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SWANSON HAMILTONIAN REVISITED THROUGH THE COMPLEX SCALING METHOD

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Abstract. In this work, we study the non-hermitian PT-symmetry Swanson Hamiltonian in the framework of the Complex Scaling Method. We show that by applying this method we can work with eigenfunctions that are square-integrable both in the PT and in the non-PT symmetry phase.

Keywords: PT-symmetric Hamiltonians, Swanson model, Complex Scaling Method.

1. Introduction

The Swanson Model has been introduced in [1] as an example of a PT-symmetry hamiltonian [2–8]. Since then it has been extensively studied, allowing for several interesting extensions [9–25]. Among recent works, let us mention an extension of the Swanson model with complex parameters [23, 25], this work introduces bicoherent-state path integration as a method to quantify non-Hermitian systems. Though the Swanson model is described by quadratic operators, the underlying physics is nevertheless very rich. Depending on the region in the model parameter space, the Swanson model is similar to the hamiltonian of a parabolic barrier or the hamiltonian of a harmonic oscillator [26]. From the mathematical point of view, it is an example of a hamiltonian with eigenfunctions that do not belong to $\mathcal{L}^2(\mathbb{R})$ in some regions of the space of parameters.

Among the methods that are employed to describe the physics of resonances with complex energy, the Complex Scaling Method (CSM) [27–32] is one of the most powerful. It has been extensively used in the description of many-body resonant states and non-resonant continuum states observed in unstable nuclei [32]. In this work, we propose the use of the CSM to describe the dynamics of the Swanson model, particularly in the region of non-PT-symmetry.

The work is organized as follows. In Section 2 we describe the application of the CSM to the Swanson Hamiltonian. We establish a similarity transformation between the transformed hamiltonian and its adjoint operator. We discuss, according to the space of parameters of the model, the possibility of having square-integrable eigenfunctions. We present the mean values of some observables. In Section 3, we analyse with an example, the survival probability as a function of time for an initial coherent state. Conclusions are drawn in Section 4.

2. Formalism

The hamiltonian of Swanson [1] is given by

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2}\right) + \hbar\alpha\ a^2 + \hbar\beta\ {a^\dagger}^2, \qquad (1)$$

with $\omega,\ \alpha,\ \beta \in \mathbb{R}$. The hamiltonian of Eq. (1) can be written in terms of the coordinate operator, $\hat{x}$, and the momentum operator, $\hat{p}$, by implementing the following representation

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{b_0} + \mathbf{i}\frac{b_0}{\hbar}\hat{p}\right), \\ a^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{b_0} - \mathbf{i}\frac{b_0}{\hbar}\hat{p}\right), \end{aligned} \qquad (2)$$

being b_0 the characteristic length of the non-interacting system. The hamiltonian in Eq. (1) reads

$$\begin{aligned} \mathrm{H}(\omega,\alpha,\beta) &= \frac{1}{2}\hbar(\omega+\alpha+\beta)\left(\frac{\hat{x}}{b_0}\right)^2 \\ &+\frac{1}{2}\hbar(\omega-\alpha-\beta)\left(\frac{b_0\ \hat{p}}{\hbar}\right)^2 \\ &+\hbar\frac{(\alpha-\beta)}{2}\left(2\ \hat{x}\frac{\mathbf{i}}{\hbar}\hat{p}+1\right). \end{aligned} \qquad (3)$$

The adjoint hamiltonian of $\mathrm{H}(\omega,\alpha,\beta)$ is $\mathrm{H}_c = \mathrm{H}(\omega,\beta,\alpha)$.

As we showed in [26], some of the eigenfunctions of Eq. (3) do not belong to the usual Hilbert space, $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$, so that we have to work in a Rigged Hilbert Space [33, 34].

An alternative approach to solve the eigenvalue problem of the hamiltonian of Eq. (1), is the use of the CSM method [27–32] . The aim of the CSM is to make a similarity transformation from the original hamiltonian to a hamiltonian which has eigenfunctions that belong to $\mathcal{L}^2(\mathbb{R})$. In the framework of the CSM, we shall introduce the transformation operator $\hat{V}(\theta) = \mathrm{e}^{-\frac{\theta}{2\hbar}(\hat{x}\hat{p}+\hat{p}\hat{x})}$ with a real scaling parameter θ:

$$\begin{aligned}\hat{V}(\theta)\hat{x}\hat{V}^{-1}(\theta) &= \mathrm{e}^{\,\mathbf{i}\theta}\,\hat{x},\\ \hat{V}(\theta)\hat{p}\hat{V}^{-1}(\theta) &= \mathrm{e}^{-\mathbf{i}\theta}\,\hat{p}.\end{aligned} \tag{4}$$

