Compactifications of rational maps, and the implicit equations of their images

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ABSTRACT

In this paper, we give different compactifications for the domain and the codomain of an affine rational map f which parameterizes a hypersurface. We show that the closure of the image of this map (with possibly some other extra hypersurfaces) can be represented by a matrix of linear syzygies. We compactify A^{n-1} into an $(n-1)$-dimensional projective arithmetically Cohen–Macaulay subscheme of some P^N. One particular interesting compactification of A^{n-1} is the toric variety associated to the Newton polytope of the polynomials defining f. We consider two different compactifications for the codomain of f: P^n and $(P^1)^n$. In both cases we give sufficient conditions, in terms of the nature of the base locus of the map, for getting a matrix representation of its closed image, without involving extra hypersurfaces. This constitutes a direct generalization of the corresponding results established by Laurent Busé and Jean-Pierre Jouanolou (2003) [12], Laurent Busé et al. (2009) [9], Laurent Busé and Marc Dohm (2007) [11], Nicolás Botbol et al. (2009) [5] and Nicolás Botbol (2009) [4].

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1. Introduction

The interest in computing explicit formulas for resultants and discriminants goes back to Bézout, Cayley, Sylvester and many others in the nineteenth century. It has increased in recent years due to the increase of computing power. Resultants have been related to the implicitization of given rational maps; in turn, both resultants and discriminants can be seen as the implicit equation of a suitable map (see [18]). Recently, rational maps have appeared in computer-engineering contexts, mostly applied to shape modeling using computer-aided design methods for curves and surfaces.

Rational algebraic curves and surfaces can be described in several different ways, the most common being parametric and implicit representations. Parametric representations describe the geometric object as the image of a rational map, whereas implicit representations describe it as the set of points verifying a certain algebraic condition, e.g. as the zeros of a polynomial equation. Both representations have a wide range of applications in Computer-Aided Geometric Design (CAGD), and depending on the problem one needs to solve, one or the other might be better suited. It is thus interesting to be able to pass from parametric representations to implicit equations. This is a classical problem, and there are numerous approaches to its solution; see [26] and [17] for a good historical overview. However, it turns out that the implicitization problem is computationally difficult.

A promising alternative suggested in [11] is to compute a so-called *matrix representation* instead, which is easier to compute but still shares some of the advantages of the implicit equation. For a given hypersurface $\mathscr{H} \subset P^n$, a matrix M with entries in the polynomial ring $k[X_0, \dots, X_n]$ is called a *representation matrix* of $\mathscr{H}$ if it is generically of full rank and if the rank of M evaluated in a point of P^n drops if and only if the point lies on $\mathscr{H}$ (see [5]). Equivalently, a matrix M represents

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$\mathscr{H}$ if and only if the greatest common divisor (gcd) of all its minors of maximal size is a power of the homogeneous implicit equation $F \in \boldsymbol{k}[X_0, \ldots, X_n]$ of $\mathscr{H}$.

In the case of a planar rational curve $\mathcal{C}$ given by a parameterization of the form $\boldsymbol{A}^1 \xrightarrow{\ f\ } \boldsymbol{A}^2$, $s \mapsto \left(\frac{f_1(s)}{f_3(s)}, \frac{f_2(s)}{f_3(s)}\right)$, where $f_i \in \boldsymbol{k}[s]$ are coprime polynomials of degree d and $\boldsymbol{k}$ is a field, a linear syzygy (or moving line) is a linear relation on the polynomials f_1, f_2, f_3, i.e. a linear form $L = h_1 X_1 + h_2 X_2 + h_3 X_3$ in the variables X_1, X_2, X_3 and with polynomial coefficients $h_i \in \boldsymbol{k}[s]$ such that $\sum_{i=1,2,3} h_i f_i = 0$. We denote by $\mathrm{Syz}(f)$ the set of all those linear syzygy forms and for any integer ν the graded part $\mathrm{Syz}(f)_\nu$ of syzygies of degree at most ν. To be precise, one should homogenize the f_i with respect to a new variable and consider $\mathrm{Syz}(f)$ as a graded module here. It is obvious that $\mathrm{Syz}(f)_\nu$ is a finite-dimensional $\boldsymbol{k}$-vector space with basis $(L_1, \ldots, L_k)$ $(k = k(\nu))$ obtained by solving a linear system. If $L_i = \sum_{|\alpha|=\nu} s^\alpha L_{i,\alpha}(X_1, X_2, X_3)$, we define the matrix $M_\nu = (L_{i,\alpha})_{1 \le i \le k, |\alpha|=\nu}$, that is, the coefficients of the L_i with respect to a $\boldsymbol{k}$-basis of $\boldsymbol{k}[s]_\nu$ form the columns of the matrix. Note that the entries of this matrix are linear forms in the variables X_1, X_2, X_3 with coefficients in the field $\boldsymbol{k}$. Let F denote the homogeneous implicit equation of the curve and $\deg(f)$ the degree of the parameterization as a rational map. Intuitively, $\deg(f)$ measures how many times the curve is traced. It is known that, for $\nu \ge d-1$, the matrix M_ν is a representation matrix; more precisely, if $\nu = d-1$, then M_ν is a square matrix, such that $\det(M_\nu) = F^{\deg(f)}$. Also, if $\nu \ge d$, then M_ν is a non-square matrix with more columns than rows, such that the greatest common divisor of its minors of maximal size equals $F^{\deg(f)}$. In other words, one can always represent the curve as a square matrix of linear syzygies. One could now actually calculate the implicit equation.

For surfaces, matrix representations have been studied in the recent article [5] for the case of two-dimensional projective toric varieties. Previous works had been done in this direction, with two main approaches: one allows the use of quadratic syzygies (or higher-order syzygies) in addition to the linear syzygies, in order to be able to construct square matrices, and the other one only uses linear syzygies as in the curve case and obtains non-square representation matrices.

The first approach using linear and quadratic syzygies (or moving planes and quadrics) has been treated in [15] for base-point-free homogeneous parameterizations and some genericity assumptions, when $\mathscr{T} = \boldsymbol{P}^2$. The authors of [10] also treat the case of toric surfaces in the presence of base points. In [1], square matrix representations of bihomogeneous parameterizations, i.e. $\mathscr{T} = \boldsymbol{P}^1 \times \boldsymbol{P}^1$, are constructed with linear and quadratic syzygies, whereas [21] gives such a construction for parameterizations over toric varieties of dimension 2. The methods using quadratic syzygies usually require additional conditions on the parameterization, and the choice of the quadratic syzygies is often not canonical.

The second approach, even though it does not produce square matrices, has certain advantages, in particular in the sparse setting that we present. In previous publications, this approach with linear syzygies, which relies on the use of the so-called approximation complexes, has been developed in the case $\mathscr{T} = \boldsymbol{P}^2$ (see for example [12,8] and [14]), and in [11] for bihomogeneous parameterizations of degree (d, d). However, for a given affine parameterization f, these two varieties are not necessarily the best choice of a compactification, since they do not always reflect well the combinatorial structure of the polynomials $f_1, \ldots, f_4$. The method was extended to a much larger class of varieties, namely toric varieties of dimension 2 (see [5]), where it was shown that it is possible to choose a "good" toric compactification of $(\boldsymbol{A}^*)^2$ depending on the polynomials $f_1, \ldots, f_4$, which makes the method applicable in cases where it failed over $\boldsymbol{P}^2$ or $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, and also, that it is significantly more efficient, leading to smaller representation matrices.

In this article, we extend the method of computing an implicit equation of a parameterized hypersurface focusing on different compactifications of the domain $\mathscr{T}$ and of the codomain, $\boldsymbol{P}^n$ and $(\boldsymbol{P}^1)^n$. Hereafter we will always assume that $\mathscr{T}$ is embedded in $\boldsymbol{P}^N$, and its coordinate ring A is n-dimensional, graded and Cohen–Macaulay.

In Section 3, we focus on the implicitization problem for a rational map $\varphi : \mathscr{T} \dashrightarrow \boldsymbol{P}^n$ defined by $n+1$ polynomials of degree d. We extend the method for projective two-dimensional toric varieties developed in [5] to a map defined over an $(n-1)$-dimensional arithmetically Cohen–Macaulay closed scheme $\mathscr{T}$ embedded in $\boldsymbol{P}^N$. We show that we can relax the hypotheses on the base locus by allowing it to be a zero-dimensional almost locally complete intersection scheme.

In order to consider more general parameterizations given by rational maps of the form $f = \left(\frac{f_1}{g_1}, \ldots, \frac{f_n}{g_n}\right)$ with different denominators $g_1, \ldots, g_n$, we develop in Section 4 the study of the $(\boldsymbol{P}^1)^n$ compactification of the codomain. With this approach, we generalize, in the spirit of [4], the method of implicitization of projective hypersurfaces embedded in $(\boldsymbol{P}^1)^n$ to general hypersurfaces parameterized by any $(n-1)$-dimensional arithmetically Cohen–Macaulay closed subscheme of $\boldsymbol{P}^N$. As in the previously mentioned works, we compute the implicit equation as the determinant of a complex which coincides with the gcd of the maximal minors of the last matrix of the complex, and we give an in-depth study of the geometry of the base locus.

Section 5 is devoted to the algorithmic approach of both cases studied in Sections 3 and 4. We show how to compute the dimension of the representation matrices obtained in both cases by means of the Hilbert functions of the ring A and its Koszul cycles. In the last part of this section, we show, for the case of toric parameterizations given from a polytope $\mathcal{N}(f)$ (see 1), how the interplay between the multiples of $\mathcal{N}(f)$ and the degree of the maps may lead to having smaller matrices.

We conclude in Section 6 by giving several examples. First, we show in a very sparse setting the advantage of not considering the homogeneous compactification of the domain when the denominators are very different. In the second example, we extend this idea to the case of a generic affine rational map in dimension 2 with fixed Newton polytope. In the last example, we give, for a parameterized toric hypersurface of $(\boldsymbol{P}^1)^n$, a detailed analysis of the relation between the nature of the base locus of a map and the extraneous factors appearing in the computed equation.

2. General setting

Throughout this section we will give a general setting for the implicitization problem of hypersurfaces. Our aim is to analyze how far these techniques from homological commutative algebra (syzygies and graded resolutions) can be applied.

Write $A^k := \text{Spec}(k[T_1, \ldots, T_k])$ for the k-dimensional affine space over k. Assume that we are given a rational map

$$f : A^{n-1} \dashrightarrow A^n : s := (s_1, \ldots, s_{n-1}) \mapsto \left(\frac{f_1}{g_1}, \ldots, \frac{f_n}{g_n} \right)(s), \tag{1}$$

where $\deg(f_i) = d_i$ and $\deg(g_i) = e_i$ are without common factors. Observe that this setting is general enough to include all classical implicitization problems. Typically, all g_i are assumed to be equal, and a few conditions on the degrees are needed, depending on the context.

We consider a rational map $\psi : \mathscr{T} \dashrightarrow \mathscr{K}$, where $\mathscr{T}$ and $\mathscr{K}$ are suitable compactifications of a suitable dense open subset of A^{n-1} and A^n, respectively, in such a way that the map f extends through $\mathscr{T}$ and $\mathscr{K}$ via ψ and that the closed image of f can be recovered from the closed image of ψ.

Assume that $\mathscr{T}$ can be embedded in some P^N, and set A for the homogeneous coordinate ring of $\mathscr{T}$. Since A^{n-1} is irreducible, so is $\mathscr{T}$; hence A is a domain. Assume also that $\mathscr{T}$ defines via ψ a hypersurface in $\mathscr{K}$; hence, $\ker(\psi^*)$ is a principal ideal, generated by the implicit equation.

Most of our results are stated for a general arithmetically Cohen–Macaulay scheme as the domain. Nevertheless, the map (1) gives rise, naturally, to a toric variety $\mathscr{T}$ on the domain (see [21, Section 2], [16], and [19, Ch. 5 & 6]) associated to the following polytope $\mathscr{N}(f)$.

Definition 1. Let f denote a map as in Eq. (1). We will write

$$\mathscr{N}(f) := \text{conv} \left(\bigcup_{i=1}^n \left(\mathscr{N}(f_i) \cup \mathscr{N}(g_i) \right) \right)$$

for the convex hull of the union of the Newton polytopes of all the polynomials defining the map f.

There is a standard way of associating a semigroup $S_{\mathscr{N}}$ to a polytope $\mathscr{N} \subset R^{n-1}$: take $\iota : R^{n-1} \hookrightarrow R^n : x \mapsto (x, 1)$, and define $S_{\mathscr{N}}$ as the semigroup generated by the lattice points in $\iota(\mathscr{N})$. Due to a theorem of Hochster, if $S_{\mathscr{N}}$ is normal then the semigroup algebra $k[S_{\mathscr{N}}]$ is Cohen–Macaulay. Unluckily, it turns out that $S_{\mathscr{N}}$ is in general not always normal. A geometric or combinatorial characterization of the normality of $k[S_{\mathscr{N}}]$ is one of the most important open problems in combinatorial algebra (see [6]).

Note that $m\mathscr{N} \times \{m\} = \{(p_1 + \cdots + p_m, m) : p_i \in \mathscr{N}\} \subset S_{\mathscr{N}} \cap (Z^{n-1} \times \{m\})$ for any $m \in N$, but in general these two sets are not equal. When this happens for all $m \in N$, we say that the polytope $\mathscr{N}$ is normal; equivalently, $(m \cdot \mathscr{N}) \cap Z^{n-1} = m \cdot (\mathscr{N} \cap Z^{n-1})$ for all $m \in N$, and in this case it follows that $k[S_{\mathscr{N}}]$ is Cohen–Macaulay.

In this article, we focus on the study of toric varieties by fixing an embedding. Changing $\mathscr{N}$ by a multiple $l \cdot \mathscr{N}$ changes the embedding; hence, we will fix the polytope. Since we also need Cohen–Macaulayness of the quotient ring by the corresponding toric ideal in several results, we will assume throughout that $\mathscr{N}$ is normal.

