

$C^{1,\alpha}$ -ESTIMATES FOR THE NEAR FIELD REFRACTOR

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ABSTRACT. We establish local $C^{1,\alpha}$ estimates for one source near field refractors under structural assumptions on the target, and with no assumptions on the smoothness of the densities.

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1. INTRODUCTION

The main purpose in this paper is to prove Hölder estimates for gradients of weak solutions to the near field refractor problem introduced in [GH14], where existence of weak solutions is proved as a consequence of a general abstract method applicable also in other situations. The set up for the problem is as follows. Suppose we have a domain $\Omega \subset S^{n-1}$ and a domain Σ contained in an n dimensional surface in \mathbb{R}^n ; here, Ω denotes the set of incident directions, and Σ denotes the target domain, receiver, or screen to be illuminated. Let n_1

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and n_2 be the indices of refraction of two homogeneous and isotropic media I and II, respectively. From a point source at the origin, surrounded by medium I, radiation emanates in each direction x with intensity $f(x)$ for $x \in \Omega$, and the target Σ is surrounded by medium II. A *near field refractor* is an optical surface \mathcal{R} , interface between media I and II, such that all rays refracted by \mathcal{R} into medium II, in accordance with the Snell law, are received at the surface Σ with prescribed radiation intensity distribution given by a measure ν . Assuming no loss of energy in this process, we have the conservation of energy equation $\int_{\Omega} f(x) dx = \nu(\Sigma)$. Under visibility assumptions on the target and conditions to avoid total reflection, existence of solutions to this problem is proved in [GH14].

The problem solved in the present paper is that weak solutions are C^1 and their gradients are locally Hölder continuous under no smoothness assumptions on the density f and the measure ν . In fact, we prove a more general result, Theorem 5.5, valid for more general near field refractors in the sense of Definition 5.1. Our assumptions are of structural nature, that is, they depend on the relative location of the target, its visibility from the cone of incident directions, and its convexity; see Section 2.1. A major difficulty with the near field refractor problem is that solutions have a complicated structure given by Descartes ovals that often require difficult analytical estimates, and it does not have an optimal mass transport structure.

To place our results in perspective we mention that regularity results for one source far field reflectors are in [CGH08], results for near field parallel refractors are in [GT15] and [AGT16], and results for generated Jacobian equations, including reflector problems, are in [GK17]. Numerical methods are developed in [LGM17] to solve the one source far field refractor problem, in [GM19] to solve the near field, and in [AG17] to solve generated Jacobian equations.

The organization of the paper is as follows. Section 2 contains structural conditions on the target as well as a discussion on them and an example. Analytical estimates for ovals and a maximum principle, Lemma 3.4, of the type developed in [Loe09] and [KM10] are contained in Section 3. More analytical estimates for derivatives of ovals are in Section 4. Section 5 contains the Hölder estimates, where the main result is Theorem 5.4 from which we deduce as consequences Theorems 5.5 and 5.6.

2. PRELIMINARIES, STRUCTURAL ASSUMPTIONS, AND EXAMPLES

Recall that a Descartes oval is the set $\mathcal{O}(Y, b) = \{X \in \mathbb{R}^n : |X| + \kappa|X - Y| = b\}$, with $\kappa|Y| < b < |Y|$. Here $\kappa = n_2/n_1$, where n_1 is the refractive index of the material inside the oval and n_2 is the refractive index of the material outside. We assume throughout that $\kappa < 1$, which is the most interesting from an optical point of view (when $\kappa > 1$ the arguments are similar). From the Snell law, a ray emanating from the origin with unit direction x is refracted at the point $X \in \mathcal{O}(Y, b)$ into the point Y provided that

$$(2.1) \quad \frac{X}{|X|} \cdot \frac{Y - X}{|Y - X|} \geq \kappa;$$

an inequality that by the equation of the oval is equivalent to $x \cdot Y \geq b$. The polar equation of the oval is $\mathcal{O}(Y, b) = \{\rho(x, Y, b)x : x \in S^{n-1}\}$ where

$$(2.2) \quad \rho(x, Y, b) = \frac{b - \kappa^2 x \cdot Y - \sqrt{(b - \kappa^2 x \cdot Y)^2 - (1 - \kappa^2)(b^2 - \kappa^2|Y|^2)}}{1 - \kappa^2}.$$

For a geometric analysis and estimates for Descartes ovals we refer to [GH14, Sec. 4]. If we specify a point X_0 on the oval $\mathcal{O}(Y, b)$, then $b = |X_0| + \kappa|X_0 - Y|$ and it will be useful to introduce the function

$$(2.3) \quad h(x, Y, X_0) = \rho(x, Y, b),$$

with the point X_0 so that $\frac{X_0}{|X_0|} \cdot \frac{Y - X_0}{|Y - X_0|} \geq \kappa$. For $\Omega \subseteq S^{n-1}$ open and constants $0 < c_1 < c_2$, we let

$$\Gamma_{c_1 c_2} = \{rx : x \in \Omega, c_1 \leq r \leq c_2\}.$$

2.1. Structural assumptions on the target Σ . We begin introducing the following notion of curve in S^{n-1} that will be used to state our assumptions.

Let $x_0, \hat{m}, \bar{m} \in S^{n-1}$ with $\bar{m} \cdot x_0 \geq \kappa$ and $\hat{m} \cdot x_0 \geq \kappa$. By definition, $[\bar{m}, \hat{m}]_{x_0}$ denotes the curve obtained intersecting the triangle with vertices \bar{m} , \hat{m} , and x_0/κ with the sphere S^{n-1} . Notice that since $\kappa < 1$, the point x_0/κ is outside the unit ball. In this triangle, the side joining \hat{m} and \bar{m} , is given by $m_\lambda = (1 - \lambda)\bar{m} + \lambda\hat{m}$, with $0 \leq \lambda \leq 1$. Each point $m \in [\bar{m}, \hat{m}]_{x_0}$ can then be obtained intersecting the line $\frac{x_0}{\kappa} + \beta \xi$ with the sphere S^{n-1} , where $\xi = m_\lambda - \frac{1}{\kappa}x_0$, $\beta \in \mathbb{R}$. Solving for β yields

$$(2.4) \quad \beta(\lambda) = \frac{-x_0 \cdot \xi - \sqrt{(x_0 \cdot \xi)^2 - (1 - \kappa^2)|\xi|^2}}{\kappa|\xi|^2},$$

since the point $\frac{x_0}{\kappa} + \beta \xi$ is inside the triangle so $0 < \beta < 1$. Therefore, we obtain the parametrization

$$(2.5) \quad [\bar{m}, \hat{m}]_{x_0} = \left\{ m(\lambda) = \frac{1}{\kappa} x_0 + \beta(\lambda) \left(m_\lambda - \frac{1}{\kappa} x_0 \right), \lambda \in [0, 1] \right\}.$$

In particular, for $m \in [\bar{m}, \hat{m}]_{x_0}$ we can write

$$(2.6) \quad m = \frac{1}{\kappa} x_0 + \bar{\beta} \left(\bar{m} - \frac{1}{\kappa} x_0 \right) + \hat{\beta} \left(\hat{m} - \frac{1}{\kappa} x_0 \right)$$

with $\bar{\beta}, \hat{\beta} \geq 0$ and $\bar{\beta} + \hat{\beta} \leq 1$; $\bar{\beta} = (1 - \lambda)\beta(\lambda)$, $\hat{\beta} = \lambda\beta(\lambda)$. Notice that $m(\lambda) \cdot x_0 \geq \kappa$ for $0 \leq \lambda \leq 1$ since $\beta(\lambda) \leq 1$ and $\kappa < 1$.

We next introduce our structural assumptions.

H.A For each $X \in \Gamma_{c_1 c_2}$, let $C_X = \left\{ Y : \frac{X}{|X|} \cdot \frac{Y - X}{|Y - X|} \geq \kappa \right\}$ be the cone with vertex X , axis $X/|X|$, and opening $\arccos \kappa$. Set

$$C_\Omega = \bigcap_{X \in \Gamma_{c_1 c_2}} C_X.$$

We assume the following:

- (a) $\Sigma \subset C_\Omega$, so (2.1) holds for all $Y \in \Sigma$ and $X \in \Gamma_{c_1 c_2}$;
- (b) For each $X \in \Gamma_{c_1 c_2}$ there exists a set $E(X) \subset \{m \in S^{n-1} : m \cdot x \geq \kappa, x = X/|X|\}$ and a continuous function $s_X : E(X) \rightarrow \mathbb{R}^+$ such that

$$\Sigma = \{X + s_X(m) m : m \in E(X)\},$$

with the set $E(X)$ satisfying $[\bar{m}, \hat{m}]_x \subset E(X)$ for all $\bar{m}, \hat{m} \in E(X)$, with $x = X/|X|$;

- (c) The family of functions $\{s_X\}_{X \in \Gamma_{c_1 c_2}}$ is uniformly Lipschitz continuous, i.e., there exists a constant $C > 0$ such that $|s_X(m_1) - s_X(m_2)| \leq C |m_1 - m_2|$ for all $m_1, m_2 \in E(X)$ and $X \in \Gamma_{c_1 c_2}$.

H.B Let $C(\kappa) = \kappa \left(\sqrt{1 + (1 + \kappa)^{-2}} - 1 \right)$. We assume $\frac{|X|}{|Y - X|} \leq C(\kappa)$ for all $X \in \Gamma_{c_1 c_2}$ and $Y \in \Sigma$. Notice that this holds if $\text{dist}(\Gamma, \Sigma) \geq c_2/C(\kappa)$.

H.C There exists a constant $0 \leq \mu < \kappa$ such that for all $X_0 \in \Gamma_{c_1 c_2}$ and $\bar{m}, \hat{m} \in E(X_0)$, the function s_{X_0} satisfies the following concavity condition

$$\frac{1}{s_{X_0}(m(\lambda))} + \frac{\mu}{|X_0|} \geq \bar{\beta}(\lambda) \left(\frac{1}{s_{X_0}(\bar{m})} + \frac{\mu}{|X_0|} \right) + \hat{\beta}(\lambda) \left(\frac{1}{s_{X_0}(\hat{m})} + \frac{\mu}{|X_0|} \right)$$

for $0 \leq \lambda \leq 1$, with $\bar{\beta}(\lambda) = (1 - \lambda)\beta(\lambda)$ and $\hat{\beta}(\lambda) = \lambda\beta(\lambda)$, $\beta(\lambda)$ defined in (2.4) (depending on x_0), and $m(\lambda)$ from (2.5).

H.D Given $X_0 \in \Gamma_{c_1 c_2}$, $\bar{Y}, \hat{Y} \in \Sigma$, let $\bar{m} = \frac{\bar{Y} - X_0}{|\bar{Y} - X_0|}$ and $\hat{m} = \frac{\hat{Y} - X_0}{|\hat{Y} - X_0|}$; $x_0 = X_0/|X_0|$.
Let $[\bar{Y}, \hat{Y}]_{X_0}$ be the curve defined by

$$[\bar{Y}, \hat{Y}]_{X_0} = \{Y(\lambda) = X_0 + s_{X_0}(m(\lambda)) m(\lambda) : \lambda \in [0, 1]\},$$

where $m(\lambda)$ is the parametrization of $[\bar{m}, \hat{m}]_{x_0}$ defined in (2.5). We assume that there exist positive constants μ_0 and C such that for all $X_0 \in \Gamma_{c_1 c_2}$, $\bar{Y}, \hat{Y} \in \Sigma$, we have

$$H^{n-1} \left(N_\mu \left(\left\{ [\bar{Y}, \hat{Y}]_{X_0} : \frac{1}{4} \leq \lambda \leq \frac{3}{4} \right\} \cap \Sigma \right) \right) \geq C \mu^{n-2} |\bar{Y} - \hat{Y}|,$$

for each $\mu \leq \mu_0$, where H^{n-1} denotes the $n - 1$ dimensional Hausdorff measure in \mathbb{R}^n and N_μ denotes the μ -neighborhood in \mathbb{R}^n .

Throughout the paper, a structural constant refers to a constant depending only on some or all of the constants in the structural conditions above.

Remark 2.1. We begin noticing that from **H.A** and **H.B** we get that s_X is bounded below:

$$(2.7) \quad s_X(m) \geq c_1/C(\kappa),$$

for all $X \in \Gamma_{c_1 c_2}$ and $m \in E(X)$.

Also from **H.A** and **H.B** we get for $\hat{Y} = X + s_X(\hat{m}) \hat{m}$ and $\bar{Y} = X + s_X(\bar{m}) \bar{m}$ that

$$(2.8) \quad |\hat{m} - \bar{m}| \leq 2 \min \left\{ \frac{1}{|\bar{Y} - X|}, \frac{1}{|\hat{Y} - X|} \right\} |\bar{Y} - \hat{Y}| \leq C |\bar{Y} - \hat{Y}|.$$

Indeed, from **H.A**

$$\hat{m} - \bar{m} = \frac{\hat{Y} - X}{|\hat{Y} - X|} - \frac{\bar{Y} - X}{|\bar{Y} - X|} = \frac{|\bar{Y} - X|(\hat{Y} - \bar{Y}) + (\bar{Y} - X)(|\bar{Y} - X| - |\hat{Y} - X|)}{|\hat{Y} - X||\bar{Y} - X|},$$

so

$$|\bar{m} - \hat{m}| \leq \frac{2|\hat{Y} - \bar{Y}|}{|\hat{Y} - X|}, \quad |\bar{m} - \hat{m}| \leq \frac{2|\hat{Y} - \bar{Y}|}{|\bar{Y} - X|}.$$

Therefore from **H.B** the desired inequality follows since $X \in \Gamma_{c_1 c_2}$.

Concerning each of our assumptions we mention the following. Assumption **H.A** guarantees that each ray from 0 striking $X \in \Gamma_{c_1 c_2}$ can be refracted into Σ and the refracted ray intersects Σ at only one point. Assumption **H.B** says that $\Gamma_{c_1 c_2}$ is sufficiently far from the target Σ^* and it will be applied to show that the ovals

*The value of the constant $C(\kappa)$ in **H.B** is only needed in Lemma 3.1.

used in the definition of refractor have controlled derivatives. Assumption **H.C** is crucial to obtain regularity of refractors and is akin to the condition (AW) first introduced in [MTW05] and later considered in [Loe09] and [KM10]. Assumption **H.D** is a form of convexity of Σ with respect to points $X \in \Gamma_{c_1 c_2}$.

Remark 2.2. We relate now the structural assumptions introduced with the following assumptions needed to prove existence of refractors [GH14, Sect. 5]:

H.1 there exists τ , with $0 < \tau < 1 - \kappa$, such that $x \cdot Y \geq (\kappa + \tau)|Y|$ for all $x \in \Omega$ and $Y \in \Sigma$;

H.2 if $0 < r_0 < \frac{\tau}{1 + \kappa} \text{dist}(0, \Sigma)$ and $Q_{r_0} = \{tx : x \in \Omega, 0 < t < r_0\}$, then given $X \in Q_{r_0}$ each ray emanating from X intersects Σ in at most one point.

