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RESEARCH ARTICLE

Untangling the Newman–Janis algorithm

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Abstract Newman–Janis algorithm for Kerr–Newman geometry is reanalyzed in the light of Cartan calculus.

Keywords Kerr–Newman · Kerr–Schild · Newman–Janis

1 Introduction

Two years after the discovery of Kerr geometry [1], Newman and Janis showed an algorithm for converting Schwarzschild geometry into Kerr geometry. They described it as a *complex coordinate transformation on the Schwarzschild metric* for "deriving" (quoted by the authors) the Kerr metric [2]. Of course, the reasons why such a short cut to Kerr solution does work can be traced to the behavior of Einstein equations [3,4]. On another hand, Newman–Janis algorithm resembled the complex shift of the origin used in electromagnetism for obtaining a magnetic dipole starting from an electric monopole [5]. Actually, Newman and Janis tried to interpret Kerr geometry by taking advantage of such analogy. Besides, Newman has shown that the Weyl tensor of Schwarzschild and Kerr solutions are just different "real slices" of a same complex field

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in complex Minkowski space–time [6] (see also Ref. [7]). More recently, Newman–Janis algorithm has been invoked to explore axially symmetric inner solutions [8–11] or vacuum solutions in theories of modified gravity [12]. Deepening the understanding of Newman–Janis algorithm can help to improve the chances of successfully applying this mechanism, or a similar one, to get axially symmetric solutions in other areas (for instance, Kaluza–Klein theory, string theory, non-Abelian BH's, alternative gravities, non-linear electrodynamics, etc.).

The route to Kerr geometry is highly simplified within the framework of Cartan calculus. We will reobtain the Kerr solution by exploiting the power of exterior calculus and keeping the focus on the Newman–Janis mechanism. In Sect. 2 we introduce a simple rule to connect two different null tetrads in Minkowski space–time; these tetrads are based, respectively, on spherical and twisted spheroidal coordinates. In Sect. 3 we review general relativity in terms of null tetrads and the spin connection as the potentials for torsion and curvature respectively. In Sect. 4 we obtain the Kerr–Newman solution in the Kerr–Schild form, and state the Newman–Janis algorithm. In Sect. 5 we display the conclusions.

2 Coordinates and tetrads in flat space

2.1 Twisted spheroidal coordinates

Let be x, y, z Cartesian coordinates in Euclidean space. The oblate spheroidal coordinates r, θ, φ (a case of ellipsoidal coordinates) are defined as

$$x = \sqrt{r^2 + a^2}\ \sin\theta\ \cos\varphi, \qquad y = \sqrt{r^2 + a^2}\ \sin\theta\ \sin\varphi, \qquad z = r\ \cos\theta. \quad (1)$$

The surfaces $r = constant$ and $\theta = constant$ are spheroids and one-sheet hyperboloids; in fact,

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1, \qquad x^2 + y^2 - z^2 \tan^2\theta = a^2 \sin^2\theta. \quad (2)$$

For θ going to $\pi/2$, the throat radii of the hyperboloid goes to a. The $z = 0$ plane is divided into two regions separated by a circle of radius a: i) for $r = 0$ it is $x^2 + y^2 = a^2\ \sin^2\theta < a^2$; ii) for $\theta = \pi/2$ it is $x^2 + y^2 = r^2 + a^2 > a^2$. If $a = 0$ the spheroidal coordinates become spherical; the surfaces $r = constant$ and $\theta = constant$ become spheres and cones.

Oblate spheroidal coordinates (1) are Boyer-Lindquist-like coordinates in Euclidean geometry since

$$dx^2 + dy^2 + dz^2 = \frac{r^2+a^2\ \cos^2\theta}{r^2 + a^2}\, dr^2 + (r^2+a^2\cos^2\theta)\, d\theta^2 + (r^2+a^2)\ \sin^2\theta\, d\varphi^2. \quad (3)$$

In this expression, let us replace the azimuth angle φ for

$$\phi = \varphi - \arctan\frac{r}{a} \quad \Rightarrow \quad d\phi = d\varphi - \frac{a\,dr}{r^2 + a^2} \tag{4}$$

(then, ϕ coincides with φ in the region $x^2 + y^2 < a^2$ where $r = 0$). By using the twisted azimuth angle ϕ, the coefficient of dr^2 in the distance (3) becomes equal to 1:

$$dx^2 + dy^2 + dz^2 = (dr + a\,\sin^2\theta\, d\phi)^2 + (r^2 + a^2\cos^2\theta)\,(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{5}$$

If $a = 0$ one still gets the Euclidean metric in spherical coordinates. In sum, to change from spherical coordinates to *twisted* spheroidal coordinates in Euclidean space we can follow the short cut

$$\begin{aligned} r^2 &\longrightarrow \rho^2 \doteq r^2 + a^2\cos^2\theta\,, \\ dr &\longrightarrow dr + a\,\sin^2\theta\, d\phi\,. \end{aligned} \tag{6}$$

2.2 Orthonormal and null tetrads

Tetrads are bases in the cotangent space; they are made up of four 1-forms. Each basis $\{\mathbf{E}^i\}$ in the cotangent space is *dual* of a vector basis $\{\mathbf{E}_j\}$ in the tangent space; this means that $\mathbf{E}^i(\mathbf{E}_j) = \delta^i_j$. Both related bases can be expanded in dual coordinate bases $\{dx^\mu\}$, $\{\partial_\mu\}$:

$$\mathbf{E}^i = E^i_\mu\, dx^\mu, \qquad \mathbf{E}_i = E^\mu_i\, \partial_\mu. \tag{7}$$

Duality implies that the matrix E^μ_i is inverse of E^i_μ.

