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THE CANONICAL INTENSIVE QUALITY OF A COHESIVE TOPOS

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ABSTRACT. We strengthen a result in [3] by proving that every pre-cohesive geometric morphism $p : \mathcal{E} \to \mathcal{S}$ has a canonical intensive quality $s : \mathcal{E} \to \mathcal{L}$. We also discuss examples among bounded pre-cohesive $p : \mathcal{E} \to \mathbf{Set}$ and, in particular, we show that if $\mathcal{E}$ is a presheaf topos then so is $\mathcal{L}$. This result lifts to Grothendieck toposes but the sites obtained need not be subcanonical. To illustrate this phenomenon, and also the subtle passage from $\mathcal{E}$ to $\mathcal{L}$, we consider a particular family of bounded cohesive toposes over $\mathbf{Set}$ and build subcanonical sites for their associated categories $\mathcal{L}$.

Contents

1. Introduction

Inside the rich Appendix C of [6] one finds a short section entitled *Topos and the Cantorian Contrast* suggesting that Cantor's work partially motivates the study of geometric morphisms $p : \mathcal{E} \to \mathcal{S}$ with enough properties to capture the idea that $\mathcal{E}$ is a topos 'of spaces' and that the extraction of pure points $p_* : \mathcal{E} \to \mathcal{S}$ unites two opposite inclusions $p^*, p^! : \mathcal{S} \to \mathcal{E}$ of 'discrete' and 'codiscrete' spaces respectively. (See [3] for a more detailed presentation of *Axiomatic Cohesion*. See also [5] and [11] for more on the relation to Cantor's work.)

Recall that a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *local* if $p_* : \mathcal{E} \to \mathcal{S}$ has a fully faithful right adjoint. As suggested in the Appendix C cited above, such a local map provides "a

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zeroeth approximation to the reconstruction" of any object X of $\mathcal{E}$ by placing it canonically in the interval

$$p^*(p_*X) \xrightarrow{\beta} X \xrightarrow{\eta} p^!(p_*X)$$

where β is the counit of $p^* \dashv p_*$ and η is the unit of $p_* \dashv p^!$. This interval and some of its properties may be conveniently expressed in an alternative way using a canonical transformation $\phi : p^* \to p^!$ recalled in the next result.

1.1. Lemma. *If p is local then the two composites below are equal*

$$p^* \xrightarrow{\eta_{p^*}} p^! p_* p^* \xrightarrow{p^! \alpha^{-1}} p^! \qquad p^* \xrightarrow{p^* \varepsilon^{-1}} p^* p_* p^! \xrightarrow{\beta_{p^!}} p^!$$

where α is the unit of $p^* \dashv p_*$ and ε is the counit of $p_* \dashv p^!$. The resulting transformation is denoted by

$$\phi : p^* \to p^!$$

and the following diagrams commute.

$$\begin{tikzcd} p^*p_* \arrow[r, "\beta"] \arrow[dr, "\phi_{p_*}"'] & 1_{\mathcal{E}} \arrow[d, "\eta"] \\ & p^! p_* \end{tikzcd} \qquad \begin{tikzcd} p_*p^* \arrow[r, "\alpha^{-1}"] \arrow[dr, "p_*\phi"'] & 1_{\mathcal{S}} \arrow[d, "\varepsilon^{-1}"] \\ & p_* p^! \end{tikzcd}$$

(Notice that the left triangle above is the "interval" quoted before the statement.)

Proof. For the first part see [3] and also [2] for a detailed proof. The equality $\phi_{p_*} = \eta \cdot \beta$ appears in the proof of Lemma 2.4(i) in [2]. On the other hand

$$p_*\phi = p_*\beta_{p^!} \cdot p_*p^*\varepsilon^{-1} = \alpha^{-1}_{p_*p^!} \cdot p_*p^*\varepsilon^{-1} = \varepsilon^{-1} \cdot \alpha^{-1},$$

so the other diagram involving ϕ commutes. ∎

A geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *connected* if $p^* : \mathcal{S} \to \mathcal{E}$ is fully faithful. Of course, any local map is connected. Such a connected p is *hyperconnected* if its counit β is monic.