We show that if τ is sufficiently small, then **H.1** and **H.2** imply **H.A** (a) and **H.B**. We first claim that there are positive constants $C_{\tau, \kappa}$ and $\hat{C}_{\tau, \kappa}$ such that if $\frac{X}{|X|} \cdot \frac{Y}{|Y|} \geq \kappa + \tau$, and $|Y| \geq C_{\tau, \kappa}|X|$, for all $Y \in \Sigma$ and $X \in \Gamma_{c_1 c_2}$, then

$$\frac{Y - X}{|Y - X|} \cdot \frac{X}{|X|} \geq \kappa, \text{ and } \frac{|X|}{|Y - X|} \leq \hat{C}_{\tau, \kappa}$$

with

$$C_{\tau, \kappa} = \frac{\sqrt{1 - \kappa^2}}{(\kappa + \tau) \sqrt{1 - \kappa^2} - \kappa \sqrt{1 - (\kappa + \tau)^2}}, \quad \hat{C}_{\tau, \kappa} = \frac{1}{C_{\tau, \kappa} - 1}.$$

Then the desired relation between the assumptions follows noticing that $C_{\tau, \kappa} \rightarrow \infty$ and $\hat{C}_{\tau, \kappa} \rightarrow 0$ as $\tau \rightarrow 0$. To prove the claim, fix X and calculate the intersection between the cones $C_1 = \left\{ Y : \frac{X}{|X|} \cdot \frac{Y}{|Y|} = \kappa + \tau \right\}$, and $C_2 = \left\{ Y : \frac{Y - X}{|Y - X|} \cdot \frac{X}{|X|} = \kappa \right\}$. From the sine law, it is easy to see that if Y is in the intersection of these cones, then $|Y| = C_{\tau, \kappa}|X|$. So $|Y| \geq C_{\tau, \kappa}|X|$ and Y is in the interior of C_1 , then Y is in the interior of C_2 , and $|Y - X| \geq |Y| - |X| \geq (C_{\tau, \kappa} - 1)|X|$.

Remark 2.3. When the target Σ is C^2 one can give a differential condition that is equivalent to **H.C**. To do this, we first need to have another parametrization of the curve $[\bar{m}, \hat{m}]_{x_0}$. For $Y \in \Sigma$, recall that from (5.5)

$$\nabla^T h(x, Y, X_0) = \nabla_x h(x, Y, X_0) - \langle \nabla_x h(x, Y, X_0), x \rangle x,$$

and since $Y = X_0 + s m$, for some $m \in E(X_0)$, we have from (4.4) that

$$\nabla^T h(x_0, Y, X_0) = \kappa |X_0| \frac{m - \langle m, x_0 \rangle x_0}{1 - \kappa \langle m, x_0 \rangle} := v = T_{X_0}(m), \quad x_0 = X_0 / |X_0|.$$

Notice that $v \perp x_0$ and $|v|^2 \leq \frac{\kappa^2 |X_0|^2}{1 - \kappa^2}$. We will write m in terms of v , with $\langle m, x_0 \rangle \geq \kappa$ and $|m| = 1$. First note that $\langle m, x_0 \rangle = \frac{|v|^2 + |X_0| \sqrt{\kappa^2 |X_0|^2 - (1 - \kappa^2) |v|^2}}{\kappa (|v|^2 + |X_0|^2)}$, and thus

$$\frac{1 - \kappa \langle m, x_0 \rangle}{\kappa |X_0|} = \frac{1 - \kappa^2}{\kappa} \frac{1}{|X_0| + \sqrt{\kappa^2 |X_0|^2 - (1 - \kappa^2) |v|^2}} := t(v).$$

We can then write $m = \langle m, x_0 \rangle x_0 + t(v) v$ and so

$$m = m(v) := \frac{1}{\kappa} x_0 + t(v)(v - X_0).$$

Given $\bar{m}, \hat{m} \in S^{n-1}$ with $\langle \bar{m}, x_0 \rangle \geq \kappa$ and $\langle \hat{m}, x_0 \rangle \geq \kappa$, let

$$\bar{v} = \kappa |X_0| \frac{\bar{m} - \langle \bar{m}, x_0 \rangle x_0}{1 - \kappa \langle \bar{m}, x_0 \rangle}, \quad \text{and} \quad \hat{v} = \kappa |X_0| \frac{\hat{m} - \langle \hat{m}, x_0 \rangle x_0}{1 - \kappa \langle \hat{m}, x_0 \rangle}.$$

Letting $v_\gamma = (1 - \gamma)\bar{v} + \gamma\hat{v}$, we show that the curve $[\bar{m}, \hat{m}]_{x_0}$ in (2.5) can be parametrized as follows:

$$\tilde{m}(\gamma) = m(v_\gamma) = \frac{1}{\kappa} x_0 + t(v_\gamma)(v_\gamma - X_0), \quad 0 < \gamma < 1,$$

that is, $\tilde{m}(\gamma) = m(\lambda)$ with the change of parameter $\lambda = \frac{\gamma t(\bar{v})}{(1 - \gamma)t(\hat{v}) + \gamma t(\bar{v})}$ (we are abusing the notation $m(\lambda)$ and $m(v)$). In fact, from the definition of $\beta(\lambda)$

$$\bar{\beta} = (1 - \lambda)\beta(\lambda) = \frac{t(v_\gamma)(1 - \gamma)}{t(\bar{v})} \quad \text{and} \quad \hat{\beta} = \lambda\beta(\lambda) = \frac{t(v_\gamma)\gamma}{t(\hat{v})};$$

see the end of the proof of Lemma 3.4 for similar calculations with $\beta(\lambda)$. Also $\bar{m} = \frac{1}{\kappa} x_0 + t(\bar{v})(\bar{v} - X_0)$ and $\hat{m} = \frac{1}{\kappa} x_0 + t(\hat{v})(\hat{v} - X_0)$. Then,

$$\begin{aligned} m(\lambda) &= \frac{1}{\kappa} x_0 + \beta(\lambda) \left((1 - \lambda)\bar{m} + \lambda\hat{m} - \frac{1}{\kappa} x_0 \right) \\ &= \frac{1}{\kappa} x_0 + \frac{1}{\kappa} t(v_\gamma) \frac{(1 - \gamma)t(\hat{v}) + \gamma t(\bar{v})}{t(\hat{v})t(\bar{v})} \left((1 - \lambda)(\kappa\bar{m} - x_0) + \lambda(\kappa\hat{m} - x_0) \right). \end{aligned}$$

Since $\kappa\bar{m} - x_0 = \kappa t(\bar{v})(\bar{v} - X_0)$ and $\kappa\hat{m} - x_0 = \kappa t(\hat{v})(\hat{v} - X_0)$, substituting and simplifying yields $m(\lambda) = \tilde{m}(\gamma)$ as desired.

Therefore, with this reparametrization of the curve $[\bar{m}, \hat{m}]_{x_0}$ assumption **H.C** is then equivalent to

$$\left(\frac{1}{s_{X_0}(\tilde{m}(\gamma))} + \frac{\mu}{|X_0|} \right) \frac{1}{t(v_\gamma)} \geq (1 - \gamma) \left(\frac{1}{s_{X_0}(\tilde{m}(0))} + \frac{\mu}{|X_0|} \right) \frac{1}{t(v_0)} + \gamma \left(\frac{1}{s_{X_0}(\tilde{m}(1))} + \frac{\mu}{|X_0|} \right) \frac{1}{t(v_1)},$$

for $0 < \gamma < 1$, with $\tilde{m}(0) = \bar{m}$, $\tilde{m}(1) = \hat{m}$, $v_0 = \bar{v}$, and $v_1 = \hat{v}$. In other words, the function

$$\Phi(v) = \left(\frac{1}{s_{X_0}(m(v))} + \frac{\mu}{|X_0|} \right) \frac{1}{t(v)}$$

is a concave function of v for $|v|^2 \leq \frac{\kappa^2 |X_0|^2}{1 - \kappa^2}$ and $v \perp x_0$, i.e., concave in a $n - 1$ -dimensional disk. Thus, when Σ is C^2 , we obtain that **H.C** is equivalent to

$$\left. \frac{d^2}{dt^2} (\Phi(v + t\xi)) \right|_{t=0} \leq 0$$

for all $v \perp x_0$ with $|v|^2 \leq \frac{\kappa^2 |X_0|^2}{1 - \kappa^2}$ and for all $\xi \perp x_0$. The domain of s_{X_0} is $E(X_0)$, and the domain of $\Phi(v)$ is $T_{X_0}(E(X_0))$. The fact that $E(X_0)$ satisfies the convexity assumption that $[\bar{m}, \hat{m}]_{x_0} \subset E(X_0)$ for all $\bar{m}, \hat{m} \in E(X_0)$ is equivalent that $T_{X_0}(E(X_0))$ is a convex set in the classical sense on the hyperplane perpendicular to x_0 .

2.2. Examples. We will construct $\Omega \subset S^{n-1}$ and a target Σ so that the structural assumptions are satisfied. Notice that if $\Omega \subset \Omega'$, then $C_{\Omega'} \subset C_{\Omega}$, with c_1, c_2 fixed. To do this construction, we will first choose Ω' and calculate $C_{\Omega'}$. We will then choose a target $\Sigma \subset C_{\Omega'}$ and next pick $\Omega \subset \Omega'$. It will then follow that $\Sigma \subset C_{\Omega}$.

Let $\theta = \arccos \kappa$ and $\Omega' = \{x \in S^{n-1} : x \cdot e_n \geq \cos(\theta/2)\}$ where e_n is the unit vector in the vertical direction x_n .

Pick constants $c_1 = 1$ and $c_2 > 1$, and let $Y_0 = 2c_2 \cos(\theta/2) e_n$. We claim that $C_{\Omega'} = \left\{ Y : \frac{Y - Y_0}{|Y - Y_0|} \cdot e_n \geq \cos(\theta/2) \right\} := E$, the cone with vertex at Y_0 direction e_n and opening $\theta/2$. To prove this, let $Y \in E$ and we want to show that $Y \in C_X$ for all $X \in \Gamma_{c_1 c_2}$ (defined with Ω'). That $Y \in E$ means $\angle(Y - Y_0, e_n) \leq \theta/2$, where \angle denotes the angle between the vectors. Obviously, $Y \in C_X$ if and only if $\angle(Y - X, X) \leq \theta$. From the choice of Y_0 , it is easy to see that $\angle(Y_0 - X, X) \leq \theta$ for all $X \in \Gamma_{c_1 c_2}$, i.e., $Y_0 \in C_X$.

We have $Y = Y_0 + v$ with $\angle(v, e_n) \leq \theta/2$. Let $\tilde{Y} = X + v$. Since $\angle(\tilde{Y} - X, X) = \angle(v, X) \leq \angle(v, e_n) + \angle(e_n, X) \leq \frac{\theta}{2} + \frac{\theta}{2}$, it follows that $\tilde{Y} \in C_X$. Then from the convexity of C_X we obtain $\frac{Y_0 + \tilde{Y}}{2} \in C_X$. Since $Y = \tilde{Y} + Y_0 - X$, it follows that $\angle(Y - X, X) = \angle(\tilde{Y} + Y_0 - 2X, X) = \angle\left(\frac{\tilde{Y} + Y_0}{2} - X, X\right) \leq \theta$. So $Y \in C_X$ and the claim is proved.

Now, we choose Σ the planar disk centered at 0 with radius R at height M , that is,

$$\Sigma = \{Y = (Y', Y_n) : |Y'| \leq R, Y_n = M\}.$$

If we pick $M = C + 2c_2 \cos(\theta/2) = C + 2c_2 \sqrt{\frac{1+\kappa}{2}}$ with C any positive constant and pick $R \leq \sqrt{\frac{1-\kappa}{1+\kappa}}C$, then it is easy to verify that $\Sigma \subset E = C_{\Omega'}$.

Next, we will choose $\Omega \subset \Omega'$ so that if $\bar{Y}, \hat{Y} \in \Sigma$, then $[\bar{m}, \hat{m}]_x \subseteq E(X)$, for all $X \in \Gamma_{c_1 c_2}$, where $\bar{m} = \frac{\bar{Y} - X}{|\bar{Y} - X|}$, $\hat{m} = \frac{\hat{Y} - X}{|\hat{Y} - X|}$ and $E(X)$ is the set of visibility directions in S^{n-1} of the target Σ from the point X . First notice that since $\cos(\theta/2) = \sqrt{\frac{1+\kappa}{2}}$ and $\sin(\theta/2) = \sqrt{\frac{1-\kappa}{2}}$, we have $x \in \Omega'$ if and only if $\frac{|x'|}{x_n} \leq \sqrt{\frac{1-\kappa}{1+\kappa}}$, with $x_n > 0$. Now define

$$\Omega = \left\{ x \in S^{n-1} : x_n > 0, \frac{|x'|}{x_n} \leq \frac{R}{M} \right\}.$$

Since $\frac{R}{M} \leq \frac{\sqrt{\frac{1-\kappa}{1+\kappa}}C}{C + 2c_2 \sqrt{\frac{1-\kappa}{1+\kappa}}}$, we obtain that $\Omega \subset \Omega'$.

If $X \in \Gamma_{c_1 c_2}$ and $\bar{Y}, \hat{Y} \in \Sigma$, then we show $[\bar{m}, \hat{m}]_x \subseteq E(X)$, where $\bar{m} = \frac{\bar{Y} - X}{|\bar{Y} - X|}$ and $\hat{m} = \frac{\hat{Y} - X}{|\hat{Y} - X|}$ and $x = \frac{X}{|X|}$. Since $\bar{Y}, \hat{Y} \in C_{\Omega}$, we have $\bar{m} \cdot x \geq \kappa$ and $\hat{m} \cdot x \geq \kappa$. We have from (2.6) that $m = \frac{1}{\kappa}x + \bar{\beta}(\bar{m} - \frac{1}{\kappa}x) + \hat{\beta}(\hat{m} - \frac{1}{\kappa}x)$ for $m \in [\bar{m}, \hat{m}]_x$, and we need to show that the ray $X + sm$ strikes Σ for some s (that is, $s = s_X(m)$). If $s = \frac{M - X_n}{m_n}$, then will show that $Y = X + \frac{M - X_n}{m_n}m \in \Sigma$. Indeed, write $Y = (Y', Y_n)$. Clearly $Y_n = M$. If $|\bar{Y}'|, |\hat{Y}'| \leq R$, will prove that $|Y'| \leq R$. We have with $m = (m', m_n)$ that

$$Y' = X' + (M - X_n) \frac{m'}{m_n} = X' + (M - X_n) \frac{\frac{1}{\kappa}x' (1 - \bar{\beta} - \hat{\beta}) + \bar{\beta}\bar{m}' + \hat{\beta}\hat{m}'}{m_n},$$

and

$$Y' = \frac{\bar{\beta}\bar{m}_n}{m_n} \left(X' + (M - X_n) \frac{\bar{m}'}{\bar{m}_n} \right) + \frac{\hat{\beta}\hat{m}_n}{m_n} \left(X' + (M - X_n) \frac{\hat{m}'}{\hat{m}_n} \right) \\ + \left(1 - \frac{\bar{\beta}\bar{m}_n + \hat{\beta}\hat{m}_n}{m_n} \right) X' + \frac{M - X_n}{m_n} \left(\frac{1}{\kappa} x' (1 - \bar{\beta} - \hat{\beta}) \right).$$

Combining the last two terms and simplifying yields

$$Y' = \frac{\bar{\beta}\bar{m}_n}{m_n} \bar{Y}' + \frac{\hat{\beta}\hat{m}_n}{m_n} \hat{Y}' + \frac{1}{\kappa} x' \frac{1 - \bar{\beta} - \hat{\beta}}{m_n} M.$$

Therefore, $|Y'| \leq R \frac{\bar{\beta}\bar{m}_n + \hat{\beta}\hat{m}_n}{m_n} + \frac{1}{\kappa} |x'| \frac{1 - \bar{\beta} - \hat{\beta}}{m_n} M \leq R$, where we have used that $|x'| \leq \frac{x_n R}{M}$ since $x \in \Omega$. Thus, $[\bar{m}, \hat{m}]_x \subseteq E(X)$.