The metric properties of a manifold can be represented by the metric tensor field $\mathbf{g}$ or, alternatively, a field of tetrads $\{\mathbf{e}^{\hat{a}}\}$ linked to the metric by the assumption of orthonormality:

$$g_{\mu\nu} = \eta_{\hat{a}\hat{b}}\, e^{\hat{a}}_\mu\, e^{\hat{b}}_\nu, \qquad g^{\mu\nu} = \eta^{\hat{a}\hat{b}}\, e^\mu_{\hat{a}}\, e^\nu_{\hat{b}}, \tag{8}$$

where $\eta_{\hat{a}\hat{b}}$ is the Minkowskian metric diag$(1, -1, -1, -1)$. We can check the orthonormality by computing the inner products between elements of the basis $\{\mathbf{e}^{\hat{a}}\}$:

$$\mathbf{e}^{\hat{a}} \cdot \mathbf{e}^{\hat{b}} = g^{\mu\nu}\, e^{\hat{a}}_\mu\, e^{\hat{b}}_\nu = \eta^{\hat{a}\hat{b}}, \tag{9}$$

because of duality. We avoid coordinate indexes by writing the relation (8) as

$$\mathbf{g} = \eta_{\hat{a}\hat{b}}\, \mathbf{e}^{\hat{a}} \otimes \mathbf{e}^{\hat{b}}, \tag{10}$$

where $\otimes$ stands for the tensor product.

At each point of the manifold there exists a continuous of orthonormal tetrad fields, all of them related through (local) Lorentz transformations L,

$$\mathbf{e}_{\hat{a}'} = L^{\hat{a}}_{\ \hat{a}'}\, \mathbf{e}_{\hat{a}}, \quad \mathbf{e}^{\hat{a}'} = L^{\hat{a}'}_{\ \hat{a}}\, \mathbf{e}^{\hat{a}}, \tag{11}$$

(duality requires that $L^{\hat{a}'}_{\ \hat{a}}$ be inverse of $L^{\hat{a}}_{\ \hat{a}'}$). This kind of ambiguity has no consequences for the metric $\mathbf{g}$ because $\eta_{\hat{a}\hat{b}}$ is Lorentz-invariant.

The link between metric and tetrad can also be established through a null tetrad. Both strategies are related, since any orthonormal tetrad $\{\mathbf{e}^{\hat{a}}\}$ defines a null tetrad $\{\mathbf{n}^a\} = \{\mathbf{l},\ \mathbf{n},\ \mathbf{m},\ \overline{\mathbf{m}}\}$:

$$\mathbf{l} = \frac{1}{\sqrt{2}}\,(\mathbf{e}^{\hat{0}} + \mathbf{e}^{\hat{1}}), \quad \mathbf{n} = \frac{1}{\sqrt{2}}\,(\mathbf{e}^{\hat{0}} - \mathbf{e}^{\hat{1}}), \quad \mathbf{m} = \frac{1}{\sqrt{2}}\,(\mathbf{e}^{\hat{2}} + i\ \mathbf{e}^{\hat{3}}),$$
$$\overline{\mathbf{m}} = \frac{1}{\sqrt{2}}\,(\mathbf{e}^{\hat{2}} - i\ \mathbf{e}^{\hat{3}}). \tag{12}$$

In fact, the tetrad (12) results to be null:

$$\mathbf{l}\cdot\mathbf{l} = 0 = \mathbf{n}\cdot\mathbf{n}, \qquad \mathbf{m}\cdot\mathbf{m} = 0 = \overline{\mathbf{m}}\cdot\overline{\mathbf{m}}; \tag{13}$$

besides it is

$$\mathbf{l}\cdot\mathbf{n} = 1 = -\mathbf{m}\cdot\overline{\mathbf{m}}, \qquad \mathbf{l}\cdot\mathbf{m} = 0 = \mathbf{n}\cdot\mathbf{m}. \tag{14}$$

The relationship between related orthonormal and null tetrads can be written as

$$\mathbf{n}^a = \Lambda^a_{\ \hat{a}}\,\mathbf{e}^{\hat{a}}, \qquad \mathbf{e}^{\hat{a}} = \Lambda^{\hat{a}}_{\ a}\,\mathbf{n}^a, \tag{15}$$

where $\Lambda^a_{\ \hat{a}}$ and its inverse matrix are

$$\Lambda^a_{\ \hat{a}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix}, \qquad \Lambda^{\hat{a}}_{\ a} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{pmatrix}. \tag{16}$$

By replacing in Eq. (10) one gets

$$\mathbf{g} = \eta_{\hat{a}\hat{b}}\,\Lambda^{\hat{a}}_{\ a}\,\Lambda^{\hat{b}}_{\ b}\,\mathbf{n}^a\otimes\mathbf{n}^b = \eta_{ab}\ \mathbf{n}^a\otimes\mathbf{n}^b\,, \tag{17}$$

where

$$\eta_{ab} = \eta_{\hat{a}\hat{b}}\,\Lambda^{\hat{a}}_{\ a}\,\Lambda^{\hat{b}}_{\ b} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \eta^{ab}\,. \tag{18}$$

In sum, the relation between metric tensor and null tetrad is

$$\mathbf{g} = \mathbf{l}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{l} - \mathbf{m}\otimes\overline{\mathbf{m}} - \overline{\mathbf{m}}\otimes\mathbf{m}\,. \tag{19}$$

For instance, according to Eq. (5), a possible null tetrad for Minkowski space–time is

$$\mathbf{l} = \frac{1}{\sqrt{2}}\,(dt + dr\, +\, a\,\sin^2\theta\, d\phi)\,, \qquad \mathbf{n} = \frac{1}{\sqrt{2}}\,(dt - dr\, -\, a\,\sin^2\theta\, d\phi)\,,$$
$$\mathbf{m} = \frac{\xi}{\sqrt{2}}\,(d\theta\, +\, i\,\sin\theta\, d\phi)\,, \qquad \overline{\mathbf{m}} = \frac{\overline{\xi}}{\sqrt{2}}\,(d\theta\, -\, i\,\sin\theta\, d\phi)\,, \quad (20)$$

where r, θ, ϕ are twisted spheroidal coordinates and $\xi \doteq r + i\,a\,\cos\theta$. Thus, the rule (6) for passing from spherical ($a = 0$) to twisted spheroidal coordinates is rephrased as

$$\text{i) in the } \{\mathbf{m}, \overline{\mathbf{m}}\} \text{ sector, replace} \qquad r \longrightarrow \xi\,,\ \overline{\xi}$$
$$\text{ii) in the } \{\mathbf{l}, \mathbf{n}\} \text{ sector, replace} \qquad dr \longrightarrow dr + a\,\sin^2\theta\, d\phi\ . \quad (21)$$

