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Balancedness of some subclasses of circular-arc graphs [1]

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Abstract

A graph is balanced if its clique-vertex incidence matrix is balanced, i.e., it does not contain a square submatrix of odd order with exactly two ones per row and per column. Interval graphs, obtained as intersection graphs of intervals of a line, are well-known examples of balanced graphs. A circular-arc graph is the intersection graph of a family of arcs on a circle. Circular-arc graphs generalize interval graphs, but are not balanced in general. In this work we characterize balanced graphs by minimal forbidden induced subgraphs restricted to graphs that belong to some classes of circular-arc graphs.

Keywords: balanced graphs, circular-arc graphs, forbidden subgraphs, perfect graphs

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1 Introduction

A $\{0,1\}$-matrix A is *balanced* if it does not contain a square submatrix of odd order with exactly two ones per row and per column. Such matrices have remarkable properties related to two fundamental combinatorial optimization problems, set packing

$$\max c^T x \text{ s.t. } Ax \leq \mathbb{1},\ x \in \{0,1\}^n \tag{1}$$

and set covering

$$\min c^T x \text{ s.t. } Ax \geq \mathbb{1},\ x \in \{0,1\}^n. \tag{2}$$

The matrix A is *perfect* (resp. *ideal*) if no integrality requirements are needed in (1) (resp. (2)) as the polytope $P(A) = \{x \in \mathbb{R}^n_+ \mid Ax \leq \mathbb{1}\}$ (resp. the polyhedron $Q(A) = \{x \in \mathbb{R}^n_+ \mid Ax \geq \mathbb{1}\}$) has integral extreme points only. The matrix A is *balanced* if and only if all its submatrices are perfect if and only if all its submatrices are ideal [10]. Well-known examples of balanced matrices are totally unimodular matrices where even no integrality requirements are needed in the above formulations for varying right hand side vectors.

A graph G is *balanced* if its clique-matrix is balanced [8]. Here, a *clique* Q in a graph $G = (V, E)$ is an inclusion-wise maximal subset of pairwise adjacent vertices and given an enumeration $Q_1, \ldots, Q_k$ of all cliques of G and an order $v_1, \ldots, v_n$ of all vertices of G, a *clique-matrix* of G is the $k \times n$ $\{0,1\}$-matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if and only if $v_j \in Q_i$. The clique-matrix of a graph is unique up to permutations of rows and/or columns.

The class of balanced graphs is closed under taking induced subgraphs. Well-known examples of balanced graphs are *bipartite graphs* (having a partition of their vertices into two stable sets) and *interval graphs* (the intersection graphs of intervals of a line), as their clique-matrices are totally unimodular and, thus, balanced.

Well-known superclasses of balanced graphs are perfect graphs and hereditary clique-Helly graphs. A graph is *perfect* if its clique-matrix is perfect [7]. Some years ago, the minimal forbidden induced subgraphs of perfect graphs were characterized [6], settling affirmatively a conjecture posed more than 40 years before by Berge [1]. The minimal forbidden induced subgraphs of perfect graphs are the chordless cycles of odd length having at least 5 vertices, called

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odd holes C_{2k+1}, and their complements, the *odd antiholes* $\overline{C}_{2k+1}$.

A graph is *hereditary clique-Helly* if in any of its induced subgraphs, every nonempty subfamily of pairwise intersecting cliques has a common vertex. It follows from [2] that balanced graphs are hereditary clique-Helly. Prisner [13] characterized hereditary clique-Helly graphs as those graphs containing no induced 0-, 1-, 2-, or 3-pyramid (see Figure 1).

Hence, no balanced graph contains an odd hole, odd antihole, or any of the pyramids as induced subgraph.