Remark 2. Given a map f as in Eq. (1), we will always assume that $\mathscr{N} := \mathscr{N}(f)$ is normal. Therefore, the coordinate ring A of $\mathscr{T}$ will be always Cohen–Macaulay; hence $\mathscr{T} \subset P^N$ will be *arithmetically Cohen–Macaulay (aCM)*. This is automatic when $n = 2$.

The polytope $\mathscr{N}(f)$ defines an $(n-1)$-dimensional projective toric variety $\mathscr{T}$ provided with an ample line bundle which defines an embedding: if $N = \#(\mathscr{N}(f) \cap Z^{n-1}) - 1$ we have $\mathscr{T} \subseteq P^N$ (see [16]). Write $\mathscr{N}'(f)$ for the smallest lattice contraction of $\mathscr{N}(f)$ (that is $\mathscr{N}(f) = d\mathscr{N}'(f)$ where $\mathscr{N}'(f)$ is a lattice polytope and d is maximal with this property), and ρ for the embedding determined by this ample sheaf. We get that the map

$$(A^*)^{n-1} \overset{\rho}{\hookrightarrow} P^{N'} : (s) \mapsto (\ldots : s^\alpha : \ldots), \tag{2}$$

where $\alpha \in \mathscr{N}'(f) \cap Z^{n-1}$ and $N' = \#(\mathscr{N}'(f) \cap Z^{n-1}) - 1$, which factorizes f through a rational map $\psi : \mathscr{T} \dashrightarrow \mathscr{K}$.

The main reason for considering projective toric varieties associated to the Newton polytope $\mathscr{N}(f)$ of f is based on the following fact.

Remark 3. Assume that f is as in Eq. (1), with $g_1 = \cdots = g_n$. Write $f_0 := g_i$ for all i. Assume also that all f_i are generic with Newton polytope $\mathscr{N}$, and hence write $\mathscr{N} := \mathscr{N}(f_i)$ for all i. Set $N := \#(\mathscr{N} \cap Z^{n-1}) - 1$ and let $\mathscr{T} \subset P^N$ be the toric variety associated to $\mathscr{N}$. Write $\phi : \mathscr{T} \dashrightarrow P^n : T \mapsto (h_0 : \cdots : h_n)$ for the map induced by f. Since the coefficients are generic, the point associated to these coefficients is not in $V(\text{Res}_{\mathscr{N}}(h_0, \ldots, h_n))$; here $V(\text{Res}_{\mathscr{N}}(h_0, \ldots, h_n))$ stands for the zero locus of the sparse resultant $\text{Res}_{\mathscr{N}}(h_0, \ldots, h_n)$ associated to $h_0, \ldots, h_n$. Hence, they have no common root in $\mathscr{T}$. Thus, ϕ has an empty base locus in $\mathscr{T}$.

If we take instead another lattice polytope $\tilde{\mathscr{N}}$ strictly containing $\mathscr{N}$, the f_i will not be generic relative to $\tilde{\mathscr{N}}$, and typically the associated map $\tilde{\phi}$ will have a non-empty base locus in the toric variety $\tilde{\mathscr{T}}$ associated to $\tilde{\mathscr{N}}$.

Henceforward, let $\mathscr{T}$ be any $(n-1)$-dimensional projective arithmetically Cohen–Macaulay scheme provided with an embedding ρ into some $\boldsymbol{P}^N$. Assume that $\mathscr{K} := \boldsymbol{P}^n$ or $\mathscr{K} := (\boldsymbol{P}^1)^n$. In both cases we consider the diagram

$$\mathscr{T} \xleftarrow{\pi_1} \mathscr{T} \times \mathscr{K} \xrightarrow{\pi_2} \mathscr{K},$$

where π_1 and π_2 are the natural projections. Since $\psi : \mathscr{T} \dashrightarrow \mathscr{K}$ is not, in principle, defined everywhere in $\mathscr{T}$, we set Ω for the open set of the definition of ψ. Precisely, we give the following definition.

Definition 4. Assume that $\mathscr{K} := \boldsymbol{P}^n$. In this setting, ψ will be denoted by φ. Hence, write $\varphi : \mathscr{T} \dashrightarrow \boldsymbol{P}^n$ given by $\boldsymbol{s} \mapsto (h_0 : \cdots : h_n)(\boldsymbol{s})$, then define the base locus of φ as the closed subscheme of $\mathscr{T}$:

$$X_{\boldsymbol{P}^n} := \operatorname{Proj}(A/(h_0, \ldots, h_n)).$$

Similarly, for $\mathscr{K} := (\boldsymbol{P}^1)^n$, ψ will be denoted by ϕ. Hence, write $\phi : \mathscr{T} \dashrightarrow (\boldsymbol{P}^1)^n$ given by $\boldsymbol{s} \mapsto (f_1 : g_1)(\boldsymbol{s}) \times \cdots \times (f_n : g_n)(\boldsymbol{s})$, then define the base locus of ϕ as the closed subscheme of $\mathscr{T}$:

$$X_{(\boldsymbol{P}^1)^n} := \operatorname{Proj}\left(A/\prod_i (f_i, g_i)\right).$$

In what follows, X will stand for $X_{\boldsymbol{P}^n}$ and $X_{(\boldsymbol{P}^1)^n}$, and will be understood, depending on the context. In any case, we call Ω the complement of the base locus, namely $\Omega := \mathscr{T} \setminus X$. Let Γ_Ω be the graph of ϕ or φ into $\subset \Omega \times \mathscr{K}$.

Clearly $\Gamma_\Omega \xrightarrow{\pi_1} \Omega$ is birational, which is in general not the case over X. As was shown in [4], the scheme structure of the base locus when $X = X_{(\boldsymbol{P}^1)^n}$ can be fairly complicated, and extraneous factors may occur when projecting on $(\boldsymbol{P}^1)^n$ via π_2 (see Section 4.2). This motivates the need for a splitting of the base locus, giving rise to families of multiprojective bundles over $\mathscr{T}$.

Due to this important difference between the projective and multiprojective cases, we need to separate the study of the two settings. In the next section, we treat the case $\mathscr{K} := \boldsymbol{P}^n$, and in Section 4 the case $\mathscr{K} := (\boldsymbol{P}^1)^n$. In both situations, we find a matrix representation of the closed image of φ and ϕ, and we compute the implicit equation and extraneous factors that occur.

3. The case $\mathscr{K} = \boldsymbol{P}^n$

In this section, we focus on the computation of the implicit equation of a hypersurface in $\boldsymbol{P}^n$, parameterized by an $(n-1)$-dimensional arithmetically Cohen–Macaulay (aCM) subscheme of some projective space $\boldsymbol{P}^N$. We generalize [12,9,5] amongst others, and we give a more general result on the acyclicity of the approximation complex of cycles, by relaxing the conditions on the base ring and on the base locus.

Henceforward, in this section, let $\mathscr{T}$ be an $(n-1)$-dimensional projective aCM closed scheme over a field $\boldsymbol{k}$, embedded in $\boldsymbol{P}_{\boldsymbol{k}}^N$, for some $N \in \boldsymbol{N}$. Write $A = \boldsymbol{k}[T_0, \ldots, T_N]/J$ for its CM coordinate ring, and J for the homogeneous defining ideal of $\mathscr{T}$. Set $\boldsymbol{T} := T_0, \ldots, T_N$ the variables in $\boldsymbol{P}^N$, and $\boldsymbol{X}$ the sequence $X_0, \ldots, X_n$ of variables in $\boldsymbol{P}^n$.

We denote by $\mathfrak{m} := A_+ = (\boldsymbol{T}) \subset A$, the maximal homogeneous irrelevant ideal of A.

Let φ be a finite map defined over a relative open set U in $\mathscr{T}$ defining a hypersurface in $\boldsymbol{P}^n$, e.g. $U = \Omega$:

$$\boldsymbol{P}^N \supset \mathscr{T} \xdashrightarrow{\varphi} \boldsymbol{P}^n : \boldsymbol{T} \mapsto (h_0 : \cdots : h_n)(\boldsymbol{T}), \tag{3}$$

where $h_0, \ldots, h_n$ are homogeneous elements of A of degree d. Set $\boldsymbol{h} := h_0, \ldots, h_n$. The map φ gives rise to a morphism of graded $\boldsymbol{k}$-algebras in the opposite sense:

$$\boldsymbol{k}[X_0, \ldots, X_n] \xrightarrow{\varphi^*} A : X_i \mapsto h_i(\boldsymbol{T}). \tag{4}$$

Since $\ker(\varphi^*)$ is a principal ideal in $\boldsymbol{k}[\boldsymbol{X}]$, write H for a generator. We proceed as in [12] or in [5] to get a matrix such that the gcd of its maximal minors gives $H^{\deg(\phi)}$, or possibly, a multiple of it.

Remark 5. Observe that, if we start with an affine setting as in (1), $\mathscr{T} \subset \boldsymbol{P}^N$ can be taken as the embedded toric variety associated to $\mathscr{N}'(f)$. In the classical implicitization problem, it is common to suppose that $g_i = g_j$ for all i and j, and $\deg(f_i) = \deg(g_i) = d$ for all i. Hence write f_0 for any of the g_i. This setting naturally gives rise to a homogeneous compactification of the codomain, defined by the embedding

$$\boldsymbol{A}^n \xhookrightarrow{j} \boldsymbol{P}^n : \boldsymbol{x} \mapsto (1 : \boldsymbol{x}). \tag{5}$$

It is clear that, for $f_0, \ldots, f_n$ taken as above, the map $f : \boldsymbol{A}^{n-1} \dashrightarrow \boldsymbol{A}^n$ of Eq. (1) compactifies via ρ and j to $\varphi : \mathscr{T} \dashrightarrow \boldsymbol{P}^n$. It is important to note that $\overline{\operatorname{im}(f)}$ can be obtained from $\overline{\operatorname{im}(\varphi)}$ and vice versa, via the classical (first variable) dehomogenization and homogenization, respectively. Finally, we want to give a matrix representation for a toric hypersurface of $\boldsymbol{P}^n$ given as the image of the toric rational map $\varphi : \mathscr{T} \dashrightarrow \boldsymbol{P}^n : \boldsymbol{T} \mapsto (h_0 : \cdots : h_n)(\boldsymbol{T})$.

Next, we introduce the homological machinery needed to deal with the computations of the implicit equations and the representation matrix of the hypersurface.

3.1. Homological algebra tools

For simplicity, we denote by T_i the classes of each variable in the quotient ring $A = \mathbf{k}[\mathbf{T}]/J$. Recall that A is canonically graded, each variable having weight 1. Let $I = (h_0, \ldots, h_n) \subset A$ be the ideal generated by the h_i's.

More precisely, we will see that the implicit equation of $\mathscr{S}$ can be recovered as the determinant of certain graded parts of the $\mathcal{Z}$-complex we define below. We denote by $\mathcal{Z}_\bullet$ the approximation complex of cycles associated to the sequence $h_0, \ldots, h_n$ over A (see [27]), as in Definition 6.

Consider the Koszul complex $(K_\bullet(\mathbf{h}, A), \delta_\bullet)$ associated to $h_0, \ldots, h_n$ over A and denote $Z_i = \ker(\delta_i)$, $B_i = \mathrm{im}(\delta_{i+1})$. It is of the form

$$K_\bullet(\mathbf{h}, A): \quad A[-(n+1)d] \xrightarrow{\delta_{n+1}} A[-nd]^{n+1} \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} A[-d]^{n+1} \xrightarrow{\delta_1} A,$$

where the differentials are matrices such that every non-zero entry is $\pm h_i$.

Write $K_i := \bigwedge^i A^{n+1}[-i \cdot d]$. Since $Z_i \subset K_i$, it keeps the shift in the degree. Note that with this notation the sequence

$$0 \to Z_i \to K_i \to B_{i-1} \to 0 \tag{6}$$

is exact graded, and no degree shift is needed.

We introduce new variables $X_0, \ldots, X_n$ with $\deg(X_i) = 1$. Since A is $\mathbf{N}$-graded, $A[\mathbf{X}]$ inherits a bigrading.

Definition 6. Denote by $\mathcal{Z}_i = Z_i[i \cdot d] \otimes_A A[\mathbf{X}]$ the ideal of cycles in $A[\mathbf{X}]$, and write $[-]$ for the degree shift in the variables T_i and $(-)$ the one in the X_i. The approximation complex of cycles $(\mathcal{Z}_\bullet(\mathbf{h}, A), \epsilon_\bullet)$, or simply $\mathcal{Z}_\bullet$, is the complex

$$\mathcal{Z}_\bullet(\mathbf{h}, A): \quad 0 \longrightarrow \mathcal{Z}_n(-n) \xrightarrow{\epsilon_n} \mathcal{Z}_{n-1}(-(n-1)) \xrightarrow{\epsilon_{n-1}} \cdots \xrightarrow{\epsilon_2} \mathcal{Z}_1(-1) \xrightarrow{\epsilon_1} \mathcal{Z}_0, \tag{7}$$

where the differentials $\epsilon_\bullet$ are obtained by replacing h_i by X_i for all i in the matrices of $\delta_\bullet$.

It is important to observe that $H_0(\mathcal{Z}_\bullet) = A[\mathbf{X}]/\mathrm{im}(\epsilon_1) \cong \mathrm{Sym}_A(I)$. Note that the degree shifts indicated in the complex above are with respect to the grading given by the X_i's, while the degree shifts with respect to the grading of A are already contained in our definition of the $\mathcal{Z}_i$'s. From now on, when we take the degree ν part of the approximation complex, denoted by $(\mathcal{Z}_\bullet)_\nu$, it should always be understood to be taken with respect to the grading induced by A.

Under certain conditions on the base locus of the map, this complex is acyclic and it provides a free $\mathbf{k}[\mathbf{X}]$-resolution of $(\mathrm{Sym}_A(I))_\nu$ for all ν. Hence, we focus on finding the acyclicity conditions for the complex $\mathcal{Z}_\bullet$. In this direction we have the following.

