ with $C$ depending only on $N$, $p_{\min}$, $p_{\max}$, $L$, $\|f\|_{L^\infty(\Omega)}$, $\|u\|_{L^\infty(\Omega'')}$, $\operatorname{dist}(\Omega', \partial\Omega'')$ and $\varepsilon$, with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$.*

*Proof.* The proof can be done following the lines of Theorem 3.2 in [27], if we let $a(x) \equiv 1$ in that proof.

In fact, we first recall that $u \in L^\infty_{\rm loc}(\Omega)$ by Remark 3.2 and we use that $\Delta_{p(x)}u \geq f$ in $\Omega$ by Lemma 3.1. Then, for $B_r(y) \subset\subset \Omega$ and $v \in W^{1,p(\cdot)}(B_r(y))$ such that $\Delta_{p(x)}v = f$ in $B_r(y)$, with $v - u \in W_0^{1,p(\cdot)}(B_r(y))$, we have $v \geq u$ in $B_r(y)$. Therefore, the application of Lemma 3.2 gives

$$J^{p,\frac{1}{\varepsilon},f}_{B_r(y)}(u) \leq J^{p,\frac{1}{\varepsilon},f}_{B_r(y)}(v).$$

Then, from (3.10) in [27], we obtain the bounds (3.11) and (3.12) in that paper, with a constant $C$ depending on $\varepsilon$. The rest of the proof follows as in [27] without changes. ☐

Hence, under the assumptions of the previous theorem we have that $u$ is continuous in $\Omega$ and therefore, $\{u > 0\}$ is open. We can now prove the following property for nonnegative local minimizers of $\mathcal{J}_\varepsilon$

**Lemma 3.3.** *Let $p, f$ and $u$ be as in Theorem 3.2. Then*

$$\Delta_{p(x)}u = f \quad \text{in } \{u > 0\}.$$

*Proof.* See Lemma 3.3 in [27]. ☐

In order to get the Lipschitz continuity we prove first the following result

**Theorem 3.3.** *Let $p, f$ and $u$ be as in Theorem 3.2. Let $\Omega' \subset\subset \Omega$. There exist constants $C > 0$, $r_0 > 0$ such that if $x_0 \in \Omega' \cap \partial\{u > 0\}$ and $r \leq r_0$ then*

$$\sup_{B_r(x_0)} u \leq Cr.$$

*The constants depend only on $N$, $p_{\min}$, $p_{\max}$, $L$, $\|f\|_{L^\infty(\Omega)}$, $\|u\|_{L^\infty(\Omega'')}$, $\operatorname{dist}(\Omega', \partial\Omega'')$ and $\varepsilon$, with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$.*

*Proof.* The proof can be done following the lines of Theorem 3.3 in [27]. In fact, we use that $u \in L^\infty_{\rm loc}(\Omega)$,

$$\Delta_{p(x)} u \geq f \quad \text{ in } \Omega, \tag{3.2}$$

$$\Delta_{p(x)} u = f \quad \text{ in } \{u > 0\}, \tag{3.3}$$

and that $u$ is a nonnegative local minimizer from below of $J^{p,\frac{1}{\varepsilon},f}$ in $\Omega$.

Although the proof of Theorem 3.3 in [27] is stated for bounded nonnegative local minimizers of the energy functional

$$\int_\Omega \Big( \frac{|\nabla v|^{p(x)}}{p(x)} + \lambda(x) \chi_{\{v>0\}} + fv \Big)\, dx$$

it only uses that $u \in L^\infty_{\rm loc}(\Omega)$ satisfies (3.2), (3.3) and that $u$ is a nonnegative local minimizer from below of that energy.

As in Theorem 3.2, in order to be able to use the local minimality from below property of $u$ for functional $J^{p,\frac{1}{\varepsilon},f}$ (and of the sucesive rescalings of it), we use (3.2) to guarantee that the comparison of the corresponding energy functionals is allowed. ☐

We are now able to prove the Lipschitz continuity of nonnegative local minimizers of $\mathcal{J}_\varepsilon$

**Corollary 3.1.** *Let $p, f$ and $u$ be as in Theorem 3.2. Then $u$ is locally Lipschitz continuous in $\Omega$. Moreover, for any $\Omega' \subset\subset \Omega$ the Lipschitz constant of $u$ in $\Omega'$ can be estimated by a constant $C$ depending only on $N$, $p_{\min}$, $p_{\max}$, $L$, $\|f\|_{L^\infty(\Omega)}$, $\|u\|_{L^\infty(\Omega'')}$, $\mathrm{dist}(\Omega', \partial\Omega'')$ and $\varepsilon$, with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$.*

*Proof.* The result is a consequence of Theorem 3.2, Lemma 3.3 and Theorem 3.3 above, and Proposition 2.1 in [26]. ☐

We also obtain

**Theorem 3.4.** *Let $p, f$ and $u$ be as in Theorem 3.2. Let $\Omega' \subset\subset \Omega$. There exist constants $c > 0$, $r_0 > 0$ such that if $x_0 \in \Omega' \cap \partial\{u > 0\}$ and $r \leq r_0$ then*

$$\sup_{B_r(x_0)} u \geq cr.$$

*The constants depend only on $N$, $p_{\min}$, $p_{\max}$, $L$, $\|f\|_{L^\infty(\Omega)}$, $\|u\|_{L^\infty(\Omega'')}$, $\mathrm{dist}(\Omega', \partial\Omega'')$ and $\varepsilon$, with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$.*

*Proof.* The proof can be done following the lines of Theorem 3.5 in [27]. We use that

$$\Delta_{p(x)} u = f \quad \text{ in } \{u > 0\}, \tag{3.4}$$

the local Lipschitz continuity of $u$ and that $u$ is a nonnegative local minimizer from above of $J^{p,\varepsilon,f}$ in $\Omega$.

Although the proof of Theorem 3.5 in [27] is stated for Lipschitz continuous nonnegative local minimizers of the energy functional

$$\int_\Omega \Big( \frac{|\nabla v|^{p(x)}}{p(x)} + \lambda(x) \chi_{\{v>0\}} + fv \Big)\, dx$$

it only uses that $u$ satisfies (3.4) and is locally Lipschitz continuous and that $u$ is a nonnegative local minimizer from above of that energy. ☐

The following result in the section is

**Theorem 3.5.** *Let* $p, f$ *and* $u$ *be as in Theorem 3.2. Let* $\Omega' \subset\subset \Omega$. *There exist constants* $\tilde{c} \in (0,1)$ *and* $\tilde{r}_0 > 0$ *such that, if* $x_0 \in \Omega' \cap \partial\{u > 0\}$ *with* $B_r(x_0) \subset \Omega'$ *and* $r \leq \tilde{r}_0$, *there holds*

$$\tilde{c} \leq \frac{|B_r(x_0) \cap \{u > 0\}|}{|B_r(x_0)|} \leq 1 - \tilde{c}.$$

*The constants depend only on* $N$, $p_{\min}$, $p_{\max}$, $L$, $\|f\|_{L^\infty(\Omega)}$, $\|u\|_{L^\infty(\Omega'')}$, $\operatorname{dist}(\Omega', \partial\Omega'')$ *and* $\varepsilon$, *with* $\Omega' \subset\subset \Omega'' \subset\subset \Omega$.

*Proof.* The lower bound follows from Lemma 2.3 in [26].

The proof of the upper bound can be done following the lines of Theorem 3.6 in [27]. In fact, we use that

$$\Delta_{p(x)} u \geq f \quad \text{in } \Omega, \tag{3.5}$$

the local Lipschitz continuity of $u$ and that $u$ is a nonnegative local minimizer from below of $J^{p,\frac{1}{\varepsilon},f}$ in $\Omega$. □

The next result gives a representation formula for nonnegative local minimizers of $\mathcal{J}_\varepsilon$. We will denote by $\mathcal{H}^{N-1} \lfloor \partial\{u > 0\}$ the measure $\mathcal{H}^{N-1}$ restricted to the set $\partial\{u > 0\}$. We define the reduced boundary as in [14], 4.5.5. (see also [11]) by, $\partial_{\text{red}}\{u > 0\} := \{x \in \Omega \cap \partial\{u > 0\} / |\nu_u(x)| = 1\}$, where $\nu_u(x)$ is the exterior unit normal to the free boundary $\Omega \cap \partial\{u > 0\}$ at the point $x \in \Omega \cap \partial\{u > 0\}$ in the measure theoretic sense (recall Definition 2.1), if such a vector exists, and $\nu_u(x) = 0$ otherwise.

**Theorem 3.6.** *Let* $p, f$ *and* $u$ *be as in Theorem 3.2. Then,*
*1)* $\mathcal{H}^{N-1}(D \cap \partial\{u > 0\}) < \infty$, *for every* $D \subset\subset \Omega$.
*2) There exist a borelian function* $q_u$ *defined on* $\Omega \cap \partial\{u > 0\}$ *such that*

$$\Delta_{p(x)} u - f\chi_{\{u>0\}} = q_u \, \mathcal{H}^{N-1} \lfloor \partial\{u > 0\},$$

*that is, for every* $\xi \in C_0^\infty(\Omega)$ *we have*

$$-\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \xi \, dx - \int_{\Omega \cap \{u>0\}} f\xi \, dx = \int_{\Omega \cap \partial\{u>0\}} q_u \xi \, d\mathcal{H}^{N-1}.$$

*3) For every* $D \subset\subset \Omega$ *there exist* $C > 0, c > 0$ *and* $r_1 > 0$ *such that*

$$cr^{N-1} \leq \mathcal{H}^{N-1}(B_r(x_0) \cap \partial\{u > 0\}) \leq Cr^{N-1}$$

*for balls* $B_r(x_0) \subset D$ *with* $x_0 \in D \cap \partial\{u > 0\}$ *and* $0 < r < r_1$ *and, in addition,*
*4)* $c \leq q_u \leq C$ *in* $D \cap \partial\{u > 0\}$.
*5)* $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$.

*The constants depend only on* $N$, $p_{\min}$, $p_{\max}$, $L$, $||f||_{L^\infty(\Omega)}$, $||u||_{L^\infty(D')}$, $\operatorname{dist}(D, \partial D')$ *and* $\varepsilon$, *with* $D \subset\subset D' \subset\subset \Omega$.

*Proof.* Assertions 1) to 4) follow from Theorem 2.1 in [26] and assertion 5) follows from the application of Theorem 3.5 and Theorem 4.5.6(3) in [14]. □

## 4. The free boundary condition for the penalized problem

We have already shown that nonnegative local minimizers of $\mathcal{J}_\varepsilon$ satisfy properties (1) and (2) in the definition of weak solution (Definition 2.2). We devote this section to discuss the fulfillment of properties (3) and (4).

We will make use of the following result which was proven in [26].

**Lemma 4.1.** *([26], Lemma 2.5) Assume that $u$ satisfies hypotheses (1) and (2) of Definition 2.2. Let $B_{\rho_k}(x_k) \subset \Omega$ be a sequence of balls with $\rho_k \to 0$, $x_k \to x_0 \in \Omega$ and $u(x_k) = 0$. Let us consider the blow-up sequence with respect to $B_{\rho_k}(x_k)$. That is,*

$$u_k(x) := \frac{1}{\rho_k} u(x_k + \rho_k x).$$

*Then, there exists a blow-up limit $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that, for a subsequence,*

1. *$u_k \to u_0$ in $C^\alpha_{\mathrm{loc}}(\mathbb{R}^N)$ for every $0 < \alpha < 1$,*
2. *$\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ locally in Hausdorff distance,*
3. *$\nabla u_k \to \nabla u_0$ uniformly on compact subsets of $\{u_0 > 0\}$,*
4. *$\nabla u_k \to \nabla u_0$ a.e. in $\mathbb{R}^N$,*
5. *If $x_k \in \partial\{u > 0\}$, then $0 \in \partial\{u_0 > 0\}$,*
6. *$\Delta_{p(x_0)} u_0 = 0$ in $\{u_0 > 0\}$,*
7. *$u_0$ is Lipschitz continuous and satisfies property (2) of Definition 2.2 in $\mathbb{R}^N$ with the same constants as $u$ in a ball $B_{\rho_0}(x_0) \subset\subset \Omega$ .*

We will need the following lemma

**Lemma 4.2.** *Let $p, f$ and $u$ be as in Theorem 3.2. Let $x_0, x_1 \in \Omega \cap \partial\{u > 0\}$. For $i = 0, 1$ let $x_{i,k} \to x_i$ with $u(x_{i,k}) = 0$ such that $B_{\rho_k}(x_{i,k}) \subset \Omega$, with $\rho_k \to 0$, and such that the blow-up sequence*

$$u_{i,k}(x) = \frac{1}{\rho_k} u(x_{i,k} + \rho_k x)$$

*has a limit $u_i(x) = \lambda_i \langle x, \nu_i \rangle^-$, with $0 < \lambda_i < \infty$ and $\nu_i$ a unit vector.*

*Then $\big(\frac{p(x_0)-1}{p(x_0)}\big)\lambda_0^{p(x_0)} = \big(\frac{p(x_1)-1}{p(x_1)}\big)\lambda_1^{p(x_1)}$.*

*Proof.* Assume that $\big(\frac{p(x_1)-1}{p(x_1)}\big)\lambda_1^{p(x_1)} < \big(\frac{p(x_0)-1}{p(x_0)}\big)\lambda_0^{p(x_0)}$, then we will perturb the local minimizer $u$ near $x_0$ and $x_1$ and get an admissible function with less energy. To this end, we take a nonnegative $C_0^\infty$ function $\phi$ supported in the unit interval, $\phi \not\equiv 0$. For $k$ large, define

$$\tau_k(x) = \begin{cases} x + \rho_k^2 \phi\Big(\dfrac{|x - x_{0,k}|}{\rho_k}\Big)\nu_0 & \text{for } x \in B_{\rho_k}(x_{0,k}), \\ \\ x - \rho_k^2 \phi\Big(\dfrac{|x - x_{1,k}|}{\rho_k}\Big)\nu_1 & \text{for } x \in B_{\rho_k}(x_{1,k}), \\ \\ x & \text{elsewhere,} \end{cases}$$

which is a diffeomorphism if $k$ is big enough. Now let

$$v_k(x) = u(\tau_k^{-1}(x)),$$

that are admissible functions. Let us also define, for $i = 0, 1$,

$$\eta_i(y) = (-1)^i \phi(|y|)\nu_i.$$

From Lemma 4.1 and Theorem 3.5 it follows that

$$\chi_{\{u_{i,k}>0\}} \to \chi_{\{y\cdot\nu_i<0\}} \text{ in } L^1(B_1(0)).$$

This gives

$$\rho_k^{-N-1}\Big(|\{v_k>0\}\cap B_{\rho_k}(x_{i,k})|-|\{u>0\}\cap B_{\rho_k}(x_{i,k})|\Big) \\ \to(-1)^i\int_{B_1(0)\cap\{y.\nu_i<0\}}\phi'(|y|)\frac{y}{|y|}.\nu_i\,dy=(-1)^i\int_{B_1(0)\cap\{y\cdot\nu_i=0\}}\phi(|y|)\,d\mathcal{H}^{N-1}(y), \tag{4.1}$$

which implies that

$$|\{v_k>0\}|-|\{u>0\}|=o(\rho_k^{N+1})$$

and therefore,

$$F_\varepsilon(|\{v_k>0\}|)-F_\varepsilon(|\{u>0\}|)=o(\rho_k^{N+1}). \tag{4.2}$$