Equation (5) shows that $dr + a\ \sin^2\theta\, d\phi$ is a unitary 1-form on each t =*const.* hypersurface, being orthogonal to $\mathbf{m}$ and $\overline{\mathbf{m}}$. Actually it is the covector associated with the unitary vector $\partial/\partial r$ tangent to the lines θ, ϕ =*const.*[1]

3 Gravity in Cartan language

Newman–Janis algorithm is not merely a way to change coordinates in flat space–time. It involves gravity; it is a short cut to change from Schwarzschild (or Reissner–Nordstrom) geometry to Kerr (or Kerr–Newman) geometry. So, we should add gravity (curvature) to the rule (21). Gravity can be added in Eq. (19) by using tetrads other than the one of Eq. (20). However, not any tetrad is allowed since the so built new metric should still accomplish Einstein equations.

3.1 Torsion and curvature

Let be $\Gamma^i_{\ jk}$ the affine connection defining the covariant derivative in a manifold. Since the derivative index k behaves tensorially under changes of basis, the $\Gamma^i_{\ jk}$'s define a set of 1-forms $\omega^i_{\ j}$,

$$\omega^i_{\ j} \doteq \Gamma^i_{\ jk}\,\mathbf{E}^k, \quad (22)$$

called the spin connection. The transformation of the spin connection under change of basis,

$$\mathbf{E}^{i'} = \Lambda^{i'}_{\ \ i}\,\mathbf{E}^i, \quad \omega^{i'}_{\ \ j'} = \Lambda^{i'}_{\ \ i}\,\omega^i_{\ j}\,\Lambda^j_{\ j'} + \Lambda^{i'}_{\ \ k}\,d\Lambda^k_{\ j'}, \quad (23)$$

allows the preservation of the tensorial character of an object under covariant differentiation. By differentiating the tetrad and the connection, one defines two tensor-valued

[1] It can be verified that lines θ, ϕ =*const.* are the *straight* lines generating the one-sheet hyperboloids of Eq. (2). They form a congruence of (geodesic) straight lines displaying the axial symmetry we will pursue for the gravitational field in Sect. 4.

2-form fields on the manifold. Torsion $\mathbf{T}^i$ is the covariant derivative of the tetrad,

$$\mathbf{T}^i \doteq D\mathbf{E}^i = d\mathbf{E}^i + \omega^i{}_j \wedge \mathbf{E}^j, \tag{24}$$

and curvature $\mathbf{R}^i{}_j$ is built with derivatives of the spin connection:

$$\mathbf{R}^i{}_j \doteq d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j. \tag{25}$$

$\mathbf{T}^i$ and $\mathbf{R}^i{}_j$ have a mixed character. On one hand they are 2-forms for each choice of their indexes. On the other hand they transform as components of tensors in the indexes $i, j, \ldots$ by virtue of the behaviors (23).

In general, the covariant derivative D of a tensor-valued p-form is a $(p+1)$-form that preserves its tensorial character thanks to the compensating terms contributed by the connection. Tensor $\mathbf{R}^i{}_j$ cannot be thought as the covariant derivative of $\omega^i{}_j$ because the connection does not transform as a tensor. Tensor $\mathbf{R}^i{}_j$ can be covariantly differentiated to obtain the (second) Bianchi identity,

$$D\mathbf{R}^i{}_j = d\mathbf{R}^i{}_j + \omega^i{}_k \wedge \mathbf{R}^k{}_j - \omega^k{}_j \wedge \mathbf{R}^i{}_k \equiv 0. \tag{26}$$

For more details about Cartan calculus see, for instance, Ref. [13].

3.2 Einstein equations

In Gravity we choose an orthonormal tetrad $\{\mathbf{e}^{\hat{a}}\}$ and the spin connection $\{\omega^{\hat{a}}{}_{\hat{b}}\}$ to play the role of potentials describing the gravitational fields (torsion and curvature). The assumed orthonormality of the tetrad establishes the link tetrad-metric; this link is invariant under local Lorentz transformations (11). On the other hand the spin connection is assumed to be *metric*, which means the vanishing of the covariant derivative of the (Lorentz) tensor-valued 0-form $\eta_{\hat{a}\hat{b}}$:

$$0 = D\eta_{\hat{a}\hat{b}} = d\eta_{\hat{a}\hat{b}} - \omega^{\hat{c}}{}_{\hat{a}}\, \eta_{\hat{c}\hat{b}} - \omega^{\hat{c}}{}_{\hat{b}}\, \eta_{\hat{a}\hat{c}}\,, \tag{27}$$

i.e.,

$$\omega_{\hat{b}\hat{a}} = -\omega_{\hat{a}\hat{b}} \tag{28}$$

(Lorentz tensor indexes are lowered with $\eta_{\hat{a}\hat{b}}$). This property also implies

$$D\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}} = 0, \tag{29}$$

where $\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}}$ is the Levi-Civita symbol, which is a tensor under Lorentz transformations.[2]

[2] For theories harboring a non-metricity field see, for instance, Ref. [14].

