

Article

# Nonlinear Fokker-Planck Equation for an Overdamped System with Drag Depending on Direction

Angel Ricardo Plastino <sup>1,\*</sup> , Roseli S. Wedemann <sup>2</sup>  and Constantino Tsallis <sup>3,4,5</sup> 

<sup>1</sup> CeBio y Departamento de Ciencias Básicas, Universidad Nacional del Noroeste de la Provincia de Buenos Aires, UNNOBA-Conicet, Roque Saenz Peña, Junin 456, Argentina

<sup>2</sup> Instituto de Matemática e Estatística, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, Rio de Janeiro 20550-900, RJ, Brazil; roseli@ime.uerj.br

<sup>3</sup> Centro Brasileiro de Pesquisas Físicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, Rio de Janeiro 22290-180, RJ, Brazil; tsallis@cbpf.br

<sup>4</sup> Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA

<sup>5</sup> Complexity Science Hub Vienna, Josefstädter Straße 39, 1080 Vienna, Austria

\* Correspondence: arplastino@unnoba.edu.ar

**Abstract:** We investigate a one-dimensional, many-body system consisting of particles interacting via repulsive, short-range forces, and moving in an overdamped regime under the effect of a drag force that depends on direction. That is, particles moving to the right do not experience the same drag as those moving to the left. The dynamics of the system, effectively described by a non-linear, Fokker–Planck equation, exhibits peculiar features related to the way in which the drag force depends on velocity. The evolution equation satisfies an  $H$ -theorem involving the  $S_q$  nonadditive entropy, and admits particular, exact, time-dependent solutions closely related, but not identical, to the  $q$ -Gaussian densities. The departure from the canonical,  $q$ -Gaussian shape is related to the fact that in one spatial dimension, in contrast to what occurs in two or more spatial dimensions, the drag's dependence on direction entails that its dependence on velocity is necessarily (and severely) non-linear. The results reported here provide further evidence of the deep connections between overdamped, many-body systems, non-linear Fokker–Planck equations, and the  $S_q$ -thermostatistics.

**Keywords:** non-linear Fokker–Planck equation; direction-dependent drag;  $H$ -theorem



**Citation:** Plastino, A.R.; Wedemann, R.S.; Tsallis, C. Nonlinear Fokker-Planck Equation for an Overdamped System with Drag Depending on Direction. *Symmetry* **2021**, *13*, 1621. <https://doi.org/10.3390/sym13091621>

Academic Editor: Alessandro Sarracino

Received: 26 July 2021

Accepted: 31 August 2021

Published: 3 September 2021

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

Non-linear Fokker–Planck equations (NLFPEs) [1] are nowadays recognized as valuable tools for understanding diverse aspects of the dynamics of complex systems. In particular, they proved to be useful for the study of type-II superconductors [2,3], granular media [4], and self-gravitating systems [5,6]. A NLFPE determines the evolution of a density  $\rho$  that, in many applications, describes the spatial distribution of particles in the system's configuration space [7]. In the power-law NLFPE, which has been studied intensively in recent years, the time derivative of the density  $\rho$  is equal to the sum of two terms: a diffusion term depending on a power of  $\rho$ , and a drift term depending linearly on  $\rho$  [8,9]. In some applications [3], the power-law diffusion term provides an effective description of the forces between the particles of the system, while the drift term accounts for the external forces acting on the particles. Non-linear diffusion is useful also for the analysis of other phenomena [10,11], such as the spread of biological populations [12,13] and the transmission of information in a neural network [14].

The non-linear, power-law, Fokker–Planck equations satisfy an  $H$ -theorem [15] involving the  $S_q$ , non-additive, entropic measures [16], whose associated maximum entropy distributions are central to various recent developments in statistical physics, complex systems theory, and related fields [17–24]. In some relevant situations, the non-linear, power-law, Fokker–Planck equations admit  $q$ -Gaussian solutions, which are densities

optimizing the  $S_q$  entropies under simple constraints [16,17]. The  $q$ -Gaussian solutions highlight the connection linking the non-linear, Fokker–Planck dynamics with the  $S_q$ -thermostatistics [8]. The connection, through its intricately ramified physical implications, is related to lines of enquiry that are currently being pursued in diverse directions [25–32]. The relation between the NLFPEs and the  $S_q$  entropies helps to explain the successful phenomenological account that the  $S_q$ -thermostatistics gives for various phenomena in complex systems [17]. As a remarkable illustration, we can mention the experiments on granular media reported by Combe et al. [4], which verified a quantitative prediction deduced from the  $S_q$ -thermostatistics. The experiments confirmed, within a 2% error and for a wide range of experimental conditions, a scale relation that had been derived theoretically in 1996, using the maximum  $S_q$ -entropy solutions of the power-law NLFPE [9].

Most works using the power-law NLFPEs to model overdamped systems of interacting particles assume that the drag force acting on each particle is a linear and isotropic function of the particle's velocity. In complex systems, however, motion in one direction may not be as easy as motion in a different direction. Simple experience using a comb can attest to that. The drag force acting on a moving particle or agent arises from its interactions with the medium in which the particle is moving. The relevant interactions may not be simple, and may lead to departures both from linearity and from isotropy. Anisotropic friction is not rare in nature. Among uncountable other illustrations, we may mention lateral force microscope experiments in silicon, Si(100) surfaces [33], and similar phenomena in bio-inspired, asymmetrically-structured surfaces [34]. In the case of over-damped systems described by NLFPEs, isotropic, non-linear drag forces were considered in [35], and non-isotropic, linear drag forces in [36]. In the present work, we investigate a system with drag forces that are both non-linear and non-isotropic. To that effect, we consider a one-dimensional system of confined and interacting particles that move under the effects of drag forces depending on direction. We explore the effects that the direction-dependent drag force has on the thermostatistics of the system. We derive a NLFPE governing the system's dynamics, and investigate its physically relevant properties. We obtain its stationary-state solution, and identify a free-energy-like quantity, related to the  $S_q$ -entropy, that satisfies an  $H$ -theorem. We prove that the NLFPE admits particular time-dependent solutions that have a form akin to  $q$ -Gaussians.