In order to estimate the other terms in $\mathcal{J}_\varepsilon$, we let $p_k^i(y)=p(x_{i,k}+\rho_k y)$, we make a change of variables and then,

$$\rho_k^{-N}\int_{B_{\rho_k}(x_{i,k})}\Big(\frac{|\nabla v_k|^{p(x)}}{p(x)}-\frac{|\nabla u|^{p(x)}}{p(x)}\Big)\,dx \\ =\int_{B_1(0)\cap\{u_{i,k}>0\}}\frac{\rho_k}{p_k^i(y)}\Big[|\nabla u_{i,k}|^{p_k^i(y)}\operatorname{div}(\eta_i)-p_k^i(y)\,|\nabla u_{i,k}|^{p_k^i(y)-2}(\nabla u_{i,k})^tD\eta_i\nabla u_{i,k}\Big]+o(\rho_k)\,dy.$$

On the other hand, by Lemma 4.1, we have

$$\nabla u_{i,k}\to\nabla u_i=-\lambda_i\nu_i\chi_{\{y\cdot\nu_i<0\}}\text{ a.e in }B_1(0),$$

and, using that $\nabla u_{i,k}$ are uniformly bounded in $B_1(0)$, we get

$$\rho_k^{-N-1}\int_{B_{\rho_k}(x_{i,k})}\Big(\frac{|\nabla v_k|^{p(x)}}{p(x)}-\frac{|\nabla u|^{p(x)}}{p(x)}\Big)\,dx\to\frac{\lambda_i^{p(x_i)}}{p(x_i)}\int_{B_1(0)\cap\{y\cdot\nu_i<0\}}\big(\operatorname{div}(\eta_i)-p(x_i)\,\nu_i^t\,D\eta_i\,\nu_i\big)\,dy.$$

As there holds that,

$$\operatorname{div}(\eta_i)-p(x_i)\,\nu_i^t\,D\eta_i\,\nu_i=(-1)^i(1-p(x_i))\frac{\phi'(|y|)}{|y|}(y\cdot\nu_i)=(1-p(x_i))\operatorname{div}(\eta_i),$$

we obtain

$$\rho_k^{-N-1}\int_{B_{\rho_k}(x_{i,k})}\Big(\frac{|\nabla v_k|^{p(x)}}{p(x)}-\frac{|\nabla u|^{p(x)}}{p(x)}\Big)\,dx\to(-1)^i\frac{(1-p(x_i))}{p(x_i)}\lambda_i^{p(x_i)}\int_{B_1(0)\cap\{y\cdot\nu_i=0\}}\phi(|y|)\,d\mathcal{H}^{N-1}(y).$$

We also observe that $|v_k-u|=O(\rho_k^2)$ in $B_{\rho_k}(x_{i,k})$. Then,

$$\rho_k^{-N-1}\Big(\int_{B_{\rho_k}(x_{i,k})}\Big(\frac{|\nabla v_k|^{p(x)}}{p(x)}-\frac{|\nabla u|^{p(x)}}{p(x)}\Big)\,dx+\int_{B_{\rho_k}(x_{i,k})}f(v_k-u)\,dx\Big) \\ \to(-1)^i\frac{(1-p(x_i))}{p(x_i)}\lambda_i^{p(x_i)}\int_{B_1(0)\cap\{y\cdot\nu_i=0\}}\phi(|y|)\,d\mathcal{H}^{N-1}(y). \tag{4.3}$$

Hence,

$$\int_\Omega\frac{|\nabla v_k|^{p(x)}}{p(x)}\,dx-\int_\Omega\frac{|\nabla u|^{p(x)}}{p(x)}+\int_\Omega fv_k\,dx-\int_\Omega fu\,dx= \\ =\rho_k^{N+1}\Big(\Big(\frac{p(x_1)-1}{p(x_1)}\Big)\lambda_1^{p(x_1)}-\Big(\frac{p(x_0)-1}{p(x_0)}\Big)\lambda_0^{p(x_0)}\Big)\int_{B_1(0)\cap\{y_1=0\}}\phi(|y|)\,d\mathcal{H}^{N-1}(y)+o(\rho_k^{N+1}). \tag{4.4}$$

Combining (4.2) and (4.4), we get, if we take $k$ large enough,

$$\mathcal{J}_\varepsilon(v_k) < \mathcal{J}_\varepsilon(u),$$

a contradiction. □

Our following result is

**Lemma 4.3.** *Let $p, f$ and $u$ be as in Theorem 3.2. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and let*

$$\lambda = \lambda(x_0) := \limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)|.$$

*Then $0 < \lambda < \infty$. Moreover, there exist sequences $y_k \in \Omega \cap \partial\{u > 0\}$ with $y_k \to x_0$, $B_{d_k}(y_k) \subset \Omega$ and $d_k \to 0$, such that the blow-up sequence $u_{d_k}(x) = \frac{1}{d_k}u(y_k + d_k x)$ has a limit $u_0$ with*

$$u_0(x) = \lambda\langle x, \nu\rangle^- + o(|x|), \tag{4.5}$$

*and $\nu = \nu(x_0)$ a unit vector.*

*Proof.* Let

$$\lambda := \limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)|.$$

Since $u \in Lip_{\text{loc}}(\Omega)$, $0 \le \lambda < \infty$. By the definition of $\lambda$ there exists a sequence $z_k \to x_0$ such that

$$u(z_k) > 0, \qquad |\nabla u(z_k)| \to \lambda.$$

Let $y_k$ be the nearest point from $z_k$ to $\Omega \cap \partial\{u > 0\}$ and let $d_k = |z_k - y_k|$.

Consider the blow-up sequence $u_{d_k}$ with respect to $B_{d_k}(y_k)$. That is, $u_{d_k}(x) = \frac{1}{d_k}u(y_k + d_k x)$. Since $u$ is locally Lipschitz, and $u_{d_k}(0) = 0$ for every $k$, there exists $u_0$, with $u_0(0) = 0$, such that (for a subsequence) $u_{d_k} \to u_0$ uniformly on compact sets of $\mathbb{R}^N$. Moreover, using Lemma 3.3 and interior Hölder gradient estimates (Theorem 1.1 in [12]) we deduce that $\nabla u_{d_k} \to \nabla u_0$ uniformly on compact subsets of $\{u_0 > 0\}$ with $|\nabla u_0| \le \lambda$ in $\mathbb{R}^N$.

Now, if $\lambda = 0$, since $u_0(0) = 0$, it follows that $u_0 \equiv 0$. This contradicts Theorem 3.4 and then, $\lambda > 0$.

Finally, using Lemma 3.3 and Theorem 3.5 and proceeding as in the proof of Theorem 5.1 in [25] we obtain that, after a rotation,

$$u_0(x) = \lambda x_1 \quad \text{in } \{x_1 \ge 0\},$$

and

$$u_0(x) = o(|x|) \quad \text{in } \{x_1 < 0\}.$$

That is, (4.5) holds. □

We will prove an identification result for the function $q_u$ given in Theorem 3.6, which holds at points $x_0 \in \partial_{\text{red}}\{u > 0\}$ that are Lebesgue points of the function $q_u$ and are such that

$$\limsup_{r\to 0} \frac{\mathcal{H}^{N-1}(\partial\{u > 0\} \cap B(x_0, r))}{\mathcal{H}^{N-1}(B'(x_0, r))} \le 1. \tag{4.6}$$

(Here $B'(x_0, r) = \{x' \in \mathbb{R}^{N-1} \,/\, |x'| < r\}$).

Notice that under our assumptions, $\mathcal{H}^{N-1}-$ *a.e.* point in $\partial_{\text{red}}\{u > 0\}$ satisfies (4.6) (see Theorem 4.5.6(2) in [14]).

We have,

**Lemma 4.4.** *Let $p, f$ and $u$ be as in Theorem 3.2. For $\mathcal{H}^{N-1}-$ a.e. point $x_0$ in $\Omega\cap\partial_{\rm red}\{u>0\}$ the following property holds:*

*If $B_{\rho_k}(x_0)\subset\Omega$ is any sequence with $\rho_k\to 0$ such that the blow-up sequence $u_k(x)=\frac{1}{\rho_k}u(x_0+\rho_k x)$ has limit $u_0$, then*

$$u_0(x)=q_u(x_0)^{\frac{1}{p(x_0)-1}}\langle\, x,\,\nu\,\rangle^- + o(|x|),$$

*where $\nu=\nu(x_0)$ is the exterior unit normal to $\partial\{u>0\}$ at $x_0$ in the measure theoretic sense.*

*Proof.* We take $x_0\in\partial_{\rm red}\{u>0\}$ and $\nu(x_0)$ the exterior unit normal to $\partial\{u>0\}$ at $x_0$ in the measure theoretic sense. We assume $\nu(x_0)=e_N$. Consider any sequence $\rho_k\to 0$ such that the blow-up sequence $u_k(x)=\frac{1}{\rho_k}u(x_0+\rho_k x)$ has a limit $u_0$.

We claim that

$$u_0>0 \quad \text{in } x_N<0, \tag{4.7}$$

$$u_0=0 \quad \text{in } x_N\geq 0. \tag{4.8}$$

In fact, from (2.1) we get

$$\chi_{\{u_k>0\}}\to\chi_{\{x_N<0\}} \text{ in } L^1_{\rm loc}(\mathbb{R}^N).$$

Thus assertion (4.8) follows. Using (2) in Lemma 4.1 and the second inequality in Theorem 3.5, we deduce that $\partial\{u_0>0\}\cap\{x_N<0\}=\emptyset$. Now, from (5) in Lemma 4.1 we obtain (4.7).

If $\xi\in C_0^\infty(\Omega)$ we have

$$-\int_{\{u>0\}}|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\xi\,dx-\int_{\{u>0\}}f\xi\,dx=\int_{\partial\{u>0\}}q_u(x)\xi d\mathcal{H}^{N-1},$$

and if we replace $\xi$ by $\xi_k(x)=\rho_k\xi(\frac{x-x_0}{\rho_k})$ with $\xi\in C_0^\infty(B_R)$, $k\geq k_0$ and we change variables, we obtain

$$-\int_{\{u_k>0\}}|\nabla u_k|^{p_k(x)-2}\nabla u_k\cdot\nabla\xi\,dx-\int_{\{u_k>0\}}f_k\xi\,dx=\int_{\partial\{u_k>0\}}q_u(x_0+\rho_k x)\xi d\mathcal{H}^{N-1},$$

where $p_k(x)=p(x_0+\rho_k x)$ and $f_k(x)=\rho_k f(x_0+\rho_k x)$. From Lemma 4.1, it follows that, for a subsequence, $|\nabla u_k|^{p_k(x)-2}\nabla u_k\to|\nabla u_0|^{p_0-2}\nabla u_0$ a.e. in $\mathbb{R}^N$, with $p_0=p(x_0)$. This, together with (4.8), gives

$$-\int_{\{u_k>0\}}|\nabla u_k|^{p_k(x)-2}\nabla u_k\cdot\nabla\xi\,dx-\int_{\{u_k>0\}}f_k\xi\,dx\to-\int_{\{x_N<0\}}|\nabla u_0|^{p_0-2}\nabla u_0\cdot\nabla\xi\,dx.$$

We now fix $r>0$ and let

$$\xi(x)=\xi_r(x)=\min\Big(2\Big(1-\frac{|x_N|}{r}\Big)^+,1\Big)\eta(x_1,...,x_{N-1}), \tag{4.9}$$

for $|x_N|\leq r$ and $\xi=0$ otherwise, where $\eta\in C_0^\infty(B'_r)$, (where $B'_r$ is a ball $(N-1)$ dimensional with radius $r$) and $\eta\geq 0$. Then, if $x_0$ is a Lebesgue point of $q_u$ satisfying (4.6), we proceed as in [5], p.121 and we get

$$\int_{\partial\{u_k>0\}}q_u(x_0+\rho_k x)\xi\,d\mathcal{H}^{N-1}\to q_u(x_0)\int_{\{x_N=0\}}\xi\,d\mathcal{H}^{N-1}.$$

It follows that

$$-\int_{\{x_N<0\}}|\nabla u_0|^{p_0-2}\nabla u_0\cdot\nabla\xi\,dx=q_u(x_0)\int_{\{x_N=0\}}\xi\,d\mathcal{H}^{N-1}. \tag{4.10}$$

From Lemma 4.1, and from (4.7) and (4.8), we know that $u_0 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$, $\Delta_{p_0} u_0 = 0$ in $\{x_N < 0\}$ and $u_0 = 0$ in $\{x_N = 0\}$. Then, boundary regularity results for the $p$-Laplacian operator give, for some $\beta > 0$, $u_0 \in C^{1,\beta}(B_2(0) \cap \{x_N \le 0\})$ and therefore, $u_0(x) = \alpha x_N^- + o(|x|)$ for some $\alpha \ge 0$. Now Theorem 3.4 implies $\alpha > 0$.

Finally, we let $\eta \in C_0^\infty(B_1')$, $\eta \ge 0$, and take $\xi$ as in (4.9) with $r = 1$. For some $r_k \to 0^+$, we define $\xi_k(x) = \xi(\frac{x}{r_k})$ and we thus obtain (4.10) with $\xi$ replaced by $\xi_k$. Changing variables and passing to the limit, we get

$$\alpha^{p_0-1} \int_{\{x_N<0\}} \xi_{x_N}\, dx = q_u(x_0) \int_{\{x_N=0\}} \xi \, d\mathcal{H}^{N-1},$$

which concludes the proof. □

Next we prove the following identification result

**Theorem 4.1.** *Let $p, f$ and $u$ be as in Theorem 3.2. There exists a constant $\lambda_u > 0$ such that*

$$\limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)| = \lambda_u^*(x_0) \quad \textit{for all } x_0 \in \Omega \cap \partial\{u > 0\}, \tag{4.11}$$

$$q_u(x_0)^{\frac{1}{p(x_0)-1}} = \lambda_u^*(x_0) \quad \textit{for } \mathcal{H}^{N-1} - \textit{a.e.}\, x_0 \in \Omega \cap \partial\{u > 0\}, \tag{4.12}$$

*where* $\lambda_u^*(x) = \Big(\frac{p(x)}{p(x)-1}\,\lambda_u\Big)^{1/p(x)}$.

*Proof.* Choose $x_1 \in \partial_{\text{red}}\{u > 0\}$ for which the conclusion of Lemma 4.4 holds. Given $x_0 \in \Omega\cap\partial\{u > 0\}$, set

$$\lambda_0 = \lambda(x_0) = \limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)|,$$

and apply Lemma 4.3 to $x_0$. We find in this way a sequence of balls $B_{d_k}(y_k) \subset \Omega$ with $y_k \in \Omega \cap \partial\{u > 0\}$, $y_k \to x_0$, and $d_k \to 0$, and a unit vector $\nu_0 = \nu(x_0)$, such that the blow-up sequence $u^0_{d_k}(x) = \frac{1}{d_k} u(y_k + d_k x)$ has a limit $u_0$ with

$$u_0(x) = \lambda_0 \langle\, x, \nu_0 \,\rangle^- + o(|x|),$$

and $0 < \lambda_0 < \infty$.