Lemma 7. *Let $m \geq n$ be non-negative integers, A be an m-dimensional graded Cohen–Macaulay ring and $I = (h_0, \ldots, h_n) \subset A$ be of codimension (hence depth) at least $n - 1$ with $\deg(h_i) = d$ for all i. Assume that $X := \mathrm{Proj}(A/I) \subset \mathscr{S}$ is locally defined by n equations (i.e. locally an almost complete intersection). Then $\mathcal{Z}_\bullet$ is acyclic.*

Proof. The proof follows ideas of [8, Lemma 2] and [11, Lemma 1]. Observe that the lemma is unaffected by an extension of the base field, so one may assume that $\mathbf{k}$ is infinite.

By [20, Theorem 12.9], we know that $\mathcal{Z}_\bullet$ is acyclic (resp. acyclic outside $V(\mathfrak{m})$) if and only if I is generated by a proper sequence (resp. X is locally defined by a proper sequence). Recall that a sequence $a_1, \ldots, a_n$ of elements in a commutative ring B is a *proper sequence* if $a_{i+1}H_j(a_1, \ldots, a_i; B) = 0$ for $i = 0, \ldots, n-1$ and $j > 0$, where the H_j's denote the homology groups of the corresponding Koszul complex.

By following the same argument of [8, Lemma 2], and since X is locally defined by n equations, one can choose $\tilde{h}_0, \ldots, \tilde{h}_n$ to be sufficiently generic linear combinations of the h_i's such that

1. $(\tilde{h}_0, \ldots, \tilde{h}_n) = (h_0, \ldots, h_n) \subset A$,
2. $\tilde{h}_0, \ldots, \tilde{h}_{n-2}$ is an A-regular sequence; hence $\tilde{h}_0 \ldots, \tilde{h}_{n-1}$ is a proper sequence in A,
3. $\tilde{h}_0, \ldots, \tilde{h}_{n-1}$ define X in codimension $n - 1$.

Note that this last condition is slightly more general (and coincides when $m = n$) than the one in [8, Lemma 2]. Set $J := (\tilde{h}_0, \ldots, \tilde{h}_{n-1})$ and write J^{um} for the unmixed part of J of codimension $n - 1$. Hence, observe that we obtain $\tilde{h}_n \in J^{um}$.

Since $\tilde{h}_n \in J^{um}$, we show that $\tilde{h}_n H_1(\tilde{h}_0, \ldots, \tilde{h}_{n-1}; A) = 0$. Applying [7, Thm. 1.6.16] to the sequence $\tilde{h}_0, \ldots, \tilde{h}_{n-1}$, we obtain that $H_1(\tilde{h}_0, \ldots, \tilde{h}_{n-1}; A) \cong \mathrm{Ext}_A^{n-1}(A/J, A)$. Taking the long exact sequence of $\mathrm{Ext}_A^\bullet(-, A)$ coming from the short exact sequence $0 \to J^{um}/J \to A/J \to A/J^{um} \to 0$, we get that

$$\cdots \longrightarrow \mathrm{Ext}_A^{n-2}(J^{um}/J, A) \longrightarrow \mathrm{Ext}_A^{n-1}(A/J, A) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}_A^{n-1}(A/J^{um}, A) \longrightarrow \mathrm{Ext}_A^{n-1}(J^{um}/J, A) \longrightarrow \cdots$$

is exact. Since A is a Cohen–Macaulay noetherian graded ring, and J^{um}/J is an $m - (n - 1)$-dimensional A-module, $\mathrm{Ext}_A^{n-1}(J^{um}/J, A)$ and $\mathrm{Ext}_A^{n-2}(J^{um}/J, A)$ vanish (see [24, Thm. 17.1]). Hence

$$\mathrm{Ext}_A^{n-1}(A/J, A) \cong \mathrm{Ext}_A^{n-1}(A/J^{um}, A);$$

thus, since $\tilde{h}_n \in J^{um}$, $\tilde{h}_n$ annihilates $\mathrm{Ext}_A^{n-1}(A/J^{um}, A)$, and hence also $\tilde{h}_n$ annihilates $H_1(\tilde{h}_0, \ldots, \tilde{h}_{n-1}; A)$, which finishes the proof. $\square$

We stress in the following remark one useful application of the previous Lemma 7.

Remark 8. Let $m \geq n$ be non-negative integers. Set $\mathscr{T}$ to be an arithmetically Cohen–Macaulay scheme over $\boldsymbol{k}$ embedded in some $\boldsymbol{P}^N$ with coordinate ring A of affine dimension m. Assume that we are given a rational map $\varphi : \mathscr{T} \dashrightarrow \boldsymbol{P}^n$ given by $n + 1$ homogeneous polynomials $h_0, \ldots, h_n \in A := \boldsymbol{k}[T_0, \ldots, T_N]/I(\mathscr{T})$. Write $\mathcal{Z}_\bullet$ for the approximation complex of cycles associated to the sequence $h_0, \ldots, h_n$. If the base locus of φ, $X \subset \mathscr{T}$, is locally defined by n equations, then $\mathcal{Z}_\bullet$ is acyclic, independent of m and N.

We translate Lemma 7 geometrically.

Corollary 9. *Assume $m = n$ to be a non-negative integer. Let $\mathscr{T}$ be an $(n - 1)$-dimensional CM closed subscheme of $\boldsymbol{P}^N$ defined by a homogeneous ideal J, and coordinate ring $A = \boldsymbol{k}[\boldsymbol{T}]/J$. Assume that we are given a rational map $\varphi : \mathscr{T} \dashrightarrow \boldsymbol{P}^n$ given by $n + 1$ homogeneous polynomials $h_0, \ldots, h_n \in A$ of degree d. Write $\mathcal{Z}_\bullet$ for the approximation complex of cycles associated to the sequence $h_0, \ldots, h_n$. If the base locus of φ, $X \subset \mathscr{T}$, is finite, and locally an almost complete intersection (defined by n equations), then $\mathcal{Z}_\bullet$ is acyclic.*

The following result establishes a vanishing criterion on the graded strands of the local cohomology of $\mathrm{Sym}_A(I)$, which ensures that the implicit equation can be obtained as a generator of the annihilator of the symmetric algebra in that degree.

Since A is a finitely generated graded Cohen–Macaulay A-module of dimension n, $H_\mathfrak{m}^i(A) = 0$ for all $i \neq n$ and $H_\mathfrak{m}^n(A) = \omega_A^\vee$, where $(-)^\vee := {}^*\mathrm{hom}_A(-, \boldsymbol{k})$ stands for the Matlis dualizing functor (see [7]). Write $a_i(M) := \inf\{\mu : (H_\mathfrak{m}^i(M))_{>\mu} = 0\}$. Hence, we set $\gamma := a_n(A) = \inf\{\mu : (\omega_A^\vee)_\mu = 0\}$, and we conclude the following result.

Theorem 10. *Let $A = \boldsymbol{k}[\boldsymbol{T}]/J$ be a CM graded ring of dimension n. Let $I = (h_0, \ldots, h_n)$ be a homogeneous ideal of A, with $\deg(h_i) = d$ for all i. Let $X := \mathrm{Proj}(A/I) \subset \mathscr{T}$ be finite and locally an almost complete intersection. Set $\nu_0 := \max\{(n - 2)d, (n - 1)d - \gamma\}$; then $H_\mathfrak{m}^0(\mathrm{Sym}_A(I))_\nu = 0$ for all $\nu \geq \nu_0$.*

Proof. For the bound on ν, consider the two spectral sequences associated to the double complex $C_\mathfrak{m}^\bullet(\mathcal{Z}_\bullet)$, both converging to the hypercohomology of $\mathcal{Z}_\bullet$. The first spectral sequence stabilizes at step two with

$$'_\infty E_q^p = {}_2' E_q^p = H_\mathfrak{m}^p(H_q(\mathcal{Z}_\bullet)) = \begin{cases} H_\mathfrak{m}^p(\mathrm{Sym}_A(I)) & \text{for } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The second has first terms $''_1 E_q^p = H_\mathfrak{m}^p(Z_q)[qd] \otimes_A A[\boldsymbol{X}](-q)$. The comparison of the two spectral sequences shows that $H_\mathfrak{m}^0(\mathrm{Sym}_A(I))_\nu$ vanishes as soon as $({}_1'' E_p^p)_\nu$ vanishes for all p; in fact, we have that

$$\mathrm{end}(H_\mathfrak{m}^0(\mathrm{Sym}_A(I))) \leq \max_{p \geq 0}\{\mathrm{end}({}_1'' E_p^p)\} = \max_{p \geq 0}\{\mathrm{end}(H_\mathfrak{m}^p(Z_p)) - pd\},$$

where we denote, for an A-module M, $\mathrm{end}(M) = \max\{\nu \mid M_\nu \neq 0\}$. Since $Z_0 \cong A$, we get $H_\mathfrak{m}^0(Z_0) = 0$. The sequence $0 \to Z_{i+1} \to K_{i+1} \to B_i \to 0$ is graded exact (see Eq. (6)); hence, from the long exact sequence of local cohomology for $i = 0$ (writing $B_0 = I$), we obtain

$$\cdots \to H_\mathfrak{m}^0(I) \to H_\mathfrak{m}^1(Z_1) \to H_\mathfrak{m}^1(K_1) \to \cdots.$$

As I is an ideal of an integral domain, $H_\mathfrak{m}^0(I) = 0$, it follows from the local cohomology of A that $H_\mathfrak{m}^1(K_1) = 0$; hence $H_\mathfrak{m}^1(Z_1)$ vanishes. By construction, $Z_{n+1} = 0$, and $B_n = \mathrm{im}(d_n) \simeq A[-d]$. Using the fact that $H_\mathfrak{m}^n(A)_\nu = 0$ for $\nu \geq -1$ (resp. $\nu \geq 0$), we can deduce that $H_\mathfrak{m}^n(Z_n)_\nu = H_\mathfrak{m}^n(B_n)_\nu = (\omega_A^\vee)[d] = 0$ if $\nu \geq d - \gamma$. Write

$$\epsilon_p := \mathrm{end}({}_1'' E_p^p) = \mathrm{end}(H_\mathfrak{m}^p(Z_p)) - pd.$$

By [13, Cor. 6.2.v], $\mathrm{end}(H_\mathfrak{m}^p(Z_p)) \leq \max_{0 \leq i \leq n-p}\{a_{p+i}(A) + (p+i+1)d\} = \max\{nd, (n+1)d - \gamma\}$, where $\gamma := -a_n(A)$, as above. Hence, $\epsilon_p := \max\{(n - p)d, (n + 1 - p)d - \gamma\}$. As ϵ_p decreases when p increases, $\epsilon_p \leq \epsilon_2 = \max\{(n - 2)d, (n - 1)d - \gamma\}$, which completes the proof. $\square$

This generalizes [12,9,5] to general $(n - 1)$-dimensional arithmetically Cohen–Macaulay schemes. Next, we recall how the homological tools developed in this part are applied for computing the implicit equation of the closed image of a rational map.

3.2. The representation matrix, the implicit equation, and the extraneous factor

It is well known that the annihilator above can be computed as the determinant (or MacRae invariant) of the complex $(\mathcal{Z}_\bullet)_{v_0}$ (see [12,9,4,5] amongst others). Hence, the determinant of the complex $(\mathcal{Z}_\bullet)_{v_0}$ is a multiple of a power of the implicit equation of $\mathcal{S}$. Indeed, we conclude the following result.

Lemma 11. *Let $\mathcal{T}$ be an $(n-1)$-dimensional CM closed subscheme of $\boldsymbol{P}^N$ defined by a homogeneous ideal J, and coordinate ring $A = \boldsymbol{k}[\boldsymbol{T}]/J$. Let $I = (h_0, \ldots, h_n)$ be a homogeneous ideal of A, with $\deg(h_i) = d$ for all i. Let $X := \operatorname{Proj}(A/I) \subset \mathcal{T}$ be finite and locally an almost complete intersection. Set $v_0 := \max\{(n-2)d, (n-1)d - \gamma\}$; then $H^0_{\mathfrak{m}}(\operatorname{Sym}_A(I))_v = 0$, for all $v \geq v_0$, and $\operatorname{ann}_{\boldsymbol{k}[X]}(\operatorname{Sym}_A(I)_v) \subset \ker(\varphi^*)$, for all $v \geq v_0$.*

Proof. The first part follows from 10. The proof of the second part can be taken verbatim from [11, Lemma 2]. $\square$

Corollary 12. *Let $\mathcal{T}$ be an $(n-1)$-dimensional CM closed subscheme of $\boldsymbol{P}^N$ defined by a homogeneous ideal J, and coordinate ring $A = \boldsymbol{k}[\boldsymbol{T}]/J$. Let $I = (h_0, \ldots, h_n)$ be an homogeneous ideal of A, with $\deg(h_i) = d$ for all i. Let $X := \operatorname{Proj}(A/I) \subset \mathcal{T}$ be finite and locally almost a complete intersection. Let v_0 be as above. For any integer $v \geq v_0$, the determinant D of the complex $(\mathcal{Z}_\bullet)_v$ of $\boldsymbol{k}[X]$-modules defines (up to multiplication with a constant) the same non-zero element in $\boldsymbol{k}[X]$. Moreover, $D = F^{\deg(\varphi)}G$, where F is the implicit equation of $\mathcal{S}$.*

Proof. It follows from Lemmas 7 and 11, and Theorem 14, by following the same lines of the proof of [12, Thm. 5.2]. $\square$

By [19, Appendix A], the determinant D can be computed either as an alternating product of subdeterminants of the differentials in $(\mathcal{Z}_\bullet)_v$ or as the greatest common divisor of the maximal-size minors of the matrix M associated to the rightmost map $(\mathcal{Z}_1)_v \to (\mathcal{Z}_0)_v$ of the $\mathcal{Z}$-complex (see Definition 6). Note that this matrix is nothing other than the matrix M_v of linear syzygies as described in the Introduction; it can be computed with the same algorithm as in [11] or [5]. Hence, if $\mathcal{T} \overset{\varphi}{\dashrightarrow} \boldsymbol{P}^n$ is as in Corollary 12, the matrix M_v of linear syzygies of $h_0, \ldots, h_n$ in degree $v \geq v_0$ is a representation matrix for the closed image of φ.