The hamiltonian of Eq. (3) is transformed as $\mathrm{H}(\theta) = \hat{V}(\theta)\mathrm{H}\hat{V}^{-1}(\theta)$:

$$\begin{aligned}\mathrm{H}(\theta) &= \mathrm{H}(\theta,\omega,\alpha,\beta)\\ &= \frac{1}{2}\hbar(\omega+\alpha+\beta)\left(\frac{e^{\,\mathbf{i}\theta}\,\hat{x}}{b_0}\right)^2\\ &\quad+\frac{1}{2}\hbar(\omega-\alpha-\beta)\left(\frac{b_0\,e^{-\mathbf{i}\theta}\,\hat{p}}{\hbar}\right)^2\\ &\quad+\hbar\frac{(\alpha-\beta)}{2}\left(2\,\hat{x}\frac{\mathbf{i}}{\hbar}\hat{p}+1\right).\end{aligned} \tag{5}$$

It is straightforward to observe that

$$\mathrm{H}^\dagger(\theta) = \mathrm{H}(-\theta,\omega,\beta,\alpha). \tag{6}$$

Notice that $\mathrm{H}(\theta)$ is not invariant under the usual PT-symmetry given by $\hat{x}\to-\hat{x}$, and $\hat{p}\to\hat{p}$, and $\mathbf{i}\to-\mathbf{i}$.

We shall introduce the following similarity transformation induced by the operator $\Upsilon(\theta) = \mathrm{e}^{-\frac{\alpha-\beta}{\omega-\alpha-\beta}\frac{\mathrm{e}^{2\mathbf{i}\theta}\mathrm{x}^2}{2\mathrm{b}_0^2}}$. It reads

$$\Upsilon(\theta)\;\mathrm{H}(\theta)\Upsilon(\theta)^{-1} = \mathfrak{h}(\theta), \tag{7}$$

where $\mathfrak{h}(\theta)$ is given by

$$\mathfrak{h}(\theta) = \frac{1}{2m}\left(\mathrm{e}^{-\mathbf{i}\theta}\hat{p}\right)^2+\frac{1}{2}k\left(\mathrm{e}^{\mathbf{i}\theta}\hat{x}\right)^2. \tag{8}$$

We have defined [26] $k = m\,\Omega^2$ and

$$\begin{aligned}m = m(\omega,\alpha,\beta,b_0) &= \frac{\hbar}{(\omega-\alpha-\beta)b_0^2}\\ \Omega = \Omega(\omega,\alpha,\beta) &= \sqrt{\omega^2-4\alpha\beta} = |\Omega|\mathrm{e}^{\mathbf{i}\phi}.\end{aligned} \tag{9}$$

Though $\mathfrak{h}(\theta)$ is a non-hermitian operator, $\mathfrak{h}^\dagger(\theta) = \mathfrak{h}(-\theta) = V(-2\theta)\mathfrak{h}(\theta)V(-2\theta)^{-1}$. Consequently

$$\begin{aligned}\Upsilon(-\theta)^{-1}\mathrm{H}^\dagger(\theta)\Upsilon(-\theta) &= \mathfrak{h}(\theta)^*,\\ (\Upsilon(-\theta)V(-2\theta))^{-1}\mathrm{H}^\dagger(\theta)(\Upsilon(-\theta)V(-2\theta)) &= \mathfrak{h}(\theta).\end{aligned} \tag{10}$$

From Eqs. (7) and (10), it results $\mathrm{H}^\dagger(\theta)S = S\mathrm{H}(\theta)$, with $S = \Upsilon(-\theta)V(-2\theta)\Upsilon(\theta)$ [35–37].

The eigenfunctions and eigenvalues of $\mathfrak{h}(\theta)$, $\phi(\theta)$ and $E(\theta)$, are related to that of H and $\mathrm{H}^\dagger$ as follows. Given $\mathfrak{h}(\theta)\phi(\theta,x) = E(\theta)\phi(\theta,x)$:

$$\begin{aligned}\mathrm{H}\quad \phi(\theta,x) &= \widetilde{E}(\theta)\quad \phi(\theta,x),\\ \mathrm{H}^\dagger\quad \psi(\theta,x) &= \overline{E}(\theta)\quad \psi(\theta,x),\end{aligned} \tag{11}$$

with

$$\begin{aligned}\widetilde{\phi}(\theta,x) &= \Upsilon(\theta)^{-1}\phi(\theta,x), & E(\theta) = E(\theta),\\ \overline{\psi}(\theta,x) &= \Upsilon(-\theta)(\phi(\theta,x))^*, & E(\theta) = E(\theta)^*.\end{aligned} \tag{12}$$

Thus, the eigenfunctions of $\mathrm{H}(\theta)$ with eigenvalue $\widetilde{E}_\nu(\theta) = E_\nu(\theta)$ are given by

$$\widetilde{\phi}_\nu(\theta,x) = \mathrm{e}^{\frac{\alpha-\beta}{\omega-\alpha-\beta}\frac{\mathrm{e}^{2\mathbf{i}\theta}\mathrm{x}^2}{2\mathrm{b}_0^2}}\;\mathcal{N}_\nu\phi_\nu(\theta,x) \tag{13}$$

with $\mathcal{N}_\nu$ a normalization constant.