1.2. Lemma. *If $p : \mathcal{E} \to \mathcal{S}$ is local then, it is hyperconnected if and only if $\phi : p^* \to p^!$ is monic.*

Proof. Follows from Lemma 1.1. See also [3] and, in particular, Lemma 2.4(i) in [2]. ∎

Monomorphicity of $\phi : p^* \to p^!$ is one of the formulations of the *Nullstellensatz* condition introduced in [3]. This is consistent with the intuition that $\beta_X : p^*(p_* X) \to X$ is the discrete *sub*space of points of X.

Recall that a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *essential* if p^* has a left adjoint, usually denoted by $p_! : \mathcal{E} \to \mathcal{S}$.

As in [5] or [8] we say that a geometric morphism is *pre-cohesive* if it is local, hyperconnected, essential, and $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products. In other words, it is a string of adjoints $p_! \dashv p^* \dashv p_* \dashv p^!$ with fully faithful $p^*, p^! : \mathcal{S} \to \mathcal{E}$ such that the Nullstellensatz holds and the leftmost adjoint preserves finite products. The intuition remains as above, except for the leftmost adjoint which sends each space X to the set $p_! X$ of 'pieces' of X. It is relevant to recall that for a string of adjoints $p_! \dashv p^* \dashv p_* \dashv p^!$ with fully faithful $p^*, p^! : \mathcal{S} \to \mathcal{E}$, the first three functors determine, in a way analogous to ϕ, a canonical transformation $\theta : p_* \to p_!$ from 'points' to 'pieces'. Then, for the whole string, ϕ is monic if and only if θ is epic. In other words, the Nullstellensatz as formulated above in terms of ϕ is equivalent to the statement that 'every piece has a point'. See [3, Definition 2(c)] and [2, Lemma 2.3].

1.3. Definition. A local geometric morphism $q : \mathcal{Q} \to \mathcal{S}$ is a *quality type* if the canonical transformation $\phi : q^* \to q^!$ is an isomorphism.

Notice that a quality type $q : \mathcal{Q} \to \mathcal{S}$ is always a pre-cohesive geometric morphism, but of a very special kind since the concepts of 'discrete' and 'codiscrete' coincide. In this case also the concepts of 'piece' and 'point' coincide, in the sense that $\theta : q_* \to q_!$ is an isomorphism.

1.4. Definition. An *intensive quality* (on p) is a quality type $q : \mathcal{Q} \to \mathcal{S}$ together with a functor $s : \mathcal{E} \to \mathcal{Q}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "s"] \arrow[dr, "p_*"'] & \mathcal{Q} \arrow[d, "q_*"] \\
& \mathcal{S}
\end{tikzcd}$$

and such that s preserves finite products and finite coproducts.

Loosely speaking, 'qualities' are functors valued in quality types. (See also the paragraph after Definition 2.3 in [10].) Intensive qualities have the distinguishing feature that they preserve points.

Theorem 2 in [3] implies that any pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ satisfying certain completeness conditions has a canonical intensive quality which is actually the direct image of a hyperconnected essential geometric morphism denoted by s and called "substance". The two aspects s_* and $s_!$ are distinguished as *rarefied* vs. *condensed* and they are viewed as "the result of partial observation of the same space (or sample of material), under extreme conditions of hot vs. cold; the canonical 'cooling' maps $s_* X \to s_! X$ gives further partial information about the specific nature of the substance of X".

As far as we know only two examples of the canonical intensive quality were calculated explicitly. That of the cohesive topos of finite reversible reflexive graphs [3, Section V] and that of the pre-cohesive topos of reflexive graphs [4]. Our purpose in the present paper is to give an elementary proof of the existence of the canonical intensive quality, and to describe it concretely in other examples of (pre-)cohesion.

In Section 2 we introduce a strengthening of the notion of local map that is opposite to quality types. We call them *intensive* maps. In Section 3 we recall the definition of Leibniz object [4] (going back to [3, Theorem 2]) and then use it to prove, in Section 4, that every local and hyperconnected geometric morphism $p : \mathcal{E} \to \mathcal{S}$ has an 'extremal' factorization $p = qs$ with s intensive and q a quality type. Following [3], the direct image of s will be called the *canonical intensive quality* of p. In Section 6 we show that if p is moreover essential then so is s. It will then be clear that our result is a strengthening of [3, Theorem 2]. The rest of the paper is devoted to examples.