## 2. The Non-Linear Fokker–Planck Equation with Power-Law Diffusion

The one-dimensional, power-law NLFPE is

$$\frac{\partial \rho}{\partial t} = \mathcal{D} \frac{\partial^2}{\partial x^2} \left[ \rho \left( \frac{\rho}{\rho_0} \right)^{1-q} \right] - \frac{\partial}{\partial x} (\rho K), \quad (1)$$

where  $\mathcal{D}$  is the diffusion constant,  $K(x)$  is the drift force, and  $q$  is a dimensionless parameter characterizing the power-law non-linearity in the diffusion term. The diffusion constant and the parameter  $q$  satisfy the inequality ( $\mathcal{D}(2-q) > 0$ ). The time-dependent density  $\rho(x, t)$  and the constant  $\rho_0$  have dimensions of inverse length. The evolution Equation (1) is sometimes cast as  $\frac{\partial P}{\partial t} = \mathcal{D} \frac{\partial^2}{\partial x^2} (P^{2-q}) - \frac{\partial}{\partial x} (PK)$ , where  $P(x, t) = \rho(x, t)/\rho_0$  is a dimensionless quantity.

In one spatial dimension, the drift force  $K(x)$  can always be expressed as the gradient of a potential function  $U(x)$ ,  $K(x) = -\frac{\partial}{\partial x} U$  (in  $N$  dimensions this is not necessarily the case, and one can have curl forces). According to the standard convention used in the literature dealing with the Fokker–Planck equation,  $K$  and  $U$  are here referred to as a “force” and a “potential”, even though these quantities do not have physical dimensions of force and energy. As we explain in the next Section,  $K$  and  $U$  are, however, proportional to a force and a potential energy, having the corresponding physical dimensions. The power-law NLFPE admits a stationary solution of the  $q$ -exponential form [17],

$$\begin{aligned}\rho_q(x) &= \rho_0 C \exp_q[-\beta U(x)] \\ &= \rho_0 C [1 - (1 - q)\beta U(x)]_+^{\frac{1}{1-q}}.\end{aligned}\quad (2)$$

The positive constants  $C$  and  $\beta$  satisfy the relation

$$(2 - q)\beta D = C^{q-1}, \quad (3)$$

and the  $q$ -exponential function is defined as

$$\exp_q(x) = [1 + (1 - q)x]_+^{\frac{1}{1-q}}, \quad (4)$$

where the  $+$  sign indicates that the function  $\exp_q(x)$  vanishes when  $1 + (1 - q)x \leq 0$ . The potential  $U(x)$  is assumed to be bounded from below so that, with an appropriate choice of the zero of energy, its minimum value is  $U_{\min} = 0$ , and  $U(x) \geq 0$  for all  $x$ . It is also assumed that the form of  $U(x)$  is such that the stationary distribution  $\rho_q(x)$  has a finite norm, so that  $\int \rho_q(x) dx = I < \infty$ . We do not require that  $I = 1$ , because in many applications  $\rho$  corresponds to the spatial distribution of particles in a multi-particle system. The set of admissible  $q$ -values for which the density  $\rho_q$  has finite norm depends on the form of the function  $U(x)$ . The density  $\rho_q(x)$  can be regarded as a density optimizing the  $q$ -entropy

$$S_q[\rho] = \frac{k}{q-1} \int \rho \left[ 1 - \left( \frac{\rho}{\rho_0} \right)^{q-1} \right] dx, \quad (5)$$

under the constraints imposed by the norm  $I$  and the mean value  $\langle U \rangle = \int \rho U dx$  of the potential  $U(x)$  [8,17]. The positive constant  $k$  determines the units and physical dimensions in which the entropy  $S_q$  is measured. In the limit  $q \rightarrow 1$ , the usual linear Fokker–Planck equation,  $\frac{\partial \rho}{\partial t} = D \frac{\partial^2}{\partial x^2} \rho - \frac{\partial}{\partial x} (\rho K)$ , is recovered, and the stationary solution (2) coincides with the well-known, Boltzmann–Gibbs, exponential density,  $\rho_{BG}(x) = A \exp[-\beta U(x)]$ . Additionally, in this  $q \rightarrow 1$  limit, relation (3) is reduced to  $\beta D = 1$ .

The NLFPE with power-law diffusion admits the free-energy functional,

$$\mathcal{F} = \langle U \rangle - \frac{D}{k} S_{q^*}[\rho], \quad (6)$$

that satisfies the  $H$ -theorem [15]

$$\frac{d\mathcal{F}}{dt} \leq 0. \quad (7)$$

The entropy  $S_{q^*}$  appearing in the definition of  $\mathcal{F}$  corresponds to the entropic parameter  $q^* = 2 - q$ . For integer values of  $q$  the NLFPE can be cast in a simpler form, because the factor  $\rho_0^{q-1}$  can be absorbed into the diffusion constant. By recourse to the re-scaled diffusion constant  $D = D\rho_0^{q-1}$ , the NLFPE can be written as

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2}{\partial x^2} (\rho^{2-q}) - \frac{\partial}{\partial x} (\rho K). \quad (8)$$

It is worth mentioning that, even though (8) looks like the NLFPE corresponding to a dimensionless density, the quantity  $\rho$  appearing in (8) still has physical dimensions of inverse length.

### 3. Direction-Dependent Drag Forces and Non-Linear Fokker–Planck Equations

The power-law NLFPE provides an effective mean-field description of confined, over-damped systems of particles interacting via short-range forces [2,3]. The particles are

confined by an external potential  $W(x)$ . The NLFPE governs the evolution of the density  $\rho$  associated with the particles' spatial distribution. The NLFPE approach to over-damped, many-particle systems led to a deeper understanding of the thermostatics of these many-body systems. Two central assumptions made in [2], and in most other works dealing with the NLFPE approach to over-damped systems, are that drag forces are isotropic and that they depend linearly on velocity. It is, therefore, natural to wonder if the NLFPE approach can still be implemented if the above assumptions are relaxed. The principal aim of the present contribution is to address this question. To this end, we consider a one-dimensional system where neither condition on the drag forces, not the isotropy nor the linearity, is fulfilled.

Our system consists of particles of mass  $m$  confined by an external potential  $W(x)$ . The particles interact via short-range, repulsive forces and move under the effect of a direction-dependent drag force. That is, the total force acting on one of the particles has three components: the force  $F_W$  derived from the confining potential, the force  $F_{\text{int}}$  due to the interaction with the other particles, and the force  $F_{\text{drag}}$  describing the direction-dependent drag. Following the treatment advanced in [3], we neglect the effects of random forces associated with standard thermal noise. The ensuing approximation is relevant both from the practical and from the theoretical points of view. On the practical side, as was shown in [3,7], there are important systems, such as systems of interacting vortices in type-II superconductors, where typical temperatures are so low that the forces associated with noise are several orders of magnitude smaller than the interaction forces, and can consequently be neglected. On the theoretical-conceptual side, as was highlighted by Nobre, Curado, and collaborators in a remarkable series of works, the study of over-damped systems without thermal noise is of considerable interest, because they still admit an effective thermostatical treatment in terms of the non-additive  $S_q$  entropies (see [3,7,27,28] and references therein).