In addition,

$$\frac{1}{s} = \frac{m_n}{M - X_n} = \frac{\frac{1}{\kappa} x_n (1 - (\bar{\beta} + \hat{\beta})) + \bar{\beta}\bar{m}_n + \hat{\beta}\hat{m}_n}{M - X_n} \\ \geq \frac{\bar{\beta}\bar{m}_n}{M - X_n} + \frac{\hat{\beta}\hat{m}_n}{M - X_n} = \frac{\bar{\beta}}{\bar{s}} + \frac{\hat{\beta}}{\hat{s}}.$$

and so the concavity assumption in **H.C** holds with $\mu = 0$.

Therefore the example described satisfies the assumptions **H.A**, and **H.C**. In order to satisfy **H.B**, it is enough to keep c_2 fixed and pick C large enough. It remains to verify that example satisfies **H.D**. For this we use the following lemma.

Lemma 2.4. *Let $\gamma : [a, b] \rightarrow \mathbb{R}^n$ be a smooth curve such that $|\gamma'(t)| = 1$ and $|\gamma''(t)| \leq M_1$ for all $t \in [a, b]$. In addition, assume $M_2|t_1 - t_2| \leq |\gamma(t_1) - \gamma(t_2)|$ for all $t_1, t_2 \in [a, b]$. Let T_t denote the hyperplane passing through $\gamma(t)$ with normal $\gamma'(t)$ and let $D_\mu(t) = B_\mu(\gamma(t)) \cap T_t$, and $N_\mu = \bigcup_{t \in [a, b]} D_\mu(t)$. Then, there exists μ_0 and C depending only on M_1, M_2 such that for $\mu \leq \mu_0$, we have $H^n(N_\mu) \geq C\mu^{n-1}|\gamma(b) - \gamma(a)|$.*

Proof. First observe that there exists μ_0 such that if $\mu \leq \mu_0$, then $D_\mu(t_1) \cap D_\mu(t_2) = \emptyset$, for $t_1 \neq t_2$.

Consider the cylinder in \mathbb{R}^n given by $D \times [a, b] = \{(x', t) : |x'| \leq \mu; t \in [a, b]\}$, where $D = \{(x', 0) : |x'| \leq \mu\}$, and define $F : D \times [a, b] \rightarrow N_\mu$ by

$$F(x', t) = \gamma(t) + A(t)(x', 0)$$

where $A(t)$ is the $n \times n$ matrix whose column vectors are $\{\eta_1(t), \dots, \eta_{n-1}(t), \gamma'(t)\}$ where $\eta_i(t)$ are chosen so that they are smooth with $A(t)A^T(t) = I$; here $(x', 0)$ is

a column vector. Notice that F is one to one and each disk $D \times \{t\}$ is mapped to $D_\mu(t)$. By the formula of change of variables

$$\begin{aligned} H^n(N_\mu) &= H^n(F(D \times [a, b])) = \int_{D \times [a, b]} |\det DF(x', t)| dx' dt \\ &\geq CH^n(D \times [a, b]) \geq C(b-a)\mu^{n-1} \geq C|\gamma(b) - \gamma(a)|\mu^{n-1}, \end{aligned}$$

provided that $|\det DF(x', t)| \geq C$ for some $C > 0$. Indeed, note that the matrix $DF(x', t)$ has column vectors given by $\eta_1(t), \dots, \eta_{n-1}(t)$ and its last column vector is $\gamma'(t) + x_1\eta'_1(t) + \dots + x_{n-1}\eta'_{n-1}(t)$, $x' = (x_1, \dots, x_{n-1})$. Therefore, we can expand $\det DF(x', t) = \det A(t) + \sum_{k=1}^{n-1} x_k \det \Lambda_k(t)$, where $\Lambda_k(t)$ is the matrix whose column vectors are $\eta_1(t), \dots, \eta_{n-1}(t), \eta'_k(t)$. It follows that $|\det DF(x', t)| \geq |\det A(t)| - \sum_{k=1}^{n-1} |x_k| |\det \Lambda_k(t)| \geq 1 - \mu_0 \sum_{k=1}^{n-1} |\det \Lambda_k(t)| \geq 1 - \mu_0 C$, with C depending only on M_1, M_2 and n . Then choosing μ_0 sufficiently small the lemma follows. \square

Finally, to verify that our example satisfies **H.D**, we notice that the curves $[\tilde{Y}, \hat{Y}]_{X_0}$ in the example satisfy the assumptions of the last lemma (the curves can be reparametrized to have $|\gamma'| = 1$) in \mathbb{R}^{n-1} so it is applicable to our case, obtaining constants that depend only on the structure. Also, varying the parameters c_2, C and R in the construction, we obtain a family of examples.

3. PRELIMINARY RESULTS FOR OVALS AND A MAXIMUM PRINCIPLE

We analyze the function $h(x, Y, X_0)$ for $Y \in \Sigma$ and $X_0 \in \Gamma_{c_1 c_2}$; $x \in S^{n-1}$, corresponding to the oval $O(Y, b)$. From **H.A**, we can write $Y = X_0 + s m$ with $m \in S^{n-1}$, $s = s_{X_0}(m) > 0$, $x_0 \cdot m \geq \kappa$, and recall that $b = |X_0| + \kappa|Y - X_0| = |X_0| + \kappa s$; $x_0 = X_0/|X_0|$. Hence

$$(3.1) \quad b - \kappa^2 x \cdot Y = |X_0| (1 - \kappa^2 x \cdot x_0) + \kappa s (1 - \kappa x \cdot m)$$

$$(3.2) \quad b^2 - \kappa^2 |Y|^2 = (1 - \kappa^2) |X_0|^2 + 2\kappa s |X_0| (1 - \kappa x_0 \cdot m).$$

Setting

$$(3.3) \quad B = \frac{b - \kappa^2 x \cdot Y}{1 - \kappa^2} \quad \text{and} \quad C = \frac{b^2 - \kappa^2 |Y|^2}{1 - \kappa^2},$$

we then can write

$$h(x, Y, X_0) = B - \sqrt{B^2 - C}.$$

In order to get to our crucial Lemma 3.4, first we need to prove three auxiliary lemmas.

Lemma 3.1. *Assume **H.A** and **H.B**. Let $\bar{Y}, \hat{Y} \in \Sigma$, and consider the ovals $\mathcal{O}(\bar{Y}, \bar{b}), \mathcal{O}(\hat{Y}, \hat{b})$. If $X_0 \in \Gamma_{c_1 c_2}$ is a common point to both ovals, then with the notation above we have*

$$\hat{B} \geq \bar{B} - \sqrt{\bar{B}^2 - \bar{C}}, \quad \forall x \in S^{n-1}.$$

Proof. Since $h(x, \bar{Y}, X_0) > 0$, it follows that $\bar{B} - \sqrt{\bar{B}^2 - \bar{C}} = \frac{\bar{C}}{\bar{B} + \sqrt{\bar{B}^2 - \bar{C}}} \leq \frac{\bar{C}}{\bar{B}}$. So, it is enough to show $\bar{C} \leq \bar{B} \hat{B}$, which is equivalent to show that

$$\begin{aligned} & \frac{(1 - \kappa^2)|X_0|^2 + 2\kappa \bar{s}|X_0|(1 - \kappa x_0 \cdot \bar{m})}{1 - \kappa^2} \\ & \leq \frac{(|X_0|(1 - \kappa^2 x \cdot x_0) + \kappa \bar{s}(1 - \kappa x \cdot \bar{m}))(|X_0|(1 - \kappa^2 x \cdot x_0) + \kappa \hat{s}(1 - \kappa x \cdot \hat{m}))}{(1 - \kappa^2)^2}, \end{aligned}$$

where we have used the notation $\bar{Y} = X_0 + \bar{s} \bar{m}$ and $\hat{Y} = X_0 + \hat{s} \hat{m}$, $\bar{s} = s_{X_0}(\bar{m})$, $\hat{s} = s_{X_0}(\hat{m}) > 0$, $x_0 = X_0/|X_0|$ with $x_0 \cdot \bar{m} \geq \kappa$, $x_0 \cdot \hat{m} \geq \kappa$. The last inequality is equivalent to

$$\begin{aligned} & |X_0|^2 \left(1 - \frac{(1 - \kappa^2 x \cdot x_0)^2}{(1 - \kappa^2)^2} \right) + \frac{2\kappa |X_0| \bar{s}(1 - \kappa x_0 \cdot \bar{m})}{1 - \kappa^2} \\ & \leq \frac{\kappa |X_0|(1 - \kappa^2 x \cdot x_0)}{(1 - \kappa^2)^2} (\bar{s}(1 - \kappa x \cdot \bar{m}) + \hat{s}(1 - \kappa x \cdot \hat{m})) \\ & \quad + \frac{\kappa^2}{(1 - \kappa^2)^2} \bar{s} \hat{s} (1 - \kappa x \cdot \bar{m})(1 - \kappa x \cdot \hat{m}). \end{aligned}$$

The left hand side of the last inequality is $\leq |X_0|^2 + 2\kappa |X_0| \bar{s}$ and the right hand side is $\geq \frac{\kappa^2}{(1 - \kappa^2)^2} \bar{s} \hat{s} (1 - \kappa)^2 = \frac{\kappa^2}{(1 + \kappa)^2} \bar{s} \hat{s}$. Therefore, if $|X_0|^2 + 2\kappa |X_0| \bar{s} \leq \frac{\kappa^2}{(1 + \kappa)^2} \bar{s} \hat{s}$, then the desired inequality follows. This is equivalent to

$$\frac{|X_0| |X_0|}{\bar{s}} + 2\kappa \frac{|X_0|}{\hat{s}} \leq \frac{\kappa^2}{(1 + \kappa)^2}$$

which follows from **H.B**. □

A second auxiliary calculus lemma is as follows.

Lemma 3.2. *Consider the two variable function $f(B, C) = B - \sqrt{B^2 - C}$ on the set $0 \leq C \leq B^2$ and $B \geq 0$. Fix (\bar{B}, \bar{C}) in that set, and suppose that $f(\bar{B}, \bar{C}) \leq \bar{B}$. Then*

$f(B, C) \leq f(\bar{B}, \bar{C})$ if and only if $C - \bar{C} \leq 2(B - \bar{B})f(\bar{B}, \bar{C})$. In addition, if $C - \bar{C} \leq 2(B - \bar{B})f(\bar{B}, \bar{C}) - E$ for some $E \geq 0$, then $f(B, C) \leq f(\bar{B}, \bar{C}) - \frac{E}{B + \sqrt{B^2 - C} - f(\bar{B}, \bar{C})}$.

Proof. Assume that $C - \bar{C} \leq 2(B - \bar{B})f(\bar{B}, \bar{C}) - E$, for some $E \geq 0$. Then

$$\begin{aligned} f(B, C) - f(\bar{B}, \bar{C}) &= \frac{C - \bar{C} - (f(B, C) + f(\bar{B}, \bar{C}))(B - \bar{B})}{\sqrt{B^2 - C} + \sqrt{\bar{B}^2 - \bar{C}}} \\ &\leq \frac{2(B - \bar{B})f(\bar{B}, \bar{C}) - E - (f(B, C) + f(\bar{B}, \bar{C}))(B - \bar{B})}{\sqrt{B^2 - C} + \sqrt{\bar{B}^2 - \bar{C}}} \\ &= \frac{(f(\bar{B}, \bar{C}) - f(B, C))(B - \bar{B}) - E}{\sqrt{B^2 - C} + \sqrt{\bar{B}^2 - \bar{C}}}. \end{aligned}$$

Therefore,

$$(f(B, C) - f(\bar{B}, \bar{C})) \left(1 + \frac{B - \bar{B}}{\sqrt{B^2 - C} + \sqrt{\bar{B}^2 - \bar{C}}} \right) \leq \frac{-E}{\sqrt{B^2 - C} + \sqrt{\bar{B}^2 - \bar{C}}}$$

which implies

$$f(B, C) \leq f(\bar{B}, \bar{C}) - \frac{E}{B + \sqrt{B^2 - C} - f(\bar{B}, \bar{C})}.$$

Conversely, assume $f(B, C) \leq f(\bar{B}, \bar{C})$, that is, $B - \sqrt{B^2 - C} \leq f(\bar{B}, \bar{C})$ which implies

$$0 \leq B - f(\bar{B}, \bar{C}) \leq \sqrt{B^2 - C},$$

where the first inequality is from the assumption. Hence,

$$C \leq 2Bf(\bar{B}, \bar{C}) - f(\bar{B}, \bar{C})^2 = 2(B - \bar{B})f(\bar{B}, \bar{C}) + \bar{C}.$$

□

The third auxiliary lemma says that the oval passing thru X_0 is enclosed by the ellipsoid with axis m and eccentricity κ passing thru X_0 when $x_0 \cdot m \geq \kappa$.

Lemma 3.3. Suppose $x_0 \cdot m \geq \kappa$ and let $Y = X_0 + s m$ with $s > 0$; $x_0 = X_0/|X_0|$. Then

$$\{X : |X| + \kappa|X - Y| \leq |X_0| + \kappa|X_0 - Y|\} \subseteq \{X : |X| - \kappa X \cdot m \leq |X_0| - \kappa X_0 \cdot m\}.$$

In particular,

$$h(x, Y, X_0) \leq \frac{|X_0|(1 - \kappa x_0 \cdot m)}{1 - \kappa x \cdot m}$$

for all $x \in S^{n-1}$.