As was done by Busé et al. in [9, Section 2], we conclude that the extraneous factor G can be described in terms of linear forms.

Proposition 13. *If the field $\boldsymbol{k}$ is algebraically closed and X is locally generated by at most n elements, then there exist linear forms $L_x \in \boldsymbol{k}[X]$, and integers e_x and d_x such that*

$$G = \prod_{x \in X} L_x^{e_x - d_x} \in \boldsymbol{k}[X].$$

Moreover, if we identify x with the prime ideal in $\operatorname{Spec}(A)$ defining the point x, e_x is the Hilbert–Samuel multiplicity $e(I_x, A_x)$, and $d_x := \dim_{A_x/xA_x}(A_x/I_x)$.

Proof. The proof goes along the same lines of [9, Prop. 5]: observe that [9, Lemma 6] is stated for a Cohen–Macaulay ring as is A in our case. $\square$

4. The case $\mathcal{K} = (\boldsymbol{P}^1)^n$

The aim of this section is to compute the implicit equation of a hypersurface in $(\boldsymbol{P}^1)^n$. There is also a big spectrum of problems that are not well adapted to taking a common denominator. Typically this process enlarges the base locus of f, which could imply having a worse compactification of the domain or an embedding into a bigger projective space. This may also increase considerably the degree of the new maps f_i, and hence the degree of f, which is harmful for the algorithmic approach.

Here we generalize the work in [4]. Hereafter in this section, let $\mathcal{T}$ be an $(n-1)$-dimensional projective arithmetically Cohen–Macaulay closed scheme over a field $\boldsymbol{k}$, embedded in $\boldsymbol{P}^N_{\boldsymbol{k}}$, for some $N \in \boldsymbol{N}$. Write $A = \boldsymbol{k}[T_0, \ldots, T_N]/J$ for its CM graded coordinate ring, and let J denote the homogeneous defining ideal of $\mathcal{T}$. Set $\boldsymbol{T} := T_0, \ldots, T_N$ the variables in $\boldsymbol{P}^N$, and $\boldsymbol{X}$ the sequence $X_1, Y_1, \ldots, X_n, Y_n$, of variables in $(\boldsymbol{P}^1)^n$. Write $\mathfrak{m} := A_+ = (\boldsymbol{T}) \subset A$ for the maximal irrelevant homogeneous ideal of A.

Let ϕ be a finite map over a relative open set U of $\mathcal{T}$ defining a hypersurface in $\boldsymbol{P}^n$:

$$\boldsymbol{P}^N \supset \mathcal{T} \overset{\phi}{\dashrightarrow} (\boldsymbol{P}^1)^n : \boldsymbol{T} \mapsto (f_1 : g_1) \times \cdots \times (f_n : g_n)(\boldsymbol{T}), \tag{8}$$

where f_i and g_i are homogeneous elements of A of degree d_i, for $i = 1, \ldots, n$. As in the section above, this map ϕ gives rise to a morphism of graded $\boldsymbol{k}$-algebras in the opposite sense:

$$\boldsymbol{k}[\boldsymbol{X}] \xrightarrow{\phi^*} A : X_i \mapsto f_i(\boldsymbol{T}), \ Y_i \mapsto g_i(\boldsymbol{T}). \tag{9}$$

Since $\ker(\phi^*)$ is a principal ideal in $\boldsymbol{k}[\boldsymbol{X}]$, write H for a generator. We proceed as in [4] to get a matrix such that the gcd of its maximal minors equals $H^{\deg(\phi)}$, or a multiple of it.

Assume that we are given a rational map like the one in (1) with $\deg(f_i) = \deg(g_i) = d_i, i = 1, \ldots, n$. Take $\mathscr{T} \subset \boldsymbol{P}^N$ as the toric embedding obtained from $\mathscr{N}'(f)$; see Definition 1. The multiprojective compactification is given by

$$\boldsymbol{A}^n \stackrel{\iota}{\hookrightarrow} (\boldsymbol{P}^1)^n : (x_1, \ldots, x_n) \mapsto (x_1 : 1) \times \cdots \times (x_n : 1). \tag{10}$$

As before, f compactifies via ρ through $\mathscr{T}$ to $\phi : \mathscr{T} \dashrightarrow (\boldsymbol{P}^1)^n$ as defined in (8).

When $\mathscr{T} = \boldsymbol{P}^{n-1}$, the relation between the tensor product of length-one approximation complexes and a Koszul complex $\mathcal{K}_\bullet$ was studied in [4] (see Section 3 in [4]). In that case, the implicit equation of ϕ is computed as a homogeneous resultant by means of a graded strand of $\mathcal{K}_\bullet$. We will recall the definition of $\mathcal{K}_\bullet$.

We associate to each pair of homogeneous polynomials f_i, g_i a linear form $L_i := Y_i f_i - X_i g_i$ in the ring $R := A[\boldsymbol{X}]$ of bidegree $(d, 1)$. Write $\mathcal{K}_\bullet$ for the Koszul complex $\mathcal{K}_\bullet(L_1, \ldots, L_n; R)$, associated to the sequence $L_1, \ldots, L_n$ and coefficients in R. The $\boldsymbol{N}^n$-graded $\boldsymbol{k}$-algebra $\mathscr{B} := \operatorname{coker}(\bigoplus_i R(-d_i, -1) \to R)$ is the multihomogeneous coordinate ring of the incidence scheme $\Gamma = \overline{\Gamma_\Omega}$. It can be easily observed that $\mathscr{B} \cong \bigotimes_A \operatorname{Sym}_A(I^{(i)}) \cong R/(L_1, \ldots, L_n)$.

We show that under certain conditions on the L_i, there exists an element in $\boldsymbol{k}[\boldsymbol{X}]$ that vanishes whenever $L_1, \ldots, L_n$ have a common root in $\mathscr{T}$. This polynomial coincides with the sparse resultant $\operatorname{Res}_{\mathscr{T}}(L_1, \ldots, L_n)$. We will see that it is not irreducible in general. In fact, it is not only a power of the implicit equation and it can also have some extraneous factors, while the generic sparse resultant is always irreducible. Those factors come from some components of the base locus of ϕ which are not necessarily a common root of all L_i: it is enough that one of them vanishes at some point p of $\mathscr{T}$ to obtain a base point of ϕ. We will give sufficient conditions for avoiding extraneous factors.

We compute the implicit equation of the closed image of ϕ as a factor of the determinant of $(\mathcal{K}_\bullet)_{(\nu,*)}$, for certain degree ν in the grading of A. As in [5], the last map of this complex of vector spaces is a matrix M_ν that represents the closed image of ϕ. Thus, we focus on the computation of the regularity of $\mathscr{B}$ in order to bound ν.

Theorem 14. *Suppose that A is Cohen–Macaulay and $\mathcal{K}_\bullet$ is acyclic. Then*

$$H_{\mathfrak{m}}^0(\mathscr{B})_\nu = 0 \text{ for all } \nu \geq \nu_0 = \left(\sum_i d_i\right) - \gamma,$$

where $\gamma := \max\{i \mid C_i \text{ contains no interior points}\}$.

Proof. Write $\mathcal{K}_q$ for the qth object in $\mathcal{K}_\bullet$. Consider the two spectral sequences associated to the double complex $C_{\mathfrak{m}}^\bullet(\mathcal{K}_\bullet)$, both converging to the hypercohomology of $\mathcal{K}_\bullet$. As $\mathcal{K}_\bullet$ is acyclic, the first spectral sequence stabilizes at the E_2-term. The second one has as the E_1-term ${}''_1E_q^p = H_{\mathfrak{m}}^p(\mathcal{K}_q)$.

Recall that, for an A-module M, $\operatorname{end}(M) = \max\{\nu \mid M_\nu \neq 0\}$. The comparison of the two spectral sequences shows that $H_{\mathfrak{m}}^0(\mathscr{B})_\nu$ vanishes as soon as $({}_1''E_p^p)_\nu$ vanishes for all p. In fact, we have

$$\operatorname{end}(H_{\mathfrak{m}}^0(\mathscr{B})) \leq \max_{p \geq 0}\{\operatorname{end}({}_1''E_p^p)\} = \max_{p \geq 0}\{\operatorname{end}(H_{\mathfrak{m}}^p(\mathcal{K}_p))\}.$$

It remains to observe that, since $\mathcal{K}_p = \bigoplus_{i_1,\ldots,i_p} A(-\sum_{j=1}^p d_{i_j}) \otimes_{\boldsymbol{k}} \boldsymbol{k}[\boldsymbol{X}](-p)$ and $\boldsymbol{k}[\boldsymbol{X}]$ is flat over $\boldsymbol{k}$,

$$\max_{p \geq 0}\{\operatorname{end}(H_{\mathfrak{m}}^p(\mathcal{K}_p))\} = \max_{p \geq 0}\left\{\max_{i_1,\ldots,i_p}\left\{\operatorname{end}\left(H_{\mathfrak{m}}^p\left(A\left(-\sum_{j=1}^p d_{i_j}\right)\right)\right)\right\}\right\}.$$

Hence, we have

$$\operatorname{end}(H_{\mathfrak{m}}^p(\mathcal{K}_p)) = \begin{cases} \operatorname{end}\left(H_{\mathfrak{m}}^n\left(\omega_A^\vee\left(-\sum_i d_i\right)\right)\right) & \text{for } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, since $(H_{\mathfrak{m}}^n(\omega_A^\vee))_\nu = 0$ for all $\nu \geq -\gamma$, we get

$$\operatorname{end}\left(H_{\mathfrak{m}}^n\left(\omega_A^\vee\left(-\sum_i d_i\right)\right)\right) = \operatorname{end}(H_{\mathfrak{m}}^n(\omega_A^\vee)) + \sum_i d_i < \sum_i d_i - \gamma. \quad \square$$

In order to compute the representation matrix $\mathit{\Xi}_\nu$ and the implicit equation of ϕ, we need to be able to get acyclicity conditions for $\mathcal{K}_\bullet$. A theorem due to Avramov (see [2]) gives necessary and sufficient conditions for $(L_1, \ldots, L_n)$ to be a regular sequence in R in terms of the depth of certain ideals of minors of the following matrix $\mathit{\Xi} := (m_{ij})_{i,j} \in Mat_{2n,n}(A)$:

$$\mathit{\Xi} = \begin{pmatrix} -g_1 & 0 & \cdots & 0 \\ f_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -g_n \\ 0 & 0 & \cdots & f_n \end{pmatrix}. \tag{11}$$

This matrix defines a map of A-modules $\psi : A^n \to \bigoplus_{i=0}^n A[x_i, y_i]_1 \cong A^{2n}$; we verify that the symmetric algebra $\text{Sym}_A(\text{coker}(\psi)) \cong A[\boldsymbol{X}]/(L_1, \ldots, L_n)$. Since $\text{Sym}_A(\text{coker}(\psi)) = \mathcal{B}$ is naturally multigraded, it can be seen as a subscheme of $\mathcal{T} \times (\boldsymbol{P}^1)^n$. This embedding is determined by the natural projection $A[\boldsymbol{X}] \to A[\boldsymbol{X}]/(L_1, \ldots, L_n)$. In fact, the graph of ϕ is an $(n-1)$-dimensional irreducible component of $\text{Proj}(\text{Sym}_A(\text{coker}(\psi))) \subset \mathcal{T} \times (\boldsymbol{P}^1)^n$ which is a projective fiber bundle outside the base locus of ϕ in $\mathcal{T}$.

Let $\varXi$ be a matrix as in (11). Write $I_r := I_r(\varXi)$ for the ideal of A generated by the $r \times r$ minors of $\varXi$, for $0 \le r \le r_0 := \min\{n+1, m\}$, and define $I_0 := A$ and $I_r := 0$ for $r > r_0$. Theorem 15 below relates both algebraic and geometric aspects. It gives conditions in terms of the ideals of minors I_r, for the complex to being acyclic, and on the equation given by the determinant of a graded branch for describing the closed image of ϕ.

4.1. The implicit equation

In this part, we gather together the facts about acyclicity of the complex $\mathcal{K}_\bullet$, and the geometric interpretation of the zeros of the ideals of minors I_r, and we show that under suitable hypotheses no extraneous factor occurs. In order to do this, we introduce some previous notation, following that of [4].

Denote by W the closed subscheme of $\mathcal{T} \subset \boldsymbol{P}^N$ given by the common zeros of all $2n$ polynomials, and write $I^{(i)}$ for the ideal (f_i, g_i) of A, and X for the base locus of ϕ, namely

$$W := \text{Proj}\left(A/\sum_i I^{(i)}\right), \quad \text{and} \quad X := \text{Proj}\left(A/\prod_i I^{(i)}\right). \tag{12}$$

Set $\alpha \subset [1, n]$, write $I^{(\alpha)} := \sum_{j \in \alpha} I^{(j)}$, and set $X_\alpha := \text{Proj}(A/I^{(\alpha)})$ and $U_\alpha := X_\alpha \setminus \bigcup_{j \notin \alpha} X_{\{j\}}$. If U_α is non-empty, consider $p \in U_\alpha$; then $\dim(\pi_1^{-1}(p)) = |\alpha|$. As the fiber over U_α is equidimensional by construction, write

$$\mathcal{E}_\alpha := \pi_1^{-1}(U_\alpha) \subset \mathcal{T} \times (\boldsymbol{P}^1)^n \tag{13}$$

for the fiber over U_α, which defines a multiprojective bundle of rank $|\alpha|$. Consequently,

$$\text{codim}(\mathcal{E}_\alpha) = n - |\alpha| + (\text{codim}_{\mathcal{T}}(U_\alpha)).$$

Recall from Definition 4 that Γ_Ω is the graph of ϕ, and set $\Gamma := \text{Biproj}(\mathcal{B})$, the incidence scheme of the linear forms L_i. We show in the following theorem that under suitable hypotheses $\Gamma = \overline{\Gamma_\Omega}$, and that $\pi_2(\Gamma) = \mathcal{H}$ is the implicit equation of the closed image of ϕ.