It can be shown that the eigenfunctions of $\mathrm{H}^\dagger(\theta)$ are

$$\overline{\psi}_\nu(\theta,x) = \mathrm{e}^{-\frac{\alpha-\beta}{\omega-\alpha-\beta}\frac{\mathrm{e}^{-2\mathbf{i}\theta}\mathrm{x}^2}{2\mathrm{b}_0^2}}\;\left(\mathcal{N}_\nu\phi_\nu(\theta,x)\right)^*, \tag{14}$$

and the corresponding eigenvalue is given by $\overline{E}_\nu(\theta) = \widetilde{E}_\nu(\theta)^*$. A similar structure for Eqs. (10)-(14) can be found in [38, 39]. Moreover, the relation between the eigenvalues, Eq. (12), is a typical feature for operators which are self-adjoint in Krein spaces [39–41].

It should be mentioned that the Hamiltonian of Eq. (5), for $\alpha = \beta = 0$ and $\omega = 1/\cos(2\theta)$, reduces to the one introduced in [23–25]. Particularly, in [25] the dynamics under the action of this hamiltonian is described for values of $\theta\in(-\pi/4,\pi/4)$. For further results, the reader is kindly referred to [23–25].

In what follows, we aim to determine the range of values of θ for which $\phi(\theta,x)$ belongs to the Hilbert space $\mathcal{L}^2(\mathbb{R})$.

2.1. Eigenfunctions and eigenvectors

For $\omega-(\alpha+\beta)\neq 0$, Eq. (8) can be also written as

$$-\frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2}+\left(\frac{1}{4}y^2-\epsilon\right)\;\phi(y) = 0, \tag{15}$$

with

$$\epsilon = \frac{E}{\hbar\Omega} = \frac{E}{\hbar|\Omega|}\mathrm{e}^{\mathbf{i}\phi} \tag{16}$$

and

$$y = \sqrt{2}\;|\sigma|\mathrm{e}^{\mathbf{i}(\theta+\gamma)}\frac{x}{b_0}, \tag{17}$$

where we have defined

$$\sigma = \left(\frac{m\Omega}{\hbar}\right)^{1/2} b_0 = \mathrm{e}^{\mathbf{i}\gamma}|\sigma|. \tag{18}$$

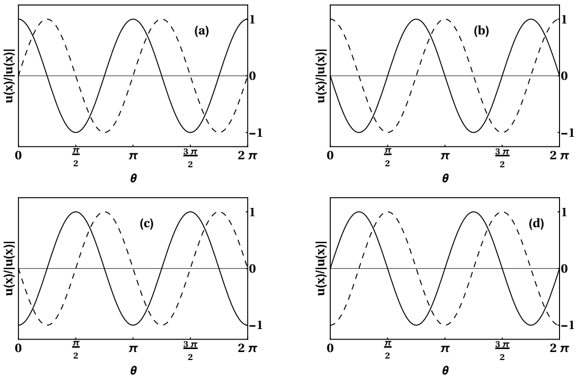

Figure 1. Effective potential of Eq. (19), $\frac{u(\theta,x)}{|u(\theta,x)|}$, for a fixed x in the regions determined by the signs of the parameters $m(\omega,\alpha,\beta,b_0)$ and $\Omega^2(\omega,\alpha,\beta,b_0)$. In Panel (a), $(sg(m), sg(\Omega^2)) = (+,+)$, Region I. For Panel (b),$(sg(m), sg(\Omega^2)) = (+,-)$, Region II. While, $(sg(m), sg(\Omega^2)) = (-,+)$ in Panel (c), Region III. In Region IV $(sg(m), sg(\Omega^2)) = (-,-)$, for Panel (d). The real part of the effective potential, $\mathrm{Re}\left(\frac{u(\theta,x)}{|u(\theta,x)|}\right)$, is displayed in solid lines, while the imaginary part of the effective potential, $\mathrm{Im}\left(\frac{u(\theta,x)}{|u(\theta,x)|}\right)$, is drawn with dashed lines.