We now consider the blow-up sequence $u^1_{d_k}(x) = \frac{1}{d_k} u(x_1 + d_k x)$ that, for a subsequence that we still call $d_k$, has a limit $u_1$. By Lemma 4.4,

$$u_1(x) = \lambda_1 \langle\, x, \nu_1 \,\rangle^- + o(|x|),$$

where $\lambda_1 = q_u(x_1)^{\frac{1}{p(x_1)-1}}$ and $\nu_1 = \nu(x_1)$ is the exterior unit normal to $\partial\{u > 0\}$ at $x_1$ in the measure theoretic sense.

We will show that an application of Lemma 4.2 to suitable blow-up sequences, constructed from $u^0_{d_k}$ and $u^1_{d_k}$, gives

$$\Big(\frac{p(x_0) - 1}{p(x_0)}\Big)\lambda_0^{p(x_0)} = \Big(\frac{p(x_1) - 1}{p(x_1)}\Big)\lambda_1^{p(x_1)}. \tag{4.13}$$

In fact, in order to obtain these blow-up sequences, we recall that

$$u^0_{d_k} \to u_0 \quad \text{and} \quad u^1_{d_k} \to u_1 \quad \text{uniformly in } B_1(0).$$

Let us take a sequence $\mu_n\to 0$ and denote

$$(u^0_{d_k})_{\mu_n}(x)=\frac{1}{\mu_n}u^0_{d_k}(\mu_n x),\quad (u_0)_{\mu_n}(x)=\frac{1}{\mu_n}u_0(\mu_n x),$$
$$(u^1_{d_k})_{\mu_n}(x)=\frac{1}{\mu_n}u^1_{d_k}(\mu_n x),\quad (u_1)_{\mu_n}(x)=\frac{1}{\mu_n}u_1(\mu_n x).$$

Then,

$$(u_0)_{\mu_n}\to u_{00}\quad\text{and}\quad (u_1)_{\mu_n}\to u_{11}\quad\text{uniformly on compact sets of }\mathbb{R}^N,$$

with $u_{00}(x)=\lambda_0\langle\, x,\,\nu_0\,\rangle^-$ and $u_{11}(x)=\lambda_1\langle\, x,\,\nu_1\,\rangle^-$.

For $i=0,1$, we have

$$\begin{aligned}(u^i_{d_k})_{\mu_n}(x)-u_{ii}(x)&=\Big(\frac{1}{\mu_n}u^i_{d_k}(\mu_n x)-\frac{1}{\mu_n}u_i(\mu_n x)\Big)\\&\quad+\big((u_i)_{\mu_n}(x)-u_{ii}(x)\big)=I+II.\end{aligned}$$

Let $m>0$ be fixed and $\delta>0$ be arbitrary. We know that $|II|<\delta$ in $B_m(0)$ if $n\geq n_i(m,\delta)$. Let us bound

$$|I|=\frac{|u^i_{d_k}(\mu_n x)-u_i(\mu_n x)|}{\mu_n}.$$

For each $n$ there exists $k_i(n)\geq n$ such that if, $k\geq k_i(n)$,

$$|u^i_{d_k}(x)-u_i(x)|\leq\frac{\mu_n}{n}\qquad\text{for }x\in B_1(0).$$

Therefore, if $k\geq k_i(n)$ with $n\geq\hat n(m)$ so that $\mu_n\leq\frac{1}{m}$ then,

$$|I|\leq\frac{1}{n}\qquad\text{for }x\in B_m(0).$$

So that if $k\geq k_i(n)$ and $n\geq\bar n_i(m,\delta)$,

$$|(u^i_{d_k})_{\mu_n}(x)-u_{ii}(x)|<2\delta\qquad\text{for }x\in B_m(0).$$

Then, if we take $k_n=\max\{k_0(n),k_1(n)\}$,

$$(u^0_{d_{k_n}})_{\mu_n}\to u_{00}\quad\text{and}\quad(u^1_{d_{k_n}})_{\mu_n}\to u_{11}\quad\text{uniformly on compact sets of }\mathbb{R}^N.$$

Now, denoting $\rho_n=d_{k_n}\mu_n$, we have that $\rho_n\to 0$ and

$$(u^0_{d_{k_n}})_{\mu_n}(x)=\frac{1}{\rho_n}u(y_{k_n}+\rho_n x),\quad u_{00}(x)=\lambda_0\langle\, x,\,\nu_0\,\rangle^-,$$
$$(u^1_{d_{k_n}})_{\mu_n}(x)=\frac{1}{\rho_n}u(x_1+\rho_n x),\quad u_{11}(x)=\lambda_1\langle\, x,\,\nu_1\,\rangle^-.$$

Consequently, the application of Lemma 4.2 to the blow-up sequences $(u^0_{d_{k_n}})_{\mu_n}$ and $(u^1_{d_{k_n}})_{\mu_n}$ gives (4.13).

To conclude the proof, we now set $\lambda_u:=\big(\frac{p(x_1)-1}{p(x_1)}\big)\lambda_1^{p(x_1)}=\big(\frac{p(x_1)-1}{p(x_1)}\big)\big(q_u(x_1)^{\frac{1}{p(x_1)-1}}\big)^{p(x_1)}$ and notice that $x_0$ was any point in $\Omega\cap\partial\{u>0\}$. We thus get (4.11).

The result (4.12) is finally obtained, if we recall 5) in Theorem 3.6 and we observe that $x_1$ is any point in $\partial_{\rm red}\{u>0\}$ for which the conclusion of Lemma 4.4 holds. ☐

Our following result is

**Theorem 4.2.** *Let $p, f$ and $u$ be as in Theorem 3.2. Let $x_0 \in \Omega \cap \partial\{u > 0\}$. Assume there is a ball $B$ contained in $\{u = 0\}$ touching $x_0$, then*

$$\limsup_{\substack{x\to x_0\\ u(x)>0}} \frac{u(x)}{dist(x, B)} = \lambda_u^*(x_0), \tag{4.14}$$

*where $\lambda_u^*(x) = \Big(\frac{p(x)}{p(x)-1}\,\lambda_u\Big)^{1/p(x)}$, with $\lambda_u$ the constant in Theorem 4.1.*

*Proof.* Let $\ell$ be the finite limit on the left hand side of (4.14) and let $y_k \to x_0$ with $u(y_k) > 0$ be such that

$$\frac{u(y_k)}{d_k} \to \ell, \quad d_k = \mathrm{dist}(y_k, B).$$

Consider the blow-up sequence $u_k$ with respect to $B_{d_k}(x_k)$, where $x_k \in \partial B$ are points with $|x_k - y_k| = d_k$, that is, $u_k(x) = \frac{u(x_k+d_kx)}{d_k}$. Choose a subsequence, still denoted by $d_k$, with blow-up limit $u_0$, such that there exists

$$e := \lim_{k\to\infty} \frac{y_k - x_k}{d_k}.$$

Using Lemma 3.3 and Theorem 3.4 and proceeding as in the proof of Theorem 5.2 in [25] we have that $u_0(x) = \ell\langle x, e\rangle^+$ and $\ell > 0$.

We now argue as in the proof of Theorem 4.1. We choose $x_1 \in \partial_{\rm red}\{u > 0\}$ for which the conclusion of Lemma 4.4 holds and as in Theorem 4.1 we find sequences $\rho_n \to 0$ and $k_n \to \infty$ such that the blow-up sequences

$$u_{0,n}(x) = \frac{1}{\rho_n} u(x_{k_n} + \rho_n x), \quad u_{1,n}(x) = \frac{1}{\rho_n} u(x_1 + \rho_n x),$$

satisfy that

$$u_{0,n} \to \ell\langle x, e\rangle^+ \quad \text{and} \quad u_{1,n} \to \lambda_1\langle\, x,\, \nu_1\,\rangle^- \quad \text{uniformly on compact sets of } \mathbb{R}^N,$$

where $\lambda_1 = q_u(x_1)^{\frac{1}{p(x_1)-1}}$ and $\nu_1 = \nu(x_1)$ is the exterior unit normal to $\partial\{u > 0\}$ at $x_1$ in the measure theoretic sense. Hence the application of Lemma 4.2 now gives

$$\big(\frac{p(x_0) - 1}{p(x_0)}\big)\ell^{p(x_0)} = \big(\frac{p(x_1) - 1}{p(x_1)}\big)\lambda_1^{p(x_1)} = \lambda_u.$$

That is, (4.14) holds. ☐

We finally have

**Theorem 4.3.** *Let $p, f$ and $u$ be as in Theorem 3.2. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $\partial\{u > 0\}$ has at $x_0$ an inward unit normal $\nu$ in the measure theoretic sense. Then,*

$$u(x) = \lambda_u^*(x_0)\langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$

*where $\lambda_u^*(x) = \Big(\frac{p(x)}{p(x)-1}\,\lambda_u\Big)^{1/p(x)}$, with $\lambda_u$ the constant in Theorem 4.1.*

*Proof.* Take $u_\lambda(x) = \frac{1}{\lambda}u(x_0 + \lambda x)$. Let $\rho > 0$ such that $B_\rho(x_0) \subset\subset \Omega$. Since $u_\lambda \in Lip(B_{\rho/\lambda})$ uniformly in $\lambda$, $u_\lambda(0) = 0$, there exist $\lambda_j \to 0$ and $U$ such that $u_{\lambda_j} \to U$ uniformly on compact sets of $\mathbb{R}^N$ with $|\nabla U(x)| \le L_0$ in $\mathbb{R}^N$ for some constant $L_0$.

Without loss of generality we assume that $x_0 = 0$, and $\nu = e_1$. From Lemma 3.3, $\Delta_{p(\lambda x)} u_\lambda = \lambda f(\lambda x)$ in $\{u_\lambda > 0\}$. Using the fact that $e_1$ is the inward normal in the measure theoretic sense, we have, for fixed $k$,

$$|\{u_\lambda > 0\} \cap \{x_1 < 0\} \cap B_k| \to 0 \quad \text{ as } \lambda \to 0.$$

Hence, $U = 0$ in $\{x_1 < 0\}$. Moreover, $U$ is nonnegative in $\{x_1 > 0\}$, $\Delta_{p_0} U = 0$ in $\{U > 0\}$ with $p_0 = p(x_0)$ and $U$ vanishes in $\{x_1 \leq 0\}$. Then, by Lemma A.1 we have that there exists $\alpha \geq 0$ such that

$$U(x) = \alpha x_1^+ + o(|x|).$$

Define $U_\lambda(x) = \frac{1}{\lambda} U(\lambda x)$, then $U_\lambda \to \alpha x_1^+$ uniformly on compact sets of $\mathbb{R}^N$.

Now, by Theorem 3.4 and Remark 2.1, we have, for some $c > 0$ and $0 < r < r_0$,

$$\frac{1}{r^N} \int_{B_r} u_{\lambda_j}\, dx \geq cr$$

and then

$$\frac{1}{r^N} \int_{B_r} U_{\lambda_j}\, dx \geq cr.$$

Therefore $\alpha > 0$.

Now, applying Lemma 4.2 in a similar way as we did in Theorems 4.1 and 4.2, we obtain that $\alpha = (\frac{p(x_0)}{p(x_0)-1}\, \lambda_u)^{1/p(x_0)} = \lambda_u^*(x_0)$, with $\lambda_u$ the constant in Theorem 4.1.

We have shown that

$$U(x) = \begin{cases} \lambda_u^*(x_0) x_1 + o(|x|) & x_1 > 0 \\ 0 & x_1 \leq 0. \end{cases}$$

Then, using that $\Delta_{p(\lambda x)} u_\lambda = \lambda f(\lambda x)$ in $\{u_\lambda > 0\}$, by interior Hölder gradient estimates (Theorem 1.1 in [12]) we have $\nabla u_{\lambda_j} \to \nabla U$ uniformly on compact subsets of $\{U > 0\}$. Then, by Theorem 4.1, $|\nabla U| \leq \lambda_u^*(x_0)$ in $\mathbb{R}^N$. As $U = 0$ on $\{x_1 = 0\}$ we have, $U \leq \lambda_u^*(x_0) x_1$ in $\{x_1 > 0\}$.

Now, proceeding as in the proof of Theorem 5.3 in [25], we conclude that $U \equiv \lambda_u^*(x_0) x_1^+$ and the result follows. □

We next obtain results on the regularity of the free boundary for nonnegative local minimizers to the energy functional $\mathcal{J}_\varepsilon$, which are a consequence of the previous results and the results in our work [26].

First, we get

**Theorem 4.4.** *Assume that* $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ *with* $\|\nabla p\|_{L^\infty} \leq L$ *and* $f \in L^\infty(\Omega)$*. Let* $u \in W^{1,p(\cdot)}(\Omega)$ *be a nonnegative local minimizer of* $\mathcal{J}_\varepsilon$*.*

*Then, u is a weak solution to the free boundary problem:* $u \geq 0$ *and*

$$(P(f, p, \lambda_u^*)) \qquad \begin{cases} \Delta_{p(x)} u = f & \text{in } \{u > 0\} \\ u = 0, \ |\nabla u| = \lambda_u^*(x) & \text{on } \partial\{u > 0\} \end{cases}$$

*where* $\lambda_u^*(x) = \left(\frac{p(x)}{p(x)-1}\, \lambda_u\right)^{1/p(x)}$*, with* $\lambda_u$ *the constant in Theorem 4.1.*

*Proof.* The result follows by applying Lemma 3.3, Corollary 3.1 and Theorems 3.3, 3.4, 4.1, 4.2 and 4.3. □

Now, we can apply the results in [26] and deduce

**Theorem 4.5.** *Let* $p$, $f$ *and* $u$ *be as in Theorem 4.4. Assume moreover that* $f \in W^{1,q}(\Omega)$ *and* $p \in W^{2,q}(\Omega)$ *with* $q > \max\{1, N/2\}$.

*Then, there is a subset* $\mathcal{R}$ *of the free boundary* $\Omega \cap \partial\{u > 0\}$ ($\mathcal{R} = \partial_{\text{red}}\{u > 0\}$) *which is locally a* $C^{1,\alpha}$ *surface, for some* $0 < \alpha < 1$, *and the free boundary condition is satisfied in the classical sense in a neighborhood of* $\mathcal{R}$. *Moreover,* $\mathcal{R}$ *is open and dense in* $\Omega \cap \partial\{u > 0\}$ *and the remainder of the free boundary has* $(N-1)-$*dimensional Hausdorff measure zero.*

*If moreover* $\nabla p$ *and* $f$ *are Hölder continuous in* $\Omega$, *then the equation is satisfied in the classical sense in a neighborhood of* $\mathcal{R}$.

*Proof.* We first observe that, by Theorem 4.4, Theorem 4.4 in [26] applies at every $x_0 \in \Omega \cap \partial_{\text{red}}\{u > 0\}$.

Finally we recall that, from 5) in Theorem 3.6, we know that $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$. □

We also obtain higher regularity from the application of Corollary 4.1 in [26]

**Corollary 4.1.** *Let* $p$, $f$ *and* $u$ *be as in Theorem 4.5. Assume moreover that* $p \in C^2(\Omega)$ *and* $f \in C^1(\Omega)$, *then* $\partial_{\text{red}}\{u > 0\} \in C^{2,\mu}$ *for every* $0 < \mu < 1$.