Theorem 15. *Let* $\phi : \mathcal{T} \dashrightarrow (\boldsymbol{P}^1)^n$ *be defined by the pairs* $(f_i : g_i)$, *not both being zero, as in* (8). *Write for* $i = 1, \ldots, n$, $L_i := f_i Y_i - g_i X_i$ *and* $\mathcal{B} := A[\boldsymbol{X}]/(L_1, \ldots, L_n)$. *Take* $\nu_0 = (\sum_i d_i) - \gamma$ *as in Theorem 14.*

1. *The following statements are equivalent:*
   (a) $\mathcal{K}_\bullet$ *is a free resolution of* $\mathcal{B}$;
   (b) $\text{codim}_A(I_r) \ge n - r + 1$ *for all* $r = 1, \ldots, n$;
   (c) $\dim(\bigcap_{\alpha \subset [1,n]}(V(\prod_{j \in \alpha} I^{(j)}))) \le |\alpha| - 2$.
2. *If any (all) of the items above are satisfied, then* M_ν *has generically maximal rank, namely* $\binom{n-1+\nu}{\nu}$. *Moreover, if for all* $\alpha \subset [1, n]$, $\text{codim}_A(I^{(\alpha)}) > |\alpha|$, *then*

   $$\det((\mathcal{K}_\bullet)_\nu) = \det(M_\nu) = H^{\deg(\phi)}, \quad \text{for } \nu \ge \nu_0,$$

   where $\det(M_\nu)$ *and* H *is the irreducible implicit equation of the closed image of* ϕ.

Proof. For proving the first part, we refer the reader to [4], where the case $\mathcal{T} = \boldsymbol{P}^{n-1}$ has been studied. It remains to observe that the only necessary condition over A is for it to be Cohen–Macaulay.

For proving the second part, the hypotheses have been taken in such a way that $\text{codim}_A(\sum_{j \in \alpha} I^{(j)}) > |\alpha|$, for all $\alpha \subset [1, n]$, which implies that $\text{codim}_{\mathcal{T}}(U_\alpha) > |\alpha|$; thus

$$\text{codim}(\mathcal{E}_\alpha) > n = \text{codim}(\Gamma_\Omega).$$

Set $\Gamma_U := \coprod_\alpha \mathcal{E}_\alpha$, and observe that $\Gamma \setminus \Gamma_U = \Gamma_\Omega$. Clearly, $\text{codim}(\Gamma_U) > n = \text{codim}(\Gamma_\Omega) = \text{codim}(\overline{\Gamma_\Omega})$.

Since $\text{Spec}(\mathcal{B})$ is a complete intersection, in $\boldsymbol{A}^{2n}$ it is unmixed and purely of codimension n. As a consequence, $\Gamma \ne \emptyset$ is also purely of codimension n. This together with the fact that $\text{codim}(\Gamma_U) > n$ implies that $\Gamma = \overline{\Gamma_\Omega}$. The graph Γ_Ω is irreducible; hence Γ as well, and its projection (the closure of the image of ϕ) is of codimension one.

It remains to observe that $\mathcal{K}_\bullet$ is acyclic, and $H_0(\mathcal{K}_\bullet) \cong \mathcal{B}$. Considering the homogeneous strand of degree $\nu > \eta$ we get the following chain of identities (see [22]):

$$\begin{aligned}
[\det((\mathcal{K}_\bullet)_\nu)] &= \text{div}_{\boldsymbol{k}[\boldsymbol{X}]}(H_0(\mathcal{K}_\bullet)_\nu) \\
&= \text{div}_{\boldsymbol{k}[\boldsymbol{X}]}(\mathcal{B}_\nu) \\
&= \sum_{\mathfrak{p} \text{ prime, } \text{codim}_{\boldsymbol{k}[\boldsymbol{X}]}(\mathfrak{p})=1} \text{length}_{\boldsymbol{k}[\boldsymbol{X}]_\mathfrak{p}}((\mathcal{B}_\nu)_\mathfrak{p})[\mathfrak{p}].
\end{aligned}$$

Our hypotheses were taken in such a way that only one prime occurs. Also, since

$$[\det((\mathcal{K}_\bullet)_\nu)] = \operatorname{div}_{\boldsymbol{k}[\boldsymbol{X}]}(\operatorname{Res}) = e \cdot [\mathfrak{q}],$$

for some integer e and $\mathfrak{q} := (H) \subset \boldsymbol{k}[\boldsymbol{X}]$, we have that

$$\sum_{\mathfrak{p} \text{ prime, } \operatorname{codim}(\mathfrak{p})=1} \operatorname{length}_{\boldsymbol{k}[\boldsymbol{X}]_\mathfrak{p}}((\mathcal{B}_\nu)_\mathfrak{p})[\mathfrak{p}] = e \cdot [\mathfrak{q}],$$

and so $[\det((\mathcal{K}_\bullet)_\nu)] = \operatorname{length}_{\boldsymbol{k}[\boldsymbol{X}]_\mathfrak{q}}((\mathcal{B}_\nu)_\mathfrak{q})[\mathfrak{q}]$. As Γ_Ω is irreducible, Γ is also irreducible, and we have

$$\operatorname{length}_{\boldsymbol{k}[\boldsymbol{X}]_\mathfrak{q}}((\mathcal{B}_\nu)_\mathfrak{q}) = \dim_{\kappa(\mathfrak{q})}(\mathcal{B}_\nu \otimes_{\boldsymbol{k}[\boldsymbol{X}]_\mathfrak{q}} \kappa(\mathfrak{q})) = \deg(\phi),$$

where $\kappa(\mathfrak{q}) := \boldsymbol{k}[\boldsymbol{X}]_\mathfrak{q}/\mathfrak{q} \cdot \boldsymbol{k}[\boldsymbol{X}]_\mathfrak{q}$, which completes the proof.   □

4.2. Analysis of the extraneous factors

Theorem 15 can be generalized (in the sense of [4, Section 4.2]) taking into account the fibers in $\mathcal{T} \times (\boldsymbol{P}^1)^n$ that give rise to extraneous factors, by relaxing the conditions on the ideals I_r stated in Theorem 15. Recall from (12) that $W := \operatorname{Proj}(A/\sum_i I^{(i)})$ and $X := \operatorname{Proj}(A/\prod_i I^{(i)})$, and that for each $\alpha \subset [1, n]$, $I^{(\alpha)} := \sum_{j \in \alpha} I^{(j)}$, $X_\alpha := \operatorname{Proj}(A/I^{(\alpha)})$ and $U_\alpha := X_\alpha \setminus \bigcup_{j \notin \alpha} X_{\{j\}}$. As was defined in (13), $\mathcal{E}_\alpha := \pi_1^{-1}(U_\alpha) \subset \mathcal{T} \times (\boldsymbol{P}^1)^n$ is a multiprojective bundle of rank $|\alpha|$ over U_α, such that $\operatorname{codim}(\mathcal{E}_\alpha) = n - |\alpha| + (\operatorname{codim}_\mathcal{T}(U_\alpha))$.

Lemma 16. *Let $\phi : \mathcal{T} \dashrightarrow (\boldsymbol{P}^1)^n$ be a rational map satisfying condition 1 in Theorem 15. Then, for all $\alpha \subset [1, n]$, $\operatorname{codim}_A(I^{(\alpha)})$ $\geq |\alpha|$.*

Proof. Same as the proof of [4]; it is only needed that A be Cohen–Macaulay.   □

We proceed as in [4], defining the basic language needed to describe the geometry of the base locus of ϕ.

Definition 17. For each $\alpha \subset [1, n]$, denote $\Theta := \{\alpha \subset [1, n] : \operatorname{codim}(I^{(\alpha)}) = |\alpha|\}$. Hence, let $I^{(\alpha)} = (\cap_{\mathfrak{q}_i \in \Lambda_\alpha} \mathfrak{q}_i) \cap \mathfrak{q}'$ be a primary decomposition, where Λ_α is the set of primary ideals of codimension $|\alpha|$, and $\operatorname{codim}_A(\mathfrak{q}') > |\alpha|$. Write $X_{\alpha,i} := \operatorname{Proj}(A/\mathfrak{q}_i)$ with $\mathfrak{q}_i \in \Lambda_\alpha$, and let $X_{\alpha,i}^{red}$ be the associated reduced variety.

Write $\alpha := \{i_1, \ldots, i_k\} \subset [1, n]$, and denote by $\pi_\alpha : (\boldsymbol{P}^1)^n \to (\boldsymbol{P}^1)^{n-|\alpha|}$ the projection given by

$$\pi_\alpha : (x_1 : y_1) \times \cdots \times (x_n : y_n) \mapsto (x_{i_{k+1}} : y_{i_{k+1}}) \times \cdots \times (x_{i_n} : y_{i_n}).$$

Define $\boldsymbol{P}_\alpha := \pi_\alpha((\boldsymbol{P}^1)^n) \cong (\boldsymbol{P}^1)^{n-|\alpha|}$, and $\phi_\alpha := \pi_\alpha \circ \phi : \mathcal{T} \dashrightarrow \boldsymbol{P}_\alpha$.

Denote by W_α the base locus of ϕ_α. Clearly $W \subset W_\alpha \subset X$ (see Eq. (12)). Denote $\mathcal{U}_\alpha := \mathcal{T} \setminus W_\alpha$, the open set where ϕ_α is well defined. Write $\Omega_\alpha := X_\alpha \cap \mathcal{U}_\alpha$ and $\Omega_{\alpha,i} := X_{\alpha,i} \cap \mathcal{U}_\alpha$. If α is empty, we set $\pi_\alpha = \operatorname{Id}_{(\boldsymbol{P}^1)^n}$, $\phi_\alpha = \phi$, $W_\alpha = W$ and $\mathcal{U}_\alpha = \Omega_\alpha = \Omega$.

We get a commutative diagram as follows:

Remark 18. *Let $p \in \mathcal{T}$ be a point; then there exists a unique pair (α, i) such that $p \in \Omega_{\alpha,i}$. If $p \in W$, then $\alpha = \emptyset$, and no i is considered.*

Proof. It is clear from the definition of Ω_α that, if $p \in W$, then $\alpha = \emptyset$, and no i needs to be considered. Hence, assume that $p \in \mathcal{T} \setminus W$. Thus, we define $\alpha := \{i \in [1, n] : f_i(p) = g_i(p) = 0\}$ which is a non-empty subset of $[1, n]$. For this set α, define ϕ_α according to Definition 17, set W_α the base locus of ϕ_α and $X_\alpha := \operatorname{Proj}(A/I^{(\alpha)})$. By definition, $p \in \Omega_\alpha := X_\alpha \setminus W_\alpha$. Since, in particular, $p \in X_\alpha$, it is one of its irreducible components that we denote by $X_{\alpha,i}$ following the notation of Definition 17. We conclude that $p \in \Omega_{\alpha,i} := X_{\alpha,i} \setminus W_\alpha$, from which we obtain the (α, i) of the statement.   □

In the following lemma we define a multiprojective bundle of rank $|\alpha|$ over $\Omega_{\alpha,i}$.

Lemma 19. *For ϕ as in Theorem 15, and for each $\alpha \in \Theta$ and each $\mathfrak{q}_i \in \Lambda_\alpha$, the following statements are satisfied:*

1. *$\Omega_{\alpha,i}$ is non-empty;*
2. *for all $p \in \Omega_{\alpha,i}$, $\dim(\pi_1^{-1}(p)) = |\alpha|$;*
3. *the restriction $\phi_{\alpha,i}$ of ϕ to $\Omega_{\alpha,i}$, defines a rational map*

$$\phi_{\alpha,i} : X_{\alpha,i} \dashrightarrow \boldsymbol{P}_\alpha \cong (\boldsymbol{P}^1)^{n-|\alpha|}; \tag{14}$$

4. *$Z_{\alpha,i} := \pi_1^{-1}(\Omega_{\alpha,i}) \xrightarrow{\pi_1} \Omega_{\alpha,i}$ defines a multiprojective bundle $\mathcal{E}_{\alpha,i}$ of rank $|\alpha|$ over $\Omega_{\alpha,i}$.*

Proof. Fix $X_{\alpha,i} \subset X_\alpha$ and write $\alpha := i_1, \ldots, i_k$. As $\Omega_{\alpha,i} = X_{\alpha,i} \setminus \bigcup_{j \notin \alpha} X_{\{j\}}$, it is an open subset of $X_{\alpha,i}$. If $\Omega_{\alpha,i} = \emptyset$, then $X_{\alpha,i} \subset \bigcup_{j \notin \alpha} X_{\{j\}}$, and as it is irreducible, there exists j such that $X_{\alpha,i} \subset X_{\{j\}}$; hence $X_{\alpha,i} \subset X_{\{j\}} \cap X_\alpha = X_{\alpha \cup \{j\}}$. Denote $\alpha' := \alpha \cup \{j\}$; it follows that $\dim(X_{\alpha'}) \geq \dim(X_{\alpha,i}) = n - |\alpha| > n - |\alpha'|$, which contradicts the hypothesis.

Let $p \in \Omega_{\alpha,i}, \pi_1^{-1}(p) = \{p\} \times \{q_{i_{k+1}}\} \times \cdots \times \{q_{i_n}\} \times (\boldsymbol{P}^1)^{|\alpha|}$, where the point $q_{i_j} \in \boldsymbol{P}^1$ is the only solution to the non-trivial equation $L_{i_j}(p, x_{i_j}, y_{i_j}) = y_{i_j} f_{i_j}(p) - x_{i_j} g_{i_j}(p) = 0$. Then we deduce that $\dim(\pi_1^{-1}(p)) = |\alpha|$, and that $\phi_{\alpha,i} : \Omega_{\alpha,i} \to \boldsymbol{P}_\alpha$, given by $p \in \Omega_{\alpha,i} \mapsto \{q_{i_{k+1}}\} \times \cdots \times \{q_{i_n}\} \in \boldsymbol{P}_\alpha$, is well defined.