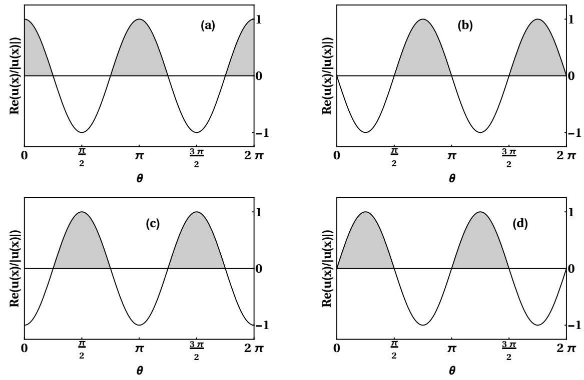

Figure 2. Real part of the effective potential of Eq. (19), $\frac{u(\theta,x)}{|u(\theta,x)|}$. The shadowed sectors correspond to the values of θ for which the solutions of Eq. (15) are square-integrable. In Panels (a), (b), c) and (d) we present the results for Regions I, II, III and IV, respectively.

Eq. (15) is the Schrödinger equation corresponding to the effective potential

$$u(\theta,x) = \frac{U(\theta,x)}{\hbar\Omega} = \mathrm{e}^{2\mathbf{i}(\theta+\gamma)}\frac{1}{2}|\sigma|^2\frac{x^2}{b_0^2}. \quad (19)$$

Solutions corresponding to Eq. (15) represent different physical systems according to the signs of $m(\omega,\alpha,\beta,b_0)$ and $\Omega^2(\omega,\alpha,\beta,b_0)$ [26]. In what follows, we shall refer to Region I when $(sg(m), sg(\Omega^2)) = (+,+)$, Region II for the case $(sg(m), sg(\Omega^2)) = (+,-)$, Region III for $(sg(m), sg(\Omega^2)) = (-,+)$, and Region IV for $(sg(m), sg(\Omega^2)) = (-,-)$, respectively.

In Figure 1 we present the behaviour of the effective potential of Eq. (19), $\frac{u(x)}{|u(x)|}$, as a function of θ, for x, $|\sigma|$ and $|\Omega|$ fixed, in the different regions of the parameter model-space. The real part of the effective potential, $\mathrm{Re}\left(\frac{u(x)}{|u(x)|}\right)$, is displayed in solid lines, while the imaginary part of the effective potential, $\mathrm{Im}\left(\frac{u(x)}{|u(x)|}\right)$, is drawn with dashed lines.

2.1.1. Discrete spectrum

For the discrete sector of the spectrum, eigenvalues and the eigenfunctions are given by

$$\begin{aligned} E_n &= \hbar\,\Omega\,[n] = \hbar\,|\Omega|\mathrm{e}^{\mathbf{i}\phi}\,[n], \\ \widetilde{\phi}_m(\theta,x) &= \mathrm{e}^{\frac{\alpha-\beta}{\omega-\alpha-\beta}\mathrm{e}^{2\mathrm{i}\theta}\frac{\mathrm{x}^2}{2\mathrm{b}_0^2}}\,\phi_m(\theta,x), \quad (20) \\ \overline{\psi}_m(\theta,x) &= \mathrm{e}^{-\frac{\alpha-\beta}{\omega-\alpha-\beta}\mathrm{e}^{-2\mathrm{i}\theta}\frac{\mathrm{x}^2}{2\mathrm{b}_0^2}}\,(\phi_m(\theta,x))^*, \quad (21) \end{aligned}$$

where $\phi_n(\theta,x)$ can be written as

$$\begin{aligned} \phi_n(\theta,x) &= \mathcal{N}_n\mathrm{e}^{-\mathrm{e}^{2\mathbf{i}(\theta+\gamma)}\frac{x^2}{2b_0^2}|\sigma|^2}\,H_n\left(\mathrm{e}^{\mathbf{i}(\theta+\gamma)}\,\frac{x}{b_0}|\sigma|\right). \\ {\mathcal{N}_n}^2 &= \frac{\mathrm{e}^{\mathbf{i}(\theta+\gamma)}}{\sqrt{\pi}n!2^n}\frac{|\sigma|}{b_0}, \quad (22) \end{aligned}$$

being $H_n(z)$ the Hermite Polynomial of order n, and $[n] = n + 1/2$.

Eigenfunctions $\phi_n(\theta,x)$ are square-integrable for θ-intervals where $\mathrm{Re}(u(\theta,x))$ takes positive values. In Figure 2, we plot $\mathrm{Re}(u(\theta,x)/|u(\theta,x)|)$ for every region, the gray regions correspond to the intervals for which the eigenfunctions are square integrable.

In Table (1), we summarize the sign of the parameter m and Ω^2, which characterize the different regions of the model, and for each region we present the values of phases γ and ϕ, and the interval where the eigenfunctions are square-integrable.