General Relativity is a theory of gravity governed by the Einstein-Hilbert Lagrangian, which is the Lorentz scalar-valued 4-form (volume) defined as

$$L \;=\; \frac{1}{32\,\pi\,G}\,\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}}\,\mathbf{e}^{\hat{a}}\wedge\mathbf{e}^{\hat{b}}\wedge\mathbf{R}^{\hat{c}\hat{d}}\,. \tag{30}$$

In Palatini approach, the action is varied independently with respect to the tetrad and the connection:

$$\delta L \;\propto\; \epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}}\,\left(2\,\delta\mathbf{e}^{\hat{a}}\wedge\mathbf{e}^{\hat{b}}\wedge\mathbf{R}^{\hat{c}\hat{d}} \;+\; \mathbf{e}^{\hat{a}}\wedge\mathbf{e}^{\hat{b}}\wedge D\delta\omega^{\hat{c}\hat{d}}\right) \tag{31}$$

(remarkably, the difference between two connections does transform as a tensor). We integrate by parts the second term to get two (vacuum) dynamical equations,

$$\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}}\,\mathbf{e}^{\hat{a}}\wedge\mathbf{T}^{\hat{b}} \;=\; 0, \tag{32}$$

$$\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}}\,\mathbf{e}^{\hat{b}}\wedge\mathbf{R}^{\hat{c}\hat{d}} \;=\; 0. \tag{33}$$

Equations (32) imply the vanishing of torsion (they are as many independent equations as independent components of the torsion). So the connection becomes the Levi-Civita connection, which is the (antisymmetric) metric connection that cancels out the torsion. Equations (33) are Einstein equations.

Einstein equations keep their form when null tetrads are used: since $\det(\Lambda^{a}{}_{\hat{a}}) = i$, it is

$$\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}} \;=\; -i\,\epsilon_{abcd}\,\Lambda^{a}{}_{\hat{a}}\,\Lambda^{b}{}_{\hat{b}}\,\Lambda^{c}{}_{\hat{c}}\,\Lambda^{d}{}_{\hat{d}}; \tag{34}$$

besides,

$$\eta^{\hat{d}\hat{e}}\,\Lambda^{d}{}_{\hat{d}} \;=\; \eta^{de}\,\Lambda^{\hat{e}}{}_{e}. \tag{35}$$

Thus, vacuum Einstein equations are as well

$$\epsilon_{abcd}\,\mathbf{n}^{b}\wedge\mathbf{R}^{cd} \;=\; 0. \tag{36}$$

4 From flat space–time to Kerr geometry

Newman–Janis algorithm connects Schwarzschild and Kerr geometries by means of the rules (21) plus a rule affecting the Newtonian gravitational potential in Schwarzschild geometry:

$$\frac{2\,M}{r} \;\longrightarrow\; \frac{2\,M\,r}{\rho^2} \;=\; M\left(\frac{1}{\xi}+\frac{1}{\bar{\xi}}\right). \tag{37}$$

To trace the reasons for this rule, we will introduce gravity by performing just a tiny change of the Minkowskian tetrad (20); we will only change the 1-form $\mathbf{l}$:

$$\mathbf{l} \;\longrightarrow\; \mathbf{l} \;+\; f(r,\theta)\,\mathbf{n}\,. \tag{38}$$

Thus, the new geometry is expressed in the Kerr–Schild form: $\mathbf{g} = \bar{\mathbf{g}} + 2\,f(r,\theta)\,\mathbf{n}\otimes\mathbf{n}$, where $\bar{\mathbf{g}}$ is the Minkowskian seed metric (19, 20) [1,15,16]. Since the function f

depends just on (r, θ), the new geometry will remain stationary and axially symmetric. Coordinates $\{u \doteq t - r,\ r,\ \theta,\ \phi\}$ are outgoing Eddington-Finkelstein-like coordinates. While $\mathbf{l}$ and $\mathbf{n}$ are null directions on an equal footing in Eq. (20) –they are covectors of $\partial/\partial t \pm \partial/\partial r$, so they represent rays of light traveling in opposite directions in Minkowski space–time (see Footnote 1)–, instead they will not be equivalent in the new geometry $\mathbf{g}$ because gravity will distinguish ingoing and outgoing rays by means of the function $f(r, \theta)$.

Function f can be regarded as well as $f(\xi, \overline{\xi})$; it cannot be arbitrarily chosen because the new geometry must verify Einstein Eqs. (36) and preserve the vanishing of torsion. Function $f(\xi, \overline{\xi})$ will play the role of gravitational potential of the new geometry, because $g_{tt} \rightarrow 1 + f(\xi, \overline{\xi})$. Newman–Janis rule (37) says that both Schwarzschild and Kerr geometries are written with the same function $f(\xi, \overline{\xi})$; the sole difference between them comes from the vanishing or not of a in $\xi = r + i\, a \cos\theta$ (apart from the explicit dependence on a of the original null forms $\mathbf{l}$ and $\mathbf{n}$). Therefore, the good working of Newman–Janis rule requires that $f(\xi, \overline{\xi})$ fulfills equations in the variables $\xi, \overline{\xi}$ that do not explicitly contain the parameter a. In fact this will be the case, as we are going to show.

4.1 Keeping the torsion null

According to the definition (24), the vanishing of torsion is expressed by the equation

$$d\mathbf{n}^a = -\omega^{ab} \wedge \mathbf{n}_b, \tag{39}$$

where indexes are lowered and raised with the metric η_{ab} and its inverse η^{ab}. So, the index 0 goes to 1, and 2 changes to 3 plus a change of sign. In particular, $\{\mathbf{n}_a\} = \{\mathbf{n},\ \mathbf{l},\ -\overline{\mathbf{m}},\ -\mathbf{m}\}$. Like $\omega_{\hat{a}\hat{b}}$, ω^{ab} is antisymmetric too. This is because the transformation (16) is constant; then the relation between $\omega_{\hat{a}\hat{b}}$ and ω_{ab} looks tensorial in Eq. (23). The antisymmetry of ω^{ab} allows to solve Eq. (39) for the components of the torsionless spin connection, so obtaining the Levi-Civita connection:

$$(\omega^{ab})^c = \frac{1}{2}\left[(d\mathbf{n}^a)^{bc} + (d\mathbf{n}^b)^{ca} - (d\mathbf{n}^c)^{ab}\right]. \tag{40}$$