The last statement follows immediately from the previous ones. $\square$

We get the following result.

Theorem 20. *Let $\phi : \mathscr{T} \dashrightarrow (\boldsymbol{P}^1)^n$ be defined by the pairs $(f_i : g_i)$, not both being zero, as in (8). Assume that $\mathrm{codim}_A(I_r) \geq n - r + 1$ for all $r = 1, \ldots, n$. Denote by H the irreducible implicit equation of the closure of its image. Then, there exist relative open subsets, $\Omega_{\alpha,i}$, of $\mathscr{T}$ such that the restriction $\phi_{\alpha,i}$ of ϕ to $\Omega_{\alpha,i}$ defines a rational map $\phi_{\alpha,i} : \Omega_{\alpha,i} \to \boldsymbol{P}_\alpha$ and positive integers $\mu_{\alpha,i}$ such that*

$$\mathrm{Res}_{\mathscr{T}}(L_0, \ldots, L_n) = H^{\deg(\phi)} \cdot \prod_{\alpha,i} (H_{\alpha,i})^{\mu_{\alpha,i} \cdot \deg(\phi_{\alpha,i})}.$$

Proof. The proof of this result follows similar lines of that of [4, Thm. 22]. Recall that $\Gamma := \mathrm{Biproj}(\mathscr{B})$, and set $\Gamma_0 := \overline{\Gamma_\Omega}$, the closure of the graph of ϕ. Applying π_2 to the decomposition $\Gamma \setminus \Gamma_U = \Gamma_0$, we see that $[\pi_2(\Gamma_U)] = [\mathrm{Res}_{\mathscr{T}}(L_0, \ldots, L_n)] - [\pi_2(\Gamma_0)]$ is the divisor associated to the extraneous factors. It is clear that $[\pi_2(\Gamma_U)]$ defines a principal divisor in $(\boldsymbol{P}^1)^n$ denoted by $G = \frac{\mathrm{Res}_{\mathscr{T}}(L_0,\ldots,L_n)}{H^{\deg(\phi)}}$, with support on $\pi_2(\Gamma \setminus \Gamma_0)$, and that Γ and Γ_0 coincide outside $X \times (\boldsymbol{P}^1)^n$.

By Lemma 19, for each α and each $\mathfrak{q}_i \in \Delta_\alpha \subset \Lambda_\alpha$, $\phi_{\alpha,i}$ defines a multiprojective bundle $\mathscr{E}_{\alpha,i}$ of rank $|\alpha|$ over $\Omega_{\alpha,i}$.

By the definition of Δ_α, $\overline{\pi_2(\mathscr{E}_{\alpha,i})}$ is a closed subscheme of $(\boldsymbol{P}^1)^n$ of codimension one. Denoting by $[\overline{\mathscr{E}_{\alpha,i}}] = \mu_{\alpha,i} \cdot [\overline{\mathscr{E}_{\alpha,i}^{red}}]$ the class of $\overline{\mathscr{E}_{\alpha,i}}$ as an algebraic cycle of codimension n in $\boldsymbol{P}^{n-1} \times (\boldsymbol{P}^1)^n$, we have $(\pi_2)_*[\overline{\mathscr{E}_{\alpha,i}}] = \mu_{\alpha,i} \cdot (\pi_2)_*[\overline{\mathscr{E}_{\alpha,i}^{red}}] = \mu_{\alpha,i} \cdot \deg(\phi_{\alpha,i}) \cdot [\mathfrak{p}_{\alpha,i}]$, where $\mathfrak{p}_{\alpha,i} := (H_{\alpha,i})$.

As in Theorem 15, we have, for $\nu > \eta$,

$$\begin{aligned}
[\det((\mathscr{K}_\bullet)_\nu)] &= \mathrm{div}_{k[\boldsymbol{X}]}(H_0(\mathscr{K}_\bullet)_\nu) \\
&= \mathrm{div}_{k[\boldsymbol{X}]}(\mathscr{B}_\nu) \\
&= \sum_{\mathfrak{p} \text{ prime, } \mathrm{codim}_{k[\boldsymbol{X}]}(\mathfrak{p})=1} \mathrm{length}_{k[\boldsymbol{X}]_\mathfrak{p}}((\mathscr{B}_\nu)_\mathfrak{p})[\mathfrak{p}].
\end{aligned}$$

We obtain that

$$[\det((\mathscr{K}_\bullet)_\nu)] = \sum_{\alpha \in \Theta} \sum_{\mathfrak{p}_{\alpha,i}} \mathrm{length}_{k[\boldsymbol{X}]_{\mathfrak{p}_{\alpha,i}}}((\mathscr{B}_\nu)_{\mathfrak{p}_{\alpha,i}})[\mathfrak{p}_{\alpha,i}] + \mathrm{length}_{k[\boldsymbol{X}]_{(H)}}((\mathscr{B}_\nu)_{(H)})[(H)].$$

In the formula above, for each $\mathfrak{p}_{\alpha,i}$, we have

$$\mathrm{length}_{k[\boldsymbol{X}]_{\mathfrak{p}_{\alpha,i}}}((\mathscr{B}_\nu)_{\mathfrak{p}_{\alpha,i}}) = \dim_{\boldsymbol{k}(\mathfrak{p}_{\alpha,i})} (\mathscr{B}_\nu \otimes_{k[\boldsymbol{X}]_{\mathfrak{p}_{\alpha,i}}} \boldsymbol{k}(\mathfrak{p}_{\alpha,i})) = \mu_{\alpha,i} \cdot \deg(\phi_{\alpha,i}),$$

where $\boldsymbol{k}(\mathfrak{p}_{\alpha,i}) := k[\boldsymbol{X}]_{\mathfrak{p}_{\alpha,i}}/\mathfrak{p}_{\alpha,i} \cdot k[\boldsymbol{X}]_{\mathfrak{p}_{\alpha,i}}$.

Consequently, we get that, for each $\alpha \in \Theta$, there is a factor of G, denoted by $H_{\alpha,i}$, that corresponds to the irreducible implicit equation of the scheme theoretic image of $\phi_{\alpha,i}$, raised to a certain power $\mu_{\alpha,i} \cdot \deg(\phi_{\alpha,i})$. $\square$

Remark 21. Observe that if $\mathrm{im}(\phi_{\alpha,i})$ is not a hypersurface in $\boldsymbol{P}_\alpha$ then $\deg(\phi_{\alpha,i})$ is 0; hence $(H_{\alpha,i})^{\mu_{\alpha,i} \cdot \deg(\phi_{\alpha,i})} = 1$. Thus $\phi_{\alpha,i}$ does not give an extraneous factor.

5. The algorithmic approach: Hilbert and Ehrhart functions

In this section, we focus on the study of the size of the matrices M_ν obtained in the two cases $\mathscr{K} = \boldsymbol{P}^n$ and $\mathscr{K} = (\boldsymbol{P}^1)^n$. Let us analyze first the case $\mathscr{K} = \boldsymbol{P}^n$, thus, where we get a map $\varphi : \mathscr{T} \dashrightarrow \boldsymbol{P}^n$ as defined in (3). Assume also that the base locus of φ is a zero-dimensional almost locally complete intersection scheme. Hence, the associated $\mathcal{Z}$-complex is acyclic. We have shown in Section 3 that the matrix M_ν is obtained as the right-most map of the $(\nu, *)$-graded strand of the approximation complex of cycles $\mathcal{Z}_\bullet(\boldsymbol{h}, A)_{(\nu,*)}$:

$$0 \to (\mathcal{Z}_n)_{(\nu,*)}(-n) \to (\mathcal{Z}_{n-1})_{(\nu,*)}(-(n-1)) \to \cdots \to (\mathcal{Z}_1)_{(\nu,*)}(-1) \xrightarrow{M_\nu} (\mathcal{Z}_0)_{(\nu,*)}.$$

Write $h_B(\mu) := \dim_{\boldsymbol{k}}(B_\mu)$ for the Hilbert function of B at μ. Since $\mathcal{Z}_i = Z_i[i \cdot d] \otimes_A A[\boldsymbol{X}] = Z_i[i \cdot d] \otimes_{\boldsymbol{k}} \boldsymbol{k}[\boldsymbol{X}]$, $(\mathcal{Z}_i)_{(\nu,*)} = (Z_i[i \cdot d])_\nu \otimes_{\boldsymbol{k}} \boldsymbol{k}[\boldsymbol{X}]$, we have $M_\nu \in \mathrm{Mat}_{h_A(\nu), h_{Z_1}(\nu+d)}(\boldsymbol{k}[\boldsymbol{X}])$.

Let $\mathscr{K} = (\boldsymbol{P}^1)^n$, and assume that we are given a map $\phi : \mathscr{T} \dashrightarrow (\boldsymbol{P}^1)^n$ as the one considered in (8), satisfying the conditions of Theorem 20. We obtain the matrix M_η computed from the Koszul complex $(\mathscr{K}_\bullet)_{(\eta,*)}$. Hence, the matrix M_η belongs to $\mathrm{Mat}_{h_A(\eta), nh_A(\eta-d)}(\boldsymbol{k}[\boldsymbol{X}])$.

Both numbers $h_A(\nu)$ and $h_{Z_1}(\nu+d)$, in the projective and multiprojective setting, can be computed easily in Macaulay2. The cost of computation depends on the ring structure of A. When A is just any finitely generated $\boldsymbol{N}$-graded Cohen–Macaulay $\boldsymbol{k}$-algebra, finding a precise theoretical estimate of these numbers would be very difficult. Also, the module structure of Z_1 can also be very intricate. Since it is an $\boldsymbol{N}$-graded sub-A-module of A^{n+1}, we have $h_{Z_1}(\nu+d) \leq (n+1)h_A(\nu+d)$.

Assume now that the ring A is the coordinate ring of a normal toric variety $\mathcal{T}$ defined from a polytope $\mathcal{N}$, as mentioned in Section 2, and later in Remark 5 and (10). In this setting, the situation above can be rephrased in a more combinatorial fashion. Let $\mathcal{N}$ be an $(n-1)$-dimensional lattice polytope, that is, a full-dimensional convex polytope in $\mathbf{R}^{n-1}$ with vertices lying in $\mathbf{Z}^{n-1}$. For any integer $k \geq 0$, the multiple $k\mathcal{N} = \{p_1 + \cdots + p_k : p_i \in \mathcal{N}\}$ is also a lattice polytope, and we can count its lattice points. The function taking each integer $k \in \mathbf{N}$ to the number $E_{\mathcal{N}}(k) = \#(k\mathcal{N}) \cap \mathbf{Z}^{n-1}$ of lattice points in the polytope $k\mathcal{N}$ is the *Ehrhart function* of $\mathcal{N}$ (see [25]). Write $E_{\mathcal{N}}^+(k) = \# \text{relint}(k\mathcal{N}) \cap \mathbf{Z}^{n-1}$, the number of integer points in the interior of $k\mathcal{N}$ (see [23] for software for computing those numbers). It is known that there is an identification between $\mathbf{k}[\text{relint}(C)]$ and ω_A; hence, this can be understood as $E_{\mathcal{N}}^+(k) = h_{\omega_A}(k)$.

Let C be the cone in $\mathbf{R}^{n-1} \times \mathbf{R}$ spanned in degree 1 by the lattice points in the polytope $\mathcal{N}$, which is normal by assumption; hence A is Cohen–Macaulay (see 2). Assume that $\mathcal{N}'$ stands for some integer contraction of $\mathcal{N}$ which is also normal and take $d \in \mathbf{N}$ such that $d\mathcal{N}' = \mathcal{N}$. Then $A' = \mathbf{k}[\mathcal{N}']$ is its Cohen–Macaulay semigroup ring. As $d\mathcal{N}' = \mathcal{N}$, we have that $E_{\mathcal{N}'}(d\mu) = E_{\mathcal{N}}(\mu)$ for all μ. Set $\gamma := a_n(A) = \inf\{\mu : (\omega_A^\vee)_\mu = 0\}$ and $\gamma' := a_n(A') = \inf\{\mu : (\omega_{A'}^\vee)_\mu = 0\}$. As $(\omega_A^\vee)_\mu = \text{Hom}_{\mathbf{k}}(M_{-\mu}, \mathbf{k})$, we have that $\gamma = \max\{i : C_i \text{ contains no interior points}\}$, where $C_i := C \cap \mathbf{Z}^{n-1} \times \{i\}$, and similarly for γ'. For a deeper understanding we refer the reader to [7, Section 5].

Both A and A' give rise to two different – but related – implicitization problems; the following result gives a condition on the rings A and A' to decide when it is algorithmically better to choose one situation or the other.

Lemma 22. *Take $\mathcal{N}$, $\mathcal{N}'$, d, γ and γ' as above. Then*

1. $\gamma \geq \gamma'$;
2. $d(\gamma' + 1) \geq \gamma + 1$.

Proof. As $d \geq 1$, we can assume that $\mathcal{N}' \subset \mathcal{N}$; hence, the first item follows. For the second item, we just need to observe that if $\mu\mathcal{N} \cap \mathbf{Z}^n$ is non-empty, then neither is $\mu d\mathcal{N}' \cap \mathbf{Z}^n$. Taking μ as the smallest positive integer with this property, and writing $\gamma = \mu + 1$, the second item follows. □

Remark 23. It is not true in general that $d(\gamma + 1) > \gamma' + 1$: take $\mathcal{N}$ as the triangle with vertices $(3, 0)$, $(0, 3)$, $(0, 0)$ and $\mathcal{N}'$ the triangle with vertices $(1, 0)$, $(0, 1)$, $(0, 0)$; hence $d = 3$, $\gamma = 0$, $\gamma' = 2$. We obtain $d(\gamma + 1) = 3 = \gamma' + 1$, which shows also that $d\gamma$ need not be bigger than γ'. Nor is it true that $d(\gamma + 1) = \gamma' + 1$, for instance, take $\mathcal{N}$ as the triangle with vertices $(4, 0)$, $(0, 4)$, $(0, 0)$ and $\mathcal{N}'$ as before. Observe that $d(\gamma + 1) = 4(0 + 1) = 4 > \gamma' + 1 = 2 + 1 = 3$.