*If* $p \in C^{m+1,\mu}(\Omega)$ *and* $f \in C^{m,\mu}(\Omega)$ *for some* $0 < \mu < 1$ *and* $m \geq 1$, *then* $\partial_{\text{red}}\{u > 0\} \in C^{m+2,\mu}$.

*Finally, if* $p$ *and* $f$ *are analytic, then* $\partial_{\text{red}}\{u > 0\}$ *is analytic.*

## 5. Behavior of minimizers for small $\varepsilon$.

In this section, since we want to analyze the dependence of problem $(P_\varepsilon)$ with respect to $\varepsilon$, we will again denote by $u_\varepsilon$ a solution to problem $(P_\varepsilon)$. We will consider nonnegative solutions $u_\varepsilon$ to $(P_\varepsilon)$. We recall that $\Omega$, $p$, $f$ and $\varphi_0$ satisfy the assumptions in Subsection 1.3.

To complete the analysis of the problem, we will now show that if $\varepsilon$ is small enough, then

$$|\{u_\varepsilon > 0\}| = \omega_0,$$

under suitable assumptions. To this end, we will prove that the constant $\lambda_{u_\varepsilon}$ in Theorem 4.1 and the function

$$\lambda^*_{u_\varepsilon}(x) = \Big(\frac{p(x)}{p(x)-1}\,\lambda_{u_\varepsilon}\Big)^{1/p(x)}$$

are bounded from above and below by positive constants independent of $\varepsilon$. We will perform this task in a series of lemmas.

As a consequence, we will finally obtain existence and regularity results for our original problem $(P)$ (Theorem 5.1 and Theorem 1.1—stated in Section 1).

We start the section by setting an assumption we are going to work with

**Definition 5.1.** Let $\kappa > 0$. Let $u \in C(\Omega)$ be a nonnegative function. We say that $u$ satisfies assumption $(H_\kappa)$ if

$$(H_\kappa) \qquad \exists \quad x_0 \in \Omega \cap \partial\{u > 0\} \text{ and } \tilde{r}_0 > 0 \quad / \quad \frac{1}{r}\Big(⨍_{B_r(x_0)} u^\gamma\,dx\Big)^{1/\gamma} \geq \kappa \quad \forall\, r \leq \tilde{r}_0,$$

where $\gamma > 0$ is the constant in Lemma 5.1 below.

In Lemmas 5.5 and 5.6 below we find conditions that guarantee that nonnegative solutions to $(P_\varepsilon)$ satisfy assumption $(H_\kappa)$, uniformly in $\varepsilon$.

We will also use

**Lemma 5.1.** *Let* $p_0 \in [p_{\min}, p_{\max}]$ *and let* $v \in W^{1,p_0}_{\text{loc}}(B_1) \cap L^\infty(B_1)$ *such that* $\Delta_{p_0} v = 0$ *in* $B_1$, $v \geq 0$. *There exist positive constants* $\gamma = \gamma(N, p_{\min})$ *and* $C = C(N, p_{\min}, p_{\max})$ *such that*

$$\inf_{B_{1/4}} v \geq C \Big( \fint_{B_{1/2}} v^\gamma \, dx \Big)^{1/\gamma}.$$

*Proof.* The result follows from Theorem 1.2 in [35]. □

Our first result in the section is

**Lemma 5.2.** *Let* $u_\varepsilon$ *be a nonnegative solution to* $(P_\varepsilon)$. *Then, there exists a constant* $C > 0$, *independent of* $\varepsilon$, *such that, for* $\varepsilon$ *small,*

$$\lambda^*_{u_\varepsilon}(x) \leq C, \quad \lambda_{u_\varepsilon} \leq C.$$

*Proof.* First we will prove that there exist $\bar{c}, \bar{C} > 0$, independent of $\varepsilon$, such that

$$\bar{c} \leq |\{u_\varepsilon > 0\}| \leq \bar{C}\varepsilon + \omega_0. \tag{5.1}$$

In fact, from 1) in Theorem 3.1, we have that $F_\varepsilon(|\{u_\varepsilon > 0\}|) \leq \bar{C}_1$ and we thus obtain the bound from above. On the other hand, we recall that 2) in Theorem 3.1 gives $\|u_\varepsilon\|_{W^{1,p(\cdot)}(\Omega)} \leq \bar{C}_2$. Now taking $1 \leq q < p_{\min}$ and using the Sobolev trace Theorem, the Hölder inequality and the embedding Theorem A.2, we get

$$\begin{aligned}\int_{\partial\Omega} \varphi_0^q \, d\mathcal{H}^{N-1} &\leq C |\{u_\varepsilon > 0\}|^{\frac{p_{\min}-q}{p_{\min}}} \|u_\varepsilon\|^q_{W^{1,p_{\min}}} \\ &\leq C |\{u_\varepsilon > 0\}|^{\frac{p_{\min}-q}{p_{\min}}} \|u_\varepsilon\|^q_{W^{1,p(\cdot)}} \leq C |\{u_\varepsilon > 0\}|^{\frac{p_{\min}-q}{p_{\min}}}.\end{aligned}$$

Hence the bound from below follows.

Next, take $D \subset\subset \Omega$ smooth, such that $\theta = |D| > \omega_0$ and $|\Omega \setminus D| < \bar{c}$, with $\bar{c}$ the lower bound in (5.1). Then,

$$|D \cap \{u_\varepsilon > 0\}| \leq \omega_0 + \bar{C}\varepsilon \leq \frac{\omega_0 + \theta}{2} < \theta,$$

for $\varepsilon$ small enough. On the other hand,

$$|D \cap \{u_\varepsilon > 0\}| \geq |\{u_\varepsilon > 0\}| - |\Omega \setminus D| \geq \bar{c} - |\Omega \setminus D| > 0.$$

Therefore, by the relative isoperimetric inequality, we have

$$\mathcal{H}^{N-1}(D \cap \partial\{u_\varepsilon > 0\}) \geq C(D, N) \min \Big\{ |D \cap \{u_\varepsilon > 0\}|, |D \cap \{u_\varepsilon = 0\}| \Big\}^{\frac{N-1}{N}} \geq c > 0.$$

Now let $w \in W^{1,p(\cdot)}(\Omega)$ be such that

$$\Delta_{p(x)} w = -\|f\|_{L^\infty(\Omega)} \quad \text{in } \Omega, \qquad w = \varphi_0 \quad \text{on } \partial\Omega.$$

We can construct such a function by a minimization argument, as that employed in Theorem 3.1. This argument also gives $\|w\|_{W^{1,p(\cdot)}(\Omega)} \leq C_0$, with $C_0$ depending only on $N$, $\Omega$, $\|\varphi_0\|_{1,p(\cdot)}$, $\|f\|_{L^\infty(\Omega)}$, $p_{\min}$, $p_{\max}$ and $L$.

From Proposition 2.1 in [36] we deduce that $w \in L^\infty_{\text{loc}}(\Omega)$ and thus, Theorem 1.1 in [12] implies that $w \in C^1(\Omega)$. On the other hand, since $\varphi_0 \geq 0$, we get $w \geq 0$ in $\Omega$. Recalling that $\varphi_0 \geq c_0 > 0$ on a subset $\mathcal{A}$ of $\partial\Omega$ of positive measure, and using Theorem 4.1 in [36], we conclude that $w > 0$ in $\Omega$.

We now obtain, using Lemma 3.1 and the fact that $w - u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega)$, that $w - u_\varepsilon \geq 0$ in $\Omega$. We also notice that $w - u_\varepsilon \in C(\Omega)$.

Now, let $D'$ be a smooth domain such that $D \subset\subset D' \subset\subset \Omega$, let $\eta$ be such that

$$\eta \in C_0^\infty(D'), \quad 0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } D,$$

and define $v = \eta(w - u_\varepsilon)$.

By a regularization argument on the function $v$ and the passage to the limit in the regularization parameter, we obtain from Theorem 3.6,

$$-\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla v\,dx - \int_{\Omega\cap\{u_\varepsilon>0\}} fv\,dx = \int_{\Omega\cap\partial\{u_\varepsilon>0\}} q_{u_\varepsilon} v\,d\mathcal{H}^{N-1}.$$

Now, if $\frac{p_{\max}}{p_{\max}-1}\lambda_{u_\varepsilon} \ge 1$ we get,

$$\begin{aligned}
C &\ge -\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla v\,dx - \int_{\Omega\cap\{u_\varepsilon>0\}} fv\,dx \\
&= \int_{\Omega\cap\partial\{u_\varepsilon>0\}} q_{u_\varepsilon} v\,d\mathcal{H}^{N-1} \ge \int_{D\cap\partial\{u_\varepsilon>0\}} q_{u_\varepsilon}(w-u_\varepsilon)\,d\mathcal{H}^{N-1} \\
&= \int_{D\cap\partial\{u_\varepsilon>0\}} \big(\tfrac{p(x)}{p(x)-1}\lambda_{u_\varepsilon}\big)^{\frac{p(x)-1}{p(x)}} w\,d\mathcal{H}^{N-1} \\
&\ge \big(\tfrac{p_{\max}}{p_{\max}-1}\lambda_{u_\varepsilon}\big)^{\frac{p_{\min}-1}{p_{\min}}} (\inf\nolimits_D w)\mathcal{H}^{N-1}(D\cap\partial\{u_\varepsilon>0\}) \ge c\big(\tfrac{p_{\max}}{p_{\max}-1}\lambda_{u_\varepsilon}\big)^{\frac{p_{\min}-1}{p_{\min}}},
\end{aligned}$$

which gives the result. Noticing that the desired result also holds if $\frac{p_{\max}}{p_{\max}-1}\lambda_{u_\varepsilon} \le 1$, we conclude the proof. ☐

As a corollary we have

**Corollary 5.1.** *Let $u_\varepsilon$ be a nonnegative solution to $(P_\varepsilon)$. Let $x_0 \in \Omega \cap \partial\{u_\varepsilon > 0\}$. Then, there exist a constant $C > 0$, independent of $\varepsilon$, and $r_0 > 0$ such that, for $r \le r_0$,*

$$|\nabla u_\varepsilon| \le C, \quad |u_\varepsilon| \le C \qquad \text{in } B_r(x_0),$$

*for $\varepsilon$ small.*

*Proof.* By Theorem 4.1, there exists $r_1 > 0$ such that, for $r \le r_1$,

$$|\nabla u_\varepsilon| \le \lambda^*_{u_\varepsilon}(x_0) + 1 \le C \quad \text{in } B_r(x_0),$$

where we can choose $C$ independent of $\varepsilon$ by Lemma 5.2, if $\varepsilon$ is small. Then,

$$|u_\varepsilon(x)| = |u_\varepsilon(x) - u_\varepsilon(x_0)| \le C \quad \text{in } B_r(x_0),$$

if $r \le r_0 = \min\{r_1, 1\}$. ☐

We will need

**Lemma 5.3.** *Assume that $1 < p_{\min} \le p(x) \le p_{\max} < \infty$, with $\|\nabla p\|_{L^\infty} \le L$, for some $L > 0$. For $x_0 \in \mathbb{R}^N$, $\mu > 0$, $A > 0$, $\delta > 0$ and $\theta > 0$, consider*

$$w(x) = A\,\frac{e^{-\mu\frac{|x-x_0|^2}{(\theta+\delta)^2}} - e^{-\mu}}{e^{-\mu\frac{\theta^2}{(\theta+\delta)^2}} - e^{-\mu}}.$$

*Assume moreover that $\delta < \theta$ and $c_1\theta \le A \le A_0$, for some $c_1 > 0$ and $A_0 > 0$. Then, given $D > 0$, there exist $\tilde\theta = \tilde\theta(p_{\min}, L)$ and $\tilde\delta = \tilde\delta(N, p_{\min}, p_{\max}, c_1, A_0, \theta, L, D)$ such that, if $\mu = |\log\delta|$, $\theta \le \tilde\theta$*

*and $\delta \le \tilde{\delta}$, there holds that*

$$\begin{cases} \Delta_{p(x)} w \ge D & \text{in } B_{\theta+\delta}(x_0) \setminus \overline{B_\theta(x_0)}, \\ w = A & \text{on } \partial B_\theta(x_0), \\ w = 0 & \text{on } \partial B_{\theta+\delta}(x_0), \end{cases}$$

*and $|\nabla w| \ge \frac{\tilde{c}}{\delta}$ in $B_{\theta+\delta}(x_0) \setminus B_\theta(x_0)$, for some positive constant $\tilde{c} = \tilde{c}(c_1, \theta)$.*

*Proof.* We denote $\theta_1 = \theta + \delta$, $\bar{w}(x) = \frac{w(x_0+\theta_1 x)}{\theta_1}$ and $\bar{p}(x) = p(x_0 + \theta_1 x)$. Then,

$$\frac{\theta_1}{2} \le \theta \le \theta_1, \tag{5.2}$$

and

$$\bar{w}(x) = M(e^{-\mu|x|^2} - e^{-\mu}) \quad \text{with} \quad M = \frac{A}{\theta_1} \frac{1}{e^{-\mu\frac{\theta^2}{\theta_1^2}} - e^{-\mu}}.$$

The calculations in the proof of Lemma B.4 in [17] show that there exist $\tilde{\mu}_0 = \tilde{\mu}_0(N, p_{\min}, p_{\max})$ and $\varepsilon_0 = \varepsilon_0(p_{\min})$ such that, if $\mu \ge \tilde{\mu}_0$ and $\|\nabla \bar{p}\|_{L^\infty} \le \varepsilon_0$, then

$$e^{\mu|x|^2}(2M\mu)^{-1}|\nabla \bar{w}|^{2-\bar{p}(x)} \Delta_{\bar{p}(x)} \bar{w} \ge \frac{p_{\min} - 1}{4}\mu - \|\nabla \bar{p}\|_{L^\infty} |\log M| \quad \text{in } B_1 \setminus B_{1/2}. \tag{5.3}$$

Notice that we have $\|\nabla \bar{p}\|_{L^\infty} \le \theta_1 L \le \varepsilon_0$, if $\theta \le \tilde{\theta}_0(p_{\min}, L)$.

We observe that

$$\frac{\delta}{\theta_1} \le 1 - \frac{\theta^2}{\theta_1^2} \le \frac{2\delta}{\theta_1}, \tag{5.4}$$

and using the inequality $\frac{1}{1-e^{-t}} \le \frac{e^t}{t}$, for $t > 0$, we obtain, if $\mu \ge 1$,

$$\frac{A_0 e^\mu}{\delta} \ge M = \frac{A}{\theta_1} \frac{e^{\mu\frac{\theta^2}{\theta_1^2}}}{1 - e^{-\mu(1-\frac{\theta^2}{\theta_1^2})}} \ge \frac{c_1}{2} e^{\frac{\mu}{4}} \ge 1 \quad \text{if} \quad \mu \ge \tilde{\mu}_1(c_1).$$

Then,

$$|\log M| = \log M \le |\log A_0| + \mu + |\log \delta|. \tag{5.5}$$

Combining (5.3) and (5.5), we get

$$e^{\mu|x|^2}(2M\mu)^{-1}|\nabla \bar{w}|^{2-\bar{p}(x)} \Delta_{\bar{p}(x)} \bar{w} \ge \frac{p_{\min} - 1}{8}\mu - L\theta_1 |\log \delta| \quad \text{in } B_1 \setminus B_{1/2}, \tag{5.6}$$

if $\theta \le \tilde{\theta}_1(p_{\min}, L)$ and $\mu \ge \tilde{\mu}_2(p_{\min}, A_0, L)$.