The simplest part of the force acting on a particle is the one due to the confining potential, which is  $F_W = -\frac{\partial W}{\partial x}$ . Let us consider the force due to the interaction with the other particles. Since the particles interact through repulsive, short-range forces, each particle interacts only with particles in its immediate neighborhood. Following [2], this intuitive notion can be expressed in a quantitative way. The repulsive force acting on a particle at  $x$  due to another one at  $x'$  is  $f(|x' - x|)(x - x')/|x' - x|$ , where  $f(r) > 0$  is assumed to be a function of  $r = |x' - x|$  decaying fast enough so that  $\int_0^\infty r f(r) dr$  converges. Given the short-range nature of the interactions, it makes sense to assume that the characteristic length scales of the system are large compared with the range of  $r$ -values for which  $f(r)$  differs substantially from zero. Consequently, when computing the force  $F_{\text{int}}(x)$  on a particle at  $x$ , originating from its interaction with other particles at different locations  $x'$ , the density  $\rho(x')$  can be approximated as  $\rho(x') = \rho(x) + (x' - x)(\frac{\partial}{\partial x} \rho)$  (see [2] for details). The force  $F_{\text{int}}(x)$  can then be expressed as

$$F_{\text{int}} = -g \frac{\partial \rho}{\partial x}, \quad (9)$$

where  $g = 2 \int_0^\infty r f(r) dr$ . The force  $F_{\text{int}}$  arises, for instance, when particles located at  $x_{1,2}$  interact through the Dirac's delta potential  $V(x_1, x_2) = g \delta(x_2 - x_1)$ .

Now we consider a drag force whose strength depends on whether the particle moves to the right or to the left. The dependence on velocity, including the dependence on direction, can be expressed concisely as

$$F_{\text{drag}} = -\frac{1}{2} [(\alpha_r + \alpha_l)v + (\alpha_r - \alpha_l)|v|], \quad (10)$$

where  $v = \dot{x}$ , and the constants  $\alpha_{r,l} > 0$  correspond to the values adopted by the drag coefficient for particles moving to the right or to the left. For  $v < 0$  one has  $F_{\text{drag}} = -\alpha_l v$ , while for  $v > 0$  one has  $F_{\text{drag}} = -\alpha_r v$ . For  $v = 0$ , of course, the drag force vanishes.

The three contributions to the force felt by a test particle lead to the equation of motion

$$m\ddot{x} = -g \frac{\partial \rho}{\partial x} - \frac{\partial W}{\partial x} - \frac{1}{2} [(\alpha_r + \alpha_l)v + (\alpha_r - \alpha_l)|v|]. \quad (11)$$

When the inertial term  $m\ddot{x}$  is negligible compared with other terms in (11), one obtains the equation of overdamped motion,

$$\frac{1}{2} [(\alpha_r + \alpha_l)v + (\alpha_r - \alpha_l)|v|] = -g \frac{\partial \rho}{\partial x} - \frac{\partial W}{\partial x}. \quad (12)$$

Solving for  $v$  one obtains

$$v = -\frac{1}{2} \left[ \frac{\alpha_r + \alpha_l}{\alpha_r \alpha_l} \left( g \frac{\partial \rho}{\partial x} + \frac{\partial W}{\partial x} \right) + \frac{\alpha_r - \alpha_l}{\alpha_r \alpha_l} \left| g \frac{\partial \rho}{\partial x} + \frac{\partial W}{\partial x} \right| \right]. \quad (13)$$

The particle density  $\rho(x, t)$  of a system of interacting particles moving according to (12) obeys a continuity equation  $\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$ , with a density current  $J = \rho v$  given by

$$J = -\frac{\rho}{2} \left[ \frac{\alpha_r + \alpha_l}{\alpha_r \alpha_l} \left( g \frac{\partial \rho}{\partial x} + \frac{\partial W}{\partial x} \right) + \frac{\alpha_r - \alpha_l}{\alpha_r \alpha_l} \left| g \frac{\partial \rho}{\partial x} + \frac{\partial W}{\partial x} \right| \right]. \quad (14)$$

The resulting continuity equation reads

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left\{ \frac{\rho}{2} \left[ \frac{\alpha_r + \alpha_l}{\alpha_r \alpha_l} \left( g \frac{\partial \rho}{\partial x} + \frac{\partial W}{\partial x} \right) + \frac{\alpha_r - \alpha_l}{\alpha_r \alpha_l} \left| g \frac{\partial \rho}{\partial x} + \frac{\partial W}{\partial x} \right| \right] \right\}, \quad (15)$$

which has the form of a non-linear Fokker–Planck equation. Making the identifications  $D \rightarrow \frac{g}{2\alpha_l}$  and  $U \rightarrow \frac{W}{2\alpha_l}$ , and introducing the dimensionless parameter

$$\varepsilon = 1 - \frac{\alpha_l}{\alpha_r}, \quad (16)$$

the Fokker–Planck Equation (15) can be written as

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left\{ \rho \left[ (2 - \varepsilon) \left( D \frac{\partial \rho}{\partial x} + \frac{\partial U}{\partial x} \right) + \varepsilon \left| D \frac{\partial \rho}{\partial x} + \frac{\partial U}{\partial x} \right| \right] \right\}. \quad (17)$$

Therefore, if one introduces direction-dependent drag forces in a many-body system of the kind considered in [2], the resulting dynamics is governed by the evolution Equation (17).

It is worth to emphasize that the direction-dependent drag forces are both anisotropic and non-linear. The parameter  $\varepsilon$  characterizes the departure from isotropy and linearity. When the parameter vanishes the drag forces become linear and isotropic, and the evolution Equation (17) reduces to the power-law NLFPE (8) with  $q = 0$ , which is relevant for the study of interacting vortices in type-II superconductors [2,3,7]. The one-dimensional instance of a drag force that depends on direction is special because it is not possible to have a one-dimensional, linear drag force that depends on direction. The only possible form of a one-dimensional, linear drag force is  $F_{\text{drag}} = -\alpha v$  (with  $\alpha > 0$ ), which obviously describes a direction-independent drag, since particles moving to the right or to the left have the same drag coefficient  $\alpha$ . To have a one-dimensional, direction-dependent drag force, it is necessary to depart from linearity. That is, in one dimension, direction-dependence of drag is necessarily accompanied by non-linearity. In  $N > 1$  spatial dimensions, the situation is different. It is possible to have a linear, anisotropic drag force of the form  $\mathbf{F}_{\text{drag}} = -\mathbf{A} \cdot \mathbf{v}$ , where  $\mathbf{A}$  is a positive-definite  $N \times N$  matrix playing a role akin to an anisotropic drag coefficient. Isotropic drag is obtained when  $\mathbf{A}$  is proportional to the identity matrix. The problem studied in the present work has the peculiarity of involving a drag force that is both anisotropic and non-linear. This kind of scenario can be extended to higher spatial dimensions. In multi-dimensional problems, however, anisotropy and non-linearity may be combined in a myriad different ways, most probably requiring a case-by-case treatment.