Proof. Let X with $|X| + \kappa|X - Y| \leq |X_0| + \kappa|X_0 - Y|$. Then

$$\begin{aligned} |X| - \kappa X \cdot m &= |X| + \kappa|X - Y| - \kappa X \cdot m - \kappa|X - Y| \leq |X_0| + \kappa|X_0 - Y| - \kappa(X \cdot m + |X - Y|) \\ &= |X_0| + \kappa|X_0 - Y| - \kappa((X - Y) \cdot m + |X - Y|) - \kappa Y \cdot m \\ &\leq |X_0| + \kappa|X_0 - Y| - \kappa Y \cdot m = |X_0| + \kappa|X_0 - Y| - \kappa(Y - X_0) \cdot m - \kappa X_0 \cdot m \\ &= |X_0| + \kappa s - \kappa s m \cdot m - \kappa X_0 \cdot m = |X_0| - \kappa X_0 \cdot m. \end{aligned}$$

□

We are now ready to prove a crucial lemma akin to [Loe09, Prop. 5.1] and [KM10, Thm. 4.10 (DMASM)] in optimal mass transport.

Lemma 3.4. *Assume H.A, H.B, and H.C. There exists a structural constant $C_0 > 0$ such that if $\bar{Y}, \hat{Y} \in \Sigma$, $X_0 \in \Gamma_{c_1 c_2}$ with $\bar{Y} = X_0 + \bar{s} \bar{m}$, $\bar{s} = s_{X_0}(\bar{m})$, $\hat{Y} = X_0 + \hat{s} \hat{m}$, $\hat{s} = s_{X_0}(\hat{m})$, then*

$$C_0 \lambda (1 - \lambda) |\bar{Y} - \hat{Y}|^2 |x - x_0|^2 + h(x, Y, X_0) \leq \max\{h(x, \bar{Y}, X_0), h(x, \hat{Y}, X_0)\}$$

for all $x \in S^{n-1}$, $Y = X_0 + s_{X_0}(m(\lambda))$, $m(\lambda) \in [\bar{m}, \hat{m}]_{x_0}$ and $0 < \lambda < 1$.

Proof. Fix $x \in S^{n-1}$ and assume without loss of generality that $h(x, \bar{Y}, X_0) \geq h(x, \hat{Y}, X_0)$, that is, $f(\hat{B}, \hat{C}) \leq f(\bar{B}, \bar{C})$. We will show that

$$C_0 \lambda (1 - \lambda) |\bar{Y} - \hat{Y}|^2 |x - x_0|^2 + h(x, Y, X_0) \leq h(x, \bar{Y}, X_0).$$

By Lemma 3.1, we have $\hat{B} \geq \bar{B} - \sqrt{\bar{B}^2 - \bar{C}} = f(\bar{B}, \bar{C})$ so we can apply Lemma 3.2 to obtain

$$\hat{C} - \bar{C} \leq 2f(\bar{B}, \bar{C})(\hat{B} - \bar{B}).$$

This means

$$2\kappa|X_0|(\hat{s}(1 - \kappa x_0 \cdot \hat{m}) - \bar{s}(1 - \kappa x_0 \cdot \bar{m})) \leq 2\kappa f(\bar{B}, \bar{C})(\hat{s}(1 - \kappa x \cdot \hat{m}) - \bar{s}(1 - \kappa x \cdot \bar{m}))$$

which is equivalently to

$$(3.4) \quad \hat{s}(|X_0|(1 - \kappa x_0 \cdot \hat{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \hat{m})) \leq \bar{s}(|X_0|(1 - \kappa x_0 \cdot \bar{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \bar{m})).$$

We will show that

$$(3.5) \quad C - \bar{C} \leq 2f(\bar{B}, \bar{C})(B - \bar{B}) - E$$

with some E to be chosen at the end, where B and C are given in (3.3) corresponding to $Y = X_0 + s_{X_0}(m(\lambda))$. To show (3.5), is equivalent to show that

$$|X_0| (s(1 - \kappa x_0 \cdot m) - \bar{s}(1 - \kappa x_0 \cdot \bar{m})) \leq f(\bar{B}, \bar{C}) (s(1 - \kappa x \cdot m) - \bar{s}(1 - \kappa x \cdot \bar{m})) - \frac{(1 - \kappa^2)E}{2\kappa},$$

for $m = m(\lambda) \in [\bar{m}, \hat{m}]_{x_0}$, $s = s_{X_0}(m(\lambda))$, and $0 < \lambda < 1$. Equivalently, we will show

$$(3.6) \quad \begin{aligned} & s (|X_0|(1 - \kappa x_0 \cdot m) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot m)) \\ & \leq \bar{s} (|X_0|(1 - \kappa x_0 \cdot \bar{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \bar{m})) - \frac{(1 - \kappa^2)E}{2\kappa}. \end{aligned}$$

Indeed, first recall that m can be written as in (2.6) with $\bar{\beta}(\lambda) = (1 - \lambda)\beta(\lambda)$ and $\hat{\beta}(\lambda) = \lambda\beta(\lambda)$ with $\beta(\lambda)$ defined in (2.4). From (2.6)

$$\begin{aligned} & s (|X_0|(1 - \kappa x_0 \cdot m) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot m)) \\ & = s (|X_0|(\bar{\beta}(1 - \kappa x_0 \cdot \bar{m}) + \hat{\beta}(1 - \kappa x_0 \cdot \hat{m})) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot m)) \\ & = s \bar{\beta} (|X_0|(1 - \kappa x_0 \cdot \bar{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \bar{m})) \\ & \quad + s \hat{\beta} (|X_0|(1 - \kappa x_0 \cdot \hat{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \hat{m})) \\ & \quad + s f(\bar{B}, \bar{C}) (\bar{\beta}(1 - \kappa x \cdot \bar{m}) + \hat{\beta}(1 - \kappa x \cdot \hat{m}) - (1 - \kappa x \cdot m)) \\ & = I + II + III. \end{aligned}$$

Again from (2.6)

$$\bar{\beta}(1 - \kappa x \cdot \bar{m}) + \hat{\beta}(1 - \kappa x \cdot \hat{m}) - (1 - \kappa x \cdot m) = (\bar{\beta} + \hat{\beta} - 1)(1 - x \cdot x_0) = \frac{1}{2}(\bar{\beta} + \hat{\beta} - 1)|x - x_0|^2.$$

From (3.4)

$$\begin{aligned} II & = s \hat{\beta} (|X_0|(1 - \kappa x_0 \cdot \hat{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \hat{m})) \\ & \leq \frac{s \hat{\beta} \bar{s}}{\hat{s}} (|X_0|(1 - \kappa x_0 \cdot \bar{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \bar{m})). \end{aligned}$$

If we now let

$$K := \frac{\mu}{|X_0|} (1 - \beta(\lambda)),$$

$\bar{\beta}(\lambda) + \hat{\beta}(\lambda) = \beta(\lambda)$, then with simplified notation **H.C** reads

$$(3.7) \quad \bar{\beta} \hat{s} + \hat{\beta} \bar{s} \leq \frac{\bar{s} \hat{s}}{s} + K \bar{s} \hat{s}.$$

We also notice that since $f(\bar{B}, \bar{C}) = h(x, \bar{Y}, X_0)$ and $x_0 \cdot \bar{m} \geq \kappa$, by Lemma 3.3

$$|X_0|(1 - \kappa x_0 \cdot \bar{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \bar{m}) \geq 0.$$

Therefore

$$\begin{aligned}
s (|X_0|(1 - \kappa x_0 \cdot m) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot m)) &= I + II + III \\
&\leq \frac{s(\bar{\beta}\hat{s} + \hat{\beta}\bar{s})}{\hat{s}} (|X_0|(1 - \kappa x_0 \cdot \bar{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \bar{m})) - s f(\bar{B}, \bar{C}) \frac{(1 - (\bar{\beta} + \hat{\beta}))}{2} |x - x_0|^2 \\
&\leq \bar{s} (|X_0|(1 - \kappa x_0 \cdot \bar{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \bar{m})) + K\bar{s}s (|X_0|(1 - \kappa x_0 \cdot \bar{m}) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot \bar{m})) \\
(3.8) \quad &- s f(\bar{B}, \bar{C}) \frac{(1 - (\bar{\beta} + \hat{\beta}))}{2} |x - x_0|^2.
\end{aligned}$$

To estimate the middle term in the last inequality we shall prove that for some $\delta > 0$

$$(3.9) \quad K\bar{s} (|X_0|(1 - \kappa \langle x_0, \bar{m} \rangle) - h(x, \bar{Y}, X_0)(1 - \kappa \langle x, \bar{m} \rangle)) \leq \delta (1 - \beta(\lambda)) |x - x_0|^2 h(x, \bar{Y}, X_0),$$

where $h(x, \bar{Y}, X_0) = f(\bar{B}, \bar{C})$. From the definition of K this inequality is equivalent to

$$(3.10) \quad \bar{s} (|X_0|(1 - \kappa \langle x_0, \bar{m} \rangle) - h(x, \bar{Y}, X_0)(1 - \kappa \langle x, \bar{m} \rangle)) \leq \frac{\delta}{\mu} |X_0| |x - x_0|^2 h(x, \bar{Y}, X_0).$$

Let

$$\Delta = |X_0|(1 - \kappa \langle x_0, \bar{m} \rangle) - h(x, \bar{Y}, X_0)(1 - \kappa \langle x, \bar{m} \rangle).$$

Writing $X = h(x, \bar{Y}, X_0)x$ with $\bar{Y} = X_0 + \bar{s}\bar{m}$, we have $|X| + \kappa|X - \bar{Y}| = |X_0| + \kappa|X_0 - \bar{Y}|$, which after simplification implies that

$$\Delta = \frac{\kappa^2|X - X_0|^2 - (|X| - |X_0|)^2}{2\kappa\bar{s}}.$$

By calculation, the right hand side of the last identity is equal to

$$\frac{|X||X_0||x - x_0|^2 - (1 - \kappa^2)|X - X_0|^2}{2\kappa\bar{s}} \leq \frac{|X||X_0||x - x_0|^2}{2\kappa\bar{s}} = \frac{h(x, \bar{Y}, X_0)|X_0||x - x_0|^2}{2\kappa\bar{s}}$$

implying (3.9) with $\delta = \mu/(2\kappa)$. Therefore inserting (3.9) in (3.8) yields

$$\begin{aligned}
&s (|X_0|(1 - \kappa x_0 \cdot m) - f(\bar{B}, \bar{C})(1 - \kappa x \cdot m)) \\
&\leq \bar{s} \Delta - \frac{1}{2} \left(1 - \frac{\mu}{\kappa}\right) s f(\bar{B}, \bar{C}) (1 - \beta(\lambda)) |x - x_0|^2.
\end{aligned}$$

Therefore we have proved (3.6) with

$$E = \left(1 - \frac{\mu}{\kappa}\right) \frac{\kappa s f(\bar{B}, \bar{C}) (1 - (\bar{\beta} + \hat{\beta})) |x - x_0|^2}{1 - \kappa^2}.$$

and consequently (3.5).

Since X_0 is on both ovals $\mathcal{O}(Y, b), \mathcal{O}(\bar{Y}, \bar{b})$, then by Lemma 3.1, $B \geq f(\bar{B}, \bar{C})$. So from (3.5) we can apply the last part of Lemma 3.2 to get

$$f(B, C) + \frac{E}{B + \sqrt{B^2 - C} - f(\bar{B}, \bar{C})} \leq f(\bar{B}, \bar{C}),$$

that is,

$$h(x, Y, X_0) + \frac{E}{B + \sqrt{B^2 - C} - f(\bar{B}, \bar{C})} \leq h(x, \bar{Y}, X_0).$$

Finally, to complete the proof of the lemma, we estimate $\frac{E}{B + \sqrt{B^2 - C} - f(\bar{B}, \bar{C})}$ from below. We shall first prove that $1 - (\bar{\beta} + \hat{\beta}) \geq C_\kappa \lambda (1 - \lambda) |\bar{m} - \hat{m}|^2$. In fact,

$$\begin{aligned} 1 - (\bar{\beta} + \hat{\beta}) &= 1 - \beta(\lambda) = \frac{\kappa|\xi|^2 + \langle x_0, \xi \rangle + \sqrt{\langle x_0, \xi \rangle^2 - (1 - \kappa^2)|\xi|^2}}{\kappa|\xi|^2} \\ &= \frac{(\kappa|\xi|^2 + \langle x_0, \xi \rangle)^2 - (\langle x_0, \xi \rangle^2 - (1 - \kappa^2)|\xi|^2)}{\kappa|\xi|^2 (\kappa|\xi|^2 + \langle x_0, \xi \rangle - \sqrt{\langle x_0, \xi \rangle^2 - (1 - \kappa^2)|\xi|^2})}. \end{aligned}$$

We have $(\kappa|\xi|^2 + \langle x_0, \xi \rangle)^2 - (\langle x_0, \xi \rangle^2 - (1 - \kappa^2)|\xi|^2) = |\xi|^2(\kappa^2|\xi|^2 + 2\kappa\langle x_0, \xi \rangle + 1 - \kappa^2) = |\xi|^2(|\kappa\xi + x_0|^2 - \kappa^2) = |\xi|^2\kappa^2(|m_\lambda|^2 - 1)$. Therefore

$$1 - \beta(\lambda) = \frac{\kappa(1 - |m_\lambda|^2)}{-\kappa|\xi|^2 - \langle x_0, \xi \rangle + \sqrt{\langle x_0, \xi \rangle^2 - (1 - \kappa^2)|\xi|^2}}.$$

Since $1 - \beta(\lambda) > 0$ and $|m_\lambda| < 1$, for $0 < \lambda < 1$, it follows that $\Delta := -\kappa|\xi|^2 - \langle x_0, \xi \rangle + \sqrt{\langle x_0, \xi \rangle^2 - (1 - \kappa^2)|\xi|^2} > 0$ and since $|\xi| \leq 1 + (1/\kappa)$, Δ is bounded above by a constant depending only on κ . Since $1 - |m_\lambda|^2 = \lambda(1 - \lambda)|\bar{m} - \hat{m}|^2$, the desired lower bound for $1 - (\bar{\beta} + \hat{\beta})$ follows.

Next, we show that $f(\bar{B}, \bar{C})$ is bounded below by a structural constant. In fact, from [GH14, first identity in (4.7)], $f(\bar{B}, \bar{C}) = h(x, \bar{Y}, X_0) \geq \frac{\bar{b} - \kappa|\bar{Y}|}{1 + \kappa}$ for all $x \in S^{n-1}$ where $\bar{b} = |X_0| + \kappa|\bar{Y} - X_0|$. So $\frac{\bar{b} - \kappa|\bar{Y}|}{1 + \kappa} \geq \frac{1 - \kappa}{1 + \kappa}|X_0| \geq c_1 \frac{1 - \kappa}{1 + \kappa}$, since $X_0 \in \Gamma_{c_1 c_2}$. Thus,

$$\begin{aligned} sf(\bar{B}, \bar{C})(1 - (\bar{\beta} + \hat{\beta}))|x - x_0|^2 &\geq Cs\lambda(1 - \lambda)|\bar{m} - \hat{m}|^2|x - x_0|^2 \\ &\geq Cs\lambda(1 - \lambda)|\bar{Y} - \hat{Y}|^2|x - x_0|^2 \quad \text{from H.A(c)} \end{aligned}$$

with $C > 0$ a structural constant.