In Section 7 we show that if the canonical $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive then the codomain of $s : \widehat{\mathcal{C}} \to \mathcal{L}$ is also a presheaf topos. We actually show that $\widehat{\mathcal{C}} \to \mathcal{L}$ is induced by a quotient $\mathcal{C} \to \mathcal{D}$ of categories that we describe explicitly. In Section 8 we illustrate the connection between Weil algebras and the Leibniz core of representable objects in the Gaeta topos associated to the field of complex numbers. The results about presheaf toposes are lifted to the level of Grothendieck toposes in Section 9. Indeed, we construct explicit sites for the relevant $\mathcal{L}$, but we also observe that the resulting sites need not be subcanonical. So we explore this phenomenon by constructing a subcanonical site in the case of a cohesive toposes described. This gives an interesting illustration of the subtle passage from $\mathcal{E}$ to $\mathcal{L}$.

2. Intensive geometric morphisms

Among other things, Johnstone's analysis [2] of the Nullstellensatz condition formulated in [3] points to certain subtle strengthenings of the concept of hyperconnected geometric morphism. Specifically, it follows from [2, Lemma 2.3] that for any local, connected and essential geometric morphism, the Nullstellensatz condition is equivalent to the 'points' functor being 'faithful on maps with discrete codomain'. Part of the argument generalizes as explained below. (For the explanation, bear in mind the definition of conservative functor as considered, for example, in [1]; that is, faithful and isomorphism-reflecting.)

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}'$ be functors. We say that G is *faithful on maps with codomain in the image of* F if for every A in $\mathcal{C}$ and $u, v : X \to FA$ in $\mathcal{D}$, $Gu = Gv$ in $\mathcal{C}'$ implies $u = v$. For brevity we say that G is F*-faithful*.

Similarly, we say that G *preserves coequalizers of maps with codomain in the image of* F if for every A in $\mathcal{C}$ and coequalizer diagram in $\mathcal{D}$ as on the left below

$$X \underset{v}{\overset{u}{\rightrightarrows}} FA \xrightarrow{q} Q \qquad\qquad GX \underset{Gv}{\overset{Gu}{\rightrightarrows}} G(FA) \xrightarrow{Gq} GQ$$

the fork on the right above is a coequalizer in $\mathcal{C}'$.

2.1. Lemma. *If $p : \mathcal{E} \to \mathcal{S}$ is a connected geometric morphism then, $p_* : \mathcal{E} \to \mathcal{S}$ is p^*-faithful if and only if p is hyperconnected and p_* preserves coequalizers of maps with codomain in the image of p^*.*

Proof. An immediate consequence of [12, Lemma 2.4]. ∎

If $p : \mathcal{E} \to \mathcal{S}$ is local then it is connected and the direct image preserves all colimits so, in this case, p is hyperconnected if and only if p_* is p^*-faithful. On the other hand, [12, Example 2.3] gives a simple example showing that hyperconnectedness is strictly weaker than the equivalent conditions of Lemma 2.1.

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}'$ be functors as before. We say that G *reflects isomorphism with codomain in the image of* F if, for every A in $\mathcal{C}$ and $u : X \to FA$ in $\mathcal{D}$, Gu an isomorphism in $\mathcal{C}'$ implies that u is an isomorphism in $\mathcal{D}$.

2.2. Lemma. *If $p : \mathcal{E} \to \mathcal{S}$ is a connected geometric morphism then the following are equivalent:*

1. *The direct image $p_* : \mathcal{E} \to \mathcal{S}$ reflects isomorphisms with codomain in the image of p^*.*
2. *For any X in $\mathcal{E}$, if the counit $\beta_X : p^*(p_*X) \to X$ of p has a retraction then β_X is an isomorphism.*

Also, if these conditions hold then p is hyperconnected.

Proof. Assume that the first item holds and let $r : X \to p^*(p_*X)$ be a retraction of β_X. As p is connected, $p_*\beta_X$ is an isomorphism. So p_*r is an isomorphism and, using our assumption, we may deduce that r is an isomorphism.