#### 4. Stationary Solutions and H-Theorem for Over-Damped Systems with Position-Dependent Drag forces

In this section, we prove that the evolution Equation (17) admits a stationary solution of the  $q$ -exponential form, and that it satisfies an  $H$ -theorem related to the  $S_q$  entropies. To this end, it is enlightening to investigate a more general evolution equation, that is defined for a continuous range of  $q$ -values, and may be connected with various applications of the  $S_q$ -thermostatistics. We consider the evolution equation,

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left\{ \frac{\rho}{2} \left[ (2 - \varepsilon) \frac{\partial}{\partial x} \left( \mathcal{D} \left( \frac{2 - q}{1 - q} \right) \left( \frac{\rho}{\rho_0} \right)^{1 - q} + U \right) + \varepsilon \left| \frac{\partial}{\partial x} \left( \mathcal{D} \left( \frac{2 - q}{1 - q} \right) \left( \frac{\rho}{\rho_0} \right)^{1 - q} + U \right) \right| \right] \right\}, \tag{18}$$

that reduces to (17) for  $q = 0$  and  $D = \mathcal{D}/\rho_0$ . The constant  $\rho_0$  with dimensions of inverse length is introduced in order to work with dimensional quantities, as was done when studying the NLFPE (1). It is assumed, again as was done with (1), that  $(2 - q)\mathcal{D} > 0$ .

On first acquaintance, the differential Equation (18) may not be pleasing to the eye. However, it has some physically relevant, nice properties that can be studied analytically and attest to its close links with the  $S_q$ -thermostatistics. In particular, it has a stationary solution of the  $q$ -exponential form, and it satisfies an  $H$ -theorem related to the  $S_q$  entropies.

##### 4.1. Stationary Solutions

We now prove that, for appropriately chosen parameters  $A$  and  $\beta$ , the  $q$ -exponential density,

$$\rho_q(x) = \rho_0 C [1 - (1 - q)\beta U(x)]_+^{\frac{1}{1 - q}}, \tag{19}$$

constitutes a stationary solution of the NLFPE (18). It follows from (19) that, within the interval where  $\rho_q \neq 0$ ,

$$\frac{C^{q - 1}}{(1 - q)\beta} \left( \frac{\rho_q}{\rho_0} \right)^{1 - q} + U = \frac{1}{(1 - q)\beta}, \tag{20}$$

implying that, for all  $x$ ,

$$\rho_q \frac{\partial}{\partial x} \left[ \frac{C^{q - 1}}{(1 - q)\beta} \left( \frac{\rho_q}{\rho_0} \right)^{1 - q} + U \right] = 0. \tag{21}$$

Choosing now values of the parameters  $A$  and  $\beta$  satisfying  $C^{q - 1} = (2 - q)\mathcal{D}\beta$ , Equation (21) can be re-expressed as

$$\rho_q \frac{\partial}{\partial x} \left[ \frac{(2 - q)\mathcal{D}}{1 - q} \left( \frac{\rho_q}{\rho_0} \right)^{1 - q} + U \right] = 0. \tag{22}$$

Comparing now (22) with (18), we can verify by inspection that the density  $\rho_q$  given by (19) is actually a stationary solution of (18). Remarkably, the stationary density (19) has the same shape as the stationary density (2) of the NLFPE (1) corresponding to systems with isotropic and linear drag forces. We see that the stationary densities of confined, over-damped systems with short-range, repulsive forces are quite robust. For one-dimensional systems, the stationary densities are not affected by the nonlinearity and the anisotropy associated with the direction-dependence of the drag forces. The robustness makes physical sense, because the stationary densities are determined by conditions that do not involve the drag forces directly. Stationary densities correspond to configurations of the system for which the interaction forces between the particles are balanced by the confining forces arising from the external potential. The robust character of the stationary densities may



be of considerable significance for the  $S_q$ -thermostatistical theory, since some of its most important applications are based on the maximum  $S_q$ -entropy, stationary densities of systems described by NLFPEs.

#### 4.2. $H$ -Theorem

Now we show that the NLFPE (18) satisfies an  $H$ -theorem. We consider the free-energy-like quantity

$$\mathcal{G} = \int \rho \left[ \frac{\mathcal{D}}{1-q} \left( \frac{\rho}{\rho_0} \right)^{1-q} + U \right] dx. \quad (23)$$

The time derivative of  $\mathcal{G}$  is

$$\frac{d\mathcal{G}}{dt} = \int \frac{\partial \rho}{\partial t} \left[ \mathcal{D} \left( \frac{2-q}{1-q} \right) \left( \frac{\rho}{\rho_0} \right)^{1-q} + U \right] dx. \quad (24)$$

We now substitute in (24) the right hand side of (18) for the time derivative  $\frac{\partial \rho}{\partial t}$ . After some calculations that include an integration by parts, one obtains

$$\begin{aligned} \frac{d\mathcal{G}}{dt} = & - \int \rho \left\{ \mathcal{D} \left( \frac{2-q}{1-q} \right) \frac{\partial}{\partial x} \left[ \left( \frac{\rho}{\rho_0} \right)^{1-q} \right] + \frac{\partial U}{\partial x} \right\}^2 dx \\ & - \frac{\varepsilon}{2} \int \rho \left( \mathcal{D} \left( \frac{2-q}{1-q} \right) \frac{\partial}{\partial x} \left[ \left( \frac{\rho}{\rho_0} \right)^{1-q} \right] + \frac{\partial U}{\partial x} \right) \times \\ & \times \left\{ - \left( \mathcal{D} \left( \frac{2-q}{1-q} \right) \frac{\partial}{\partial x} \left[ \left( \frac{\rho}{\rho_0} \right)^{1-q} \right] + \frac{\partial U}{\partial x} \right) + \left| \mathcal{D} \left( \frac{2-q}{1-q} \right) \frac{\partial}{\partial x} \left[ \left( \frac{\rho}{\rho_0} \right)^{1-q} \right] + \frac{\partial U}{\partial x} \right| \right\} dx \\ & \leq 0, \end{aligned} \quad (25)$$

which is the  $H$ -theorem satisfied by the Fokker–Planck Equation (18). The equality  $\frac{d\mathcal{G}}{dt} = 0$  holds only for the stationary solutions of (18). There is a close connection between the  $H$ -theorem (25) and the  $S_q$  entropies. The  $H$ -theorem can be expressed as

$$\frac{d}{dt} \left[ \langle U \rangle - \frac{\mathcal{D}}{k} S_{q^*}[\rho] \right] \leq 0, \quad (26)$$

where  $q^* = 2 - q$ . We see here another remarkable invariance of the NLFPEs describing over-damped systems. The forms of the  $H$ -theorem, and of the associated free-energy-like functional, are preserved when one considers the anisotropy and non-linearity associated by direction-depending drag forces. One should be aware, however, that the rate of change  $d\mathcal{G}/dt$  of the free energy does depend on the parameter  $\varepsilon$  characterizing the amount of anisotropy exhibited by the drag forces.