It remains to estimate $B + \sqrt{B^2 - C} - f(\bar{B}, \bar{C})$ from above. We have from (3.1) that $B + \sqrt{B^2 - C} - f(\bar{B}, \bar{C}) \leq B + \sqrt{B^2 - C} \leq 2B \leq C (|X_0| + \kappa s)$. Since $s = s_{X_0}(m) = |Y - X_0|$ we obtain from **H.B** that $|X_0| + \kappa s \leq C s$ with a structural constant $C > 0$. Therefore

$$\frac{E}{B + \sqrt{B^2 - C} - f(\bar{B}, \bar{C})} \geq C \lambda (1 - \lambda) |\bar{Y} - \hat{Y}|^2 |x - x_0|^2$$

for $0 < \lambda < 1$ with $C > 0$ a structural constant (since $\mu < \kappa$). The proof of the lemma is then complete. \square

4. ESTIMATES FOR DERIVATIVES OF OVALS

We analyze now the derivatives of the function $h(x, Y, X_0)$ for $Y \in \Sigma$ and $X_0 \in \Gamma_{c_1 c_2}$. To differentiate the function h with respect to the variables x and Y we will extend $h(x, Y, X_0)$ for x in a neighborhood of the unit ball and Y in a neighborhood of Σ . In order to do this, we first need to bound from below the quantity inside the square root in (2.2).

Lemma 4.1. *Let $X_0 \in \Gamma_{c_1 c_2}$. There exist $\epsilon > 0$ sufficiently small depending only on κ and constants C_0, C_1 depending only on κ and c_2 such that if $b = |X_0| + \kappa|Y - X_0| < |Y|$ and $|Y| \geq C_1$ then*

$$(4.1) \quad (b - \kappa^2 x \cdot Y)^2 - (1 - \kappa^2)(b^2 - \kappa^2|Y|^2) \geq C_0 \quad \text{for all } |x| \leq 1 + \epsilon.$$

Then by continuity there is a small neighborhood V of Y such that (4.1) holds for all $Y \in V$ with a smaller positive constant C . This implies that under this configuration, the formula defining h in (2.3) can be extended for $|x| \leq 1 + \epsilon$ and $Y \in V$.

Proof. By calculation

$$(4.2) \quad \Delta(t) := (b - \kappa^2 t)^2 - (1 - \kappa^2)(b^2 - \kappa^2|Y|^2) = \kappa^2 \left((b - t)^2 + (1 - \kappa^2)(|Y|^2 - t^2) \right).$$

From the estimate for the ovals [GH14, first identity in (4.7)], $|X_0| = h(x_0, Y, X_0) \geq \frac{b - \kappa|Y|}{1 + \kappa}$, so $b \leq \kappa|Y| + (1 + \kappa)|X_0|$. Therefore $|Y| - b \geq (1 - \kappa)|Y| - (1 + \kappa)|X_0|$. Clearly, the last quantity is non negative if $|Y| \geq \frac{1 + \kappa}{1 - \kappa}|X_0|$.

We have $\min_{|x| \leq 1 + \epsilon} \Delta(x \cdot Y) = \min_{-(1 + \epsilon)|Y| \leq t \leq (1 + \epsilon)|Y|} \Delta(t)$. The function $\Delta(t)$ is decreasing in the interval $(-\infty, b/\kappa^2)$. Let $\epsilon > 0$ be such that $\kappa(1 + \epsilon) < 1$. Since $b > \kappa|Y|$ we then have $[-(1 + \epsilon)|Y|, (1 + \epsilon)|Y|] \subset (-\infty, b/\kappa^2)$. Therefore

$$\min_{-(1 + \epsilon)|Y| \leq t \leq (1 + \epsilon)|Y|} \Delta(t) = \Delta((1 + \epsilon)|Y|)$$

Let us estimate $\Delta((1 + \epsilon)|Y|)$ from below:

$$\begin{aligned}
 & \Delta((1 + \epsilon)|Y|) \\
 &= \kappa^2 \left((b - (1 + \epsilon)|Y|)^2 + (1 - \kappa^2)|Y|^2 \left(1 - (1 + \epsilon)^2 \right) \right) \\
 &= \kappa^2 \left(b^2 - 2b|Y| - 2b\epsilon|Y| + \left(1 + \kappa^2\epsilon(2 + \epsilon) \right) |Y|^2 \right) \\
 &= \kappa^2 \left((|Y| - b)^2 + \kappa^2\epsilon(2 + \epsilon)|Y|^2 - 2b\epsilon|Y| \right) \\
 &\geq \kappa^2 \left(((1 - \kappa)|Y| - (1 + \kappa)|X_0|)^2 + \kappa^2\epsilon(2 + \epsilon)|Y|^2 - 2(\kappa|Y| + (1 + \kappa)|X_0|)\epsilon|Y| \right) \\
 &= \kappa^2 \left((1 - \kappa)^2|Y|^2 - 2(1 - \kappa^2)|Y||X_0| + (1 + \kappa)^2|X_0|^2 + \kappa^2\epsilon(2 + \epsilon)|Y|^2 - 2\kappa\epsilon|Y|^2 - 2(1 + \kappa)\epsilon|X_0||Y| \right) \\
 &= \kappa^2 \left(((1 - \kappa)^2 + \kappa^2\epsilon(2 + \epsilon) - 2\kappa\epsilon)|Y|^2 - 2(1 - \kappa^2 + (1 + \kappa)\epsilon)|Y||X_0| + (1 + \kappa)^2|X_0|^2 \right) \\
 &= \kappa^2 \left(\alpha_1|Y|^2 - 2\alpha_2|Y||X_0| + (1 + \kappa)^2|X_0|^2 \right).
 \end{aligned}$$

From the choice of ϵ , $\alpha_2 \leq (1 - \kappa^2)\frac{1 + \kappa}{\kappa} := \beta_2$ and taking ϵ small we have $\alpha_1 \geq (1 - \kappa)^2/2 := \beta_1$. Hence

$$\begin{aligned}
 \Delta((1 + \epsilon)|Y|) &\geq \kappa^2 \left(\beta_1|Y|^2 - 2\beta_2|Y||X_0| + (1 + \kappa)^2|X_0|^2 \right) \\
 &\geq \kappa^2 \left(\beta_1|Y|^2 - \beta_2 \left(\delta|Y|^2 + \frac{|X_0|^2}{\delta} \right) + (1 + \kappa)^2|X_0|^2 \right) \\
 &= \kappa^2 \left((\beta_1 - \delta\beta_2)|Y|^2 - \left(\frac{\beta_2}{\delta} - (1 + \kappa)^2 \right) |X_0|^2 \right), \quad \delta > 0.
 \end{aligned}$$

We now choose $\delta > 0$ sufficiently small depending only on κ such that

$$(\beta_1 - \delta\beta_2)|Y|^2 - \left(\frac{\beta_2}{\delta} - (1 + \kappa)^2 \right) |X_0|^2 \geq C_1(\kappa)|Y|^2 - C_2(\kappa)|X_0|^2,$$

for some C_i positive constants. Thus

$$\Delta((1 + \epsilon)|Y|) \geq \left(\sqrt{C_1(\kappa)}|Y| + \sqrt{C_2(\kappa)}|X_0| \right) \left(\sqrt{C_1(\kappa)}|Y| - \sqrt{C_2(\kappa)}|X_0| \right),$$

and the desired inequality follows. \square

With Lemma 4.1 in hand we proceed to prove estimates for h and its derivatives.

Lemma 4.2. *There exists a structural constant $C > 0$ such that if $Y \in \Sigma$, $t > 0$ and $(1 + t)X_0 \in \Gamma_{c_1 c_2}$, then $0 \leq h(x, Y, (1 + t)X_0) - h(x, Y, X_0) \leq Ct|X_0|$.*

Proof. If $b(t) = (1+t)|X_0| + \kappa|Y - (1+t)X_0|$, then $0 \leq b(t) - b(0) \leq (1+\kappa)t|X_0|$.[†] Let $Q(t) = (b(t) - \kappa^2 x \cdot Y)^2 - (1 - \kappa^2)(b(t)^2 - \kappa^2 |Y|^2)$. We have

$$\begin{aligned} Q(0) - Q(t) &= (b(0) - \kappa^2 x \cdot Y)^2 - (b(t) - \kappa^2 x \cdot Y)^2 - (1 - \kappa^2)(b(0)^2 - b(t)^2) \\ &= \kappa^2 (b(0)^2 - b(t)^2) - 2\kappa^2 x \cdot Y (b(0) - b(t)) \\ &= \kappa^2 (b(0) - b(t)) (b(0) + b(t) - 2x \cdot Y). \end{aligned}$$

From the definition of h

$$\begin{aligned} &h(x, Y, (1+t)X_0) - h(x, Y, X_0) \\ &= \frac{b(t) - \kappa^2 x \cdot Y - \sqrt{Q(t)}}{1 - \kappa^2} - \frac{b(0) - \kappa^2 x \cdot Y - \sqrt{Q(0)}}{1 - \kappa^2} \\ &= \frac{b(t) - b(0) + \sqrt{Q(0)} - \sqrt{Q(t)}}{1 - \kappa^2} \\ &= \frac{1}{1 - \kappa^2} \left(\frac{(b(t) - b(0)) (\sqrt{Q(0)} + \sqrt{Q(t)}) + Q(0) - Q(t)}{\sqrt{Q(0)} + \sqrt{Q(t)}} \right) \\ &= \frac{1}{1 - \kappa^2} \left(\frac{(b(t) - b(0)) (\sqrt{Q(0)} + \sqrt{Q(t)}) + \kappa^2 (b(0) - b(t)) (b(0) + b(t) - 2x \cdot Y)}{\sqrt{Q(0)} + \sqrt{Q(t)}} \right) \\ &= \frac{(b(t) - b(0)) (\sqrt{Q(t)} + \sqrt{Q(0)} + \kappa^2 (b(0) + b(t) - 2x \cdot Y))}{(1 - \kappa^2) (\sqrt{Q(t)} + \sqrt{Q(0)})}. \end{aligned}$$

Since $\rho(x, Y, b)$ is increasing in b and $b(0) < b(t)$, it follows from (2.3) that $h(x, Y, X_0) \leq h(x, Y, (1+t)X_0)$. From Lemma 4.1 the denominator in the last string of expressions is bounded away from zero and we obtain

$$0 \leq h(x, Y, (1+t)X_0) - h(x, Y, X_0) \leq C (b(t) - b(0)) \leq C t |X_0|.$$

□

Lemma 4.3. *Suppose H.A and H.B hold. There exist a structural constant $C > 0$ such that if $\bar{Y}, Y \in \Sigma$ and $X_0 \in \Gamma_{c_1 c_2}$, then $|\nabla_x h(x_0, Y, X_0) - \nabla_x h(x_0, \bar{Y}, X_0)| \leq C |Y - \bar{Y}|$.*

[†]We can see $b(0) \leq b(t)$ for $t > 0$ because this is equivalent to $(1+t)|X_0| + \kappa|Y - (1+t)X_0| \geq |X_0| + \kappa|Y - X_0|$ which is equivalent to show $t|X_0| + \kappa|Y - (1+t)X_0| \geq \kappa|Y - X_0|$. But by triangle inequality $\kappa|Y - (1+t)X_0| \geq \kappa|Y - X_0| - \kappa t|X_0|$ which implies $t|X_0| + \kappa|Y - (1+t)X_0| \geq t|X_0| + \kappa|Y - X_0| - \kappa t|X_0| = (1 - \kappa)t|X_0| + \kappa|Y - X_0| > \kappa|Y - X_0|$ since $\kappa < 1$.

Proof. Let $Y = X_0 + sm$ and $\bar{Y} = X_0 + \bar{s}\bar{m}$ and $b = |X_0| + \kappa s$. From Lemma 4.1 we can take derivatives of h with respect to x for x in a neighborhood of the unit ball, and by calculation

$$(4.3) \quad \frac{\partial h}{\partial x_i}(x, Y, X_0) = \frac{\kappa^2 h(x, Y, X_0) Y_i}{\sqrt{(b - \kappa^2 x \cdot Y)^2 - (1 - \kappa^2)(b^2 - \kappa^2 |Y|^2)}} = \frac{\kappa^2 h(x, Y, X_0) Y_i}{\sqrt{\Delta(x \cdot Y)}}.$$

So at $x = x_0$

$$(4.4) \quad \frac{\partial h}{\partial x_i}(x_0, Y, X_0) = \frac{\kappa^2 |X_0| Y_i}{\sqrt{(b - \kappa^2 x_0 \cdot Y)^2 - (1 - \kappa^2)(b^2 - \kappa^2 |Y|^2)}} = \frac{\kappa^2 |X_0| Y_i}{\kappa s(1 - \kappa x_0 \cdot m)},'$$

since $\sqrt{(b - \kappa^2 x_0 \cdot Y)^2 - (1 - \kappa^2)(b^2 - \kappa^2 |Y|^2)} = \kappa s(1 - \kappa x_0 \cdot m)$ from (3.1) and (3.2).

Therefore

$$\begin{aligned} & \nabla_x h(x_0, Y, X_0) - \nabla_x h(x_0, \bar{Y}, X_0) \\ &= \kappa |X_0| \left(\frac{Y}{|Y - X_0|(1 - \kappa x_0 \cdot m)} - \frac{\bar{Y}}{|\bar{Y} - X_0|(1 - \kappa x_0 \cdot \bar{m})} \right) \\ &= \kappa |X_0| \left(\frac{Y - \bar{Y}}{|Y - X_0|(1 - \kappa x_0 \cdot m)} + \bar{Y} \left(\frac{1}{|Y - X_0|(1 - \kappa x_0 \cdot m)} - \frac{1}{|\bar{Y} - X_0|(1 - \kappa x_0 \cdot \bar{m})} \right) \right) \\ &= \kappa |X_0| (A + B). \end{aligned}$$

From (2.7), $|A| \leq C |Y - \bar{Y}|$. In addition

$$\begin{aligned} |B| &\leq C \left| |\bar{Y} - X_0|(1 - \kappa x_0 \cdot \bar{m}) - |Y - X_0|(1 - \kappa x_0 \cdot m) \right| \\ &= C \left| |\bar{Y} - X_0| - |Y - X_0| + \kappa |Y - X_0| x_0 \cdot m - \kappa |\bar{Y} - X_0| x_0 \cdot \bar{m} \right| \\ &= C \left| |\bar{Y} - X_0| - |Y - X_0| + \kappa |Y - X_0| x_0 \cdot (m - \bar{m}) + \kappa x_0 \cdot \bar{m} (|Y - X_0| - |\bar{Y} - X_0|) \right| \\ &\leq C |Y - \bar{Y}| \end{aligned}$$

since $|m - \bar{m}| \leq C |Y - \bar{Y}|$ from (2.8). □

Lemma 4.4. *There exists a structural constant M such that if $X_0 \in \Gamma_{c_1 c_2}$, $Y \in \Sigma$ and $x_0 = X_0/|X_0|$, then*

$$|h(x, Y, X_0) - h(x_0, Y, X_0) - \langle \nabla_x h(x_0, Y, X_0), x - x_0 \rangle| \leq M |x - x_0|^2$$

for all $x \in S^{n-1}$.