For the converse let A in $\mathcal{S}$ and let $u : X \to p^*A$ be such that p_*u is an isomorphism. Then the top map in the naturality square

$$\begin{tikzcd} p^*(p_*X) \arrow[r, "p^*(p_*u)"] \arrow[d, "\beta"'] & p^*(p_*(p^*A)) \arrow[d, "\beta"] \\ X \arrow[r, "u"'] & p^*A \end{tikzcd}$$

is an isomorphism and, as p is connected, the right vertical map is also an isomorphism. So β has a retraction and hence, by hypothesis, it is an isomorphism. Therefore u is an isomorphism.

Assume now that the equivalent conditions hold and let the diagram left square below be a pullback

$$\begin{tikzcd} K \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & p^*(p_*X) \arrow[d, "\beta"] \\ p^*(p_*X) \arrow[r, "\beta"'] & X \end{tikzcd} \qquad \begin{tikzcd} p_*K \arrow[r, "p_*\pi_1"] \arrow[d, "p_*\pi_0"'] & p_*(p^*(p_*X)) \arrow[d, "p_*\beta"] \\ p_*(p^*(p_*X)) \arrow[r, "p_*\beta"'] & p_*X \end{tikzcd}$$

so the right square above is a pullback. As p is connected, $p_*\beta_X$ is an isomorphism, so $p_*\pi_0$ and $p_*\pi_1$ are isomorphisms. Then π_0 and π_1 are isomorphisms by hypothesis, so β_X is monic. ■

2.3. DEFINITION. A geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is *intensive* if it is connected and it satisfies the equivalent conditions of Lemma 2.2.

Intensive morphisms are opposite to quality types in the following sense.

2.4. LEMMA. *If $p : \mathcal{E} \to \mathcal{S}$ is both intensive and a quality type then p is an equivalence.*

PROOF. If p is a quality type then $\phi : p^* \to p^!$ is an isomorphism. In particular, by Lemma 1.1, $\phi_{p_*} = \eta_X \beta_X$ is an isomorphism for every X in $\mathcal{E}$. That is, β_X is split monic so, as p is intensive, β_X is an isomorphism for every X. Hence, both the unit and the counit of p are isomorphisms. ■

We will prove in Theorem 4.3 that every local hyperconnected geometric morphism factors as an intensive one followed by a quality type.

3. Leibniz objects

Fix a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ with counit denoted by β.

3.1. LEMMA. *If p is connected and p_* is p^*-faithful then, for every X in $\mathcal{E}$, the counit $p^*(p_*X) \to X$ has at most one retraction.*

PROOF. Let $u, v : X \to p^*(p_*X)$ be retractions of $\beta_X : p^*(p_*X) \to X$. As $p_*\beta$ is an isomorphism and p_* is p^*-faithful, $u = v$. ■

Assume from now on that p is connected and p_* is p^*-faithful.

As in the case of pre-cohesive geometric morphisms, we think of the objects of $\mathcal{E}$ as spaces and the objects of $\mathcal{S}$ as sets. The inclusion $p^* : \mathcal{S} \to \mathcal{E}$ may be identified with the inclusion of the full subcategory of discrete spaces and, for that reason, we may call $\beta : p^*(p_*X) \to X$ the *discrete core* of X.

3.2. DEFINITION. An object X in $\mathcal{E}$ is *Leibniz* if $\beta_X : p^*(p_*X) \to X$ has a retraction.

Intuitively, a space is Leibniz if it has no figures that substantiate cohesion between different points. Indeed, such an object should present no obstacle for a retraction onto its discrete subspace of points. Consistent with this intuition is the idea that a Leibniz object may be pictured as a collection of 'infinitesimal blobs', each blob with exactly one point. For example, for the pre-cohesive $\widehat{\Delta_1} \to \mathbf{Set}$, the Leibniz reflexive graphs are the graphs such that each connected component has exactly one node. Each node may have non-trivial loops. In the topos $\widehat{\Delta}$ of simplicial sets, an object L is Leibniz if and only if every element of $s \in L[n]$ has exactly one node, that is, for every two maps $k, l : [0] \to [n]$, $s \cdot k = s \cdot l$.

Examples aside, the above intuition suggests that discrete objects and objects with only one point should be Leibniz.