#### 5. An Example with a Time-Dependent Solution Having Asymmetric $q$ -Gaussian Form

We now consider an exact analytical solution, of an asymmetric,  $q$ -Gaussian form, for the dynamics of an un-confined system of particles with repulsive, short-range interactions doing over-damped motion under direction-dependent drag forces. The system evolves according to the non-linear diffusion equation,

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left\{ \rho \left[ (2-\varepsilon)D \frac{\partial \rho}{\partial x} + \varepsilon \left| D \frac{\partial \rho}{\partial x} \right| \right] \right\}, \quad (27)$$

which constitutes a particular instance of the NLFPE (17), corresponding to a vanishing confining potential,  $U(x) = 0$ . At locations where  $\frac{\partial \rho}{\partial x} > 0$ , the right-hand side of (27) is

$$\frac{\partial}{\partial x} \left[ 2D\rho \frac{\partial \rho}{\partial x} \right], \quad (28)$$

which corresponds to the power-law, non-linear, diffusion equation with diffusion coefficient  $D$ . On the other hand, at locations where  $\frac{\partial \rho}{\partial x} < 0$ , the right-hand side of (27) is

$$\frac{\partial}{\partial x} \left[ 2(1 - \varepsilon) D \rho \frac{\partial \rho}{\partial x} \right], \tag{29}$$

which corresponds to an effective diffusion coefficient equal to  $(1 - \varepsilon) D$ .

The general strategy for constructing an exact solution of (27) is based on matching two evolving half  $q$ -Gaussians of different widths, given by  $A(t) \exp_q(-\beta_1(t)x^2)$  and  $A(t) \exp_q(-\beta_2(t)x^2)$ . For  $x > 0$  the solution is described by  $A(t) \exp_q(-\beta_1(t)x^2)$ , and for  $x < 0$  by  $A(t) \exp_q(-\beta_2(t)x^2)$  (notice that  $A(t)$  is the same in both parts). The time-dependent parameters  $A(t)$  and  $\beta_{1,2}(t)$  evolve in such a way that the right-part ( $x > 0$ ) of the solution satisfies a power-law diffusion equation with effective diffusion constant  $(1 - \varepsilon)D$ , while the left-part ( $x < 0$ ) satisfies a similar equation with effective diffusion constant  $D$ . The right and the left parts have to match each other properly at  $x = 0$ , so that the combined density satisfies the differential Equation (27) also at that point. We shall prove that the two halves of the solution do match in an appropriate way, provided that the initial values of the parameters  $\beta_{1,2}$  satisfy the relation  $\beta_2(0) = (1 - \varepsilon)\beta_1(0)$ .

Replacing the ansatz,

$$\rho(x, t) = A(t) \left[ 1 - \frac{1}{4}(1 - q) \left( \beta_1(t)(x + |x|)^2 + \beta_2(x - |x|)^2 \right) \right]_+^{\frac{1}{1-q}}, \tag{30}$$

into the diffusion Equation (27), one verifies that, for  $q = 0$ , the ansatz (30) is a solution of (27) provided that the time-dependent parameters  $A$  and  $\beta_{1,2}$  adopt adequate initial conditions  $A(0)$  and  $\beta_{1,2}(0)$ , and satisfy an appropriate set of coupled ordinary differential equations of motion. For the rest of this section we consider the case  $q = 0$  of (30). First, notice that

$$\begin{aligned} \frac{\partial \rho}{\partial x} &= -2A\beta_1x \quad \text{for } x \geq 0, \\ \frac{\partial \rho}{\partial x} &= -2A\beta_2x \quad \text{for } x < 0. \end{aligned} \tag{31}$$

Inserting the ansatz (30) into the diffusion Equation (27), and using the relations (31), one verifies that (30) satisfies (27), if the time-dependent parameters  $A(t)$  and  $\beta_{1,2}(t)$  comply with the set of coupled, ordinary differential equations

$$\begin{aligned} \frac{dA}{dt} &= -4(1 - \varepsilon)D A^2 \beta_1, \\ \frac{d\beta_1}{dt} &= -8(1 - \varepsilon)D A \beta_1^2, \\ \frac{d\beta_2}{dt} &= -8 D A \beta_2^2. \end{aligned} \tag{32}$$

We assume that  $A(0) > 0$  and  $\beta_{1,2}(0) > 0$ . The equations of motion (32) guarantee that  $A(t) > 0$  and  $\beta_{1,2}(t) > 0$  for all times  $t > 0$ . It follows from the first two equations in (32) that

$$2 \frac{A}{\beta_1} \frac{dA}{dt} - \left( \frac{A}{\beta_1} \right)^2 \frac{d\beta_1}{dt} = \frac{d}{dt} \left( \frac{A^2}{\beta_1} \right) = 0, \tag{33}$$

and

$$\frac{A(t)}{A(0)} = \left( \frac{\beta_1(t)}{\beta_1(0)} \right)^{\frac{1}{2}}. \tag{34}$$

Combining the second and third equations in (32) one gets



$$\frac{1}{\beta_1^2} \frac{d\beta_1}{dt} = (1 - \varepsilon) \frac{1}{\beta_2^2} \frac{d\beta_2}{dt}, \tag{35}$$

implying that  $\frac{d}{dt}(\beta_1^{-1} - (1 - \varepsilon)\beta_2^{-1}) = 0$ . Therefore, if the relation

$$\beta_2 = (1 - \varepsilon) \beta_1 \tag{36}$$

holds at  $t = 0$ , it holds at all times. Equation (36) implies that, for solutions of (32) complying with the initial conditions  $\beta_2(0) = (1 - \varepsilon) \beta_1(0)$ , the quantities  $A(t)$  and  $\beta_2(t)$  satisfy also the differential equation

$$\frac{dA}{dt} = -4 D A^2 \beta_2, \tag{37}$$

from which the relation (34) can be extended to

$$\frac{A(t)}{A(0)} = \left( \frac{\beta_1(t)}{\beta_1(0)} \right)^{\frac{1}{2}} = \left( \frac{\beta_2(t)}{\beta_2(0)} \right)^{\frac{1}{2}}. \tag{38}$$