If we now take $\mu = |\log \delta|$, then we deduce from (5.6)

$$e^{\mu|x|^2}(2M\mu)^{-1}|\nabla \bar{w}|^{2-\bar{p}(x)} \Delta_{\bar{p}(x)} \bar{w} \ge \frac{p_{\min} - 1}{16} |\log \delta| \quad \text{in } B_1 \setminus B_{1/2},$$

if $\theta \le \tilde{\theta}_2(p_{\min}, L)$ and $\delta \le \tilde{\delta}_0(N, p_{\min}, p_{\max}, c_1, A_0, L)$. As a consequence, in $B_1 \setminus B_{1/2}$,

$$\begin{aligned} \Delta_{\bar{p}(x)} \bar{w} &\ge (2M\mu e^{-\mu|x|^2})^{\bar{p}(x)-1} |x|^{\bar{p}(x)-2} \frac{p_{\min} - 1}{16} |\log \delta| \\ &\ge \frac{p_{\min} - 1}{16} \Big( \frac{c_1 \theta_1}{2\delta} e^{-\frac{2\delta|\log \delta|}{\theta_1}} \Big)^{\bar{p}(x)-1} (1/2)^{p_{\max}-2} |\log \delta| \\ &\ge \frac{p_{\min} - 1}{16} \Big( \frac{c_1 \theta}{2\delta} e^{-\frac{2\delta|\log \delta|}{\theta}} \Big)^{\bar{p}(x)-1} (1/2)^{p_{\max}-2} |\log \delta|. \end{aligned}$$

Here we have used (5.2) and (5.4), the inequality $1-e^{-t}\le t$, for $t>0$ and the choice $\mu=|\log\delta|$ we have made.

We now fix $\theta$ as small as needed for the previous steps to hold. Then, if $\delta\le\tilde\delta_1(p_{\min},p_{\max},c_1,\theta,D)$, we have

$$\Delta_{\bar p(x)}\bar w\ge\frac{p_{\min}-1}{16}\Big(\frac{c_1\theta}{4\delta}\Big)^{p_{\min}-1}\big(1/2\big)^{p_{\max}-2}|\log\delta|\ge\theta_1 D\quad\text{ in } B_1\setminus B_{1/2},$$

which implies

$$\Delta_{p(x)}w\ge D\quad\text{in } B_{\theta+\delta}(x_0)\setminus B_\theta(x_0).$$

Finally we have

$$|\nabla\bar w|\ge 2M\mu e^{-\mu|x|^2}\frac12\ge\frac{c_1\theta}{2\delta}e^{-\frac{2\delta|\log\delta|}{\theta}}\frac12\ge\frac{c_1\theta}{8\delta}\qquad\text{in } B_1\setminus B_{1/2},$$

if $\delta\le\tilde\delta_2(\theta)$. We thus conclude

$$|\nabla w|\ge\frac{c_1\theta}{8\delta}\quad\text{in } B_{\theta+\delta}(x_0)\setminus B_\theta(x_0).$$

The proof is now complete. □

Now we prove a positivity result that will be used later. Recall that we have assumed that there is a nonempty relatively open subset $\mathcal A$ of $\partial\Omega$ of class $C^2$ such that $u_\varepsilon\ge c_0$ on $\mathcal A$, for some positive constant $c_0$.

**Lemma 5.4.** *Let $u_\varepsilon$ be a nonnegative solution to $(P_\varepsilon)$. For every $\varepsilon>0$ there exists a neighborhood of $\mathcal A$ in $\Omega$ such that $u_\varepsilon>0$ in this neighborhood.*

*Proof.* Let $y_0\in\mathcal A$. Let us prove that $\operatorname{dist}(y_0,\Omega\cap\partial\{u_\varepsilon>0\})>0$. Assume it is 0. Let $\theta>0$ be such that, for some $z_0$, the ball $\overline{B_\theta(z_0)}\cap\overline\Omega=\{y_0\}$ and, for $\delta>0$, let $w$ be the solution to

$$\begin{cases}\Delta_{p(x)}w\ge f & \text{in}\quad B_{\theta+\delta}(z_0)\setminus\overline{B_\theta(z_0)},\\ w=c_0 & \text{on}\quad \partial B_\theta(z_0),\\ w=0 & \text{on}\quad \partial B_{\theta+\delta}(z_0),\end{cases}$$

constructed in Lemma 5.3 for $A=A_0=c_0$, $c_1=1$ and $D=||f||_{L^\infty}$. Moreover, we take $\theta=\theta(p_{\min},L,c_0)$ and $\delta\le\bar\delta(N,p_{\min},p_{\max},c_0,L,||f||_{L^\infty})$ as indicated in that lemma. We make $\delta$ small enough so that, in addition, $B_{\theta+\delta}(z_0)\cap\partial\Omega\subset\mathcal A$. By construction we have $0<w\le c_0$ in $B_{\theta+\delta}(z_0)\setminus B_\theta(z_0)$.

Let

$$\begin{cases}v=\max\{u_\varepsilon,w\} & \text{in}\quad B_{\theta+\delta}(z_0)\cap\overline\Omega,\\ v=u_\varepsilon & \text{in}\quad \overline\Omega\setminus B_{\theta+\delta}(z_0).\end{cases}$$

Then, $v\in W^{1,p(x)}(\Omega)$ is admissible, so that

$$\begin{aligned}0&\le\mathcal J_\varepsilon(v)-\mathcal J_\varepsilon(u_\varepsilon)\\&=\int_\Omega\Big(\frac{|\nabla v|^{p(x)}}{p(x)}-\frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}\Big)\,dx+\int_\Omega f(v-u_\varepsilon)\,dx+F_\varepsilon(|\{v>0\}|)-F_\varepsilon(\{u_\varepsilon>0\}|)\\&=\int_{\Omega\cap B_{\theta+\delta}(z_0)}\Big(\frac{|\nabla v|^{p(x)}}{p(x)}-\frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}\Big)\,dx+\int_{\Omega\cap B_{\theta+\delta}(z_0)}f(v-u_\varepsilon)\,dx\\&\quad+F_\varepsilon(|\{v>0\}|)-F_\varepsilon(\{u_\varepsilon>0\}|).\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\Omega\cap B_{\theta+\delta}(z_0)}\Big(\frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}-\frac{|\nabla v|^{p(x)}}{p(x)}\Big)\,dx+\int_{\Omega\cap B_{\theta+\delta}(z_0)}f(u_\varepsilon-v)\,dx\\
&\qquad\le F_\varepsilon(|\{v>0\}|)-F_\varepsilon(\{u_\varepsilon>0\}|)\le\varepsilon^{-1}\big|\Omega\cap\{v>0\}\cap\{u_\varepsilon=0\}\big|\\
&\qquad=\varepsilon^{-1}|\Omega\cap B_{\theta+\delta}(z_0)\cap\{u_\varepsilon=0\}|=\varepsilon^{-1}|V|,
\end{aligned}$$

where we have called $V=\Omega\cap B_{\theta+\delta}(z_0)\cap\{u_\varepsilon=0\}$.

Observe that, by the positive density of $\{u_\varepsilon=0\}$ at the free boundary (Theorem 3.5) and, since $\operatorname{dist}(y_0,\Omega\cap\partial\{u_\varepsilon>0\})=0$, there holds that $|V|>0$.

On the other hand, using that $\Delta_{p(x)}w\ge f$ and the definition of $v$, we have

$$\begin{aligned}
&-\int_{\Omega\cap B_{\theta+\delta}(z_0)}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla(v-u_\varepsilon)\,dx\\
&=-\int_{\Omega\cap B_{\theta+\delta}(z_0)\cap\{u_\varepsilon<w\}}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla(v-u_\varepsilon)\,dx\\
&=-\int_{\Omega\cap B_{\theta+\delta}(z_0)\cap\{u_\varepsilon<w\}}|\nabla w|^{p(x)-2}\nabla w\cdot\nabla(w-u_\varepsilon)\,dx\\
&\ge\int_{\Omega\cap B_{\theta+\delta}(z_0)\cap\{u_\varepsilon<w\}}f(w-u_\varepsilon)\,dx=\int_{\Omega\cap B_{\theta+\delta}(z_0)}f(v-u_\varepsilon)\,dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\Omega\cap B_{\theta+\delta}(z_0)}\Big(\frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}-\frac{|\nabla v|^{p(x)}}{p(x)}\Big)\,dx+\int_{\Omega\cap B_{\theta+\delta}(z_0)}f(u_\varepsilon-v)\,dx\\
&\ge\int_{\Omega\cap B_{\theta+\delta}(z_0)}\Big(\frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}-\frac{|\nabla v|^{p(x)}}{p(x)}-|\nabla v|^{p(x)-2}\nabla v\cdot\nabla(u_\varepsilon-v)\Big)\,dx\\
&\ge\int_V|\nabla w|^{p(x)}\Big(1-\frac{1}{p(x)}\Big)\,dx\ge c\big(\min_{\overline{\Omega\cap B}_{\theta+\delta}(z_0)}|\nabla w|^{p(x)}\big)|V|.
\end{aligned}$$

Observe that, by Lemma 5.3, $|\nabla w|\ge\bar c\delta^{-1}$ for a positive constant $\bar c=\bar c(p_{\min},L,c_0)$. So that, we deduce that $\delta\ge c_\varepsilon>0$ and this is a contradiction to the fact that $\delta$ is any small enough positive constant and $c_\varepsilon$ is independent of $\delta$.

Therefore, $\operatorname{dist}(y_0,\Omega\cap\partial\{u_\varepsilon>0\})>0$. So that, there is a neighborhood of $\mathcal{A}$ in $\Omega$ where either $u_\varepsilon\equiv0$ or $u_\varepsilon>0$. Since $u_\varepsilon\ge c_0>0$ in $\mathcal{A}$, we have that $u_\varepsilon>0$ in that neighborhood of $\mathcal{A}$ in $\Omega$ and the lemma is proved. ☐

Let us show conditions implying assumption $(H_\kappa)$. The first one is

**Lemma 5.5.** *There exist $\sigma_0>0$ and $\kappa>0$ such that if $||f^+||_{L^\infty}\le\sigma_0$ and $\varepsilon$ is small enough, then any nonnegative solution $u_\varepsilon$ to $(P_\varepsilon)$ satisfies assumption $(H_\kappa)$.*

*Proof.* We recall that we have assumed that there is a nonempty relatively open subset $\mathcal{A}$ of $\partial\Omega$ of class $C^2$ such that $u_\varepsilon\ge c_0$ on $\mathcal{A}$, for some positive constant $c_0$. We will use the following fact that we have proved in Lemma 5.4: For every $\varepsilon>0$ there is a neighborhood of $\mathcal{A}$ in $\Omega$ where $u_\varepsilon>0$.

Let $y_0\in\mathcal{A}$ and let $D_t$ with $0\le t\le1$ be a continuous and increasing family of open sets with smooth boundary and (uniformly in $t$) bounded curvatures, such that $D_0$ is an exterior tangent ball to $\Omega$ at $y_0$, $D_0=B_{r_0}(z_0)$, $D_t=B_{r_0+t}(z_0)$ if $0<t\le\eta$, for some $\eta>0$ small. $D_0\subset\subset D_t$ for

$t > 0$, $D_t \cap \partial\Omega \subset \mathcal{A}$, and the measure of $D_1$ is large enough so that there is a free boundary point of $u_\varepsilon$ in $D_1$ for every $\varepsilon$ small enough (here we use the upper uniform bound in (5.1)).

Now, for $0 < t \le 1$, take $w_t$ such that

$$\begin{cases} \Delta_{p(x)} w_t = f^+ & \text{in } D_t \setminus \overline{D_0}, \\ w_t \equiv c_0 & \text{in } \overline{D_0}, \\ w_t \equiv 0 & \text{in } {D_t}^c. \end{cases} \tag{5.7}$$

So that, $w_t \le c_0$ in $D_t \setminus \overline{D_0}$.

Since the domains $D_t$ have smooth boundaries, by Theorem 4.1 in [13] and Theorem 1.2 in [12] we know that $w_t \in C^1(\overline{D_t \setminus D_0})$. Moreover, by Lemma 5.3, there holds that there exists a positive constant $c$, such that

$$|\nabla w_t(x)| \ge c, \quad \text{for every} \quad t \in (0, \eta] \quad \text{and every} \quad x \in \partial D_t, \quad \text{if } r_0 \text{ and } \eta \text{ are small.} \tag{5.8}$$

Let us see that there exist positive constants $c$ and $\sigma_0$, such that

$$|\nabla w_t(x)| \ge c, \quad \text{for every} \quad t \in (0, 1] \quad \text{and every} \quad x \in \partial D_t, \quad \text{if } ||f^+||_{L^\infty} \le \sigma_0. \tag{5.9}$$

By the observation above, we only have to prove it for $t \ge \eta$. In fact, if this is not the case, there exist $f_n \in L^\infty(\Omega)$ with $||f_n^+||_{L^\infty} \le \frac{1}{n}$, and sequences $\{t_n\} \subset [\eta, 1]$ and $\bar{x}_n \in \partial D_{t_n}$, such that $|\nabla w_n(\bar{x}_n)| \le 1/n$, where we denote $w_n$ the solution to (5.7) for $f = f_n$ and $t = t_n$. By taking subsequences, we may assume that $t_n \to t_0 \in [\eta, 1]$ and $\bar{x}_n \to \bar{x}_0$.

Using that $D_\eta \subset D_t \subset D_1$, for $t \ge \eta$, and with similar energy estimates as those in Theorem 3.1, we get $||w_n||_{W^{1,p(\cdot)}(D_{t_n})} = ||w_n||_{W^{1,p(\cdot)}(\mathbb{R}^N)} \le C$.

Now, since the domains $D_t$ have uniformly bounded curvatures, the regularity estimates in [13] and [12] give $||w_n||_{C^{1,\alpha}(\overline{D_{t_n} \setminus D_0})} \le C$ and then, for a subsequence, there holds that $w_n \to w_0$ in $C^1_{\text{loc}}(D_{t_0} \setminus \overline{D_0})$. So that, $\Delta_{p(x)} w_0 = 0$ in $D_{t_0} \setminus \overline{D_0}$. We also have $||w_n||_{W^{1,\infty}(\mathbb{R}^N)} \le C$. Hence, for a subsequence, $w_n \to w_0$ uniformly on compact sets of $\mathbb{R}^N$. Then, $w_0 \equiv 0$ in $D^c_{t_0}$, $w_0 \equiv c_0$ in $\overline{D_0}$ and $0 < w_0 < c_0$ in $D_{t_0} \setminus \overline{D_0}$.

From the fact that $\bar{x}_n \in \partial D_{t_n}$, we deduce that $\bar{x}_0 \in \partial D_{t_0}$. Using again that the domains $D_t$ have uniformly bounded curvatures, we find $r_1 > 0$ and points $\bar{y}_n$ such that $B_{r_1}(\bar{y}_n) \subset D_{t_n} \setminus \overline{D_0}$ and $\overline{B_{r_1}(\bar{y}_n)} \cap \partial D_{t_n} = \{\bar{x}_n\}$. Then, for a subsequence, $\bar{y}_n \to \bar{y}_0$ with $B_{r_1}(\bar{y}_0) \subset D_{t_0} \setminus \overline{D_0}$ and $\overline{B_{r_1}(\bar{y}_0)} \cap \partial D_{t_0} = \{\bar{x}_0\}$.