Proof. We first calculate $\frac{\partial^2}{\partial x_j \partial x_i}$. From (4.2) and (4.3)

$$\begin{aligned}
& \frac{\partial^2 h}{\partial x_j \partial x_i}(x, Y, X_0) \\
&= \frac{\partial}{\partial x_j} \left(\kappa^2 h(x, Y, X_0) Y_i \Delta(x \cdot Y)^{-1/2} \right) \\
&= \kappa^4 h(x, Y, X_0) Y_i Y_j \Delta(x \cdot Y)^{-1} - \frac{\kappa^2}{2} h(x, Y, X_0) Y_i \Delta(x \cdot Y)^{-3/2} \frac{\partial \Delta}{\partial x_j} \\
&= \kappa^4 h(x, Y, X_0) Y_i Y_j \Delta(x \cdot Y)^{-1} - \frac{\kappa^2}{2} h(x, Y, X_0) Y_i \Delta(x \cdot Y)^{-3/2} (-2 \kappa^2 (b - \kappa^2 x \cdot Y) Y_j) \\
&= \kappa^4 h(x, Y, X_0) Y_i Y_j \Delta(x \cdot Y)^{-1} \left(1 + \frac{b - \kappa^2 x \cdot Y}{\sqrt{\Delta(x \cdot Y)}} \right).
\end{aligned}$$

From Lemma 4.1 we obtain that $\left| \frac{\partial^2 h}{\partial x_j \partial x_i}(x, Y, X_0) \right| \leq C_1$ for all x in a neighborhood of the unit ball $|x| \leq 1$. From Taylor's formula

$$h(x, Y, X_0) = h(x_0, Y, X_0) + \nabla_x h(x_0, Y, X_0) \cdot (x - x_0) + \frac{1}{2} \left\langle D_x^2 h(\xi, Y, X_0)(x - x_0), x - x_0 \right\rangle$$

with ξ between x_0 and x . The lemma then follows. \square

Lemma 4.5. *Suppose H.A and H.B hold. There exists a structural constant $C > 0$ such that if $X_0 \in \Gamma_{c_1 c_2}$ and $\bar{Y}, Y \in \Sigma$, then*

$$|h(x, Y, X_0) - h(x, \bar{Y}, X_0)| \leq C |Y - \bar{Y}| |x - x_0|,$$

for $x \in S^{n-1}$.

Proof. Using Lemma 4.1 we shall first estimate the derivatives of h with respect to Y_k ; $Y = (Y_1, \dots, Y_n)$. Recall $h(x, Y, X_0) = \frac{1}{1 - \kappa^2} (b - \kappa^2 x \cdot Y - \sqrt{\Delta(x \cdot Y, b, |Y|)})$ where $\Delta(t, b, |Y|) = \Delta(t)$ is given by (4.2) and $b = |X_0| + \kappa |X_0 - Y|$. By Lemma 4.1, h can be differentiated with respect to Y_k since is defined in an open neighborhood of the target Σ . Then

$$\frac{\partial h}{\partial Y_k} = \frac{1}{1 - \kappa^2} \left(\frac{\partial b}{\partial Y_k} - \kappa^2 x_k - \frac{1}{2} \Delta^{-1/2} \frac{\partial \Delta}{\partial Y_k} \right).$$

Now

$$\frac{\partial \Delta}{\partial Y_k} = 2(b - \kappa^2 x \cdot Y) \left(\frac{\partial b}{\partial Y_k} - \kappa^2 x_k \right) - (1 - \kappa^2) \left(2b \frac{\partial b}{\partial Y_k} - 2\kappa^2 Y_k \right),$$

and $\frac{\partial b}{\partial Y_k} = -\kappa \frac{X_0^k - Y_k}{|X_0 - Y|}$. We next differentiate (4.3) with respect to Y_k . Recall that from Lemma 4.1, the right hand side of (4.3) is well defined for x in a neighborhood of the unit ball and for Y in a neighborhood of the target Σ . We then have

$$\frac{\partial^2 h}{\partial Y_k \partial x_i}(x, Y, X_0) = \kappa^2 \frac{\partial h}{\partial Y_k} Y_i \Delta^{-1/2} + \kappa^2 h \delta_{ik} \Delta^{-1/2} - \frac{1}{2} \kappa^2 h Y_i \Delta^{-3/2} \frac{\partial \Delta}{\partial Y_k}.$$

From Lemma 4.1, $\Delta \geq C$ so $\frac{\partial h}{\partial Y_k}$ is bounded, and therefore $\frac{\partial^2 h}{\partial Y_k \partial x_i}(x, Y, X_0)$ is also bounded.

Therefore we can write for some $\tilde{Y} \in \overline{\tilde{Y}Y}$, the straight segment, and for some $\tilde{x} \in \overline{x_0\tilde{x}}$

$$\begin{aligned} h(x, Y, X_0) - h(x, \tilde{Y}, X_0) &= \sum_{k=1}^n \frac{\partial h}{\partial Y_k}(x, \tilde{Y}, X_0)(Y_k - \tilde{Y}_k) \\ &= \sum_{k=1}^n \left(\frac{\partial h}{\partial Y_k}(x, \tilde{Y}, X_0) - \frac{\partial h}{\partial Y_k}(x_0, \tilde{Y}, X_0) \right) (Y_k - \tilde{Y}_k) \\ &= \sum_{k,l=1}^n \frac{\partial^2 h}{\partial Y_k \partial x_l}(\tilde{x}, \tilde{Y}, X_0)(x_l - x_l^0)(Y_k - \tilde{Y}_k) \end{aligned}$$

where we have used that $h(x_0, Y, X_0) = |X_0|$, for all Y so $\frac{\partial h}{\partial Y_k}(x_0, \tilde{Y}, X_0) = 0$. It remains to show that $\Delta(x \cdot \tilde{Y}, \tilde{b}, |\tilde{Y}|) \geq C$ and $\Delta(\tilde{x} \cdot \tilde{Y}, \tilde{b}, |\tilde{Y}|) \geq C$ so the application of the mean value theorem above is justified and we can apply the bounds for the derivatives. We have $\tilde{Y} = (1 - \lambda)\tilde{Y} + \lambda Y$ for some $\lambda \in [0, 1]$. From **H.A**, we can write $Y = X_0 + s m$ and $\tilde{Y} = X_0 + \bar{s} \bar{m}$ with $x_0 \cdot \bar{m} \geq \kappa$, $x_0 \cdot m \geq \kappa$, $x_0 = X_0/|X_0|$. So $\tilde{Y} = X_0 + (1 - \lambda)\bar{s} \bar{m} + \lambda s m := X_0 + w$, and

$$|\tilde{Y}|^2 - b^2 = |X_0 + w|^2 - (|X_0| + \kappa |w|)^2 = |w|^2(1 - \kappa^2) + 2 X_0 \cdot w - 2 \kappa |X_0| |w|$$

and

$$X_0 \cdot w = (1 - \lambda)\bar{s} X_0 \cdot \bar{m} + \lambda s X_0 \cdot m \geq (1 - \lambda)\bar{s}\kappa |X_0| + \lambda s \kappa |X_0| \geq \kappa |X_0| |w|.$$

Thus

$$\begin{aligned} |\tilde{Y}|^2 - b^2 &\geq (1 - \kappa^2)|w|^2 \\ &= (1 - \kappa^2) \left((1 - \lambda)^2 \bar{s}^2 + 2(1 - \lambda)\lambda \bar{s} s m \cdot \bar{m} + \lambda^2 s^2 \right) := (1 - \kappa^2) \varphi(\lambda). \end{aligned}$$

Since $\bar{m} \cdot x_0 \geq \kappa$ and $m \cdot x_0 \geq \kappa$ with $\kappa < 1$, it follows that $\bar{m} \cdot m \geq -\delta$ for some $0 < \delta = \delta(\kappa) < 1$. Then

$$\varphi(\lambda) \geq (1 - \lambda)^2 \bar{s}^2 - 2(1 - \lambda)\lambda \bar{s} s \delta + \lambda^2 s^2,$$

where the last expression attains its minimum when $\lambda = \frac{\bar{s}^2 + \delta \bar{s} s}{\bar{s}^2 + 2\delta \bar{s} s + s^2}$. Since \bar{s}, s are bounded, at this minimum the expression is larger than or equal to $C(1 - \delta^2) \min\{\bar{s}^2, s^2\}$, with $C > 0$ structural. From (2.7) we then obtain

$$|\tilde{Y}|^2 - b^2 \geq C > 0.$$

Using the argument the proof of Lemma 4.1 with $\epsilon = 0$, it follows that $\Delta(x \cdot \tilde{Y}, b, |\tilde{Y}|)$ and $\Delta(\tilde{x} \cdot \tilde{Y}, b, |\tilde{Y}|)$ are both greater than or equal to $\kappa^2(|\tilde{Y}| - b)^2$ obtaining the desired estimate. □

5. $C^{1,\alpha}$ ESTIMATES

We now turn to the definition of refractor and prove our main theorem.

Definition 5.1. We say $u : \Omega \rightarrow [c_1, c_2]$ is a refractor from Ω to Σ if for each $x_0 \in \Omega$, there exists $Y \in \Sigma$ such that

$$u(x) \geq h(x, Y, X_0)$$

for all $x \in \Omega$ with $X_0 = u(x_0)x_0$. If this holds, then we say $Y \in \partial u(x_0)$. Notice that $X = u(x)x \in \Gamma_{c_1 c_2}$ for all $x \in \Omega$.

We will show $u \in C^{1,\alpha}(\Omega)$, which will follow from the following two lemmas.

Lemma 5.2. Assume **H.A**, **H.B**, and **H.C**, and let u be a refractor from Ω to Σ . There exist structural constants K_1, K_2 such that if $B_{2\delta} \cap S^{n-1} \subseteq \Omega$, $\bar{x}, \hat{x} \in B_\delta \cap S^{n-1}$, $\bar{Y} \in \partial u(\bar{x})$ and $\hat{Y} \in \partial u(\hat{x})$, with $|\bar{Y} - \hat{Y}| \geq |\bar{x} - \hat{x}|$, then, there exists $x_0 \in B_\delta \cap S^{n-1}$ such that, letting $X_0 = u(x_0)x_0$, if $Y(\lambda) \in [\bar{Y}, \hat{Y}]_{x_0}$ we have

$$u(x) \geq h(x, Y(\lambda), X_0) + K_1 \lambda (1 - \lambda) |\bar{Y} - \hat{Y}|^2 |x - x_0|^2 - K_2 |\bar{x} - \hat{x}| |\bar{Y} - \hat{Y}|^2$$

for all $x \in \Omega$, $0 < \lambda < 1$.

Proof. Let $\bar{X} = u(\bar{x})\bar{x}$ and $\hat{X} = u(\hat{x})\hat{x}$. We have $u(x) \geq h(x, \bar{Y}, \bar{X})$ and $u(x) \geq h(x, \hat{Y}, \hat{X})$, for all $x \in \Omega$. Let $\varphi(x) = h(x, \bar{Y}, \bar{X}) - h(x, \hat{Y}, \hat{X})$. Since $\varphi(\bar{x}) \geq 0$ and $\varphi(\hat{x}) \leq 0$, by continuity there exists $x_0 \in [\bar{x}, \hat{x}]$, the geodesic segment in the unit sphere, such

that $h(x_0, \bar{Y}, \bar{X}) = h(x_0, \hat{Y}, \hat{X}) := \rho_0$. Set $\tilde{X}_0 = \rho_0 x_0$ and $X_0 = u(x_0)x_0$ and notice $\rho_0 \leq u(x_0)$ and by definition of refractor $C_1 \leq u(x_0) \leq C_2$, i.e., $X_0 \in \Gamma_{C_1 C_2}$. Also, the oval with focus \bar{Y} that passes through \bar{X} then also passes through \tilde{X}_0 , i.e., $h(x, \bar{Y}, \bar{X}) = h(x, \bar{Y}, \tilde{X}_0)$ for all $x \in S^{n-1}$; and similarly $h(x, \hat{Y}, \hat{X}) = h(x, \hat{Y}, \tilde{X}_0)$. Hence $h(x_0, \bar{Y}, \bar{X}) = h(x_0, \bar{Y}, \tilde{X}_0) = |\tilde{X}_0|$. From [GH14, first identity in (4.7)], $h(x_0, \bar{Y}, \bar{X}) \geq \frac{b - \kappa |\bar{Y}|}{1 + \kappa}$ where $b = |\bar{X}| + \kappa |\bar{X} - \bar{Y}|$. Therefore, $h(x_0, \bar{Y}, \bar{X}) \geq \frac{1 - \kappa}{1 + \kappa} |\bar{X}| = \frac{1 - \kappa}{1 + \kappa} u(\bar{x}) \geq \frac{1 - \kappa}{1 + \kappa} C_1$. Thus, $|\tilde{X}_0| \geq \frac{1 - \kappa}{1 + \kappa} C_1$.

We claim

$$u(x_0) - \rho_0 \leq C |\bar{x} - \hat{x}| |\bar{Y} - \hat{Y}|$$

for some structural constant C . Suppose for a moment the claim holds true. We can write $X_0 = (1 + t)\tilde{X}_0 \in \Gamma_{C_1 C_2}$ with $t = \frac{u(x_0) - \rho_0}{\rho_0}$. So applying Lemma 4.2 yields

$$h(x, \bar{Y}, \bar{X}) = h(x, \bar{Y}, \tilde{X}_0) \geq h(x, \bar{Y}, X_0) - C(u(x_0) - \rho_0) \geq h(x, \bar{Y}, X_0) - C|\bar{x} - \hat{x}| |\bar{Y} - \hat{Y}|$$

and

$$h(x, \hat{Y}, \hat{X}) = h(x, \hat{Y}, \tilde{X}_0) \geq h(x, \hat{Y}, X_0) - C(u(x_0) - \rho_0) \geq h(x, \hat{Y}, X_0) - C|\bar{x} - \hat{x}| |\bar{Y} - \hat{Y}|$$

for all $x \in \Omega$. Thus

$$\begin{aligned} u(x) &\geq \max\{h(x, \bar{Y}, \tilde{X}_0), h(x, \hat{Y}, \tilde{X}_0)\} \\ &\geq \max\{h(x, \bar{Y}, X_0), h(x, \hat{Y}, X_0)\} - C|\bar{x} - \hat{x}| |\bar{Y} - \hat{Y}| \\ &\geq h(x, Y(\lambda), X_0) + K_1 \lambda(1 - \lambda) |\bar{Y} - \hat{Y}|^2 |x - x_0|^2 - K_2 |\bar{x} - \hat{x}| |\bar{Y} - \hat{Y}|, \end{aligned}$$

where in the last inequality we have used Lemma 3.4 and renamed the resulting constants.