The above calculations guarantee that the ansatz (30) satisfies the differential Equation (27) for  $x < 0$  and for  $x > 0$ . To prove that the ansatz constitutes a full solution to (27), we have now to prove that (27) is also satisfied at  $x = 0$ . First, notice that both  $\rho(x)$  and  $\frac{\partial \rho}{\partial x}$  are continuous at  $x = 0$ ,

$$\begin{aligned} \lim_{x \rightarrow 0_+} \rho &= \lim_{x \rightarrow 0_-} \rho = A(t), \\ \lim_{x \rightarrow 0_+} \left( \frac{\partial \rho}{\partial x} \right) &= \lim_{x \rightarrow 0_-} \left( \frac{\partial \rho}{\partial x} \right) = 0. \end{aligned} \tag{39}$$

We now consider the behavior of the left and right-hand sides of Equation (27) at  $x = 0$ . The left-hand side is simply given by  $\left( \frac{\partial \rho}{\partial t} \right)_{x=0} = \frac{dA}{dt}$ . As for the right-hand side of (27) at  $x = 0$ , we have

$$\lim_{x \rightarrow 0_+} \frac{\partial}{\partial x} \left\{ \rho \left[ (2 - \varepsilon) D \frac{\partial \rho}{\partial x} + \varepsilon \left| D \frac{\partial \rho}{\partial x} \right| \right] \right\} = -4(1 - \varepsilon) D A^2 \beta_1, \tag{40}$$

and

$$\lim_{x \rightarrow 0_-} \frac{\partial}{\partial x} \left\{ \rho \left[ (2 - \varepsilon) D \frac{\partial \rho}{\partial x} + \varepsilon \left| D \frac{\partial \rho}{\partial x} \right| \right] \right\} = -4 D A^2 \beta_2. \tag{41}$$

The relation (36) between  $\beta_1$  and  $\beta_2$  implies that the limits appearing in (40) and (41) have the same value, and it follows from (32) that both limits are equal to  $\left( \frac{\partial \rho}{\partial t} \right)_{x=0}$ . Therefore, the ansatz (30) constitutes a solution of the non-linear diffusion Equation (27).

We shall refer to the density (30) as an “asymmetric  $q$ -Gaussian”. It has cut-off points at  $x_+ = \beta_1^{-\frac{1}{2}}$  and  $x_- = -\beta_2^{-\frac{1}{2}}$  (such that the density vanishes outside the interval  $[x_-, x_+]$ ). The normalization of (30) is given (for  $q = 0$ ) by

$$N = \int_{x_-}^{x_+} \rho(x, t) dx = \frac{2A}{3} \left( \beta_1^{-\frac{1}{2}} + \beta_2^{-\frac{1}{2}} \right). \tag{42}$$

It follows from the relations (38) that  $dN/dt = 0$  and normalization is conserved. The conservation of  $N$  can also be derived directly from (27) for general solutions of the non-linear diffusion equation. Taking (34) into account, (32) leads to

$$\frac{dA}{dt} = -4(1 - \varepsilon)D \left( \frac{\beta_1(0)}{A(0)^2} \right) A^4, \tag{43}$$

whose solution is

$$A(t) = A(0) \left[ 1 + \frac{t}{\tau} \right]^{-\frac{1}{3}}, \tag{44}$$

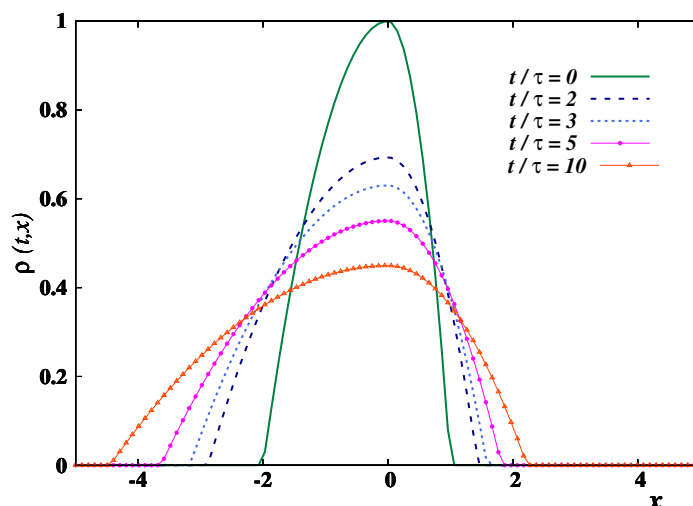
with

$$\tau = \frac{1}{12(1 - \varepsilon)D\beta_1(0)A(0)}. \tag{45}$$

The time dependence of  $\beta_1$  is then

$$\beta_1(t) = \beta_1(0) \left[ 1 + \frac{t}{\tau} \right]^{-\frac{2}{3}}. \tag{46}$$

As the time-dependent density  $\rho$  evolves, both  $A$  and  $\beta_1$  decay in a  $q$ -exponential way, with effective values of  $q$  given by  $q_A^{(\text{relax})} = 4$  and  $q_\beta^{(\text{relax})} = 5/2$  (see (44) and (46)). The evolution of the asymmetric,  $q$ -Gaussian density (30) is illustrated in Figure 1, where the density  $\rho$  is plotted for different values of  $t/\tau$ .



**Figure 1.** Time evolution of the asymmetric,  $q$ -Gaussian solution (30) to the non-linear diffusion Equation (27). The density  $\rho(x, t)$  is depicted as a function of  $x$  for different values of  $t/\tau$ . The solution corresponds to  $D = 1$  and  $q = 0$ , and to initial conditions given by  $A(0) = 1, \beta_1(0) = 1, \beta_2(0) = 1/4$ . The density  $\rho$  is measured in units of  $A(0)$ , and the coordinate  $x$  in units of  $\beta_1(0)^{-1/2}$ .

For the asymmetric,  $q$ -Gaussian solution the quantity satisfying the  $H$ -theorem evolves according to

$$\mathcal{G} = \frac{4}{5}DNA = \frac{4}{5}DNA(0) \left[ 1 + \frac{t}{\tau} \right]^{-\frac{1}{3}}, \tag{47}$$

meaning that it also decays in a  $q$ -exponential fashion, with  $q_{\mathcal{G}}^{(\text{relax})} = q_A^{(\text{relax})}$ . The asymptotic decay of  $\mathcal{G}$  with time is, therefore, as  $t^{-1/3}$ .