Since the null tetrad (20) satisfies

$$\begin{aligned} d\mathbf{l} &= -\,d\mathbf{n} = -\frac{1}{\sqrt{2}}\left(\frac{1}{\xi} - \frac{1}{\overline{\xi}}\right) \mathbf{m} \wedge \overline{\mathbf{m}}, \\ d\mathbf{m} &= \frac{1}{\sqrt{2}\,\xi}\,(\mathbf{l} - \mathbf{n}) \wedge \mathbf{m} - \frac{1}{\sqrt{2}\,\overline{\xi}}\left(\cot\theta - \frac{2i}{\xi}\, a\, \sin\theta\right) \mathbf{m} \wedge \overline{\mathbf{m}}, \end{aligned} \tag{41}$$

then by replacing these 2-forms in Eq. (40) we get the connection for the Minkowskian basis (20):

$$\omega^{ab} = \begin{pmatrix} 0 & 0 & \frac{\mathbf{m}}{\sqrt{2}\,\bar{\xi}} & \frac{\overline{\mathbf{m}}}{\sqrt{2}\,\xi} \\ \ldots & 0 & -\frac{\mathbf{m}}{\sqrt{2}\,\bar{\xi}} & -\frac{\overline{\mathbf{m}}}{\sqrt{2}\,\xi} \\ \ldots & \ldots & 0 & d\left[\ln\frac{\xi}{\bar{\xi}}\right] - \frac{\cot\theta}{\sqrt{2}}\left(\frac{\mathbf{m}}{\xi} - \frac{\overline{\mathbf{m}}}{\bar{\xi}}\right); \\ \ldots & \ldots & \ldots & 0 \end{pmatrix}. \tag{42}$$

As can be seen, there does not exist a short cut to obtain the result (42) from the Minkowskian spin connection in ($a = 0$) spherical coordinates, since no trace of $\ln(\xi/\bar{\xi})$ remains in the $a = 0$ case.

If the geometry is modified by a change $\delta\mathbf{n}^a$ of the null tetrad, then a change of the spin connection must happen as well to preserve the vanishing of torsion. The new spin connection $\omega^{ab} + \delta\omega^{ab}$ could be computed by using again the Eq. (40). However, the issue could also be considered at the level of Eq. (39), which implies a relation between $\delta\mathbf{n}^a$ and $\delta\omega^{ab}$ in order to keep the torsion null:

$$d\,\delta\mathbf{n}^a = -\,\omega^{ab} \wedge \delta\mathbf{n}_b - \delta\omega^{ab} \wedge \mathbf{n}_b - \delta\omega^{ab} \wedge \delta\mathbf{n}_b. \tag{43}$$

As said, the change of tetrad we are going to introduce is $\delta\mathbf{l} = f\ \mathbf{n}$, $\delta\mathbf{n}^\alpha = 0$ ($\alpha \neq 0$). We will argue that $\delta\omega^{ab}$ should be linear in f; besides, as a solution of Einstein equations, f should be proportional to some integration constant measuring the strength of the gravitational field. Since Eq. (43) has to be satisfied for any value of the integration constant, then the quadratic term must separately cancel out:

$$\delta\omega^{a1} \wedge \mathbf{n} = 0\,. \tag{44}$$

Equation (43) for $a = 0$ then becomes

$$df \wedge \mathbf{n} + f\ d\mathbf{n} = -\,\delta\omega^{0\alpha} \wedge \mathbf{n}_\alpha, \tag{45}$$

where we have used that $\omega^{01} = 0$. Equation (43) for $a = \alpha$ is

$$f\ \omega^{\alpha 1} \wedge \mathbf{n} = \delta\omega^{b\alpha} \wedge \mathbf{n}_b \tag{46}$$

(notice that $\mathbf{n}_1 = \mathbf{l}$ is the 1-form belonging to the original tetrad, since the change $\delta\mathbf{l}$ is separately written). The unknowns $\delta\omega^{b\alpha}$ must fulfill

$$\delta\omega^{02} = \overline{\delta\omega^{03}}, \qquad \delta\omega^{12} = \overline{\delta\omega^{13}}, \qquad \delta\omega^{23} = \overline{\delta\omega^{32}} = -\overline{\delta\omega^{23}}, \tag{47}$$

as it results from the complex behavior of the null tetrad. One can start by using Eq. (44) to eliminate a term in the Eq. (46) for $\alpha = 1$; it results

$$0 \;=\; \delta\omega^{21} \wedge \overline{\mathbf{m}} \;+\; \delta\omega^{31} \wedge \mathbf{m}. \tag{48}$$

We will try the solution $\delta\omega^{21} = 0 = \delta\omega^{31}$: by replacing it in Eq. (46) with $\alpha = 2$, we solve $\delta\omega^{32}$. Besides, df in Eq. (45) is

$$df \;=\; \partial_\xi f \; d\xi \;+\; \partial_{\overline{\xi}} f \; d\overline{\xi}, \tag{49}$$

where

$$d\xi \;=\; dr - i\; a\; \sin\theta\; d\theta \;=\; \frac{\mathbf{l} - \mathbf{n}}{\sqrt{2}} \;-\; i\, \sqrt{2}\, a\; \sin\theta\; \frac{\overline{\mathbf{m}}}{\overline{\xi}}. \tag{50}$$

It is easy to verify that the solution $\delta\omega^{ab}$ to Eqs. (45), (46) is

$$\delta\omega^{ab} \;=\; \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\left(\partial_\xi f + \partial_{\overline{\xi}} f\right)\mathbf{n} & \frac{f\;\mathbf{m}}{\sqrt{2}\;\xi} + \frac{i\sqrt{2}a\;\sin\theta}{\overline{\xi}}\;\partial_\xi f\;\mathbf{n} & \frac{f\;\overline{\mathbf{m}}}{\sqrt{2}\;\overline{\xi}} - \frac{i\sqrt{2}a\;\sin\theta}{\xi}\;\partial_{\overline{\xi}} f\;\mathbf{n} \\ \ldots & 0 & 0 & 0 \\ \ldots & \ldots & 0 & \frac{f}{\sqrt{2}}\left(\frac{1}{\xi} - \frac{1}{\overline{\xi}}\right)\mathbf{n} \\ \ldots & \ldots & \ldots & 0 \end{pmatrix} \tag{51}$$

and Eq. (44) is accomplished too. Although $\delta\omega^{ab}$ explicitly depends on the parameter a, we are going to show that the equations $f(\xi, \overline{\xi})$ accomplishes do not contain a in an explicit way.