It then remains to prove the claim. Since $x_0 \in [\bar{x}, \hat{x}]$, we can write $x_0 = \frac{(1 - t)\bar{x} + t\hat{x}}{|(1 - t)\bar{x} + t\hat{x}|} := \frac{x_t}{|x_t|}$, for some $t \in [0, 1]$. If $Y_0 \in \partial u(x_0)$, then

$$\begin{aligned} u(x) &\geq h(x, Y_0, X_0) \geq h(x_0, Y_0, X_0) + \langle \nabla_x h(x_0, Y_0, X_0), x - x_0 \rangle - M|x - x_0|^2 \\ &= u(x_0) + \langle \nabla_x h(x_0, Y_0, X_0), x - x_0 \rangle - M|x - x_0|^2, \end{aligned}$$

from Lemma 4.4. Therefore

$$\begin{aligned} (5.1) \quad &(1 - t)u(\bar{x}) + tu(\hat{x}) \\ &\geq u(x_0) + \langle \nabla_x h(x_0, Y_0, X_0), x_t - x_0 \rangle - M \left((1 - t)|\bar{x} - x_0|^2 + t|\hat{x} - x_0|^2 \right). \end{aligned}$$

By calculation

$$(5.2) \quad (1-t)|\bar{x} - x_0|^2 + t|\hat{x} - x_0|^2 = 2(1-|x_t|) = 2 \frac{1-|x_t|^2}{1+|x_t|} \leq 2|\bar{x} - \hat{x}|^2.$$

From (4.3) $|\nabla_x h(x_0, Y_0, X_0)| \leq C$ and since $|x_t - x_0| \leq 2|\bar{x} - \hat{x}|^2$ it then follows from (5.1) that

$$u(x_0) \leq (1-t)u(\bar{x}) + tu(\hat{x}) + C|\bar{x} - \hat{x}|^2.$$

Next, since as proved above, $|\tilde{X}_0| \geq \frac{1-\kappa}{1+\kappa} C_1$, we can apply Lemma 4.4 with X_0 replaced by \tilde{X}_0 to obtain

$$\begin{aligned} u(\bar{x}) &= h(\bar{x}, \bar{Y}, \tilde{X}_0) \leq h(x_0, \bar{Y}, \tilde{X}_0) + \langle \nabla_x h(x_0, \bar{Y}, \tilde{X}_0), \bar{x} - x_0 \rangle + M|\bar{x} - x_0|^2 \\ &= \rho_0 + \langle \nabla_x h(x_0, \bar{Y}, \tilde{X}_0), \bar{x} - x_0 \rangle + M|\bar{x} - x_0|^2, \end{aligned}$$

and similarly

$$u(\hat{x}) \leq \rho_0 + \langle \nabla_x h(x_0, \hat{Y}, \tilde{X}_0), \hat{x} - x_0 \rangle + M|\hat{x} - x_0|^2.$$

Therefore

$$\begin{aligned} (1-t)u(\bar{x}) + tu(\hat{x}) &\leq \rho_0 + (1-t)\langle \nabla_x h(x_0, \bar{Y}, \tilde{X}_0), \bar{x} - x_0 \rangle + t\langle \nabla_x h(x_0, \hat{Y}, \tilde{X}_0), \hat{x} - x_0 \rangle \\ &\quad + M\left((1-t)|\bar{x} - x_0|^2 + t|\hat{x} - x_0|^2\right). \end{aligned}$$

The last term is bounded above by $2M|\bar{x} - \hat{x}|^2$. To estimate the middle term we write

$$\bar{x} - x_0 = \frac{\bar{x}(|x_t| - 1) - t(\hat{x} - \bar{x})}{|x_t|}, \quad \hat{x} - x_0 = \frac{\hat{x}(|x_t| - 1) + (1-t)(\hat{x} - \bar{x})}{|x_t|},$$

so

$$\begin{aligned} &(1-t)\langle \nabla_x h(x_0, \bar{Y}, \tilde{X}_0), \bar{x} - x_0 \rangle + t\langle \nabla_x h(x_0, \hat{Y}, \tilde{X}_0), \hat{x} - x_0 \rangle \\ &= \frac{(1-t)t}{|x_t|} \left\langle \nabla_x h(x_0, \hat{Y}, \tilde{X}_0) - \nabla_x h(x_0, \bar{Y}, \tilde{X}_0), \hat{x} - \bar{x} \right\rangle \\ &\quad + \frac{|x_t| - 1}{|x_t|} \left(\left\langle (1-t)\nabla_x h(x_0, \bar{Y}, \tilde{X}_0), \bar{x} \right\rangle + \left\langle t\nabla_x h(x_0, \hat{Y}, \tilde{X}_0), \hat{x} \right\rangle \right). \end{aligned}$$

Since $|\tilde{X}_0| \geq \frac{1-\kappa}{1+\kappa} C_1$, from (5.2) and (4.3) the absolute value of the last term is $\leq C|\bar{x} - \hat{x}|^2$; and we can apply Lemma 4.3 to obtain that the absolute value of the first term is bounded by $C|\bar{Y} - \hat{Y}||\bar{x} - \hat{x}|$. Since $|\bar{x} - \hat{x}| \leq |\bar{Y} - \hat{Y}|$, the claim is proved, and the lemma follows. \square

Now using Lemmas 5.2 and 4.5, we obtain the following.

Lemma 5.3. *Under the hypotheses of Lemma 5.2, there exist structural constants K_1, K_2, K_3 and $x_0 \in B_\sigma \cap S^{n-1}$ such that for all $Y(\lambda) \in [\bar{Y}, \hat{Y}]_{x_0}$, $Y \in \Sigma$ and $x \in \Omega$,*

$$u(x) \geq h(x, Y, X_0) + K_1 \lambda (1 - \lambda) |\bar{Y} - \hat{Y}|^2 |x - x_0|^2 - K_2 |Y - Y(\lambda)| |x - x_0| - K_3 |\bar{Y} - \hat{Y}| |\bar{x} - \hat{x}|$$

where $X_0 = u(x_0)x_0$, $0 < \lambda < 1$.

Our main theorem is then the following.

Theorem 5.4. *Suppose that **H.A**, **H.B**, **H.C**, and **H.D** hold. Let u be a refractor from Ω to Σ and assume that there is a constant C such that for all balls B_σ such that $B_\sigma \cap S^{n-1} \subseteq \Omega$, we have*

$$(5.3) \quad H^{n-1} \left(\partial u \left(B_\sigma \cap S^{n-1} \right) \right) \leq C \sigma^{n-1}.$$

where H^{n-1} is the $(n-1)$ -dimensional Hausdorff measure in \mathbb{R}^n .

Assume $B_{2\delta} \cap S^{n-1} \subseteq \Omega$. There exist constants \tilde{C}_1, \tilde{C}_2 depending on δ and structure, such that if $\bar{x}, \hat{x} \in B_\delta \cap S^{n-1}$, $\bar{Y} \in \partial u(\bar{x})$, $\hat{Y} \in \partial u(\hat{x})$ with $|\bar{Y} - \hat{Y}| \geq \tilde{C}_1 |\bar{x} - \hat{x}|$, then $|\bar{Y} - \hat{Y}| \leq \tilde{C}_2 |\bar{x} - \hat{x}|^\alpha$ where $\alpha = \frac{1}{4n-5}$, $n > 1$.

Proof. By Lemma 5.3, there exists $x_0 \in [\bar{x}, \hat{x}] \subseteq B_\delta$, such that for all $Y(\lambda) \in [\bar{Y}, \hat{Y}]_{x_0}$ with $\frac{1}{4} \leq \lambda \leq \frac{3}{4}$, for all $Y \in \Sigma$ and for all $x \in \Omega$, we have

$$u(x) \geq h(x, Y, X_0) + K_1 |\bar{Y} - \hat{Y}|^2 |x - x_0|^2 - K_2 |Y - Y(\lambda)| |x - x_0| - K_3 |\bar{Y} - \hat{Y}| |\bar{x} - \hat{x}|,$$

where $X_0 = u(x_0)x_0$ and $K_i, i = 1, 2, 3$ are structural constants. Let

$$t_0 = \frac{K_2 |Y - Y(\lambda)| + \sqrt{K_2^2 |Y - Y(\lambda)|^2 + 4K_1 K_3 |\bar{Y} - \hat{Y}|^3 |\bar{x} - \hat{x}|}}{2K_1 |\bar{Y} - \hat{Y}|^2}.$$

If $|x - x_0| \geq t_0$, then $K_1 |\bar{Y} - \hat{Y}|^2 |x - x_0|^2 - K_2 |Y - Y(\lambda)| |x - x_0| - K_3 |\bar{Y} - \hat{Y}| |\bar{x} - \hat{x}| \geq 0$. Let $\mu = \sqrt{|\bar{Y} - \hat{Y}|^3 |\bar{x} - \hat{x}|}$ and suppose $|Y - Y(\lambda)| \leq \mu$, then

$$t_0 \leq \frac{K_2 + \sqrt{K_2^2 + 4K_1 K_3}}{2K_1} \sqrt{\frac{|\bar{x} - \hat{x}|}{|\hat{Y} - \bar{Y}|}} := K \sqrt{\frac{|\bar{x} - \hat{x}|}{|\hat{Y} - \bar{Y}|}} := \sigma.$$

Let $C \geq 1$ be large enough constant depending on δ and the structural constants such that $\frac{K}{\sqrt{C}} \leq \frac{\delta}{2}$ and $\frac{(\text{diam}(\Sigma))^2}{\sqrt{C}} \leq \mu_0$, with μ_0 the constant in **H.D**. Set $\tilde{C}_1 := C$.

If $|\tilde{Y} - \hat{Y}| \geq \tilde{C}_1 |\bar{x} - \hat{x}|$, then

$$t_0 \leq \sigma \leq \frac{\delta}{2}$$

and

$$\mu \leq \frac{|\tilde{Y} - \hat{Y}|^2}{\sqrt{\tilde{C}_1}} \leq \frac{(\text{diam}(\Sigma))^2}{\sqrt{\tilde{C}_1}} \leq \mu_0.$$

Let $Y \in \Sigma$ and $|Y - Y(\lambda)| \leq \mu$ for some $\frac{1}{4} \leq \lambda \leq \frac{3}{4}$. We will show that

$$(5.4) \quad Y \in \partial u \left(B(x_0, \sigma) \cap S^{n-1} \right).$$

Notice that $B(x_0, \sigma) \cap S^{n-1} \subseteq B_{2\delta} \cap S^{n-1} \subseteq \Omega$, and if $|x - x_0| \geq \sigma$ and $x \in \Omega$, then $u(x) \geq h(x, Y, X_0)$. If $X = u(x)x$, then this implies that X is outside the region enclosed by the oval $\mathcal{O}(Y, |X_0| + \kappa|X_0 - Y|)$ thorough X_0 and focus Y which implies that the oval through X with focus Y encloses $\mathcal{O}(Y, |X_0| + \kappa|X_0 - Y|)$. Therefore $|X| + \kappa|X - Y| \geq |X_0| + \kappa|X_0 - Y|$ for $|x - x_0| \geq \sigma$ and $x \in \Omega$, and by continuity

$$\inf\{|X| + \kappa|X - Y| : X = u(x)x, x \in \Omega\} = |\tilde{X}| + \kappa|\tilde{X} - Y|$$

for some $\tilde{X} = u(\tilde{x})\tilde{x}$ with $\tilde{x} \in \bar{B}(x_0, \sigma) \cap S^{n-1}$. So each $X = u(x)x$, with $x \in \Omega$, is outside the interior of the region enclosed by oval $\mathcal{O}(Y, |\tilde{X}| + \kappa|\tilde{X} - Y|)$ which implies that $u(x) \geq h(x, Y, \tilde{X})$, for all $x \in \Omega$. Since $u(\tilde{x}) = |\tilde{X}|$ we obtain that $Y \in \partial u(\tilde{x})$ and (5.4) is proved.

Therefore

$$N_\mu \left(\left\{ [Y, \hat{Y}]_{X_0} : \frac{1}{4} \leq \lambda \leq \frac{3}{4} \right\} \right) \cap \Sigma \subset \partial u(B(x_0, \sigma) \cap S^{n-1}).$$

Taking H^{n-1} -measures on both sides, using **H.D** on the left hand side and (5.3) on the right hand side yields

$$C_\star \mu^{n-2} |\tilde{Y} - \hat{Y}| \leq C^\star \sigma^{n-1}$$

which from the definitions of μ and σ implies $|\tilde{Y} - \hat{Y}| \leq \tilde{C}_2 |\bar{x} - \hat{x}|^\alpha$, with \tilde{C}_2 an structural constant. \square

We can now deduce Hölder estimates for the gradients of refractors.

Theorem 5.5. *If **H.A**, **H.B**, **H.C**, and **H.D** hold, and u is a refractor from Ω to Σ in the sense of Definition 5.1 satisfying (5.3), then $u \in C_{\text{loc}}^{1,\alpha}(\Omega)$.*

Proof. Let $x_0 \in \Omega$. We first show that $\partial u(x_0)$ is singleton. Fix $\delta > 0$ such that $B(x_0, 2\delta) \cap S^{n-1} \subseteq \Omega$ and suppose $Y_0, Y_1 \in \partial u(x_0)$, with $Y_1 \neq Y_0$. Let $\bar{x} \in B(x_0, \delta) \cap S^{n-1}$ and $\bar{Y} \in \partial u(\bar{x})$. By Theorem 5.4, $|\bar{Y} - Y_0| \leq C|\bar{x} - x_0|^\alpha$ and $|\bar{Y} - Y_1| \leq C|\bar{x} - x_0|^\alpha$ where the constant C depends on δ . Hence, $|Y_1 - Y_0| \leq 2C|\bar{x} - x_0|^\alpha$, so if we take \bar{x} close enough to x_0 we get a contradiction.