3.3. Lemma. *The following hold:*

1. *For every A in $\mathcal{S}$, p^*A is Leibniz.*

2. *For every X in $\mathcal{E}$, if $p_*X = 1$ then X is Leibniz.*

3. *If X is Leibniz then the retraction $X \to p^*(p_*X)$ is universal from X to $p^* : \mathcal{S} \to \mathcal{E}$.*

Proof. Discrete objects are Leibniz because $\beta_{p^*A} : p^*(p_*(p^*A)) \to p^*A$ is an isomorphism for every A in $\mathcal{S}$.

For X in $\mathcal{E}$, if $p_*X = 1$ then the discrete core $p^*(p_*X) \to X$ has an evident retraction.

Assume now that X is Leibniz and that $r : X \to p^*(p_*X)$ is the unique retraction for $\beta_X : p^*(p_*X) \to X$. For any $f : X \to p^*A$, the diagram on the left below commutes

$$\begin{tikzcd} p_*X \arrow[d, "p_*r"'] \arrow[dr, "id"] \arrow[drr, "p_*f"] & & \\ p_*(p^*(p_*X)) \arrow[r, "p_*\beta"'] & p_*X \arrow[r, "p_*f"'] & p_*(p^*A) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "r"'] \arrow[drr, "f"] & & \\ p^*(p_*X) \arrow[r, "\beta"'] & X \arrow[r, "f"'] & p^*A \end{tikzcd}$$

using again that $p_*\beta$ is an isomorphism (with retract p_*r). So the diagram on the right above commutes because p_* is p^*-faithful. As $p^* : \mathcal{S} \to \mathcal{E}$ is fully faithful, r has the correct universal property. ∎

The full subcategory of Leibniz objects will be denoted by $s^* : \mathcal{L} \to \mathcal{E}$.

Observe that item 3 in the previous lemma produces a natural transformation $r : s^* \to p^*p_*s^* : \mathcal{L} \to \mathcal{E}$, retraction of $\beta \cdot s^*$.

3.4. Lemma. *The composite functor $q_* = p_*s^* : \mathcal{L} \to \mathcal{S}$ has a left adjoint $q^* : \mathcal{S} \to \mathcal{L}$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{E} & \mathcal{L} \arrow[l, "s^*"'] \\ \mathcal{S} \arrow[u, "p^*"] \arrow[ur, "q^*"'] & \end{tikzcd}$$

and, moreover, q^* is also right adjoint to q_*.

Proof. As discrete objects are Leibniz (Lemma 3.3) there exists a unique functor $q^* : \mathcal{S} \to \mathcal{L}$ such that the triangle in the statement commutes. Moreover, the following calculation

$$\mathcal{L}(q^*A, X) \cong \mathcal{E}(s^*(q^*A), s^*X) \cong \mathcal{E}(p^*A, s^*X) \cong \mathcal{S}(A, p_*(s^*X))$$

shows that the composite $p_*s^* : \mathcal{L} \to \mathcal{S}$ is right adjoint to q^*.

Lemma 3.3 provides the first step in the following sequence of natural isomorphisms

$$\mathcal{E}(s^*L, p^*A) \cong \mathcal{S}(p_*(s^*L), A) = \mathcal{S}(q_*L, A)$$

with L in $\mathcal{L}$ and A in $\mathcal{S}$; but we also have

$$\mathcal{L}(L, q^*A) \cong \mathcal{E}(s^*L, s^*(q^*A)) = \mathcal{E}(s^*L, p^*A)$$

so the result follows. ∎

Hence, if $\mathcal{L}$ were a topos, then $q : \mathcal{L} \to \mathcal{S}$ would be a quality type. For instance, the comment after [3, Theorem 2] implies that for a pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ satisfying sufficient completeness conditions, $\mathcal{L}$ is indeed a topos.