Now let $\tilde{w}_n(x) = w_n(x + \bar{y}_n)$, $\tilde{f}_n(x) = f_n^+(x + \bar{y}_n)$ and $\tilde{p}_n(x) = p(x + \bar{y}_n)$. Then $\Delta_{\tilde{p}_n(x)} \tilde{w}_n = \tilde{f}_n$ in $B_{r_1}$. We have $||\tilde{w}_n||_{C^{1,\alpha}(\overline{B_{r_1}})} \le C$, therefore $\tilde{w}_n \to \tilde{w}_0$ and $\nabla \tilde{w}_n \to \nabla \tilde{w}_0$ uniformly on $\overline{B_{r_1}}$ with $\tilde{w}_0(x) = w_0(x + \bar{y}_0)$. This implies that $\nabla w_n(\bar{x}_n) \to \nabla w_0(\bar{x}_0)$ and thus $|\nabla w_0(\bar{x}_0)| = 0$.

But $\Delta_{p(x)} w_0 = 0$ in $B_{r_1}(\bar{y}_0)$, $w_0 > 0$ in $B_{r_1}(\bar{y}_0)$ and $w_0(\bar{x}_0) = 0$ with $\bar{x}_0 \in \partial B_{r_1}(\bar{y}_0)$. This in contradiction with Hopf's Lemma (Theorem 4.2 in [36]). So (5.9) follows.

Now, let $t \in (0, 1]$ be the first time such that $D_t$ touches the free boundary. Let $x_0 \in \Omega \cap \partial D_t \cap \partial\{u_\varepsilon > 0\}$. So that, since $w_t \le c_0$ and $D_t \cap \partial\Omega \subset \mathcal{A}$, by comparison in $D_t \cap \Omega$, $w_t \le u_\varepsilon$ in $D_t \cap \Omega$ and thus $w_t \le u_\varepsilon$ in $\Omega$. Therefore, for $r$ small enough, (5.9) gives

$$\left( \fint_{B_r(x_0)} u_\varepsilon^\gamma \, dx \right)^{1/\gamma} \ge \left( \fint_{B_r(x_0)} {w_t}^\gamma \, dx \right)^{1/\gamma} \ge r\bar{c}, \tag{5.10}$$

with $\bar{c}$ independent of $\varepsilon$, where $\gamma$ is the constant in Lemma 5.1. That is, $u_\varepsilon$ satisfies assumption $(H_\kappa)$ with $\kappa = \bar{c}$, if $||f^+||_{L^\infty} \le \sigma_0$. □

Another condition implying $(H_\kappa)$ is

**Lemma 5.6.** *Assume $\mathcal{A} = \partial\Omega$. There exist $\sigma_1 > 0$ and $\kappa > 0$ such that if $\omega_0 \le \sigma_1$ and $\varepsilon$ is small enough, then any nonnegative solution $u_\varepsilon$ to $(P_\varepsilon)$ satisfies assumption $(H_\kappa)$.*

*Proof.* Since we have assumed $\mathcal{A} = \partial\Omega$, we know that $u_\varepsilon > 0$ in a neighborhood of $\partial\Omega$ by Lemma 5.4.

From (5.1) we know that

$$|\{u_\varepsilon > 0\}| \le \bar{C}\varepsilon + \omega_0 \le 2\omega_0,$$

if $\varepsilon$ is small enough. For $\delta_0 > 0$, to be fixed later, we define $\Omega_{\delta_0} = \{x \in \Omega \,/\, \mathrm{dist}(x, \partial\Omega) < \delta_0\}$.

Then, if $2\omega_0 < |\Omega_{\delta_0}|$, there is a free boundary point of $u_\varepsilon$ in $\Omega_{\delta_0}$ for every $\varepsilon$ small enough.

Let $y_0^\varepsilon \in \partial\Omega$ be the closest point to $\Omega \cap \partial\{u_\varepsilon > 0\}$. Then, $0 < \mathrm{dist}(y_0^\varepsilon, \Omega \cap \partial\{u_\varepsilon > 0\}) < \delta_0$. As in Lemma 5.5 we consider a family $D_t$, with $0 \le t \le \eta$, such that $D_0$ is an exterior tangent ball to $\Omega$ at $y_0^\varepsilon$, $D_0 = B_{r_0}(z_0^\varepsilon)$, $D_t = B_{r_0+t}(z_0^\varepsilon)$ if $0 < t \le \eta$, for some $\eta > 0$ small, with $r_0$ and $\eta$ independent of $\varepsilon$.

Now, for $0 < t \le \eta$, we take $w_t$ satisfying (5.7), and as in Lemma 5.5 we get (5.8), with $c$ independent of $\varepsilon$, for $r_0$ and $\eta$ small, independent of $\varepsilon$.

We now fix $0 < \delta_0 < \eta$. Let $t \in (0, \eta]$ be the first time such that $D_t$ touches the free boundary, and let $x_0 \in \Omega \cap \partial D_t \cap \partial\{u_\varepsilon > 0\}$. Then, as in Lemma 5.5, we obtain (5.10) at $x_0$, for $r$ small enough, with $\bar{c}$ independent of $\varepsilon$, and $\gamma$ the constant in Lemma 5.1. That is, $u_\varepsilon$ satisfies assumption $(H_\kappa)$ with $\kappa = \bar{c}$, if $\omega_0 \le \sigma_1$, for a suitable constant $\sigma_1$ independent of $\varepsilon$. □

We will need

**Lemma 5.7.** *Assume that $1 < p_{\min} \le p(x) \le p_{\max} < \infty$ with $p(x)$ Lipschitz continuous and $\|\nabla p\|_{L^\infty} \le L$, for some $L > 0$. For $x_0 \in \mathbb{R}^N$, $\mu > 0$, $r > 0$, $A > 0$, consider*

$$w(x) = A\left(\frac{e^{-\mu\frac{|x-x_0|^2}{r^2}} - e^{-\mu}}{e^{-\mu/16} - e^{-\mu}}\right).$$

*Assume moreover that $c_1 r \le A \le A_0$, for some $c_1 > 0$ and $A_0 > 0$. Then, given $D > 0$, there exist $\tilde{\mu} = \tilde{\mu}(N, p_{\min}, p_{\max})$ and $\tilde{r} = \tilde{r}(p_{\min}, p_{\max}, L, D, c_1, A_0, \mu)$ such that, if $\mu \ge \tilde{\mu}$ and $r \le \tilde{r}$, there holds that*

$$\begin{cases} \Delta_{p(x)} w \ge D & \text{in } B_r(x_0) \setminus \overline{B_{r/4}(x_0)}, \\ w = A & \text{on } \partial B_{r/4}(x_0), \\ w = 0 & \text{on } \partial B_r(x_0). \end{cases}$$

*Proof.* The result is proven in Lemma 2.2 in [25] for the case $c_1 = 1$. For arbitrary $c_1 > 0$, the proof follows, with minor modifications, as that in [25]. □

We will also need

**Proposition 5.1.** *Assume that $1 < p_{\min} \le p(x) \le p_{\max} < \infty$ with $p(x)$ Lipschitz continuous and $\|\nabla p\|_{L^\infty} \le L$, for some $L > 0$, and $f \in L^\infty(\Omega)$. Let $u \in C(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ be nonnegative and $\Delta_{p(x)} u \ge f$ in $\Omega$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and assume that $|\nabla u| \le L_1$ in $B_{r_1}(x_0) \subset \Omega$, for some $L_1 > 0$, and that assumption $(H_\kappa)$ holds at $x_0$, for some $\kappa > 0$.*

*For $0 < r \le r_1$, let $v$ be the solution to*

$$\Delta_{p(x)} v = f \quad \text{in } B_r(x_0), \qquad v = u \quad \text{on } \partial B_r(x_0).$$

*Then, there exist positive constants $C$ and $r_0$, such that, if $r \leq r_0$,*

$$\int_{B_r(x_0)\cap\{p(x)\geq 2\}} |\nabla u - \nabla v|^{p(x)}\,dx + \int_{B_r(x_0)\cap\{p(x)<2\}} (|\nabla u| + |\nabla v|)^{p(x)-2}|\nabla u - \nabla v|^2\,dx \geq C|B_r(x_0)\cap\{u=0\}|,$$

*where $C = C(N, p_{\min}, p_{\max}, \kappa, L_1)$ and $r_0 = r_0(N, p_{\min}, p_{\max}, \kappa, L_1, L, ||f||_{L^\infty(\Omega)}, r_1, \tilde{r}_0)$, with $\tilde{r}_0$ such that $(H_\kappa)$ holds.*

*Proof.* For $0 < r \leq r_1$, let us take $u_r(x) = \frac{1}{r}u(x_0 + rx)$ and $v_r(x) = \frac{1}{r}v(x_0 + rx)$. Then there holds that $\Delta_{p_r(x)}u_r \geq f_r$ in $B_1$ and

$$\Delta_{p_r(x)}v_r = f_r \quad \text{in } B_1, \qquad v_r = u_r \quad \text{on } \partial B_1, \tag{5.11}$$

with $p_r(x) = p(x_0 + rx)$, $f_r(x) = rf(x_0 + rx)$. Also, assumption $(H_\kappa)$ at $x_0$ implies

$$\left(\fint_{B_{1/2}} u_r^\gamma\,dx\right)^{1/\gamma} = \frac{1}{r}\left(\fint_{B_{r/2}(x_0)} u^\gamma\,dy\right)^{1/\gamma} \geq \frac{\kappa}{2}, \tag{5.12}$$

if $r \leq \tilde{r}_0$.

We fix $z$ such that $|z| \leq \frac{1}{2}$ and we consider a change of variables from $B_1$ into itself such that $z$ becomes the new origin. We call $u_r^z(x) = u_r\big((1-|x|)z + x\big)$, $v_r^z(x) = v_r\big((1-|x|)z + x\big)$. Observe that this change of variables leaves the boundary fixed.

Given $\xi \in \partial B_1$, we define

$$s_\xi = \inf\Big\{s \,/\, \frac{1}{8} \leq s \leq 1 \quad \text{and} \quad u_r^z(s\xi) = 0\Big\},$$

if this set is nonempty and $s_\xi = 1$ otherwise.

Now, for $\mathcal{H}^{N-1}$- almost every $\xi \in \partial B_1$, if $s_\xi < 1$, we have

$$v_r^z(s_\xi \xi) = \int_{s_\xi}^1 \frac{d}{ds}(u_r^z - v_r^z)(s\xi)\,ds \leq \int_{s_\xi}^1 |\nabla(u_r^z - v_r^z)(s\xi)|\,ds. \tag{5.13}$$

Let us assume that the following inequality holds

$$v_r^z(s_\xi\xi) \geq \overline{C}(N, p_{\min}, p_{\max})(1 - s_\xi)\kappa. \tag{5.14}$$

We denote $\overline{C} = \overline{C}(N, p_{\min}, p_{\max})$ and $p_r^z(x) = p_r\big((1-|x|)z + x\big)$.

Let $s \in [s_\xi, 1]$ be such that $|\nabla(u_r^z - v_r^z)(s\xi)| \geq \frac{\overline{C}}{2}\kappa$. Then,

$$\frac{|\nabla(u_r^z - v_r^z)(s\xi)|}{\frac{\overline{C}}{2}\kappa} \leq \frac{|\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}}{\big(\frac{\overline{C}}{2}\kappa\big)^{p_r^z(s\xi)}} \leq \tilde{C}|\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)},$$

where $\tilde{C} = \tilde{C}(N, p_{\min}, p_{\max}, \kappa)$. Thus,

$$\int_{s_\xi}^1 |\nabla(u_r^z - v_r^z)(s\xi)|\,ds \leq \frac{\overline{C}}{2}\kappa(1 - s_\xi) + \frac{\overline{C}}{2}\kappa\tilde{C}\int_{s_\xi}^1 |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds. \tag{5.15}$$

Putting together (5.13), (5.14) and (5.15), we get

$$\frac{\overline{C}}{2}\kappa(1 - s_\xi) \leq \frac{\overline{C}}{2}\kappa\tilde{C}\int_{s_\xi}^1 |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds.$$

That is,

$$\hat{C}(1-s_\xi) \le \int_{s_\xi}^1 |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds, \tag{5.16}$$

where $\hat{C} = \hat{C}(N, p_{\min}, p_{\max}, \kappa)$. Note that this inequality also holds if $s_\xi = 1$.

Let us define $A_1^\xi = \{s \in [s_\xi, 1] \,/\, p_r^z(s\xi) < 2\}$ and $A_2^\xi = \{s \in [s_\xi, 1] \,/\, p_r^z(s\xi) \ge 2\}$. Then,

$$\int_{s_\xi}^1 |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds = \int_{A_1^\xi} |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds + \int_{A_2^\xi} |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds. \tag{5.17}$$

Let $0 < \eta < 1$ to be chosen later. Then, by Young's inequality, we obtain

$$\begin{aligned}\int_{A_1^\xi} |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds \le & \frac{C}{\eta^{2/p_{\min}}} \int_{A_1^\xi} (|\nabla u_r^z(s\xi)| + |\nabla v_r^z(s\xi)|)^{p_r^z(s\xi)-2} |\nabla(u_r^z - v_r^z)(s\xi)|^2\,ds \\ & + C\eta \int_{A_1^\xi} (|\nabla u_r^z(s\xi)| + |\nabla v_r^z(s\xi)|)^{p_r^z(s\xi)}\,ds,\end{aligned}$$

where $C = C(N, p_{\min}, p_{\max})$.

Since, $|\nabla v_r^z|^q \le C(|\nabla u_r^z - \nabla v_r^z|^q + |\nabla u_r^z|)^q)$, for any $q > 1$, with $C = C(q)$, we have

$$\begin{aligned}\int_{A_1^\xi} |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds \le & \frac{C_0}{\eta^{2/p_{\min}}} \int_{A_1^\xi} (|\nabla u_r^z(s\xi)| + |\nabla v_r^z(s\xi)|)^{p_r^z(s\xi)-2} |\nabla(u_r^z - v_r^z)(s\xi)|^2\,ds \\ & + C_0\eta \int_{A_1^\xi} |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds + C_0\eta \int_{A_1^\xi} |\nabla u_r^z(s\xi)|^{p_r^z(s\xi)}\,ds,\end{aligned}$$

where $C_0 = C_0(N, p_{\min}, p_{\max})$. Then, taking $\eta$ such that $1 - C_0\eta \ge \frac{1}{2}$, we obtain

$$\begin{aligned}&\int_{A_1^\xi} |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds \\ &\le \frac{2C_0}{\eta^{2/p_{\min}}} \int_{A_1^\xi} (|\nabla u_r^z(s\xi)| + |\nabla v_r^z(s\xi)|)^{p_r^z(s\xi)-2} |\nabla(u_r^z - v_r^z)(s\xi)|^2\,ds + 2C_0\eta \int_{A_1^\xi} |\nabla u_r^z(s\xi)|^{p_r^z(s\xi)}\,ds \\ &\le \frac{2C_0}{\eta^{2/p_{\min}}} \int_{A_1^\xi} (|\nabla u_r^z(s\xi)| + |\nabla v_r^z(s\xi)|)^{p_r^z(s\xi)-2} |\nabla(u_r^z - v_r^z)(s\xi)|^2\,ds + C_1\eta(1-s_\xi),\end{aligned} \tag{5.18}$$

where we have used that $|\nabla u| \le L_1$ in $B_{r_1}(x_0)$. Here $C_1 = C_1(N, p_{\min}, p_{\max}, L_1)$.