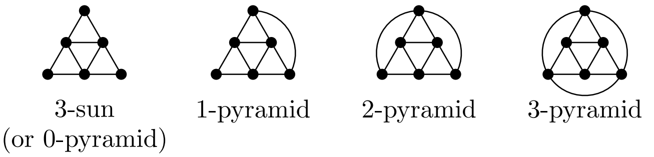

Fig. 1. The pyramids

In addition, balanced graphs were characterized by means of forbidden induced subgraphs in [4]. For a graph $G = (V, E)$ and $W \subseteq V$, let $N(W) = \bigcap_{w \in W} N(w)$ and use $N(e)$ as shorthand for $N(\{u, v\})$ for an edge $e = uv$. An *unbalanced cycle* of G is an odd cycle $C = (V', E')$ such that, for each edge $e \in E'$, there exists a (possibly empty) complete subgraph W_e of G such that $W_e \subseteq N(e) \setminus V'$ and $N(W_e) \cap N(e) \cap V' = \emptyset$. Note that the subsets W_e and W_f for different edges $e, f \in E'$ may overlap. An *extended odd sun* is a graph G with an unbalanced cycle C such that $V = V' \cup \bigcup_{e \in E'} W_e$ and $|W_e| \leq |N(e) \cap V'|$ for each edge e of C. The smallest extended odd suns are C_5 and the pyramids.

The characterization of balancedness by forbidden subgraphs is as follows.

Theorem 1.1 ([4]) *A graph is balanced if and only if it has no unbalanced cycle, or, equivalently, if and only if it contains no induced extended odd sun.*

However, the above characterization is not by *minimal* forbidden induced subgraphs because some extended odd suns contain some other extended odd suns as proper induced subgraphs, as Figure 2 shows.

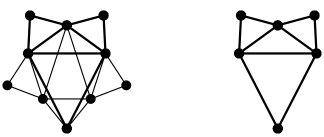

Fig. 2. On the left, an extended odd sun that is not minimal. Bold lines correspond to the edges of a proper induced extended odd sun, depicted on the right.

Thus, excluding extended odd suns suffices to guarantee balancedness, but it is not necessary to exclude all of them. We address the problem to find the *minimal forbidden induced subgraphs*, i.e., those graphs that are not balanced but all their proper induced subgraphs are balanced. This problem is still open, and only some partial results are known. A characterization of strongly chordal graphs [9] showed that they are balanced. The balanced graphs which are chordal have been characterized in [12]. Minimal forbidden induced subgraph characterizations of balanced graphs restricted to the following graph classes are found in [5]: P_4-tidy graphs, paw-free graphs, line graphs, and complements of line graphs. In this paper, we study balanced graphs restricted to some subclasses of circular-arc graphs.

A *circular-arc* (CA) graph is the intersection graph of a family of open arcs on a circle Such a family of arcs is called a *CA model* of the graph. Clearly, CA graphs can be seen as an extension of the well-known class of interval graphs. While interval graphs form a subclass of balanced graphs, this is not the case for CA graphs. Note that CA graphs are neither perfect nor hereditary clique-Helly in general as odd holes, odd antiholes, and pyramids can be easily seen to be CA graphs. Hence, the study of balancedness of CA graphs is in order. Perfectness of CA graphs was addressed in [14].

Our aim is to present minimal forbidden induced subgraph characterizations of balanced graphs within a superclass of Helly circular-arc graphs and the classes of claw-free circular-arc graphs and gem-free circular-arc graphs.

2 Balancedness of Some Classes of Circular-Arc Graphs

2.1 Balancedness of a Superclass of Helly Circular-Arc Graphs

A *Helly circular-arc* (HCA) graph is a circular-arc graph admitting a circular-arc model that satisfies the Helly property, i.e., every subset of pairwise intersecting arcs have a point in common. Note that in [11] it is shown that a CA graph is HCA if and only if it does not contain an *obstacle*, which is a graph H with a clique $Q = \{v_1, v_2, \ldots, v_t\}$ where $t \geq 3$ such that for each $i = 1, \ldots, t$ at least one of the following assertions holds (where indices are considered modulo t):

(i) $N(w_i) \cap Q = Q \setminus \{v_i, v_{i+1}\}$, for some $w_i \in V(H) \setminus Q$.

(ii) $N(u_i) \cap Q = Q \setminus \{v_i\}$ and $N(z_i) \cap Q = Q \setminus \{v_{i+1}\}$, for some adjacent vertices $u_i, z_i \in V(H) \setminus Q$.