Let $Y \in \partial u(x_0)$. We first claim that for any $\eta \perp x_0$, $|\eta| = 1$, we have $D_\eta u(x_0) = \langle \nabla h(x_0, Y, X_0), \eta \rangle$, where $X_0 = u(x_0)x_0$. To see this, let c be any curve such that $c(0) = x_0$ and $c'(0) = \eta$ and $c(t) \in B(x_0, \delta) \cap S^{n-1}$ for all t near 0. Since u is a refractor

$$u(c(t)) - u(x_0) \geq h(c(t), Y, X_0) - h(x_0, Y, X_0)$$

for all t near 0. Let $Y(t) \in \partial u(c(t))$ and $X(t) = u(c(t))c(t)$. Since $u(x) \geq h(x, Y(t), X(t))$ for all $x \in \Omega$, we get

$$u(x_0) - u(c(t)) \geq h(x_0, Y(t), X(t)) - h(c(t), Y(t), X(t))$$

for all t near zero. Therefore, we have for all $t > 0$ small

$$\frac{h(c(t), Y, X_0) - h(x_0, Y, X_0)}{t} \leq \frac{u(c(t)) - u(x_0)}{t} \leq \frac{h(c(t), Y(t), X(t)) - h(x_0, Y(t), X(t))}{t}.$$

Note that for each t

$$\frac{h(c(t), Y(t), X(t)) - h(x_0, Y(t), X(t))}{t} = \langle \nabla h(\bar{x}, Y(t), X(t)), \frac{c(t) - c(0)}{t} \rangle$$

for some $\bar{x} \in [x_0, c(t)]$. From Theorem 5.4, $Y(t) \rightarrow Y$ as $t \rightarrow 0$, and $X(t) \rightarrow X_0$ by continuity of u . Letting $t \rightarrow 0$ the claim follows.

Define $\tilde{u}(X) = u(X/|X|)$ for X with $X/|X| \in \Omega$. We will show that for each $x_0 \in \Omega$

$$(5.5) \quad \nabla \tilde{u}(x_0) = \nabla^T h(x_0, Y, X_0) := \nabla h(x_0, Y, X_0) - \langle \nabla h(x_0, Y, X_0), x_0 \rangle x_0.$$

Indeed, let $c(t) = \frac{x_0 + te_i}{|x_0 + te_i|}$ and notice that $c(0) = x_0$ and $c'(0) = e_i - \langle x_0, e_i \rangle x_0$. Since $\frac{\tilde{u}(x_0 + te_i) - \tilde{u}(x_0)}{t} = \frac{u(c(t)) - u(x_0)}{t}$, letting $t \rightarrow 0$ and using the first part we get

$$\frac{\partial \tilde{u}}{\partial x_i}(x_0) = \langle \nabla h(x_0, Y, X_0), e_i - \langle x_0, e_i \rangle x_0 \rangle$$

and the desired formula follows.

Next, let $\bar{x}, \hat{x} \in B(x_0, \delta) \cap S^{n-1} \subset \Omega$, and let $\bar{Y} \in \partial u(\bar{x})$ and $\hat{Y} \in \partial u(\hat{x})$. We shall prove that

$$(5.6) \quad |\nabla \tilde{u}(\bar{x}) - \nabla \tilde{u}(\hat{x})| \leq C |\bar{x} - \hat{x}|^\alpha.$$

First notice that

$$|\nabla^T h(\bar{x}, \bar{Y}, \bar{X}) - \nabla^T h(\hat{x}, \hat{Y}, \hat{X})| \leq 2 |\nabla h(\bar{x}, \bar{Y}, \bar{X}) - \nabla h(\hat{x}, \hat{Y}, \hat{X})| + C |\bar{x} - \hat{x}|,$$

since $|\nabla h(\bar{x}, \bar{Y}, \bar{X})|$ is bounded. Next write

$$\begin{aligned} |\nabla h(\bar{x}, \bar{Y}, \bar{X}) - \nabla h(\hat{x}, \hat{Y}, \hat{X})| &\leq |\nabla h(\bar{x}, \bar{Y}, \bar{X}) - \nabla h(\bar{x}, \hat{Y}, \bar{X})| \\ &\quad + |\nabla h(\bar{x}, \hat{Y}, \bar{X}) - \nabla h(\hat{x}, \hat{Y}, \bar{X})| \\ &\quad + |\nabla h(\hat{x}, \hat{Y}, \bar{X}) - \nabla h(\hat{x}, \hat{Y}, \hat{X})|. \end{aligned}$$

First, by Lemma 4.3 $|\nabla h(\bar{x}, \bar{Y}, \bar{X}) - \nabla h(\bar{x}, \hat{Y}, \bar{X})| \leq C |\bar{Y} - \hat{Y}|$. Second, that $|\nabla h(\bar{x}, \hat{Y}, \bar{X}) - \nabla h(\hat{x}, \hat{Y}, \bar{X})| \leq C |\bar{x} - \hat{x}|$ follows using the mean value theorem in x from the estimates in the proof of Lemma 4.4, i.e., from (4.1), (4.2) and (4.3). For the third term, from (4.3) we can write

$$\begin{aligned} &\nabla h(\hat{x}, \hat{Y}, \bar{X}) - \nabla h(\hat{x}, \hat{Y}, \hat{X}) \\ &= \frac{\kappa^2 h(\hat{x}, \hat{Y}, \bar{X}) \hat{Y}}{\sqrt{(\bar{b} - \kappa^2 x \cdot \hat{Y})^2 - (1 - \kappa^2)(\bar{b}^2 - \kappa^2 |\hat{Y}|^2)}} - \frac{\kappa^2 h(\hat{x}, \hat{Y}, \hat{X}) \hat{Y}}{\sqrt{(\hat{b} - \kappa^2 x \cdot \hat{Y})^2 - (1 - \kappa^2)(\hat{b}^2 - \kappa^2 |\hat{Y}|^2)}}, \end{aligned}$$

where $\bar{b} = |\bar{X}| + \kappa |\hat{Y} - \bar{X}|$ and $\hat{b} = |\hat{X}| + \kappa |\hat{Y} - \hat{X}|$. Since $\hat{Y} \in \Sigma$ and $\bar{X}, \hat{X} \in \Gamma_{C_1 C_2}$, and noticing that $|\bar{b} - \hat{b}| \leq C_\kappa |\bar{X} - \hat{X}|$, it follows from Definitions (2.2), (2.3), and Lemma 4.1 that

$$|\nabla h(\hat{x}, \hat{Y}, \bar{X}) - \nabla h(\hat{x}, \hat{Y}, \hat{X})| \leq C |\bar{X} - \hat{X}|.$$

Therefore

$$|\nabla \tilde{u}(\bar{x}) - \nabla \tilde{u}(\hat{x})| \leq C (|\bar{x} - \hat{x}| + |\bar{Y} - \hat{Y}| + |\bar{X} - \hat{X}|).$$

We also have $|\bar{X} - \hat{X}| = |u(\bar{x})\bar{x} - u(\hat{x})\hat{x}| \leq C_1 |\bar{x} - \hat{x}| + |u(\bar{x}) - u(\hat{x})| \leq C |\bar{x} - \hat{x}|$, since u is Lipschitz. From Theorem 5.4 we then obtain (5.6) and the proof is complete. \square

5.1. Regularity of weak solutions. We now apply Theorem 5.5 to show that weak solutions to the near field refractor problem defined with the tracing map are $C_{loc}^{1,\alpha}$. Existence of weak solutions is proved in [GH14].

Recall that the tracing mapping \mathcal{T}_u is defined as follows: given $Y \in \Sigma$, $\mathcal{T}_u(Y) = \{x \in \Omega : Y \in \partial u(x)\}$. A weak solution u to the refractor problem from Ω to Σ satisfies

$$(5.7) \quad \mu(\mathcal{T}_u(B)) = \nu(B), \quad \text{for all Borel } B \subset \Sigma.$$

Here $\mu = f(x) dx$ with $f \in L^1(\Omega)$, $f > 0$ a.e., and ν is a measure on the target Σ so that the energy conservation condition $\int_{\Omega} f(x) dx = \nu(\Sigma)$ holds.

Theorem 5.6. *Assume that H.A, H.B, H.C, and H.D hold and the target Σ is differentiable. If $f \in L^\infty(\Omega)$, $\nu \ll H^{n-1}$, and $H^{n-1} = g dv$ with $0 \leq g(x) \leq \alpha$ for a.e. $x \in \Sigma$, then each weak solution u to (5.7) satisfies (5.3), and therefore from Theorem 5.5 $u \in C_{loc}^{1,\alpha}$.*

Proof. Since Σ is differentiable, then the visibility condition implies that the tangent plane to Σ at each point cannot intersect the interior of Γ_{c_1, c_2} . Indeed, suppose the tangent plane T_Y to Σ at Y intersects Γ_{c_1, c_2} at X_0 and with a ball $B(X_0, \epsilon) \subset \Gamma_{c_1, c_2}$. The segment from X_0 to Y is on T_Y and by visibility for each $X \in B(X_0, \epsilon)$, the segment from X to Y intersects Σ only at Y . This implies that Σ cannot be differentiable at Y , because if Σ were differentiable at Y , then $T_Y \cap C = \{Y\}$ with C the cone with vertex Y and base $B(X_0, \epsilon)$, but $\overline{X_0 Y} \subset T_Y \cap C$.

Now let

$$S^* = \{Y \in \Sigma : Y \in \partial u(\bar{x}) \cap \partial u(\hat{x}), \bar{x} \neq \hat{x} \in \Omega\}.$$

We shall prove that $H^{n-1}(S^*) = 0$.

Define $u^* : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$u^*(Y) = \min \{|X| + \kappa|X - Y| : X = u(x)x, x \in \Omega\}.$$

It is easy to see that u^* is Lipschitz in \mathbb{R}^n .

If $\bar{Y} \in \partial u(\bar{x})$, $X = u(x)x$, $\bar{X} = u(\bar{x})\bar{x}$, with u a refractor, then X is outside the interior of the region enclosed by the oval $\mathcal{O}(\bar{Y}, b)$ with $b = |\bar{X}| + \kappa|\bar{X} - \bar{Y}|$. This means that the region enclosed by an oval passing through X with focus \bar{Y} contains $\mathcal{O}(\bar{Y}, b)$, that is,

$$|X| + \kappa|X - \bar{Y}| \geq |\bar{X}| + \kappa|\bar{X} - \bar{Y}|.$$

Hence

$$u^*(\bar{Y}) = |\bar{X}| + \kappa|\bar{X} - \bar{Y}|$$

and so

$$u^*(Y) \leq u^*(\bar{Y}) + \kappa|\bar{X} - Y| - \kappa|\bar{X} - \bar{Y}|,$$

for all $Y \in \mathbb{R}^n$. In particular, if $Y_0 \in S^*$ and say $Y_0 \in \partial u(\bar{x}) \cap \partial u(\hat{x})$, we then have

$$u^*(Y) \leq u^*(Y_0) + \kappa|\bar{X} - Y| - \kappa|\bar{X} - Y_0|,$$

and

$$u^*(Y) \leq u^*(Y_0) + \kappa|\hat{X} - Y| - \kappa|\hat{X} - Y_0|,$$

$\hat{X} = u(\hat{x})\hat{x}$, for all $Y \in \mathbb{R}^n$.

Let $O \subseteq \mathbb{R}^{n-1}$ be open and let $\psi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ be Lipschitz such that $\Sigma = \psi(\bar{O})$ and ψ is one to one in \bar{O} . Set $\tilde{S} = \psi^{-1}(S^*)$. We show that $H^{n-1}(\tilde{S}) = 0$.

Define $h(Y') = u^*(\psi(Y'))$. Since u^* is Lipschitz, h is Lipschitz in \mathbb{R}^{n-1} . We claim that h is not differentiable in \tilde{S} . Let $Y'_0 \in \tilde{S}$, so $Y_0 = \psi(Y'_0) \in S^*$, that is, there are $\bar{x} \neq \hat{x}$ in Ω with $Y_0 \in \partial u(\bar{x}) \cap \partial u(\hat{x})$. Then $h(Y') \leq h(Y'_0) + |\bar{X} - \psi(Y')| - |\bar{X} - \psi(Y'_0)|$ and $h(Y') \leq h(Y'_0) + |\hat{X} - \psi(Y')| - |\hat{X} - \psi(Y'_0)|$ for all $Y' \in \mathbb{R}^n$ with $\bar{X} = u(\bar{x})\bar{x}$ and $\hat{X} = u(\hat{x})\hat{x}$. If h were differentiable at Y'_0 , then we would have

$$\nabla_{Y'}(|\bar{X} - \psi(Y')|) = \nabla_{Y'}(|\hat{X} - \psi(Y')|)$$

at $Y' = Y'_0$. Thus $D\psi(Y'_0)^T \frac{Y_0 - \bar{X}}{|Y_0 - \bar{X}|} = D\psi(Y'_0)^T \frac{Y_0 - \hat{X}}{|Y_0 - \hat{X}|}$. Letting $w = \frac{Y_0 - \bar{X}}{|Y_0 - \bar{X}|} - \frac{Y_0 - \hat{X}}{|Y_0 - \hat{X}|}$, yields $D\psi(Y'_0)^T w = 0$. If v_k denote the columns of $D\psi(Y'_0)$, this means that $\langle v_k, w \rangle = 0$, for $1 \leq k \leq n-1$. Since the v_k 's span the tangent plane to Σ at Y_0 , we get that w is normal to the tangent plane to Σ at Y_0 . In particular, the line $Y_0 + t \left(\frac{Y_0 - \bar{X}}{|Y_0 - \bar{X}|} + \frac{Y_0 - \hat{X}}{|Y_0 - \hat{X}|} \right)$ is contained in the tangent plane to Σ at Y_0 . But it is easy to see that this line intersects the straight segment $[\bar{X}, \hat{X}]$, which implies that either both \bar{X} and \hat{X} are on the tangent plane or they are on opposite sides of the tangent plane. In either case, since \bar{X} and \hat{X} are on the graph of u , the tangent plane intersects the graph of u , which contradicts our initial assumption.

Since h is Lipschitz we obtain that $H^{n-1}(\tilde{S}) = 0$. This implies, since ψ is Lipschitz, that $H^{n-1}(S^*) = 0$ as we wanted to show.

From the assumption, $\nu \ll H^{n-1}$, we will show first that $\nu(\partial u(B)) \leq \mu(B)$ for each $B \subset \Omega$ Borel set. If

$$S = \{x \in \Omega : \text{there exists } \bar{x} \neq x, \bar{x} \in \Omega \text{ such that } \partial u(x) \cap \partial u(\bar{x}) \neq \emptyset\},$$

let us see that $\mu(S) = 0$. Indeed, since $\mathcal{T}_u(S^*) = S$, from the definition of weak solution $\nu(S^*) = \mu(\mathcal{T}_u(S^*)) = \mu(S)$. Since $H^{n-1}(S^*) = 0$, we then get $\mu(S) = 0$. On the other hand, $\mathcal{T}_u(\partial u(B)) \subset B \cup S$ so $\nu(\partial u(B)) = \mu(\mathcal{T}_u(\partial u(B))) \leq \mu(B \cup S) \leq \mu(B)$ and we are done.

Therefore, to conclude the proof of the theorem, we prove that u verifies (5.3). Indeed, for each ball B_σ with $B_\sigma \cap S^{n-1} \subset \Omega$ we have

$$\begin{aligned} H^{n-1}(\partial u(B_\sigma \cap S^{n-1})) &\leq \alpha \nu(\partial u(B_\sigma \cap S^{n-1})) \leq \alpha \mu(B_\sigma \cap S^{n-1}) \\ &\leq \alpha \|f\|_\infty \text{surface area}(B_\sigma \cap S^{n-1}) \leq C \sigma^{n-1}. \end{aligned}$$

□

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