In Regions (I) and (III), we can define two well-defined $\theta-$domains: $I_1 = [-\pi,-3\pi/4)\cup(-\pi/4,\pi/4)\cup(3\pi/4,\pi]$ and $I_2 = (-3\pi/4,-\pi/4)\cup(\pi/4,3\pi/4)$. While, in Regions (II) and (IV), the θ-domains are: $I_3 = (-\pi,-\pi/2)\cup(0,\pi/2)$ and $I_4 = (-\pi/2,0)\cup(\pi/2,\pi)$. The intervals repeat themselves periodically, with period π.

In the domains summarized in Table 1, eigenfunctions $\{\overline{\psi}_\nu(\theta,x), \widetilde{\phi}_\nu(\theta,x)\}$ form a biorthogonal complete set.

$$\int_{-\infty}^{\infty}(\overline{\psi}_m(\theta,x))^*\widetilde{\phi}_n(\theta,x)\,\mathrm{d}x = \int_{-\infty}^{\infty}\phi_m(\theta,x)\phi_n(\theta,x)\,\mathrm{d}x = \delta_{mn}. \quad (23)$$

It should be noticed that in all regions, the $\theta-$domains of positive spectrum are different from the domains with negative spectrum. They represent different physical boundary conditions.

	sg(m)	sg(Ω^2)	γ	ϕ	I	θ_c
I	+	+	0	0	I_1	$\pm\pi/4$
			$\pi/2$	π	I_2	
III	-	+	$\pi/2$	0	I_2	
			0	π	I_1	
II	+	-	$\pi/4$	$\pi/2$	I_4	0
			$-\pi/4$	$-\pi/2$	I_3	$\pm\pi/2$
IV	-	-	$-\pi/4$	$\pi/2$	I_3	π
			$\pi/4$	$-\pi/2$	I_4	

Table 1. Values of the characteristic parameters for the different model-space regions. In columns 2 and 3 we give the sign of m and Ω^2, respectively. Phases γ and ϕ, for the different regions, are given in columns 4 and 5, respectively. In column 6 we present the θ-interval for which the different eigenfunctions are square-integrable. In the Table $I_1 = [-\pi, -3\pi/4) \cup (-\pi/4, \pi/4) \cup (3\pi/4, \pi]$, $I_2 = (-3\pi/4, -\pi/4) \cup (\pi/4, 3\pi/4)$, $I_3 = (-\pi, -\pi/2) \cup (0, \pi/2)$ and $I_4 = (-\pi/2, 0) \cup (\pi/2, \pi)$. In the last column, we give the values of θ_c for which the eigenfunctions of the continuous spectrum are square-integrable. The intervals repeat themselves periodically, with period π.

2.1.2. Continuous spectrum

The eigenfunctions associated to the continuous spectrum [26, 42–45] are given, in terms of the eigenfunctions of $\mathfrak{h}(\theta)$ of Eq. (8), $\phi^E_\pm(\theta, x)$, by

$$\widetilde{\phi}^E_\pm(\theta,x) = e^{\frac{\alpha-\beta}{\omega-\alpha-\beta} e^{2i\theta} \frac{x^2}{2b_0^2}} \phi^E_\pm(\theta,x), \quad (24)$$

$$\overline{\psi}^E_\pm(\theta,x) = e^{-\frac{\alpha-\beta}{\omega-\alpha-\beta} e^{-2i\theta} \frac{x^2}{2b_0^2}} (\phi^E_\pm(\theta,x))^*, \quad (25)$$

with

$$\phi^E_\pm(\theta,x) = \mathcal{C}\,\Gamma(\nu+1) D_{-\nu-1}\left(\mp\sqrt{-2}e^{i(\theta+\gamma)}|\sigma|\frac{x}{b_0}\right). \quad (26)$$

being $D_{-\nu-1}(y)$ the parabolic cylinder functions and $\nu = \epsilon - \frac{1}{2}$. The normalization constant takes the value $\mathcal{C} = \frac{e^{i\pi/8} i^{\nu/2}}{\left(\frac{|\sigma|}{b_0} e^{i(\theta+\gamma)}\right)^{1/2} \pi 2^{3/4}}$.

The biorthogonality and the completeness relation can be written as

$$\int_{-\infty}^{\infty} (\overline{\psi}^E_\pm(\theta,x))^* \widetilde{\phi}^{E'}_\pm(\theta,x) dx = \delta(E-E'),$$

$$\sum_{s=\pm} \int_{-\infty}^{\infty} (\overline{\psi}^E_s(\theta,x))^* \widetilde{\phi}^E_s(\theta,x) dE = \delta(x-x'). \quad (27)$$

The possible values that the parameter θ can take to fulfill the requirements of biorthogonality and completeness of Eq. (27), θ_c, are presented in the last column of Table 1.