Lemma 24. *Let $f_0, \ldots, f_n$ be with generic coefficients and the same denominator, $\mathcal{N} := \mathcal{N}(f_0, \ldots, f_n)$, and $\mathcal{N}'$ and d such that $d\mathcal{N}' = \mathcal{N}$. Take $\nu_0 = (n-1) - \gamma$ as the bound established in [5, Thm. 11], and $\nu_0' = d(n-1) - \gamma'$. Write $\delta := d(\gamma+1) - (\gamma'+1)$. Then $E_{\mathcal{N}}(\nu_0) > E_{\mathcal{N}'}(\nu_0')$ if and only if $\delta > d - 1$.*

Proof. We have seen that $E_{\mathcal{N}'}(d\nu_0) = E_{\mathcal{N}}(\nu_0)$; hence, it is enough to compare $E_{\mathcal{N}'}(d\nu_0)$ and $E_{\mathcal{N}'}(\nu_0')$. Writing $d\gamma = \gamma' + \delta - (d-1)$, we have

$$E_{\mathcal{N}'}(d\nu_0) = E_{\mathcal{N}'}(d(n-1) - d\gamma) = E_{\mathcal{N}'}(d(n-1) - \gamma' + \delta - (d-1)),$$

from where we deduce that $E_{\mathcal{N}}(\nu_0) > E_{\mathcal{N}'}(\nu_0')$ if and only if $\delta > d - 1$. □

Corollary 25. *Let $f : \mathbf{A}^{n-1} \dashrightarrow \mathbf{A}^n$ be a rational map with generic coefficients and the same denominator, $\mathcal{N} := \mathcal{N}(f)$, $\mathcal{N}'$ normal polytopes and d such that $d\mathcal{N}' = \mathcal{N}$. Let $\mathcal{T}$ and $\mathcal{T}'$ be the arithmetically Cohen–Macaulay toric varieties defined from $\mathcal{N}$ and $\mathcal{N}'$, respectively, and $\varphi : \mathcal{T} \subset \mathbf{P}^{E_{\mathcal{N}}(1)} \dashrightarrow \mathbf{P}^n$ and $\varphi' : \mathcal{T}' \subset \mathbf{P}^{E_{\mathcal{N}'}(1)} \dashrightarrow \mathbf{P}^n$. Take ν_0, ν_0' and δ as above. Write M_{ν_0} and $M_{\nu_0'}'$ as the representation matrices of $\text{im}(\varphi)$ and $\text{im}(\varphi')$, respectively. Then $\#\text{rows}(M_{\nu_0}) > \#\text{rows}(M_{\nu_0'}')$ if and only if $\delta > d - 1$.*

In the second case, given a map $\phi : \mathcal{T} \dashrightarrow (\mathbf{P}^1)^n$ as in Theorem 20, we obtain the matrix M_ν as the right-most matrix from the Koszul complex $(\mathcal{K}_\bullet)_{(\nu,*)}$:

$$0 \to A_{\nu-nd} \otimes_{\mathbf{k}} \mathbf{k}[\mathbf{X}](-n) \to \cdots \to (A_{\nu-d})^n \otimes_{\mathbf{k}} \mathbf{k}[\mathbf{X}](-1) \xrightarrow{M_\nu} A_\nu \otimes_{\mathbf{k}} \mathbf{k}[\mathbf{X}] \to 0.$$

It is clear that M_ν is a $\dim_{\mathbf{k}}(A_\nu)$ by $\dim_{\mathbf{k}}((A_{\nu-d})^n)$ matrix. As $\bigoplus_{k \geq 0} \langle C_k \rangle_{\mathbf{k}} = \mathbf{k}[C]$, which is canonically isomorphic to A, and also $\dim_{\mathbf{k}}(A_\nu) = E_{\mathcal{N}}(\nu)$ and $\dim_{\mathbf{k}}((A_{\nu-d})^n) = nE_{\mathcal{N}}(\nu - d)$, hence

$$M_\nu \in \text{Mat}_{E_{\mathcal{N}}(\nu), nE_{\mathcal{N}}(\nu-d)}(\mathbf{k}[\mathbf{X}]). \tag{15}$$

6. Examples

In this section, we show, in a few examples, how the theory developed in earlier sections works. We first analyze two concrete examples of parameterized surfaces, given as the image of a rational map defined by rational functions with different denominators. There, we show the advantage of not taking a common denominator, and hence, we regard their images in $(\mathbf{P}^1)^3$. Later we show how the method is well adapted for generic rational affine maps with a fixed polytope.

In the last part we show for the setting where $\mathcal{K} = (\mathbf{P}^1)^n$, for small n, how the splitting of the base locus is done in order to obtain the family of multiprojective bundles.

Example 26. We consider here an example of a very sparse parameterization where the multihomogeneous compactification of the codomain is significantly better than the homogeneous compactification. Take $n = 3$, and consider the affine map

$$f : \mathbf{A}^2 \dashrightarrow \mathbf{A}^3 : (s, t) \mapsto \left(\frac{st^6 + 2}{st^5 - 3st^3}, \frac{st^6 + 3}{st^4 + 5s^2t^6}, \frac{st^6 + 4}{2 + s^2t^6} \right).$$

Observe that in this case there is no smallest multiple of the Newton polytope $\mathcal{N}(f)$ with integer vertices; hence, $\mathcal{N}(f) = \mathcal{N}'(f)$ as can be seen in the picture below.
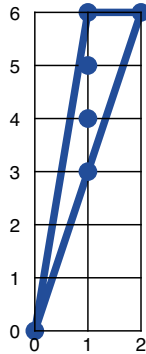
Computing in Macaulay2, we get that the homogeneous coordinate ring is

$$A = \frac{k[T_0, \ldots, T_5]}{(T_3^2 - T_2T_4, T_2T_3 - T_1T_4, T_2^2 - T_1T_3, T_1^2 - T_0T_5)}.$$

When $\mathbf{A}^3$ is compactified into $\mathbf{P}^3$, we obtain from f a new map $\varphi : \mathcal{T} \dashrightarrow \mathbf{P}^3$ by replacing (s, t) by $T_0, \ldots, T_5$, and taking a common denominator. We can easily see that taking a common denominator leads to polynomials of degree up to 23 and that the Newton polytope of the four new polynomials contains 26 integer points instead of 6. Again computing in Macaulay2, for $\nu_0 = 2$, the matrix M_ν has 351 rows and about 500 columns. It can be verified that this compactification gives a base point which is not locally a complete intersection, but locally an almost complete intersection, giving rise to extraneous factors.

On the other hand, compactifying $\mathbf{A}^3$ into $(\mathbf{P}^1)^3$, we get the map

$$\phi : \mathcal{T} \dashrightarrow \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$$
$$(T_0, \ldots, T_5) \mapsto (2T_0 + T_4 : -3T_1 + T_3)(3T_0 + T_4 : T_2 + 5T_5)(4T_0 + T_4 : 2T_0 + T_5).$$

Computations in Macaulay2 give that for $\nu_0 = 3$ the matrix M_{ν_0} is of size 34×51. Since there are no base points with two-dimensional fibers, we get no extraneous factors, and hence $H^{\deg(\phi)}$ can be computed as $\frac{(34 \times 34)\text{-matrix} \cdot (1 \times 1)\text{-matrix}}{(17 \times 17)\text{-matrix}}$, getting an equation of degree $(6, 6, 6)$.

Example 27. This example shows how the methods work in the generic case with a fixed polytope. We begin by taking $\mathcal{N}$ as a normal lattice polytope in $\mathbf{R}^{n-1}$. For the sake of clarity, we will treat a particular case in small dimension. Hence, set $n = 3$, and consider $\mathcal{N}$ as in the drawing below. It will remain clear that this example can be generalized to any dimension and any polytope with integer vertices.
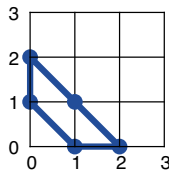
Assume that we are given six generic polynomials $f_1, f_2, f_3, g_1, g_2, g_3$ with support in $\mathcal{N}$; hence we get an affine rational map $f : \mathbf{A}^2 \dashrightarrow \mathbf{A}^3$ given by $(s, t) \mapsto (\frac{f_1}{g_1}, \frac{f_2}{g_2}, \frac{f_3}{g_3})$. We write $f_i = \sum_{(a,b) \in \mathcal{N}} U_{(a,b),i} \cdot s^a t^b$, and $g_i = \sum_{(a,b) \in \mathcal{N}} V_{(a,b),i} \cdot s^a t^b$. Set $\mathcal{U} := \{U_{(a,b),i}, V_{(a,b),i} : \text{ for all } (a, b) \in \mathcal{N}, \text{ and } i = 1, 2, 3\}$, the set of coefficients, and define $\mathbf{k} := \mathbf{Z}[\mathcal{U}]$.

Now we focus on computing the implicit equation of a convenient compactification for the map. Let $\mathcal{T}$ be the toric variety associated to the Newton polytope $\mathcal{N}$, embedded in $\mathbf{P}^4$. We will compare how the method works in the $\mathbf{P}^3$ and $(\mathbf{P}^1)^3$ compactifications of $\mathbf{A}^3$ with domain $\mathcal{T}$. One key point to remark is that these two maps have no base points, since we are taking the toric compactification associated to $\mathcal{N}$ and generic coefficients; hence, we will not have any extraneous factors.

In the first case, we take a common denominator obtaining four polynomials with generic coefficients in the polytope $3\mathcal{N}$. If we consider the smallest multiple, we recover the polytope $\mathcal{N}$, and maps of degree 3. We obtain in this case that

f factorizes through $\mathscr{T} \subset \boldsymbol{P}^4$ via $\varphi : \mathscr{T} \dashrightarrow \boldsymbol{P}^3$ given by four polynomials of degree 3 in the variables $T_0, \ldots, T_4$. From Lemma 11, we take $\nu_0 := \max\{3, 6 - \gamma\}$. Since $2\mathcal{N}$ has integer interior points but $\mathcal{N}$ does not, $\gamma = 1$; thus $\nu_0 = 5$. Now, since X is empty in $\mathscr{T}$, from Lemma 7, the complex $\mathcal{Z}_\bullet$ is acyclic.

From Theorem 12, we see that the implicit equation can be computed as the determinant of the complex $(\mathcal{Z}_\bullet)_\nu$ for $\nu \geq \nu_0$, or as the gcd of the maximal minors of the right-most map $(\mathcal{Z}_1)_5(-1) \xrightarrow{M_\nu} (\mathcal{Z}_0)_5$. We can easily compute the dimension of A_5, by the formula $\#(k \cdot \mathcal{N}) = (k + 1)(k + 1 + k/2)$. When $k = 5$, we get $\#(5 \cdot \mathcal{N}) = 51$; hence $(\mathcal{Z}_0)_5 = \boldsymbol{k}^{51}[\boldsymbol{X}]$. Since M_ν gives a surjective map, $(\mathcal{Z}_1)_5(-1)$ has dimension bigger than or equal to 51.

Instead of taking a common denominator, we can proceed by compactifying $\boldsymbol{A}^3$ into $(\boldsymbol{P}^1)^3$. In this case we get a map $\phi : \mathscr{T} \dashrightarrow (\boldsymbol{P}^1)^3$ given by three pairs of linear functions on the variables $T_0, \ldots, T_4$.

From Theorem 14, we take $\nu \geq \nu_0 = 1 + 1 + 1 - 1 = 2$. Now, since the polynomials f_i and g_i have generic coefficients, $L_i := Y_i f_i - X_i g_i$ does as well; thus $\mathcal{K}_\bullet := \mathcal{K}_\bullet(L_1, L_2, L_3; A[\boldsymbol{X}])$ is acyclic. From Lemma 15, the implicit equation can be computed as the determinant of the complex $(\mathcal{K}_\bullet)_\nu$ for $\nu \geq 2$, or as the gcd of the maximal minors of the right-most map $A_1^3[\boldsymbol{X}](-1) \xrightarrow{M_2} A_2[\boldsymbol{X}]$. Since $\dim(A_0) = 1$, $\dim(A_1) = 5$ and $\dim(A_2) = 12$, we get the complex $\boldsymbol{k}^3[\boldsymbol{X}](-2) \to \boldsymbol{k}^{15}[\boldsymbol{X}](-1) \xrightarrow{M_2} \boldsymbol{k}^{12}[\boldsymbol{X}]$. Thus, the implicit equation can be computed as the gcd of the maximal minors of a (12×14)-matrix, or as $\frac{\det(12 \times 12\text{-matrix})}{\det(3 \times 3\text{-matrix})}$.

Example 28. Assume we are given four tuples of polynomials f_i, g_i, for $i \in [1, 4]$, in three variables s, t, u. Let them be $f_1 = s + tu^2, g_1 = u^2, f_2 = st, g_2 = u^2, f_3 = su^2, g_3 = t, f_4 = stu^2, g_4 = 1$. They define a rational map $f : \boldsymbol{A}^3 \dashrightarrow \boldsymbol{A}^4$ given by $(s, t, u) \mapsto (f_1/g_1, f_2/g_2, f_3/g_3, f_4/g_4)$.

In the spirit of this article, we compactify $\boldsymbol{A}^3$ into the toric variety associated to the smallest multiple of the Newton polytope the input polynomials define. It is easy to see that this polytope $\mathcal{N}$ is a $(1 \times 1 \times 2)$-parallelepiped, and $\mathscr{T} \cong (\boldsymbol{P}^1)^3 \subset \boldsymbol{P}^{11}$.