3.5. LEMMA. *Assume that $p = gc$ with $c : \mathcal{E} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{S}$. If c is connected and g is a quality type then $c^* : \mathcal{G} \to \mathcal{E}$ factors through $s^* : \mathcal{L} \to \mathcal{E}$.*

PROOF. It is not hard to show that the retraction of $\beta \cdot c^* = \beta_c c^* \cdot c^* \beta_g c_* c^*$ is $c^* \phi_g^{-1} g_* c_* c^* \cdot c^* \eta_g c_* c^* \cdot c^* \alpha_c$, where α_c is the unit of $c^* \dashv c$, η_g the unit of $g_* \dashv g^!$, ϕ_g the arrow that witnesses that g is a quality type, η_g is the unit of $g_* \dashv g^!$ and β_c and β_g are the counits of $c^* \dashv c_*$ and $g^* \dashv g_*$ respectively. ■

4. The canonical intensive quality

Let $p : \mathcal{E} \to \mathcal{S}$ be a local and hyperconnected geometric morphism. As observed after Lemma 2.1, it follows that p_* is p^*-faithful so we may consider the subcategory $s^* : \mathcal{L} \to \mathcal{E}$ of Leibniz objects discussed in Section 3.

For any X in $\mathcal{E}$ we let the following square

$$\begin{tikzcd} \Lambda X \arrow[r, "\lambda'_X"] \arrow[d, "\lambda_X"'] & p^*(p_* X) \arrow[d, "\phi_{p_* X}"] \\ X \arrow[r, "\eta"'] & p^!(p_* X) \end{tikzcd}$$

be a pullback. Notice that $\lambda_X : \Lambda X \to X$ is monic because ϕ is. We call λ_X the *Leibniz core* of X. The terminology is justified by the next result.

4.1. LEMMA. *For every X in $\mathcal{E}$ the following hold:*

1. *The map $p_* \lambda : p_*(\Lambda X) \to p_* X$ is an isomorphism in $\mathcal{S}$.*
2. *ΛX is Leibniz.*
3. *X is Leibniz if and only if $\lambda : \Lambda X \to X$ is an isomorphism.*

PROOF. As p_* preserves finite limits, $p_* \lambda$ is the pullback of $p_* \phi$ along $p_* \eta$, but $p_* \phi$ is an isomorphism by Lemma 1.1. To prove the second item observe that $p^* p_* \lambda_X = \lambda' \cdot \beta_{\Lambda X}$ (apply $p^* p_*$ to the pullback square that defines ΛX, thus writing $p^* p_\lambda$ in terms of λ', ϕ and $(p_* \eta)^{-1}$, use the definition of ϕ in terms of β and ε and that $p^* p_* \beta = \beta_{p^* p_*}$). Thus as a retraction of $\beta_{\Lambda X}$ we have $p^*(p_* \lambda)^{-1} \cdot \lambda'_X$.

If λ_X is an isomorphism then X is Leibniz by the second item. To prove the converse assume $r : X \to p^*(p_* X)$ is a retraction of β_X witnessing that X is Leibniz. Lemma 1.1

implies that the diagram on the left below commutes

$$\begin{tikzcd} p^*(p_*X) \arrow[rr, "\beta"] \arrow[d, "\beta"'] \arrow[dr, "id"] \arrow[drr, "\phi_{p_*}"] & & X \arrow[d, "\eta"] \\ X \arrow[r, "r"'] & p^*(p_*X) \arrow[r, "\phi_{p_*}"'] & p^!(p_*X) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "r"] \arrow[dr, "\eta"'] & p^*(p_*X) \arrow[d, "\phi_{p*}"] \\ & p^!(p_*X) \end{tikzcd}$$

and, as $\beta_X : p^*(p_*X) \to X$ is dense (w.r.t. the subtopos $p_* \dashv p^!$) and $p^!(p_*X)$ is a sheaf, the triangle on the right above commutes. Then, the pullback property defining ΛX implies that $\lambda : \Lambda X \to X$ has a section. So, as it is monic, λ is an isomorphism. ∎

The intuition may be better described by a quotation from [4] (with a minor adjustment in notation): "The Leibniz Core of a space X is the union ΛX of all its generalized points; this is obtained as the right adjoint of the inclusion functor from the subcategory of those spaces that look like clouds of Leibnizian monads. The more general figures that substantiate cohesion between points are omitted in the reduction from X to ΛX, but each point may have self-cohesion (which is retained in ΛX)".

The assignment $X \mapsto \Lambda X$ is clearly functorial and the resulting functor $\Lambda : \mathcal{E} \to \mathcal{E}$ preserves finite limits because the functors p^*, p_* and $p^!$ preserve finite limits and ΛX is built as a finite limit. It is also easy to check that $\lambda_X : \Lambda X \to X$ is natural in X.