In the framework of the CSM, the continuous spectrum lies along the line 2θ. In Regions II and IV, the $2\theta_c = \pm\,\pi$ so that $E \in (-\infty, +\infty)$ Meanwhile, in Region I and III, $2\theta_c = \pm\,\frac{\pi}{2}$, so that E takes imaginary values. Consequently, the parameter ν associated to the order of the eigenfunctions of Eq. (26) takes the value $\nu = -i|\epsilon| - \frac{1}{2}$.

If we look at the effective potential $u(\theta, x)$, the values of θ_c correspond to the values of θ for which $Re(u(\theta, x)) = 0$.

2.1.3. Particular cases

Case (a): $\Omega = 0$**.**

When $\Omega = 0$ and $\omega - (\alpha + \beta) \neq 0$, the problem reduces to that of a free particle of energy $E = \varepsilon\, e^{-2i\theta}$. Eq. (8) reduces to

$$-\frac{\hbar^2}{2m\, e^{2i\theta}} \frac{d^2 f(x)}{dx^2} = E\ f(x), \quad (28)$$

the wave function can be written as $f(x) = Ae^{ikx} + Ae^{-ikx}$, with $k = \sqrt{\frac{2\varepsilon}{\hbar(\omega-\alpha-\beta)b_0^2}}$.

Case (b): $\omega - (\alpha + \beta) = 0,\ \alpha \neq \beta$**.**

To study this case we have to look at Eq. (5). If $\omega - (\alpha + \beta) = 0$, it reads

$$H(\theta) = \hbar(\alpha+\beta)\left(\frac{e^{\,i\theta}\,\hat{x}}{b_0}\right)^2 + \hbar\frac{(\alpha-\beta)}{2}\left(2\,\hat{x}\frac{i}{\hbar}\hat{p} + 1\right), \quad (29)$$

$$f(x) = e^{-e^{2i\theta}\frac{\hat{x}^2}{4b_0^2}\frac{\alpha+\beta}{\alpha-\beta}} x^{-\frac{1}{2}+\frac{\varepsilon e^{-2i\theta}}{\hbar(\alpha-\beta)}}. \quad (30)$$

In Table 2 we present the values of E for which the wavefunction $f(x)$ is square-integrable.

$(\alpha+\beta)/(\alpha-\beta)$	$(\alpha-\beta)$	$\cos(2\theta)$	ε
+	+	I_1	$\frac{\varepsilon\|\cos(2\theta)\|}{\hbar\|\alpha-\beta\|} < \frac{1}{2}$
-	-	I_2	
+	-	I_1	$\frac{\varepsilon\|\cos(2\theta)\|}{\hbar\|\alpha-\beta\|} > \frac{1}{2}$
-	+	I_2	

Table 2. Regions for which the wave function of Eq. (30) is square-integrable.

2.2. Mean values of observables

To compute the mean values, we use operators $\hat{P}$ and $\hat{X}$ defined as [19, 46, 47]

$$\begin{aligned} \hat{P} &= \Upsilon^{-1} V(\theta+\gamma)\hat{p}V(\theta+\gamma)^{-1}\Upsilon \\ &= e^{-i(\theta+\gamma)}\hat{p} + i\hbar e^{i(\theta+\gamma)}\frac{\alpha-\beta}{(\omega-\alpha-\beta)b_0^2}\hat{x}, \\ \hat{X} &= \Upsilon^{-1} V(\theta+\gamma)\hat{x}V(\theta+\gamma)^{-1}\Upsilon \\ &= e^{i(\theta+\gamma)}\hat{x}, \end{aligned} \quad (31)$$