Now, from (5.16), (5.17) and (5.18), we get

$$\begin{aligned}\hat{C}(1-s_\xi) \le & \int_{A_2^\xi} |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds \\ & + \frac{2C_0}{\eta^{2/p_{\min}}} \int_{A_1^\xi} (|\nabla u_r^z(s\xi)| + |\nabla v_r^z(s\xi)|)^{p_r^z(s\xi)-2} |\nabla(u_r^z - v_r^z)(s\xi)|^2\,ds + C_1\eta(1-s_\xi).\end{aligned}$$

If we now take $\eta$ such that $\hat{C} - C_1\eta \ge \frac{1}{2}$, we get

$$\begin{aligned}(1-s_\xi) \le & 2\int_{A_2^\xi} |\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds \\ & + \frac{4C_0}{\eta^{2/p_{\min}}} \int_{A_1^\xi} (|\nabla u_r^z(s\xi)| + |\nabla v_r^z(s\xi)|)^{p_r^z(s\xi)-2} |\nabla(u_r^z - v_r^z)(s\xi)|^2\,ds,\end{aligned}$$

which gives

$$\begin{aligned}C_2(1-s_\xi) \le \int_{A_2^\xi} &|\nabla(u_r^z - v_r^z)(s\xi)|^{p_r^z(s\xi)}\,ds\\ &+\int_{A_1^\xi}(|\nabla u_r^z(s\xi)|+|\nabla v_r^z(s\xi)|)^{p_r^z(s\xi)-2}|\nabla(u_r^z-v_r^z)(s\xi)|^2\,ds,\end{aligned}$$

where $C_2 = C_2(N, p_{\min}, p_{\max}, \kappa, L_1)$. Then, integrating over $\partial B_1$, we obtain

$$\begin{aligned}C_3|B_1\cap(B_{1/8})^c\cap\{u_r^z=0\}| \le &\int_{B_1\cap\{p_r^z(x)\ge 2\}} |\nabla u_r^z - \nabla v_r^z|^{p_r^z(x)}\,dx\\ &+\int_{B_1\cap\{p_r^z(x)<2\}}(|\nabla u_r^z|+|\nabla v_r^z|)^{p_r^z(x)-2}\quad|\nabla u_r^z-\nabla v_r^z|^2\,dx,\end{aligned}$$

where $C_3 = C_3(N, p_{\min}, p_{\max}, \kappa, L_1)$. We now deduce that

$$\begin{aligned}C_4|B_1\cap(B_{1/4}(z))^c\cap\{u_r=0\}| \le &\int_{B_1\cap\{p_r(y)\ge 2\}} |\nabla u_r - \nabla v_r|^{p_r(y)}\,dy\\ &+\int_{B_1\cap\{p_r(y)<2\}}(|\nabla u_r|+|\nabla v_r|)^{p_r(y)-2}\quad|\nabla u_r-\nabla v_r|^2\,dy,\end{aligned}\tag{5.19}$$

where $C_4 = C_4(N, p_{\min}, p_{\max}, \kappa, L_1)$. If we now consider (5.19) for $z_1$ and $z_2$ in $B_{1/2}$ with $B_{1/4}(z_1)\cap B_{1/4}(z_2) = \emptyset$ and we add both inequalities, we obtain

$$\begin{aligned}C_5|B_1\cap\{u_r=0\}| \le &\int_{B_1\cap\{p_r(y)\ge 2\}} |\nabla u_r - \nabla v_r|^{p_r(y)}\,dy\\ &+\int_{B_1\cap\{p_r(y)<2\}}(|\nabla u_r|+|\nabla v_r|)^{p_r(y)-2}\quad|\nabla u_r-\nabla v_r|^2\,dy,\end{aligned}\tag{5.20}$$

where $C_5 = C_5(N, p_{\min}, p_{\max}, \kappa, L_1)$. Now, making the change of variables $x = ry + x_0$ in (5.20), we get the desired result.

Therefore we only have to prove (5.14). Let us show first that

$$v_r \ge \frac{1}{4}C(N, p_{\min}, p_{\max})\kappa \quad \text{in } B_{1/4},\tag{5.21}$$

where $C(N, p_{\min}, p_{\max})$ is the constant in Lemma 5.1, if $r$ is small enough.

Suppose (5.21) does not hold. Then, there exist $r_k \to 0$ and $x_k \in B_{1/4}$ such that $v_{r_k}(x_k) \le \frac{1}{4}C(N, p_{\min}, p_{\max})\kappa$. We denote $v_k = v_{r_k}$, $p_k = p_{r_k}$, $f_k = f_{r_k}$ and $u_k = u_{r_k}$. Since $|\nabla u_k| \le L_1$ in $B_1$ and $u_k(0) = 0$, then $|u_k| \le L_1$ in $B_1$. Then, for a subsequence, $u_k \to u_0$ uniformly on compacts of $B_1$.

Since there holds (5.11) with $|\nabla p_k(x)| \le Lr_k$, $||f_k||_{L^\infty} \le r_k||f||_{L^\infty}$ and $|u_k| \le L_1$ in $B_1$, then Lemma 3.2 and Remark 3.4 in [27] give $|v_k| \le M$ in $B_1$, for $k$ large enough. Now by Theorem 1.1 in [12], for some $0 < \alpha < 1$, $||v_k||_{C^{1,\alpha}(\Omega')} \le C_{\Omega'}$, for every $\Omega' \subset\subset B_1$.

Then, for a subsequence, $v_k \to v_0$ in $C^{1,\alpha}_{\text{loc}}(B_1)$. Also, for a subsequence, $x_k \to \bar{x} \in \overline{B}_{1/4}$, with $v_0(\bar{x}) \le \frac{1}{4}C(N, p_{\min}, p_{\max})\kappa$. There holds that $v_0 \ge 0$, $\Delta_{p_0} v_0 = 0$ in $B_1$, for $p_0 = p(x_0) = \lim_{k\to\infty} p(x_0 + r_k x)$.

From the comparison principle, $v_k \geq u_k$ in $B_1$ and thus $v_0 \geq u_0$ in $B_1$. Then, Lemma 5.1 gives

$$\begin{aligned}\inf_{B_{1/4}} v_0 &\geq C(N, p_{\min}, p_{\max}) \Big( \fint_{B_{1/2}} v_0^\gamma \, dx \Big)^{1/\gamma} \\ &\geq C(N, p_{\min}, p_{\max}) \left( \fint_{B_{1/2}} u_0^\gamma \, dx \right)^{1/\gamma} \geq C(N, p_{\min}, p_{\max}) \frac{\kappa}{2},\end{aligned}$$

where we have used (5.12). Since $\bar{x} \in B_{1/4}$, we have

$$C(N, p_{\min}, p_{\max}) \frac{\kappa}{4} \geq v_0(\bar{x}) \geq \inf_{B_{1/4}} v_0 \geq C(N, p_{\min}, p_{\max}) \frac{\kappa}{2},$$

a contradiction. Then (5.21) holds.

Now, if $|(1 - s_\xi) z + s_\xi \xi| \leq \frac{1}{4}$, the application of (5.21) gives

$$v_r^z(s_\xi \xi) = v_r((1 - s_\xi) z + s_\xi \xi) \geq \frac{1}{4} C(N, p_{\min}, p_{\max}) \kappa.$$

If $|(1 - s_\xi) z + s_\xi \xi| \geq \frac{1}{4}$ we prove by a comparison argument that inequality (5.14) also holds. In fact, let $\tilde{w}_r$ be the solution to

$$\begin{cases} \Delta_{p(x)} \tilde{w}_r \geq ||f||_{L^\infty} & \text{in } B_r(x_0) \setminus \overline{B_{r/4}(x_0)}, \\ \tilde{w}_r = \frac{1}{4} C(N, p_{\min}, p_{\max}) \kappa r & \text{on } \partial B_{r/4}(x_0), \\ \tilde{w}_r = 0 & \text{on } \partial B_r(x_0), \end{cases}$$

constructed in Lemma 5.7, for $A = \frac{1}{4} C(N, p_{\min}, p_{\max}) \kappa r$, $c_1 = A_0 = \frac{1}{4} C(N, p_{\min}, p_{\max}) \kappa$, $D = ||f||_{L^\infty}$ and $\mu = \tilde{\mu}(N, p_{\min}, p_{\max})$, with $r \leq 1$ and $r \leq \bar{r}(N, p_{\min}, p_{\max}, L, ||f||_{L^\infty}, c_1)$ as indicated in that lemma. Then, $w_r(x) = \frac{1}{r} \tilde{w}_r(x_0 + rx)$ satisfies

$$\begin{cases} \Delta_{p_r(x)} w_r \geq r ||f||_{L^\infty} & \text{in } B_1 \setminus \overline{B_{1/4}}, \\ w_r = \frac{1}{4} C(N, p_{\min}, p_{\max}) \kappa & \text{on } \partial B_{1/4}, \\ w_r = 0 & \text{on } \partial B_1. \end{cases}$$

We use again (5.21). By comparison and the construction of $w_r$,

$$v_r \geq w_r \geq \check{C}(1 - |x|) \frac{1}{4} C(N, p_{\min}, p_{\max}) \kappa \quad \text{in} \quad B_1 \setminus B_{1/4},$$

where $\check{C} = \check{C}(N, p_{\min}, p_{\max})$. Therefore

$$v_r^z(s_\xi \xi) \geq \check{C} \Big( 1 - |(1 - s_\xi) z + s_\xi \xi| \Big) \frac{1}{4} C(N, p_{\min}, p_{\max}) \kappa \geq \check{C}(1 - s_\xi) \frac{1}{8} C(N, p_{\min}, p_{\max}) \kappa$$

since $|z| \leq \frac{1}{2}$. So that (5.14) holds for every $s_\xi \geq \frac{1}{8}$.

This completes the proof. ☐

As a consequence, we obtain

**Lemma 5.8.** *Let $u_\varepsilon$ be a nonnegative solution to $(P_\varepsilon)$ satisfying assumption $(H_\kappa)$, for $\kappa > 0$. Then, there exists $c = c(\kappa) > 0$, independent of $\varepsilon$, such that*

$$\lambda_{u_\varepsilon}^*(x) \geq c, \quad \lambda_{u_\varepsilon} \geq c,$$

*for $\varepsilon$ small.*

*Proof.* Let $x_0 \in \Omega \cap \partial\{u_\varepsilon > 0\}$ be such that $(H_\kappa)$ holds at $x_0$.

For $r$ small, let $v_0$ be the solution to

$$\begin{cases} \Delta_{p(x)} v_0 = f & \text{in } B_r(x_0) \\ v_0 = u_\varepsilon & \text{on } \partial B_r(x_0), \end{cases} \tag{5.22}$$

then, $v_0 \geq u_\varepsilon$ and thus $v_0 \geq 0$ in $B_r(x_0)$. In particular $v_0 > 0$ in $B_r(x_0) \cap \{u_\varepsilon > 0\}$. Let

$$\delta_r = |B_r(x_0) \cap \{v_0 > 0\} \cap \{u_\varepsilon = 0\}|. \tag{5.23}$$

We claim that $\delta_r > 0$. If not, $v_0 = 0$ in $B_r(x_0) \cap \{u_\varepsilon = 0\}$. Then we have

$$\begin{cases} \Delta_{p(x)} v_0 = \Delta_{p(x)} u_\varepsilon & \text{in } B_r(x_0) \cap \{u_\varepsilon > 0\}, \\ v_0 = u_\varepsilon & \text{on } \partial\big(B_r(x_0) \cap \{u_\varepsilon > 0\}\big), \end{cases}$$

implying that $v_0 = u_\varepsilon$ in $B_r(x_0) \cap \{u_\varepsilon > 0\}$ and thus, $v_0 \equiv u_\varepsilon$ in $B_r(x_0)$. But $v_0 \in C^1(B_r(x_0))$ and then $u_\varepsilon \in C^1(B_r(x_0))$. This contradicts the results in Theorems 3.4 and 3.5, satisfied at $x_0 \in \Omega \cap \partial\{u_\varepsilon > 0\}$ and thus, $\delta_r > 0$.

Next, let $u_\varepsilon^s(x) = s u_\varepsilon(x) + (1-s) v_0(x)$. By using (5.22) and the inequalities in (1.2), we get

$$\begin{aligned} \int_{B_r(x_0)} \Big( \frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)} - \frac{|\nabla v_0|^{p(x)}}{p(x)} \Big)\, dx + & \int_{B_r(x_0)} f(u_\varepsilon - v_0)\, dx \\ = \int_0^1 \frac{ds}{s} \int_{B_r(x_0)} & \Big( |\nabla u_\varepsilon^s|^{p(x)-2} \nabla u_\varepsilon^s - |\nabla v_0|^{p(x)-2} \nabla v_0 \Big) \cdot \nabla(u_\varepsilon^s - v_0)\, dx \\ & \geq C \int_{B_r(x_0) \cap \{p(x) \geq 2\}} |\nabla u_\varepsilon - \nabla v_0|^{p(x)}\, dx \\ & \quad + C \int_{B_r(x_0) \cap \{p(x) < 2\}} |\nabla u_\varepsilon - \nabla v_0|^2 \Big( |\nabla u_\varepsilon| + |\nabla v_0| \Big)^{p(x)-2}\, dx, \end{aligned}$$

where $C = C(p_{\min}, p_{\max}, N)$.

Now, from Corollary 5.1 and Proposition 5.1, we get, if $r$ is small enough,

$$\int_{B_r(x_0)} \Big( \frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)} - \frac{|\nabla v_0|^{p(x)}}{p(x)} \Big)\, dx + \int_{B_r(x_0)} f(u_\varepsilon - v_0)\, dx \geq \tilde{c} |B_r(x_0) \cap \{u_\varepsilon = 0\}| \geq \tilde{c} \delta_r, \tag{5.24}$$

where $\tilde{c}$ is a positive constant independent of $\varepsilon$, and $\delta_r$ is as in (5.23).

Consider now a free boundary point $x_1$ away from $x_0$. We can choose $x_1 \in \partial_{\text{red}}\{u_\varepsilon > 0\}$. We will use that Theorem 4.3 applies at $x_1$, so

$$\frac{1}{\rho} u_\varepsilon(x_1 + \rho x) \to \lambda^*_{u_\varepsilon}(x_1) \langle x, \nu_{u_\varepsilon}(x_1) \rangle^- \quad \text{as } \rho \to 0, \tag{5.25}$$

where $\nu_{u_\varepsilon}(x_1)$ is the exterior unit normal to $\partial\{u_\varepsilon > 0\}$ at $x_1$.