4.2 Newman–Janis rules

Function $f(\xi, \overline{\xi})$, which describes the gravitational field of the solution under consideration, is dictated by Einstein equations. The modified geometry, characterized by $\mathbf{n}^b + \delta\mathbf{n}^b$, $\omega^{ab} + \delta\omega^{ab}$, has to fulfill Einstein Eqs. (36):

$$\epsilon_{abcd}\; (\mathbf{n}^b + \delta\mathbf{n}^b) \wedge \mathbf{R}^{cd}(\omega + \delta\omega) \;=\; 0. \tag{52}$$

By expanding the curvature (25) for the new spin connection one obtains

$$\mathbf{R}^{cd}(\omega + \delta\omega) \;=\; \mathbf{R}^{cd}(\omega) \;+\; D\,\delta\omega^{cd} \;+\; \delta\omega^{ce} \wedge \delta\omega_e{}^{d}, \tag{53}$$

where D is the covariant derivative defined by the original spin connection. So long as we start from Minkowski space–time, then it is $\mathbf{R}^{cd}(\omega) = 0$. Besides, it can be easily verified that the changes (38), (51) satisfy

$$\epsilon_{abcd}\; \delta\mathbf{n}^b \wedge \delta\omega^{ce} \wedge \delta\omega_e{}^{d} \;=\; 0. \tag{54}$$

In fact, it is

$$\delta\omega^{ce} \wedge \delta\omega_e{}^d = \begin{pmatrix} 0 & 0 & \mathbf{A} & \overline{\mathbf{A}} \\ 0 & 0 & 0 & 0 \\ -\mathbf{A} & 0 & 0 & 0 \\ -\overline{\mathbf{A}} & 0 & 0 & 0 \end{pmatrix}, \tag{55}$$

where

$$\mathbf{A} = \frac{f}{2\,\xi}\left(\partial_\xi f + \partial_{\overline{\xi}} f - \frac{f}{\xi} + \frac{f}{\overline{\xi}}\right)\mathbf{n}\wedge\mathbf{m}. \tag{56}$$

So, according to Eq. (55), not only the index b, but c or d should be zero to have a non-null contribution to Eq. (54); since b, c, d are antisymmetrized, then Eq. (54) is satisfied.

Among the remaining terms in Eq. (52), one of them is linear in f and the other ones are quadratic in f. They should separately cancel out:

$$\epsilon_{abcd}\,\mathbf{n}^b \wedge D\,\delta\omega^{cd} = 0, \tag{57}$$

$$\epsilon_{abcd}\,(\mathbf{n}^b \wedge \delta\omega^{ce} \wedge \delta\omega_e{}^d + \delta\mathbf{n}^b \wedge D\,\delta\omega^{cd}) = 0. \tag{58}$$

Let us begin by considering the contraction of these equations with $\mathbf{n}^a$, which amounts the preservation of the value of the Lagrangian (i.e., the preservation of the null scalar curvature). Equation (58) is automatically accomplished when contracted with $\mathbf{n}^a$. In fact, $\epsilon_{abcd}\,\mathbf{n}^a \wedge \mathbf{n}^b \wedge \delta\omega^{ce} \wedge \delta\omega_e{}^d$ vanishes because (c, d) should be $(0, 2)$ or $(0, 3)$ (see Eq. (55)); so, a or b should be 1, what cancels out the expression (see Eq. (56)). Besides, $\epsilon_{abcd}\,\mathbf{n}^a \wedge \delta\mathbf{n}^b \wedge D\,\delta\omega^{cd}$ vanishes as a consequence of the Eq. (57). On the other hand, the contraction of Eq. (57) with $\mathbf{n}^a$ becomes

$$D(\epsilon_{abcd}\,\mathbf{n}^a \wedge \mathbf{n}^b \wedge \delta\omega^{cd}) = 0, \tag{59}$$

where we have used that $D\,\mathbf{n}^b = 0$ (null torsion) and $D\epsilon_{abcd} = 0$ (metricity). Notice that

$$\epsilon_{abcd}\,\mathbf{n}^a \wedge \mathbf{n}^b \wedge \delta\omega^{cd} = 2\sqrt{2}\left[\left(\frac{1}{\xi} + \frac{1}{\overline{\xi}}\right) f + \partial_\xi f + \partial_{\overline{\xi}} f\right]\mathbf{n}\wedge\mathbf{m}\wedge\overline{\mathbf{m}}\,, \tag{60}$$

which is a scalar-valued 3-form. So, no difference exists between D and d in Eq. (59), which reads

$$d\left[\left(\frac{1}{\xi} + \frac{1}{\overline{\xi}}\right) f + \partial_\xi f + \partial_{\overline{\xi}} f\right]\wedge\mathbf{n}\wedge\mathbf{m}\wedge\overline{\mathbf{m}} + \left[\left(\frac{1}{\xi} + \frac{1}{\overline{\xi}}\right) f + \partial_\xi f + \partial_{\overline{\xi}} f\right] d(\mathbf{n}\wedge\mathbf{m}\wedge\overline{\mathbf{m}}) = 0\,. \tag{61}$$

We use Eqs. (41) and (50) to obtain an equation for f:

$$\frac{f}{\xi\,\overline{\xi}} + \left(\frac{1}{\xi} + \frac{1}{\overline{\xi}}\right)\left(\partial_\xi + \partial_{\overline{\xi}}\right) f + \frac{1}{2}\left(\partial_\xi + \partial_{\overline{\xi}}\right)^2 f = 0\,. \tag{62}$$