that satisfy

$$[\hat{X}, \hat{P}] = \mathbf{i}\hbar. \tag{32}$$

For the discrete spectrum of H, it can be proved that

$$\begin{aligned}
\langle m|\hat{P}|n\rangle &= \int_{-\infty}^{\infty} (\overline{\psi}_m(\theta,x))^* \ \hat{P}\ \widetilde{\phi}_n(\theta,x)\mathrm{dx} \\
&= \int_{-\infty}^{\infty} \phi_m(\theta,x)\ \mathrm{e}^{-\mathbf{i}(\theta+\gamma)}\hat{p}\ \phi_n(\theta,x)\mathrm{dx} \\
&= \frac{\mathbf{i}\hbar}{\sqrt{2}b_{0r}}\left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right), \\
\langle m|\hat{P}^2|n\rangle &= \int_{-\infty}^{\infty} (\overline{\psi}_m(\theta,x))^* \ \hat{P}^2\ \widetilde{\phi}_n(\theta,x)\mathrm{dx} \\
&= \int_{-\infty}^{\infty} \phi_m(\theta,x)\ \mathrm{e}^{-2\mathbf{i}(\theta+\gamma)}\hat{p}^2\ \phi_n(\theta,x)\mathrm{dx} \\
&= \frac{\mathbf{i}\hbar}{\sqrt{2}b_{0r}}\left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right) \\
&= -\frac{\hbar^2}{2b_{0r}^2}\Big(\sqrt{(n+2)(n+1)}\delta_{m,n+2} \\
&\qquad -(2n+1)\delta_{m,n} \\
&\qquad +\sqrt{n(n-1)}\delta_{m,n-2}\Big),
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
\langle m|\hat{X}|n\rangle &= \int_{-\infty}^{\infty} (\overline{\psi}_m(\theta,x))^* \ \hat{X}\ \widetilde{\phi}_n(\theta,x)\mathrm{dx} \\
&= \int_{-\infty}^{\infty} \phi_m^{\pm}(\theta,x)\ \mathrm{e}^{\mathbf{i}(\theta+\gamma)}\hat{x}\ \phi_n^{\pm}(\theta,x)\mathrm{dx} \\
&= \frac{b_{0r}}{\sqrt{2}}\left(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right), \\
\langle m|\hat{X}^2|n\rangle &= \int_{-\infty}^{\infty} (\overline{\psi}_m(\theta,x))^* \ \hat{X}^2\ \widetilde{\phi}_n(\theta,x)\mathrm{dx} \\
&= \int_{-\infty}^{\infty} \phi_m(\theta,x)\ \mathrm{e}^{2\mathbf{i}(\theta+\gamma)}\hat{x}^2\ \phi_n(\theta,x)\mathrm{dx} \\
&= \frac{b_{0r}^2}{2}\Big(\sqrt{(n+2)(n+1)}\delta_{m,n+2} \\
&\qquad +(2n+1)\delta_{m,n} \\
&\qquad +\sqrt{n(n-1)}\delta_{m,n-2}\Big),
\end{aligned} \tag{34}$$

with $b_{0r} = b_0/|\sigma|$.

2.3. Time dependent mean values

From the Schrödinger equation

$$\mathbf{i}\hbar\frac{\partial}{\partial t}\widetilde{\Phi}_n(\theta,x,t) = H(\theta)\widetilde{\Phi}_n(\theta,x,t), \tag{35}$$

it results

$$\widetilde{\Phi}_n(\theta,x,t) = \mathrm{e}^{-\mathbf{i}\widetilde{E}_n\frac{t}{\hbar}}\widetilde{\phi}_n(\theta,x). \tag{36}$$

In the same way

$$\mathbf{i}\hbar\frac{\partial}{\partial t}\overline{\Psi}_n(\theta,x,t) = H(\theta)^\dagger\overline{\psi}_n(\theta,x,t), \tag{37}$$

it results

$$\overline{\Psi}_n(\theta,x,t) = \mathrm{e}^{-\mathbf{i}\overline{E}_n\frac{t}{\hbar}}\overline{\psi}_n(\theta,x). \tag{38}$$

2.3.1. Regions I and III: Real spectrum

In Regions I and III, the discrete eigenvalues of $H(\theta)$ take the values $E_n^{\pm} = \pm\hbar|\Omega|[n]$, with eigenfunctions $\widetilde{\phi}_n^{\pm}(\theta,x)$. In Region I, the eigenfunctions of the positive (negative) are square integrable in interval I_1 (I_2), see Table 1. Meanwhile, in Region III, tthe eigenfunctions of the positive (negative) are square integrable in interval I_2 (I_1). Consequently the time evolution of the states is given by

$$\begin{aligned}
\widetilde{\Phi}_n^{\pm}(\theta,x,t) &= \mathrm{e}^{-\mathbf{i}\frac{\widetilde{E}_n t}{\hbar}}\widetilde{\phi}_n(\theta,x), \\
&= \mathrm{e}^{\mp\mathbf{i}(n+\frac{1}{2})|\Omega|t}\widetilde{\phi}_n^{\pm}(\theta,x). \\
\overline{\Psi}_n^{\pm}(\theta,x,t) &= \mathrm{e}^{-\mathbf{i}\frac{\overline{E}_n t}{\hbar}}\overline{\psi}_n(\theta,x), \\
&= \mathrm{e}^{\mp\mathbf{i}(n+\frac{1}{2})|\Omega|t}\overline{\psi}_n^{\pm}(\theta,x),
\end{aligned} \tag{39}$$

and then

$$\langle m|\hat{O}|n\rangle = \mathrm{e}^{\mp\mathbf{i}(n-m)|\Omega|t}\int_{-\infty}^{\infty}(\overline{\psi}_m^{\pm}(\theta,x))^*\hat{O}\widetilde{\phi}_n^{\pm}(\theta,x)\mathrm{dx}. \tag{40}$$