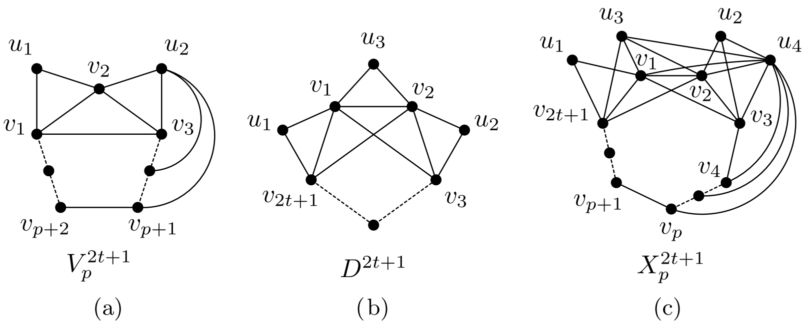

Fig. 3. Families of minimally not balanced HCA graphs: (a) The family V_p^{2t+1}. The dotted paths joining v_3 and v_{p+1} and joining v_{p+2} and v_1 represent chordless even paths, not simultaneously empty. All vertices of the dotted path joining v_3 to v_{p+1} are adjacent to u_2. (b) The family D^{2t+1}. The dotted path joining v_3 and v_{2t+1} represents a non-empty even path of length $2t-2$. (c) The family X_p^{2t+1}. The dotted paths joining v_4 and v_p and joining v_{p+1} and v_{2t+1} represent any chordless even paths, both of them possibly empty, even simultaneously. The vertices of the dotted path joining v_4 to v_p are all adjacent to u_4.

In order to give a minimal forbidden induced subgraphs characterization of balancedness within HCA graphs, we introduce the following graph families.

- For each $t \geq 2$ and even p with $2 \leq p \leq 2t$, we define the graph V_p^{2t+1} with vertex set $\{v_1, v_2, \ldots, v_{2t+1}, u_1, u_2\}$, such that $v_1 v_2 \ldots v_{2t+1} v_1$ is a cycle whose only chord is $v_1 v_3$, $N(u_1) = \{v_1, v_2\}$, and $N(u_2) = \{v_2, v_3, \ldots, v_{p+1}\}$.
- For $t \geq 2$, let the graph D^{2t+1} have vertex set $\{v_1, v_2, \ldots, v_{2t+1}, u_1, u_2, u_3\}$, such that $v_1 v_2 \ldots v_{2t+1} v_1$ is a cycle whose only chords are $v_{2t+1} v_2$ and $v_1 v_3$, $N(u_1) = \{v_{2t+1}, v_1\}$, $N(u_2) = \{v_2, v_3\}$, and $N(u_3) = \{v_1, v_2\}$.
- For $t \geq 2$ and each even p with $4 \leq p \leq 2t$, the graph X_p^{2t+1} has vertex set $\{v_1, v_2, \ldots, v_{2t+1}, u_1, u_2, u_3, u_4\}$ such that $v_1 v_2 \ldots v_{2t+1} v_1$ is a cycle whose only chords are $v_{2t+1} v_2$ and $v_1 v_3$, $N(u_1) = \{v_{2t+1}, v_1\}$, $N(u_2) = \{v_2, v_3, u_4\}$, $N(u_3) = \{v_{2t+1}, v_1, v_2, u_4\}$, and $N(u_4) = \{v_1, v_2, v_3, \ldots, v_p, u_2, u_3\}$.

In the three above families of graphs, $C = v_1 v_2 \ldots v_{2t+1} v_1$ is an unbalanced cycle and consequently all their members are not balanced. The three graph families are schematically represented in Figure 3.

As one of our main results, we characterize those HCA graphs that are balanced by minimal forbidden induced subgraphs as follows

Theorem 2.1 *Let G be a HCA graph. G is balanced if and only if G contains none of the following as induced subgraph:*

(i) *odd holes,* $\overline{C_7}$*,*

(ii) *3-sun, 1-pyramid, 2-pyramid,*

(iii) V_p^{2t+1}, D^{2t+1}, X_p^{2t+1} *for any* $t \geq 2$ *and any valid* p*.*

The proof of Theorem 2.1 is based on Theorem 1.1, i.e., we show that any minimal odd extended sun H which is a HCA graph is one of the graphs of (i), (ii), or (iii). The crucial part of the proof is showing that if H is not an odd hole and has an unbalanced cycle C of length at least 7 then C has at most two chords, each of which is short (i.e., joining two vertices at distance two within C) and if C has two chords then they cross (where two chords ab and cd of C *cross* if a, c, b, d appear in that order in C).