For  $\varepsilon = 0$ , the drag forces become linear and isotropic, the condition (36) reduces to  $\beta_1 = \beta_2$ , and the density (30) coincides with the well-known, time-dependent,  $q$ -Gaussian solution of the power-law diffusion equation.

## 6. Exact Solutions for Non-Linear Diffusion with General $q$ -Values

For general  $q$ -values satisfying  $(2 - q)\mathcal{D} > 0$ , the non-linear diffusion equation,

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left\{ \frac{\rho}{2} \left[ (2 - \varepsilon) \frac{\partial}{\partial x} \left( \mathcal{D} \left( \frac{2 - q}{1 - q} \right) \left( \frac{\rho}{\rho_0} \right)^{1 - q} \right) + \varepsilon \left| \frac{\partial}{\partial x} \left( \mathcal{D} \left( \frac{2 - q}{1 - q} \right) \left( \frac{\rho}{\rho_0} \right)^{1 - q} \right) \right| \right] \right\}, \quad (48)$$

also admits exact solutions of the asymmetric,  $q$ -Gaussian form (30). For the solution to be normalizable it is required that  $q < 3$ . It is possible to prove, by recourse to arguments similar to those developed in the previous subsection, that the ansatz (30) is a solution of the non-linear diffusion equation, if the initial values of the parameters  $\beta_{1,2}$  satisfy  $\beta_2 = (1 - \varepsilon)\beta_1$ , and the time dependence of  $A(t)$  and  $\beta_{1,2}(t)$  obeys the set of coupled, ordinary differential equations

$$\begin{aligned} \frac{dA}{dt} &= -2(2 - q)(1 - \varepsilon)\mathcal{D}\beta_1 A \left( \frac{A}{\rho_0} \right)^{1 - q}, \\ \frac{d\beta_1}{dt} &= -4(2 - q)(1 - \varepsilon)\mathcal{D}\beta_1^2 \left( \frac{A}{\rho_0} \right)^{1 - q}, \\ \frac{d\beta_2}{dt} &= -4(2 - q)\mathcal{D}\beta_2^2 \left( \frac{A}{\rho_0} \right)^{1 - q}. \end{aligned} \quad (49)$$

For  $\varepsilon = 0$ , the asymmetric,  $q$ -Gaussian form (30) becomes symmetric, and the above equations coincide with those determining the  $q$ -Gaussian solution of the non-linear Fokker–Planck equation [8] for the case of zero potential. The differential Equations (49) imply that the condition  $\beta_2(t) = (1 - \varepsilon)\beta_1(t)$  is preserved during the evolution, and also that  $A(t)/A(0) = (\beta_1(t)/\beta_1(0))^{1/2} = (\beta_2(t)/\beta_2(0))^{1/2}$ . This, in turn, guarantees that the density's normalization  $N$ , which is proportional to  $A \left( \beta_1^{-1/2} + \beta_2^{-1/2} \right)$ , is constant in time.

## 7. Conclusions

We have investigated one-dimensional systems of confined particles with short-range, repulsive interactions, that perform over-damped motion, under direction-dependent drag forces. That is, particles moving to the right experience drag forces of different strength than those moving to the left. We obtained a family of non-linear Fokker–Planck equations that govern the dynamics of these systems, and explored their main properties. We found that the Fokker–Planck equations have maximum  $S_q$ -entropy, stationary solutions, and that the equations comply with an  $H$ -theorem. There exists a free-energy-like functional, equal to a linear combination of an entropy  $S_{q^*}$  (with  $q^* = 2 - q$ ) and the mean value of the confining potential, whose time derivative is always non-positive. We also obtained, for the case of unconfined systems, analytic, time-dependent solutions of the corresponding non-linear diffusion equation. The solutions exhibit the form of asymmetric,  $q$ -Gaussian densities and include instances with  $q > 1$ , for which the  $q$ -Gaussians have long tails that decay asymptotically as power-laws.

The present work provides further evidence of the strong links between the thermostatics of over-damped, many-body systems, the associated non-linear Fokker–Planck equations, and the  $S_q$  entropies. For the systems studied here, the links are preserved even when considering drag forces that are both anisotropic and non-linear.

**Author Contributions:** Conceptualization, A.R.P., R.S.W. and C.T.; Investigation, A.R.P., R.S.W. and C.T.; Writing—review and editing, A.R.P., R.S.W. and C.T. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was partially funded by the Brazilian science funding agencies CNPq, CAPES and FAPERJ.