In order to detect the extraneous factor that occurs, consider the rational map

$$\begin{aligned} \tilde{\phi} : (\boldsymbol{P}^1)^3 &\dashrightarrow (\boldsymbol{P}^1)^4 \\ (s : s') \times (t : t') \times (u : u') &\mapsto (\tilde{f}_1 : \tilde{g}_1) \times (\tilde{f}_2 : \tilde{g}_2) \times (\tilde{f}_3 : \tilde{g}_3) \times (\tilde{f}_4 : \tilde{g}_4), \end{aligned}$$

where $(\tilde{-})$ means homogenizing with respect to the degree $(1, 1, 2)$ with the variables s', t' and u'.

We easily observe that the base locus has codimension 2; in fact, many lines occur in the base locus. There are

1. four lines $\mathcal{L}_1 = (1 : 0) \times (t : t') \times (1 : 0)$, $\mathcal{L}_2 = (1 : 0) \times (t : t') \times (0 : 1)$, $\mathcal{L}_3 = (0 : 1) \times (t : t') \times (1 : 0)$, $\mathcal{L}_4 = (0 : 1) \times (t : t') \times (0 : 1)$;
2. three lines $\mathcal{L}_5 = (1 : 0) \times (1 : 0) \times (u : u')$, $\mathcal{L}_6 = (1 : 0) \times (0 : 1) \times (u : u')$, $\mathcal{L}_7 = (0 : 1) \times (1 : 0) \times (u : u')$; and
3. three lines $\mathcal{L}_8 = (s : s') \times (1 : 0) \times (1 : 0)$, $\mathcal{L}_9 = (s : s') \times (1 : 0) \times (0 : 1)$, $\mathcal{L}_{10} = (s : s') \times (0 : 1) \times (0 : 1)$;
4. seven points of intersection of the previous lines: $\mathcal{L}_1 \cap \mathcal{L}_5 \cap \mathcal{L}_8 = \{(1 : 0) \times (1 : 0) \times (1 : 0)\}$, $\mathcal{L}_1 \cap \mathcal{L}_6 = \{(1 : 0) \times (0 : 1) \times (1 : 0)\}$, $\mathcal{L}_2 \cap \mathcal{L}_5 \cap \mathcal{L}_9 = \{(1 : 0) \times (1 : 0) \times (0 : 1)\}$, $\mathcal{L}_2 \cap \mathcal{L}_6 \cap \mathcal{L}_{10} = \{(1 : 0) \times (0 : 1) \times (0 : 1)\}$, $\mathcal{L}_3 \cap \mathcal{L}_7 \cap \mathcal{L}_8 = \{(0 : 1) \times (1 : 0) \times (1 : 0)\}$, $\mathcal{L}_4 \cap \mathcal{L}_7 \cap \mathcal{L}_9 = \{(0 : 1) \times (1 : 0) \times (0 : 1)\}$ and $\mathcal{L}_4 \cap \mathcal{L}_{10} = \{(0 : 1) \times (0 : 1) \times (0 : 1)\}$.

Over those lines the fiber is of dimension 2, except over the points of intersection of them.

In the language of Section 4.2, we have that $W = \emptyset$. The set Θ formed by the sets $\alpha \subset [1, 4]$ giving fibers of dimension $|\alpha|$ is

$$\Theta = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 4\}, \{2, 3, 4\}\}.$$

Recall that this does not imply that every $\alpha \in \Theta$ will give an extraneous factor (see Remark 21). We clarify this.

As we have mentioned, the base locus is a union of lines with non-trivial intersection. Take $\alpha = \{1, 2\}$. Set theoretically $X_\alpha = \mathcal{L}_1 \sqcup \mathcal{L}_4$, and hence there are two irreducible components of X_α, namely $X_{\alpha,1} = \mathcal{L}_1$ and $X_{\alpha,2} = \mathcal{L}_4$. The line $X_{\alpha,1} = \mathcal{L}_1$ only intersects $\mathcal{L}_5, \mathcal{L}_6$ and $\mathcal{L}_8$; hence

$$\Omega_{\alpha,1} = \mathcal{L}_1 \setminus (\mathcal{L}_5 \cap \mathcal{L}_6 \cap \mathcal{L}_8) = \{(1 : 0) \times (t : t') \times (1 : 0) : t \neq 0 \text{ and } t' \neq 0\}.$$
$$\Omega_{\alpha,2} = \mathcal{L}_4 \setminus (\mathcal{L}_7 \cap \mathcal{L}_9 \cap \mathcal{L}_{10}) = \{(0 : 1) \times (t : t') \times (0 : 1) : t \neq 0 \text{ and } t' \neq 0\}.$$

Since $\alpha = \{1, 2\}$, the linear forms $L_1(p, \boldsymbol{X})$ and $L_2(p, \boldsymbol{X})$ vanish identically for all $p \in X_\alpha$, while $L_3(p, \boldsymbol{X}) = f_3(p)Y_3 - g_3(p)X_3 = t'Y_3$ and $L_4(p, \boldsymbol{X}) = tY_4$ for $p \in X_{\alpha,1}$. It is easy to note that none of them vanish if and only if $p \in \Omega_{\alpha,1}$. We get that $L_3(p, \boldsymbol{X}) = tX_3$ and $L_4(p, \boldsymbol{X}) = t'X_4$ for $p \in X_{\alpha,2}$.

Finally, for $\alpha = \{1, 2\}$, we obtain two multiprojective bundles $\mathcal{E}_{\alpha,i}$ over $\Omega_{\alpha,i}$, for $i = 1, 2$,

$$\mathcal{E}_{\alpha,1} : \{(1 : 0) \times (t : t') \times (1 : 0) \times (\boldsymbol{P}^1)^2 \times (t' : 0) \times (t : 0) : t \neq 0, t' \neq 0\} \xrightarrow{\pi_1} \Omega_{\alpha,1},$$
$$\mathcal{E}_{\alpha,2} : \{(0 : 1) \times (t : t') \times (0 : 1) \times (\boldsymbol{P}^1)^2 \times (0 : t) \times (0 : t') : t \neq 0, t' \neq 0\} \xrightarrow{\pi_1} \Omega_{\alpha,2}.$$

Observe that $\mathrm{im}(\phi_{\alpha,1}) = \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times (1 : 0) \times (1 : 0)$; hence it does not define a hypersurface. Thus, $\phi_{\alpha,1}$ does not contribute with an extraneous factor. The same holds for $\phi_{\alpha,2}$.

The situation is similar when $\alpha \in \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$, but quite different for $\alpha = \{3, 4\}$. Take $\alpha = \{3, 4\}$; the linear forms $L_3(p, \boldsymbol{X})$ and $L_4(p, \boldsymbol{X})$ vanish identically for all $p \in X_\alpha$. Take $X_{\alpha,1} = \mathcal{L}_2$ and $X_{\alpha,2} = \mathcal{L}_3$. Define $\Omega_{\alpha,1} := \mathcal{L}_3 \setminus \{(0 : 1) \times (0 : 1) \times (1 : 0), (0 : 1) \times (1 : 0) \times (1 : 0)\}$, and observe that $\phi_{\alpha,1} : \Omega_{\alpha,1} \dashrightarrow \boldsymbol{P}_\alpha$ defines a hypersurface given by the equation $(X_2 = 0)$. Hence, when $\alpha = \{3, 4\}$, $\phi_{\alpha,1}$ does give an extraneous factor.

Now, let us take $\alpha = \{1, 2, 3\}$ in order to illustrate a different situation. Verifying with the seven points listed above, we see that $X_\alpha = \{(1 : 0) \times (0 : 1) \times (1 : 0)\} \cup \{(0 : 1) \times (0 : 1) \times (0 : 1)\}$. Hence, there are two irreducible components $X_{\alpha,1} = \{(1 : 0) \times (0 : 1) \times (1 : 0)\}$ and $X_{\alpha,2} = \{(0 : 1) \times (0 : 1) \times (0 : 1)\}$, and clearly $\Omega_{\alpha,i} = X_{\alpha,i}$ for $i = 1, 2$. Thus, we get the trivial bundles

$$\mathscr{E}_{\alpha,1} : \{(1 : 0) \times (1 : 0) \times (1 : 0) \times (\boldsymbol{P}^1)^3 \times (1 : 0) \ : \ t \neq 0 \text{ and } t' \neq 0\} \xrightarrow{\pi_1} \Omega_{\alpha,1},$$

$$\mathscr{E}_{\alpha,2} : \{(0 : 1) \times (0 : 1) \times (0 : 1) \times (\boldsymbol{P}^1)^3 \times (0 : 1) \ : \ t \neq 0 \text{ and } t' \neq 0\} \xrightarrow{\pi_1} \Omega_{\alpha,2}.$$

These two bundles give rise to the factors Y_4 and X_4. We conclude by a similar argument that the extraneous factor is

$$G = Y_1^2 X_2 Y_2 Y_3^2 X_4 Y_4.$$

The degree of the multihomogeneous resultant $\mathrm{Res}_{\mathcal{N}}(L_1, L_2, L_3, L_4)$ in the coefficients of each L_i, as polynomials in s, s', t, t', u and u', is equal to $3 \cdot 1 \cdot 1 \cdot 2 = 6$ for all $i = 1, \ldots, 4$ by [19, Prop. 2.1, Ch. 13]. So, the total degree of $\det((\mathcal{K}_\bullet)_\nu)$ is $24 = 4 \cdot 6$. Indeed, the irreducible implicit equation is

$$H = X_4^2 Y_1^2 Y_2^2 Y_3^2 + 2 X_4 X_2 X_3 Y_1^2 Y_2 Y_3 Y_4 - X_4 X_1^2 X_3 Y_2^2 Y_3 Y_4 + X_2^2 X_3^2 Y_1^2 Y_4^2,$$

and $\deg(\phi) = 2$. Thus, $\det((\mathcal{K}_\bullet)_\nu) = H^2 \cdot G$ for $\nu \gg 0$.

Let us change now our analysis, and consider the (smallest multiple of the) Newton polytope $\mathcal{N}$ of f_i and g_i for $i = 1, 2, 3, 4$. We easily see that $\mathcal{N}$ is a parallelepiped with opposite extremes in the points $(0, 0, 0)$ and $(1, 1, 2)$. For a suitable labeling of the points in $\mathcal{N} \cap \boldsymbol{Z}^3$ by $\{T_i\}_{i=0,\ldots,11}$, we have that the toric ideal that defines the toric embedding of $(\boldsymbol{A}^*)^3 \xhookrightarrow{\iota} \boldsymbol{P}^{11}$ is

$J := I(\mathscr{T}) = (T_9 T_{10} - T_8 T_{11}, T_7 T_{10} - T_6 T_{11}, T_5 T_{10} - T_4 T_{11}, T_3 T_{10} - T_2 T_{11}, T_1 T_{10} - T_0 T_{11}, T_9^2 - T_7 T_{11}, T_8 T_9 - T_6 T_{11}, T_5 T_9 - T_3 T_{11}, T_4 T_9 - T_2 T_{11}, T_3 T_9 - T_1 T_{11}, T_2 T_9 - T_0 T_{11}, T_8^2 - T_6 T_{10}, T_7 T_8 - T_6 T_9, T_5 T_8 - T_2 T_{11}, T_4 T_8 - T_2 T_{10}, T_3 T_8 - T_0 T_{11}, T_2 T_8 - T_0 T_{10}, T_1 T_8 - T_0 T_9, T_5 T_7 - T_1 T_{11}, T_4 T_7 - T_0 T_{11}, T_3 T_7 - T_1 T_9, T_2 T_7 - T_0 T_9, T_5 T_6 - T_0 T_{11}, T_4 T_6 - T_0 T_{10}, T_3 T_6 - T_0 T_9, T_2 T_6 - T_0 T_8, T_1 T_6 - T_0 T_7, T_3 T_4 - T_2 T_5, T_1 T_4 - T_0 T_5, T_3^2 - T_1 T_5, T_2 T_3 - T_0 T_5, T_2^2 - T_0 T_4, T_1 T_2 - T_0 T_3).$
This computation has been done in Macaulay2 using the code [3].

The inclusion $\iota : (\boldsymbol{A}^*)^3 \hookrightarrow \boldsymbol{P}^{11}$ defines a graded morphism of graded rings $\iota^* : \boldsymbol{k}[T_0, \ldots, T_{11}]/J \to \boldsymbol{k}[s, t, u]$. This morphism maps $T_1 + T_{10} \mapsto f_1, T_7 \mapsto g_1, T_4 \mapsto f_2, T_7 \mapsto g_2, T_6 \mapsto f_3, T_5 \mapsto g_3, T_0 \mapsto f_4$, and $T_{11} \mapsto g_4$.

Hence, for $\alpha = \{1, 2\}$, we have that

$$X_\alpha = \mathrm{Proj}(\boldsymbol{k}[T_0, \ldots, T_{11}]/(J + (T_1 + T_{10}, T_4, T_7))).$$

Using Macaulay2, we can compute the primary decomposition of the radical ideal of $(T_1 + T_{10}, T_4, T_7)$ in $A := \boldsymbol{k}[T_0, \ldots, T_{11}]/J$, obtaining the two irreducible components $X_{\alpha,1}$ and $X_{\alpha,2}$. Precisely,

$$X_{\alpha,1} = \mathrm{Proj}(\boldsymbol{k}[T_0, \ldots, T_{11}]/(J + (T_{10}, T_8, T_7, T_6, T_4, T_2, T_1, T_0))), \ \text{and}$$

$$X_{\alpha,2} = \mathrm{Proj}(\boldsymbol{k}[T_0, \ldots, T_{11}]/(J + (T_{11}, T_7, T_6, T_5, T_4, T_1 + T_{10}, T_0))).$$

After embedding $(\boldsymbol{P}^1)^3$ in $\boldsymbol{P}^{11}$ via ι, we get that $X_{\alpha,1} = \iota_*(L_1)$ and $X_{\alpha,2} = \iota_*(L_2)$, which coincides with the situation described above for $\mathscr{T} = \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$.

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