In fact, we can extend Theorem 2.1 to a superclass of HCA graphs, namely the class of CA graphs containing no induced net, S_4, or U_4 (see Figure 4).

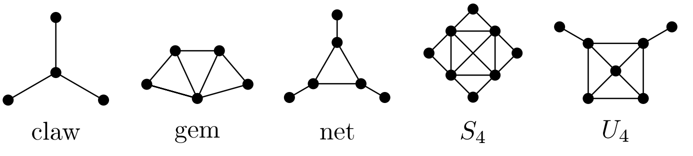

Fig. 4. Some small graphs

For each $t \geq 3$, let S_t denote the *complete t-sun* consisting of a clique $Q_t = \{w_1, w_2, \ldots, w_t\}$ and vertices $v_1, v_2, \ldots, v_t$ such that v_i is adjacent to exactly w_i and w_{i+1} for each $i = 1, 2, \ldots, t$ (subindices are considered modulo t). The complement of S_t is denoted by $\overline{S_t}$. Note that obstacles for HCA graphs as defined in [11] are not necessarily *minimal* (i.e., there are obstacles that contain proper induced obstacles) and even there are minimal obstacles that are not CA graphs. The following lemma characterizes those minimal obstacles that are CA graphs and contain no induced 1-pyramid or 2-pyramid.

Lemma 2.2 *Let* H *be a* {*1-pyramid,2-pyramid*}*-free minimal obstacle which is a CA graph. Then,* H *is 3-pyramid,* U_4*, or* $\overline{S_t}$ *for some* $t \geq 3$*.*

Lemma 2.2 and the characterization of HCA graphs given in [11] can be used to extend Theorem 2.1 to {net,S_4,U_4}-free CA graphs as follows.

Corollary 2.3 *Let* G *be a* {*net,*S_4*,*U_4}*-free CA graph.* G *is balanced if and only if* G *contains none of the following as induced subgraph:*

(i) *odd holes,* $\overline{C_7}$*,*

(ii) *the pyramids,*

(iii) V_p^{2t+1}, D^{2t+1}, X_p^{2t+1} *for any* $t \geq 2$ *and any valid* p*.*

2.2 Balancedness of Claw-Free Circular-Arc Graphs

Next, we characterize the CA graphs without induced claw that are balanced (see Figure 4 for a claw). Note that the class of claw-free CA graphs encompasses all *proper circular-arc graphs*, i.e., all CA graphs admitting a CA model in which no arc properly contains another.

Our result relies on Corollary 2.3 and a result derived from [3] that a claw-free CA graph containing an induced net is either a net, or contains an induced 3-sun, or has true twins (i.e., two adjacent vertices with the same closed neighborhoods).

Theorem 2.4 *Let G be a claw-free CA graph. G is balanced if and only if G contains none of the following as induced subgraph:*

(i) *odd holes, $\overline{C_7}$,*

(ii) *the pyramids.*

The list of minimal forbidden induced subgraphs implies further:

Corollary 2.5 *A claw-free CA graph G is balanced if and only if G is perfect and hereditary clique-Helly.*

2.3 Balancedness of Gem-Free Circular-Arc Graphs

In this subsection we will give a minimal forbidden induced subgraph characterization of those gem-free CA graphs that are balanced (see Figure 4 for a gem). Our prove relies on Corollary 2.3 and the following lemma.

Lemma 2.6 *Let G be a gem-free CA graph that contains an induced net or an induced U_4. Then, G has either true twins or a cutpoint.*

Here, a cutpoint is a vertex whose removal increases the number of connected components of the graph.

We characterize balanced graphs among gem-free CA graphs as follows.

Theorem 2.7 *Let G be a gem-free CA graph. G is balanced if and only if G contains none of the following as induced subgraph:*

(i) *odd holes,*

(ii) *3-pyramid.*

The list of minimal forbidden induced subgraphs implies further:

Corollary 2.8 *A gem-free CA graph G is balanced if and only if G is perfect and hereditary clique-Helly.*

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