Let us take

$$\tau_\rho(x) = \begin{cases} x - \rho^2 \phi\left( \dfrac{|x - x_1|}{\rho} \right) \nu_{u_\varepsilon}(x_1) & \text{for } x \in B_\rho(x_1), \\ x & \text{elsewhere}, \end{cases} \tag{5.26}$$

where $\phi$ is a nonnegative $C_0^\infty$ function supported in the unit interval, $\phi \not\equiv 0$.

Now take $v_\rho(\tau_\rho(x)) = u_\varepsilon(x)$ in $B_\rho(x_1)$. Since there holds (5.25), we can use the arguments in Lemma 4.2, where a similar construction was carried out.

We choose $\rho$ small such that

$$\delta_r = |\{u_\varepsilon > 0\} \cap B_\rho(x_1)| - |\{v_\rho > 0\} \cap B_\rho(x_1)| = \tilde{C}\rho^{N+1} + o_\varepsilon(\rho^{N+1}), \tag{5.27}$$

if $r$ is small enough. Here $\tilde{C} > 0$ is independent of $\varepsilon$ and the last inequality follows from (4.1) in Lemma 4.2.

We next define

$$v = \begin{cases} v_0 & \text{in } B_r(x_0) \\ v_\rho & \text{in } B_\rho(x_1) \\ u_\varepsilon & \text{elsewhere.} \end{cases}$$

Then, $v \in W^{1,p(x)}(\Omega)$ is an admissible function and we have

$$|\{v > 0\}| = |\{u_\varepsilon > 0\}|. \tag{5.28}$$

On the other hand as in (4.3) in Lemma 4.2, we have, as $\rho \to 0$,

$$\rho^{-N-1}\Big(\int_{B_\rho(x_1)} \Big(\frac{|\nabla v_\rho|^{p(x)}}{p(x)} - \frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}\Big)\,dx + \int_{B_\rho(x_1)} f(v_\rho - u_\varepsilon)\,dx\Big)$$
$$\to \frac{(p(x_1)-1)}{p(x_1)}\lambda^*_{u_\varepsilon}(x_1)^{p(x_1)} \int_{B_1(0)\cap\{y\cdot\nu_{u_\varepsilon}(x_1)=0\}} \phi(|y|)\,d\mathcal{H}^{N-1}(y) = \lambda_{u_\varepsilon}\hat{c},$$

with $\hat{c} > 0$ independent of $\varepsilon$. Then,

$$\int_{B_\rho(x_1)} \Big(\frac{|\nabla v_\rho|^{p(x)}}{p(x)} - \frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}\Big)\,dx + \int_{B_\rho(x_1)} f(v_\rho - u_\varepsilon)\,dx = \lambda_{u_\varepsilon}\hat{c}\rho^{N+1} + o_\varepsilon(\rho^{N+1}).$$

But as (5.27) shows that $\delta_r$ has the same order of $\rho^{N+1}$, uniformly in $\varepsilon$,

$$\int_{B_\rho(x_1)} \Big(\frac{|\nabla v_\rho|^{p(x)}}{p(x)} - \frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}\Big)\,dx + \int_{B_\rho(x_1)} f(v_\rho - u_\varepsilon)\,dx \le \tilde{k}\lambda_{u_\varepsilon}\delta_r + o_\varepsilon(\delta_r), \tag{5.29}$$

with $\tilde{k} > 0$ independent of $\varepsilon$.

Therefore by (5.24), (5.29) and (5.28), we have

$$0 \le \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) \le -\tilde{c}\delta_r + \tilde{k}\lambda_{u_\varepsilon}\delta_r + o_\varepsilon(\delta_r)$$

and then $\lambda_{u_\varepsilon} \ge c > 0$. □

With these uniform bounds on $\lambda_{u_\varepsilon}$, we can prove the following partial existence and regularity result for our original problem $(P)$. We point out that our proof is different from the ones in previous articles, since we do not use the regularity of the free boundary of the solutions of the penalized problems $(P_\varepsilon)$ to prove existence of a solution to problem $(P)$. We only use that there exists a free boundary point satisfying Theorem 4.3 and that there hold Lemmas 5.2 and 5.8, that do not use the regularity of the free boundary either.

**Theorem 5.1.** *Let $\kappa > 0$. There exists $\varepsilon_0 = \varepsilon_0(\kappa) > 0$ such that, if $u_\varepsilon$ is a nonnegative solution to $(P_\varepsilon)$ satisfying assumption $(H_\kappa)$ and $\varepsilon < \varepsilon_0$, there holds that $|\{u_\varepsilon > 0\}| = \omega_0$. Therefore, $u = u_\varepsilon$ is a nonnegative solution to problem $(P)$.*

*In the situation above, the regularity results in Corollary 3.1, Theorem 4.5 and Corollary 4.1 apply to the solution $u$ and to any other nonnegative solution to $(P)$.*

*Proof.* Let us show that $|\{u_\varepsilon > 0\}| = \omega_0$. Arguing by contradiction, we assume first that $|\{u_\varepsilon > 0\}| > \omega_0$. Let $x_1 \in \partial_{\rm red}\{u_\varepsilon > 0\}$. We will proceed as in the proof of Lemma 5.8. Given $\delta > 0$, we perturb the domain $\{u_\varepsilon > 0\}$ in a neighborhood of $x_1$, decreasing its measure by $\delta$. We choose $\delta$ small so that the measure of the perturbed set is still larger than $\omega_0$. We take $v_\rho(\tau_\rho(x)) = u_\varepsilon(x)$, and we let

$$v = \begin{cases} v_\rho & \text{in } B_\rho(x_1)\\ u_\varepsilon & \text{elsewhere,}\end{cases}$$

where $\tau_\rho$ is the function that we have considered in (5.26) in Lemma 5.8.

Arguing as in Lemma 5.8 and using Lemma 5.2, we get, for a constant $C > 0$ independent of $\varepsilon$,

$$\begin{aligned} 0 \le \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) &= \int_\Omega \frac{|\nabla v|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}\,dx + \int_\Omega f(v - u_\varepsilon)\,dx\\ &\quad + F_\varepsilon(|\{v > 0\}|) - F_\varepsilon(|\{u_\varepsilon > 0\}|)\\ &\le \tilde{k}\lambda_{u_\varepsilon}\delta + o_\varepsilon(\delta) - \frac{1}{\varepsilon}\delta = (C - \frac{1}{\varepsilon})\delta + o_\varepsilon(\delta) < 0,\end{aligned}$$

if $\varepsilon < \varepsilon_0$ and then $\delta < \delta_0(\varepsilon)$. A contradiction.

Now assume that $|\{u_\varepsilon > 0\}| < \omega_0$. We proceed as in the previous case but this time we perturb the set $\{u_\varepsilon > 0\}$ in a neighborhood of $x_1$ increasing its measure by $\delta$. We choose $\delta$ small so that the measure of the perturbed set is still smaller than $\omega_0$. That is, we take

$$\tau_\rho(x) = \begin{cases} x + \rho^2\phi\left(\dfrac{|x - x_1|}{\rho}\right)\nu_{u_\varepsilon}(x_1) & \text{for } x \in B_\rho(x_1),\\ x & \text{elsewhere,}\end{cases}$$

where $\nu_{u_\varepsilon}(x_1)$ is the exterior unit normal to $\partial\{u_\varepsilon > 0\}$ at $x_1$ and $\phi$ is a nonnegative $C_0^\infty$ function supported in the unit interval, $\phi \not\equiv 0$. We take $v_\rho(\tau_\rho(x)) = u_\varepsilon(x)$ and

$$v = \begin{cases} v_\rho & \text{in } B_\rho(x_1)\\ u_\varepsilon & \text{elsewhere,}\end{cases}$$

and we choose $\rho$ small such that

$$\delta = |\{v > 0\}| - |\{u_\varepsilon > 0\}| = \tilde{C}\rho^{N+1} + o_\varepsilon(\rho^{N+1}), \tag{5.30}$$

if $r$ is small. Here $\tilde{C} > 0$ is independent of $\varepsilon$ and the last inequality follows from (4.1) in Lemma 4.2. We can argue here again as in Lemma 4.2, since $x_1 \in \partial_{\rm red}\{u_\varepsilon > 0\}$ and then, by Theorem 4.3, $\frac{1}{\rho}u_\varepsilon(x_1 + \rho x) \to \lambda^*_{u_\varepsilon}(x_1)\langle x, \nu_{u_\varepsilon}(x_1)\rangle^-$, as $\rho \to 0$.

Now (5.30) gives

$$F_\varepsilon(|\{v > 0\}|) - F_\varepsilon(|\{u_\varepsilon > 0\}|) = \varepsilon\delta. \tag{5.31}$$

On the other hand, as in (4.3) in Lemma 4.2, we get, as $\rho \to 0$,

$$\begin{aligned} &\rho^{-N-1}\Big(\int_{B_\rho(x_1)}\Big(\frac{|\nabla v_\rho|^{p(x)}}{p(x)} - \frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}\Big)\,dx + \int_{B_\rho(x_1)} f(v_\rho - u_\varepsilon)\,dx\Big)\\ &\qquad \to \frac{(1 - p(x_1))}{p(x_1)}\lambda^*_{u_\varepsilon}(x_1)^{p(x_1)}\int_{B_1(0)\cap\{y\cdot\nu_{u_\varepsilon}(x_1)=0\}}\phi(|y|)\,d\mathcal{H}^{N-1}(y) = -\lambda_{u_\varepsilon}\hat{c},\end{aligned}$$

with $\hat{c} > 0$ independent of $\varepsilon$. Therefore,

$$\int_{B_\rho(x_1)}\Big(\frac{|\nabla v_\rho|^{p(x)}}{p(x)} - \frac{|\nabla u_\varepsilon|^{p(x)}}{p(x)}\Big)\,dx + \int_{B_\rho(x_1)} f(v_\rho - u_\varepsilon)\,dx = -\lambda_{u_\varepsilon}\hat{c}\rho^{N+1} + o_\varepsilon(\rho^{N+1}). \tag{5.32}$$

We now combine (5.31) and (5.32), and use that $\delta$ has the same order of $\rho^{N+1}$ uniformly in $\varepsilon$ by (5.30). We then apply Lemma 5.8 and obtain, for a constant $\hat{C} > 0$ independent of $\varepsilon$,

$$0 \leq \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) \leq (-\hat{C} + \varepsilon)\delta + o_\varepsilon(\delta) < 0, \tag{5.33}$$

if $\varepsilon < \varepsilon_1$ and then $\delta < \delta_0(\varepsilon)$. Again a contradiction that shows that $|\{u_\varepsilon > 0\}| = \omega_0$.

Therefore, $u = u_\varepsilon$ is a nonnegative solution to problem $(P)$.

For the regularity results satisfied by this solution $u$ and any other nonnegative solution to $(P)$, we refer to the last part of the proof of Theorem 1.1. ☐

As a corollary, we can now prove the main result in the paper, Theorem 1.1—stated in Section 1—of existence and regularity of solution to our original problem $(P)$

**Proof of Theorem 1.1.** If $f \leq 0$, by Theorem 3.1 and Remark 3.2 there exists a nonnegative solution $u_\varepsilon$ to $(P_\varepsilon)$, for every $\varepsilon > 0$. Then, by Lemma 5.5 and Theorem 5.1 there exists a nonnegative solution $u$ to problem $(P)$.

In particular, this nonnegative solution $u$ to problem $(P)$ is a nonnegative solution to $(P_\varepsilon)$, for $\varepsilon$ small, satisfying $|\{u > 0\}| = \omega_0$.

Now let $\bar{u}$ be any solution to $(P)$. Then, $\mathcal{J}_\varepsilon(\bar{u}) = \mathcal{J}(\bar{u}) = \mathcal{J}(u) = \mathcal{J}_\varepsilon(u)$ and therefore, $\bar{u}$ is a solution to $(P_\varepsilon)$, for $\varepsilon$ small. Then, by Remark 3.2 $\bar{u}$ is nonnegative, and finally, the regularity results for $\bar{u}$ follow from the application of Corollary 3.1, Theorem 4.5 and Corollary 4.1. ☐

## 6. Conclusions

In this section we include some final comments regarding our results. Namely, under the assumptions of our main result:

- We proved existence of a nonnegative solution to problem $(P)$.
- We proved the nonnegativity and regularity of the solution $u$ and of $\partial\{u > 0\}$ for *any* solution $u$ to problem $(P)$.
- We remark that we did not use the regularity of the free boundary of the solutions to the penalized problem $(P_\varepsilon)$ in the existence proof for problem $(P)$, as was the case in previous articles.
- We remark that, in several domain optimization problems the regularity of the boundary of the optimal configuration was a necessary tool in order to derive geometric properties such as symmetries, for instance. This makes the knowledge of its regularity a very important result.
- We remark that we have shown that any solution to problem $(P)$ is a solution to the penalized problem $(P_\varepsilon)$ and thus, the penalized problem provides *all* the solutions to problem $(P)$.

## Appendix A

In Section 1 we included some preliminaries on Lebesgue and Sobolev spaces with variable exponent. For the sake of completeness we collect here some additional results on these spaces as well as some other results that are used throughout the paper.

**Proposition A.1.** *There holds*

$$\min\Big\{\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\min}},\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\max}}\Big\} \le \|u\|_{L^{p(\cdot)}(\Omega)}$$
$$\le \max\Big\{\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\min}},\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\max}}\Big\}.$$

Some important results for these spaces are

**Theorem A.1.** *Let $p'(x)$ such that*

$$\frac{1}{p(x)}+\frac{1}{p'(x)}=1.$$

*Then $L^{p'(\cdot)}(\Omega)$ is the dual of $L^{p(\cdot)}(\Omega)$. Moreover, if $p_{\min}>1$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive.*

**Theorem A.2.** *Let $q(x)\le p(x)$. If $\Omega$ has finite measure, then $L^{p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ continuously.*

We also have the following Hölder's inequality

**Theorem A.3.** *Let $p'(x)$ be as in Theorem A.1. Then there holds*

$$\int_\Omega |f||g|\,dx \le 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

*for all $f\in L^{p(\cdot)}(\Omega)$ and $g\in L^{p'(\cdot)}(\Omega)$.*

The following version of Poincare's inequality holds

**Theorem A.4.** *Let $\Omega$ be bounded. Assume that $p(x)$ is log-Hölder continuous in $\Omega$ (that is, $p$ has a modulus of continuity $\omega(r)=C(\log\frac{1}{r})^{-1}$). For every $u\in W_0^{1,p(\cdot)}(\Omega)$, the inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)}\le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

*holds with a constant $C$ depending only on $N$,* diam$(\Omega)$ *and the log-Hölder modulus of continuity of $p(x)$.*

For the proof of these results and more about these spaces, see [10], [19], [21], [31] and the references therein.

We will also need

**Lemma A.1.** *Let $1<p_0<+\infty$. Let $u$ be Lipschitz continuous in $\overline{B_1^+}$, $u\ge 0$ in $B_1^+$, $\Delta_{p_0}u=0$ in $\{u>0\}$ and $u=0$ on $\{x_N=0\}$. Then, in $B_1^+$ $u$ has the asymptotic development*

$$u(x)=\alpha x_N+o(|x|),$$

*with $\alpha\ge 0$.*

*Proof.* See [6] for $p_0=2$, [9] for $1<p_0<+\infty$ and [29] for a more general operator. □

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