This equation just constrains f to preserve the vanishing scalar curvature. So, it is useful not only for vacuum Einstein equations but also for traceless sources. The equation does not completely determine f. Since just the operator $\partial_\xi + \partial_{\overline{\xi}} = 2\, \partial_{\xi+\overline{\xi}}|_{\xi-\overline{\xi}}$ appears in this linear and homogeneous equation for f, then the solution f is undetermined by a factor $g(\xi - \overline{\xi})$. The general solution of this equation is

$$f(\xi, \overline{\xi}) = \left[\frac{Q^2}{\xi\,\overline{\xi}} - M\left(\frac{1}{\xi} + \frac{1}{\overline{\xi}}\right)\right] g(\xi - \overline{\xi}), \tag{63}$$

where M, Q^2 are integration constants. For a constant g, one would obtain the Kerr–Newman metric,

$$ds^2 = dt^2 - (dr + a\,\sin^2\theta\, d\phi)^2 - (r^2 + a^2\cos^2\theta)\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$-\left[M\left(\frac{1}{\xi} + \frac{1}{\overline{\xi}}\right) - \frac{Q^2}{\xi\,\overline{\xi}}\right]\left(dt - dr - a\,\sin^2\theta\, d\phi\right)^2, \tag{64}$$

expressed in coordinates similar to the ones Kerr used in his original article [1,17] ($u \doteq t - r$, r, θ, ϕ are outgoing Eddington-Finkelstein-like coordinates; ingoing coordinates would result if the starting point at Eq. (38) were $\mathbf{n} \to \mathbf{n} + f\,\mathbf{l}$). Other coordinatizations can be found in Ref. [18]. In Eq. (63) one recognizes the Newman–Janis rules to pass from Reissner-Nordström to Kerr–Newman:

iii) in the gravitational potential, replace $\frac{2}{r} \longrightarrow \frac{1}{\xi} + \frac{1}{\overline{\xi}}$,

iv) in the electric term, replace $\frac{1}{r^2} \longrightarrow \frac{1}{\xi\,\overline{\xi}}$.

It could be said that Newman–Janis algorithm does work because the rest of Einstein equations constrain the function $g(\xi - \overline{\xi})$ in Eq. (63) to be a constant. Otherwise, function $f(\xi, \bar{\xi})$ would contain a dependence on $\xi - \overline{\xi} = 2\,i\,a\,\cos\theta$ with no trace of it in Schwarzschild or Reissner–Nordstrom solutions.

4.3 More equations for $f(\xi, \bar{\xi})$

The story has not finished yet, because f must still accomplish the rest of the equations involved in Eqs. (57) and (58). Since we are trying with vacuum solutions, the electric term in Eq. (63) should be suppressed by Einstein equations. Besides, function $g(\xi - \overline{\xi})$ should be constrained to be a constant. One can easily verify that three of the four equations in Eq. (58) –those for $a = 0, 2, 3$– are trivially fulfilled by the matrices (42), (51) and (55). Since we have already worked a combination of the equations taking part in Eq. (58), it only remains to satisfy Eq. (57), which reads

$$DV_a = dV_a - \omega^b_{\ a} \wedge V_b = 0, \quad \text{where} \quad V_a \doteq \epsilon_{abcd}\,\mathbf{n}^b \wedge \delta\omega^{cd}\,. \tag{65}$$

According to the expression (51) for $\delta\omega^{ab}$, the covector-valued 2-form V_a is

$$V_a \;=\; 2\,\epsilon_{a\beta 0\gamma}\,\mathbf{n}^\beta \wedge \delta\omega^{0\gamma} \;+\; 2\,\epsilon_{ab23}\,\mathbf{n}^b \wedge \delta\omega^{23}\,. \tag{66}$$

i) $a = 0$

$V_0 = 0$. Thus the 0-component of $DV_a = 0$ is:

$$0 = \;\omega^b{}_0 \wedge V_b \;=\; \omega^{21} \wedge V_2 + \;\omega^{31} \wedge V_3\;, \tag{67}$$

where $\omega^{21} = \frac{\mathbf{m}}{\sqrt{2}\,\bar{\xi}} = \overline{\omega^{31}}$, and

$$-\,V_2 \;=\; \overline{V_3} \;=\; \sqrt{2}\left(\frac{f}{\bar{\xi}} + \partial_\xi f + \partial_{\bar{\xi}} f\right)\,\mathbf{n}\wedge\overline{\mathbf{m}}\,. \tag{68}$$

Thus, $f(\xi, \bar{\xi})$ must fulfill

$$\left(\frac{1}{\xi^2} + \frac{1}{\bar{\xi}^2}\right)\,f \;+\; \left(\frac{1}{\xi} + \frac{1}{\bar{\xi}}\right)\left(\partial_\xi + \partial_{\bar{\xi}}\right)\,f \;=\; 0, \tag{69}$$

whose general solution does not contain the electric charge term:

$$f \;=\; \left(\frac{1}{\xi} + \frac{1}{\bar{\xi}}\right)\; g(\xi - \bar{\xi}). \tag{70}$$

ii) $a = 1$

The 1-component of $DV_a = 0$ is:

$$0 \;=\; dV_1 - \omega^{10}\wedge V_1 - \omega^{20}\wedge V_2 - \omega^{30}\wedge V_3 \;=\; dV_1 + \omega^{21}\wedge V_2 + \omega^{31}\wedge V_3 \tag{71}$$