**Acknowledgments:** We acknowledge partial financial support from the Brazilian science funding agencies CNPq, CAPES and FAPERJ.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Franck, T.D. *Nonlinear Fokker-Planck Equations: Fundamentals and Applications*; Springer: Berlin/Heidelberg, Germany, 2005.
2. Andrade, J.S., Jr.; da Silva, G.F.T.; Moreira, A.A.; Nobre, F.D.; Curado, E.M.F. Thermostatistics of Overdamped Motion of Interacting Particles. *Phys. Rev. Lett.* **2010**, *105*, 260601. [[CrossRef](#)]
3. Ribeiro, M.S.; Nobre, F.D.; Curado, E.M.F. Time evolution of interacting vortices under overdamped motion. *Phys. Rev. E* **2012**, *85*, 021146. [[CrossRef](#)]
4. Combe, G.; Richefeu, V.; Stasiak, M.; Atman, A.P.F. Experimental validation of a nonextensive scaling law in confined granular media. *Phys. Rev. Lett.* **2015**, *115*, 238301. [[CrossRef](#)]
5. Shiino, M. Stability analysis of mean-field-type nonlinear Fokker-Planck equations associated with a generalized entropy and its application to the self-gravitating system. *Phys. Rev. E* **2003**, *67*, 056118. [[CrossRef](#)] [[PubMed](#)]
6. Chavanis, P.H. Generalized thermodynamics and Fokker-Planck equations: Applications to stellar dynamics and two-dimensional turbulence. *Phys. Rev. E* **2003**, *68*, 036108. [[CrossRef](#)] [[PubMed](#)]
7. Curado, E.M.F.; Souza, A.M.C.; Nobre, F.D.; Andrade, R.F.S. Carnot cycle for interacting particles in the absence of thermal noise. *Phys. Rev. E* **2014**, *89*, 022117. [[CrossRef](#)] [[PubMed](#)]
8. Plastino, A.R.; Plastino, A. Non-extensive statistical mechanics and generalized Fokker-Planck equation. *Physica A* **1995**, *222*, 347–354. [[CrossRef](#)]
9. Tsallis, C.; Bukman, D.J. Anomalous diffusion in the presence of external forces: Exact time-dependent solutions and their thermostatical basis. *Phys. Rev. E* **1996**, *54*, R2197–R2200. [[CrossRef](#)]
10. Frank, T.D.; Friedrich, R. Estimating the nonextensivity of systems from experimental data: A nonlinear diffusion equation approach. *Physica A* **2005**, *347*, 65–76. [[CrossRef](#)]
11. Ubriaco, M.R. A simple mathematical model for anomalous diffusion via Fisher's information theory. *Phys. Lett. A* **2009**, *373*, 4017–4021. [[CrossRef](#)]
12. Mendes, G.A.; Ribeiro, M.S.; Mendes, R.S.; Lenzi, E.K.; Nobre, F.D. Nonlinear Kramers equation associated with nonextensive statistical mechanics. *Phys. Rev. E* **2015**, *91*, 052106. [[CrossRef](#)] [[PubMed](#)]
13. Chavanis, P.H. Nonlinear mean field Fokker-Planck equations. Application to the chemotaxis of biological populations. *Eur. Phys. J. B* **2008**, *62*, 179–208. [[CrossRef](#)]
14. Wedemann, R.S.; Plastino, A.R. A nonlinear Fokker-Planck description of continuous neural network dynamics. In *Artificial Neural Networks and Machine Learning—ICANN 2019, Lecture Notes in Computer Science*; Tetko, I., Kůrková, V., Karpov, P., Theis, F., Eds.; Springer: Cham, Switzerland, 2019; Volume 11727, pp. 43–56.
15. Schwämmle, V.; Nobre, F.D.; Curado, E.M.F. Consequences of the  $H$  theorem from nonlinear Fokker-Planck equations. *Phys. Rev. E* **2007**, *76*, 041123. [[CrossRef](#)]
16. Tsallis, C. Possible Generalization of Boltzmann-Gibbs Statistics. *J. Stat. Phys.* **1988**, *52*, 479–487. [[CrossRef](#)]
17. Tsallis, C. *Introduction to Nonextensive Statistical Mechanics, Approaching a Complex World*; Springer: New York, NY, USA, 2009.
18. Beck, C. Generalised information and entropy measures in physics. *Contemp. Phys.* **2009**, *50*, 495–510. [[CrossRef](#)]
19. Naudts, J. *Generalised Thermostatistics*; Springer: London, UK, 2011.
20. Pluchino, A.; Rapisarda, A. Metastability in the Hamiltonian mean field model and Kuramoto model. *Physica A* **2006**, *365*, 184–189. [[CrossRef](#)]
21. Tempesta, P. Group entropies, correlation laws, and zeta functions. *Phys. Rev. E* **2011**, *84*, 021121. [[CrossRef](#)] [[PubMed](#)]
22. Livadiotis, G.; McComas, D.J. Understanding Kappa distributions: A toolbox for space science and astrophysics. *Space Sci. Rev.* **2013**, *175*, 183–214. [[CrossRef](#)]
23. Ruiz, G.; Tirnakli, U.; Borges, E.P.; Tsallis, C. Statistical characterization of discrete conservative systems: The web map. *Phys. Rev. E* **2017**, *96*, 042158. [[CrossRef](#)] [[PubMed](#)]
24. Rodríguez, A.; Nobre, F.D.; Tsallis, C.  $d$ -Dimensional classical Heisenberg model with arbitrarily-ranged interactions: Lyapunov exponents and distributions of momenta and energies. *Entropy* **2019**, *21*, 31. [[CrossRef](#)]
25. Conroy, J.M.; Miller, H.G. Determining the Tsallis parameter via maximum entropy. *Phys. Rev. E* **2015**, *91*, 052112. [[CrossRef](#)]
26. Asgarani, S. Families of Fokker-Planck equations and the associated entropic form for a distinct steady-state probability distribution with a known external force field. *Phys. Rev. E* **2015**, *91*, 022104. [[CrossRef](#)]
27. Nobre, F.D.; Curado, E.M.F.; Souza, A.M.C.; Andrade, R.F.S. Consistent thermodynamic framework for interacting particles by neglecting thermal noise. *Phys. Rev. E* **2015**, *91*, 022135. [[CrossRef](#)]
28. Ribeiro, M.S.; Nobre, F.D. Repulsive particles under a general external potential: Thermodynamics by neglecting thermal noise. *Phys. Rev. E* **2016**, *94*, 022120. [[CrossRef](#)] [[PubMed](#)]

29. Czégel, D.; Balogh, S.; Pollner, P.; Palla, G. Phase space volume scaling of generalized entropies and anomalous diffusion scaling governed by corresponding non-linear Fokker-Planck equations. *Sci. Rep.* **2018**, *8*, 1883. [[CrossRef](#)] [[PubMed](#)]
30. Lenzi, E.K.; Lenzi, M.K.; Ribeiro, H.V.; Evangelista, L.R. Extensions and solutions for nonlinear diffusion equations and random walks. *Proc. R. Soc. A* **2019**, *475*, 20190432. [[CrossRef](#)] [[PubMed](#)]
31. Fuentes, J.; López, J.L.; Obregón, O. Generalized Fokker-Planck equations derived from nonextensive entropies asymptotically equivalent to Boltzmann-Gibbs. *Phys. Rev. E* **2020**, *102*, 012118. [[CrossRef](#)] [[PubMed](#)]
32. Santos, L. Microscopic dynamics of nonlinear Fokker-Planck equations. *Phys. Rev. E* **2021**, *103*, 032106. [[CrossRef](#)] [[PubMed](#)]
33. Weymouth, A.J.; Meuer, D.; Mutombo, P.; Wutscher, T.; Ondracek, M.; Jelinek, P.; Giessibl, F.J. Atomic Structure Affects the Directional Dependence of Friction. *Phys. Rev. Lett.* **2013**, *111*, 126103. [[CrossRef](#)]
34. Tramsen, H.T.; Gorb, S.N.; Zhang, H.; Manoonpong, P.; Dai, Z.; Heepe, L. Inversion of friction anisotropy in a bio-inspired asymmetrically structured surface. *J. R. Soc. Interface* **2018**, *15*, 20170629. [[CrossRef](#)]
35. Plastino, A.R.; Wedemann, R.S.; Curado, E.M.F.; Nobre, F.D.; Tsallis, C. Nonlinear drag forces and the thermostatics of overdamped motion. *Phys. Rev. E* **2018**, *98*, 012129. [[CrossRef](#)] [[PubMed](#)]
36. Wedemann, R.S.; Plastino, A.R. Thermostatics of overdamped motion with anisotropic drag forces. *Eur. Phys. J. Spec. Top.* **2020**, *229*, 809–818. [[CrossRef](#)]