4.2. Lemma. *The pair (Λ, λ) is a cartesian idempotent comonad on $\mathcal{E}$ with monic counit λ. Moreover, the full subcategory of coalgebras coincides with $s^* : \mathcal{L} \to \mathcal{E}$. Hence, $\mathcal{L}$ is a topos and, moreover s^* is the inverse image of an intensive geometric morphism $s : \mathcal{E} \to \mathcal{L}$.*

Proof. By Lemma 4.1, $\lambda_\Lambda : \Lambda(\Lambda X) \to \Lambda X$ is an isomorphism. If we define the comultiplication as $\lambda_\Lambda^{-1} : \Lambda X \to \Lambda(\Lambda X)$ then it is straightforward to check that $(\Lambda, \lambda, \lambda_\Lambda^{-1})$ is a comonad. Coassociativity follows from naturality of λ and the counit axiom follows because, as λ is monic, $\lambda_\Lambda = \Lambda\lambda : \Lambda(\Lambda X) \to \Lambda X$.

Lemma 4.1 implies that X is a coalgebra if and only if X is Leibniz. That is, $\mathcal{L}$ is the topos of coalgebras for (Λ, λ) and $s^* : \mathcal{L} \to \mathcal{E}$ is the inverse image of a hyperconnected geometric morphism $s : \mathcal{E} \to \mathcal{L}$ with counit λ. To prove that s is intensive let X in $\mathcal{E}$ and assume that $\lambda_X : s^*(s_*X) = \Lambda X \to X$ has a retraction, say, r. Then the functor p_* sends the composite

$$X \xrightarrow{r} \Lambda X \xrightarrow{\lambda'_X} p^*(p_*X) \xrightarrow{\beta_X} X$$

to the identity (use the fact that $p_* r = (p_*\lambda)^{-1}$, that composed with $p_*\lambda'$ is $(p_*\varphi)^{-1}_{p_*X} \cdot p_*\eta$ and that $\beta p_* \cdot \varphi_{p_*} = \varepsilon_{p_*}$); since p_* is p^*-faithful, r is mono, thus λ_X is an isomorphism ∎

4.3. Theorem. *If $p : \mathcal{E} \to \mathcal{S}$ is a local hyperconnected geometric morphism then there exists a factorization $p = qs$ with s connected and q a quality type which is 'extremal' in the sense that: for any factorization $p = gc$ with c connected and g a quality type, the full subcategory c^* of $\mathcal{E}$ factors through s^*. Moreover, s is intensive and s^* coincides with the full subcategory of Leibniz objects of p.*

PROOF. Let $s^* : \mathcal{L} \to \mathcal{S}$ be the full subcategory of Leibniz objects. Lemma 3.4 implies that $q : \mathcal{L} \to \mathcal{S}$ is a quality type. Also, as the diagram on the left below commutes

$$\begin{tikzcd} \mathcal{E} & \mathcal{L} \arrow[l, "s^*"'] \\ \mathcal{S} \arrow[u, "p^*"] \arrow[ur, "q^*"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \arrow[r, "s_*"] \arrow[d, "p_*"'] & \mathcal{L} \arrow[dl, "q_*"] \\ \mathcal{S} & \end{tikzcd}$$

the diagram on the right above also commutes. That is, $p = qs : \mathcal{E} \to \mathcal{S}$. Lemma 4.2 implies that s is intensive and Lemma 3.5 implies that the factorization $p = qs$ is 'extremal'. ∎

Notice that s_* preserves finite coproducts because hyperconnected morphisms are pure [1, C3.4.12(ii)] so s_* is an intensive quality in the sense of Definition 1.4. Following [3], we call $s_* : \mathcal{E} \to \mathcal{L}$ the *canonical intensive quality* of p.

5. Leibniz objects and codiscrete objects

Let $p : \mathcal{E} \to \mathcal{S}$ be a hyperconnected and local geometric morphism and let $s : \mathcal{E} \to \mathcal{L}$ be its canonical intensive quality. The presence of codiscrete objects allow us to say a few more things about Leibniz objects.