(see Eq. (42)). The last two terms cancel out because the function f verifies the Eq. (67). So, the Eq. (71) says that V_1 is a closed 2-form: $dV_1 = 0$, where

$$\begin{aligned} V_1 &= 2\,\epsilon_{1203}\,\mathbf{m}\wedge\delta\omega^{03} \;+\; 2\,\epsilon_{1302}\,\overline{\mathbf{m}}\wedge\delta\omega^{02} \;+\; 2\,\epsilon_{1023}\,\mathbf{l}\wedge\delta\omega^{23} \\ &= 2(\partial_\xi f\; d\xi - \partial_{\bar{\xi}} f\; d\bar{\xi})\wedge\mathbf{n} \\ &\quad -\sqrt{2}\,\left(\frac{f}{\xi} - \frac{f}{\bar{\xi}} + \partial_\xi f - \partial_{\bar{\xi}} f\right)\,\mathbf{l}\wedge\mathbf{n} + \sqrt{2} f\,\left(\frac{1}{\xi} + \frac{1}{\bar{\xi}}\right)\,\mathbf{m}\wedge\overline{\mathbf{m}} \end{aligned} \tag{72}$$

and $d\xi$ is given in Eq. (50). Let us examine the component $\mathbf{l}\wedge\mathbf{n}\wedge\mathbf{m}$ of the 3-form dV_1. Noticeably, $d\mathbf{n}$, $d(\mathbf{l}\wedge\mathbf{n})$, $d(\mathbf{m}\wedge\overline{\mathbf{m}})$ and $d\xi\wedge d\bar{\xi}\wedge\mathbf{n}$ do not contribute to such component. Instead,

$$d\bar{\xi}\wedge\mathbf{l}\wedge\mathbf{n} \;=\; \frac{i\sqrt{2}\,a\,\sin\theta}{\xi}\,\mathbf{l}\wedge\mathbf{n}\wedge\mathbf{m}\,. \tag{73}$$

Therefore the component $\mathbf{l} \wedge \mathbf{n} \wedge \mathbf{m}$ of dV_1 vanishes if

$$\partial_{\bar{\xi}} \left[\left(\frac{1}{\xi} - \frac{1}{\bar{\xi}} \right) f + \partial_{\xi} f - \partial_{\bar{\xi}} f \right] = 0 \,. \tag{74}$$

By replacing the result (70) in Eq. (74) one gets

$$\partial_{\bar{\xi}} \left[2 \left(\frac{1}{\xi} + \frac{1}{\bar{\xi}} \right) g'(\xi - \bar{\xi}) \right] = 0 \,. \tag{75}$$

A similar complex conjugate equation is obtained by analyzing the component $\mathbf{l} \wedge \mathbf{n} \wedge \overline{\mathbf{m}}$ of dV_1. Therefore, function g is constant. The reader can verify that $f(\xi, \bar{\xi}) = -M\,(\frac{1}{\xi} + \frac{1}{\bar{\xi}})$ also cancels the rest of the components of dV_1 as well as those of DV_a for $a = 2, 3$.

5 Conclusion

Reissner–Nordstrom geometry written in Kerr–Schild form,

$$\mathbf{g} \;=\; \mathbf{l} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{l} - \mathbf{m} \otimes \overline{\mathbf{m}} - \overline{\mathbf{m}} \otimes \mathbf{m} - 2 \left(\frac{2M}{r} - \frac{Q^2}{r^2} \right) \mathbf{n} \otimes \mathbf{n}, \tag{76}$$

where $\mathbf{l} = 2^{-\frac{1}{2}}(dt + dr)$, $\mathbf{n} = 2^{-\frac{1}{2}}(dt - dr)$, $\mathbf{m} = 2^{-\frac{1}{2}} r(d\theta + i\,\sin\theta\; d\phi)$, is promoted to Kerr–Newman geometry through the rules

i) in the $\{\mathbf{m}, \overline{\mathbf{m}}\}$ sector, replace $\quad r \longrightarrow \xi\,,\ \bar{\xi} \quad$ (where $\xi \doteq r + i\,a\,\cos\theta$)

ii) in the $\{\mathbf{l}, \mathbf{n}\}$ sector, replace $\quad dr \longrightarrow dr + a\,\sin^2\theta\; d\phi\;,$

iii, iv) replace $\quad \dfrac{2M}{r} - \dfrac{Q^2}{r^2} \longrightarrow M \left(\dfrac{1}{\xi} + \dfrac{1}{\bar{\xi}} \right) - \dfrac{Q^2}{\xi\,\bar{\xi}}\;.$

Newman–Janis algorithm results to be a simple rule because Einstein equations constrain function $g(\xi - \bar{\xi})$ in Eqs. (63) and (70) to be a constant. Thus, any possible dependence on $\xi - \bar{\xi} = 2i\,a\,\cos\theta$ is excluded from Kerr–Newman geometry; otherwise, Kerr–Newman geometry would contain a dependence on a variable leaving no trace in the $a = 0$ Schwarzschild geometry. It can be concluded that Newman–Janis algorithm seems to be linked to particular features of Einstein's theory that could hardly be replicated in other theories.

As a final remark, notice that the $\mathbf{n} \otimes \mathbf{n}$ term in Eq. (76) displays non-diagonal components g_{rt}. The usual diagonal form of Reissner–Nordstrom metric tensor is obtained by means of a redefinition of t. Likewise, Kerr–Newman metric in Boyer-Lindquist coordinates is reached not only by undoing the twisting (4) but by redefining the time too:

$$d\varphi \doteq d\phi + \frac{a}{r^2 + a^2 - 2Mr + Q^2}\, dr, \tag{77}$$

$$d\tilde{t} \doteq dt + \frac{2Mr - Q^2}{r^2 + a^2 - 2Mr + Q^2}\, dr. \tag{78}$$

Thus, the Kerr–Newman geometry (64) in Boyer–Lindquist coordinates is

$$ds^2 = d\tilde{t}^2 - \frac{2Mr - Q^2}{\rho^2} \left(d\tilde{t} - a \sin^2\theta \, d\varphi\right)^2 - \frac{dr^2}{\frac{r^2+a^2}{\rho^2} - \frac{2Mr-Q^2}{\rho^2}} - \rho^2 \, d\theta^2 - \left(r^2 + a^2\right) \sin^2\theta \, d\varphi^2, \tag{79}$$

where $\rho^2, \doteq r^2 + a^2 \cos^2\theta$. For $a = 0$, the usual form of Reissner–Nordstrom geometry is recovered.

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