2.3.2. Region II and IV: Complex spectrum

In Regions II and IV, the discrete eigenvalues of $H(\theta)$ take the values $E_n^{\pm} = \pm\mathbf{i}\hbar|\Omega|[n]$, with eigenfunctions $\widetilde{\phi}_n^{\pm}(\theta,x)$. In Region II, the eigenfunctions of the positive (negative) are square integrable in interval I_4 (I_3). Meanwhile, in Region III, the eigenfunctions of the positive (negative) are square integrable in interval I_3 (I_4). So that the time evolution of the eigenfunctions are given by

$$\begin{aligned}
\widetilde{\Phi}_n^{\pm}(\theta,x,t) &= \mathrm{e}^{-\mathbf{i}\frac{E_n t}{\hbar}}\widetilde{\phi}_n^{\pm}(\theta,x), \\
&= \mathrm{e}^{\pm(n+\frac{1}{2})|\Omega|t}\widetilde{\phi}_n(\theta,x).
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
\overline{\Psi}_n^{\pm}(\theta,x,t) &= \mathrm{e}^{-\mathbf{i}\frac{E_n^* t}{\hbar}}\overline{\psi}_n(\theta,x), \\
&= \mathrm{e}^{\mp(n+\frac{1}{2})|\Omega|t}\overline{\psi}_n^{\pm}(\theta,x).
\end{aligned} \tag{42}$$

As a result

$$\langle m|\hat{O}|n\rangle = \mathrm{e}^{\pm(n-m)|\Omega|t}\int_{-\infty}^{\infty}(\overline{\psi}_m^{\pm}(\theta,x))^*\hat{O}\widetilde{\phi}_n^{\pm}(\theta,x)\mathrm{dx}. \tag{43}$$

3. Results and discussion

In order to evaluate the benefits of the present approach, let us consider the time evolution of a given initial state when the parameters of the model correspond to Region II.

In [26] we have analysed the Swanson model by solving its eigenvalue problem in the Rigged Hilbert Space. We have found that in Region II the hamiltonian was similar to the one of a particle in a parabolic barrier. In the framework of the CSM, we model the effective interaction by a complex potential. This fact resembles the spirit of the Optical Potential in Nuclear Physics [48, 49], as the potential seen by an incident nucleon on a nucleus is modeled by a complex effective potential accounting for the loss of flux due to the interaction of an incident particle with the nucleons of the nucleus.

We shall consider the solutions with eigenvalues $E_n = -\mathbf{i}\hbar|\Omega|(n + 1/2)$, which evolve in time as $\mathrm{e}^{-|\Omega|(n+1/2)t}$. They correspond to the boundary problem for $0 < t < \infty$. In this case $\gamma = -\pi/4$ and $\theta \in I_3$.

For simplicity, let us assume that the initial state is a coherent state of the form

$$\phi_I(z,\theta,x) = \mathrm{e}^{-|z|^2/2}\sum_{k=0}^{\infty}\frac{z^k}{\sqrt{k!}}\phi_k(\theta,x), \tag{44}$$

where $\widetilde{\phi}_k(\theta,x)$ is the k-eigenfunction of $\mathrm{H}(\theta)$. The survival probability of the state can be computed as

$$\begin{aligned} p(t) &= \left|\int_{-\infty}^{\infty}(\overline{\psi}_I(z,\theta,x))^*\mathrm{e}^{-\mathbf{i}\mathrm{H}(\theta)\mathrm{t}/\hbar}\phi_\mathrm{I}(\mathrm{z},\theta,\mathrm{x})\mathrm{dx}\right|^2 \\ &= \left|\mathrm{e}^{-|z|^2-|\Omega|t/2}\sum_{k=0}^{\infty}\frac{(|z|^2\mathrm{e}^{-|\Omega|\mathrm{t}})^\mathrm{k}}{k!}\right|^2 \\ &= \mathrm{e}^{-|\Omega|t+2|z|^2(\mathrm{e}^{-|\Omega|\mathrm{t}}-1)}. \end{aligned} \tag{45}$$

Notice that, in this particular case, $p(t)$ is independent of the parameter θ.

4. Conclusions

In this work we analyse the advantages of the CSM for describing the dynamics of a non-hermitian system when the eigenfunctions of the problem do not belong to $\mathcal{L}^2(\mathbb{R})$. We have shown that we can cast the original problem into a complex potential, which includes absorption and dissipation effects according to the sign of its imaginary component. We have shown that for a range of values of θ in the different regions of the model, the resulting eigenfunctions are square-integrable. This feature facilitates the study of the dynamics of the system from the computational point of view. The price we have to pay is the lack of PT-symmetry invariance of the transformed hamiltonian.

Work is in progress concerning the application of the CSM to a more involved problem as the one presented in [50].

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