Intuitively, points of codiscrete objects in $\mathcal{E}$ lack self-cohesion and, therefore so do their subobjects. Hence, if we omit the "more general figures that substantiate cohesion between points" we should get a discrete object. The next result makes this argument precise.

5.1. COROLLARY. *If X in $\mathcal{E}$ is separated for the subtopos $p_* \dashv p^!$ then $s^* s_* X$ is discrete.*

PROOF. Let $X \in \mathcal{E}$ be separated for $p_* \dashv p^!$. By Lemma 1.1 we have that the following square

$$\begin{tikzcd} p^* p_* X \arrow[r, "\beta_X"] \arrow[d, "1_{p^* p_* X}"'] & X \arrow[d, "\eta_X"] \\ p^* p_* X \arrow[r, "\phi_{p^* X}"'] & p^! p_* X \end{tikzcd}$$

commutes. Since η_X is monic (because X is separated), it follows that the above square is a pullback. Thus $s^* s_* X \simeq p^* p_* X$ is discrete. ∎

Let Ω be the subobject classifier of $\mathcal{E}$. Intuitively, it is unlikely that there will be much self-cohesion around $\bot : 1 \to \Omega$. On the other hand, consider the 'Leibnizian monad' around $\top : 1 \to \Omega$. That is, consider the following pullback diagram

$$\begin{tikzcd} J \arrow[r] \arrow[d, "!"'] & \Omega \arrow[d, "\eta_\Omega"] \\ p^! p_* 1 \arrow[r, "p^! p_* \top"'] & p^! p_* \Omega \end{tikzcd}$$

in $\mathcal{E}$. Notice that the subobject $J \to \Omega$ is the well-known classifier of dense subobjects (w.r.t. $p_* \dashv p^!$). Taking points of the above pullback square (i.e. applying p_*) we may conclude that $p_* J \to p_* p^! p_* 1 = 1$ is an iso, so J is Leibniz by Lemma 3.3.

Intuition suggests that if $J \to \Omega$ is to classify something non trivial then there must be some non trivial cohesion in the Leibnizian monad J. Indeed:

5.2. Corollary. *The following are equivalent:*

1. *$s^* s_* \Omega$ is discrete.*
2. *J is terminal.*
3. *$p : \mathcal{E} \to \mathcal{S}$ is an equivalence.*
4. *$q : \mathcal{L} \to \mathcal{S}$ is an equivalence.*

Proof. Notice first that, as J is Leibniz, we have a monomorphism $J \cong s^*(s_* J) \to s^*(s_* \Omega)$.

To prove that the first item implies the second assume that $s^* s_* \Omega$ is discrete. Since subobjects of discrete objects are discrete, the monic $J \to s^*(s_* \Omega)$ implies that J is discrete and hence terminal.

The second item implies the third because if J is terminal then the only dense subobjects are identities and so every object is a sheaf. That is, $p^! : \mathcal{S} \to \mathcal{E}$ is an equivalence.

The third item trivially implies the fourth. Finally, if $q : \mathcal{L} \to \mathcal{S}$ is an equivalence then every Leibniz object is discrete. In particular, $s^*(s_* \Omega)$ is discrete. ∎

For the rest of the section we assume that the reader is familiar with De Morgan toposes [1, Section D4.6]. During CT2017 in Vancouver, the first author presented some of the results in this paper. At the same meeting, D. J. Myers proposed to the authors the conjecture that the topos $\mathcal{L}$ of Leibniz objects associated to a pre-cohesive topos (over a base $\mathcal{S}$) is De Morgan. The intuition he transmitted was using reflexive graphs: for any subobject of a Leibniz graph, its Heyting complement must contain all loops around the underlying points, therefore it must have a Boolean complement. Before the meeting ended we offered Myers a proof of the conjecture under the assumption that the base $\mathcal{S}$ is Boolean. We present below an improvement of that result, with a slightly more efficient proof.

5.3. Lemma. *For any hyperconnected geometric morphism $g : \mathcal{G} \to \mathcal{S}$ the following hold:*

1. *If $\mathcal{G}$ is De Morgan then so is $\mathcal{S}$.*
2. *If the subobject classifier of $\mathcal{G}$ is Leibniz and $\mathcal{S}$ is De Morgan then $\mathcal{G}$ is De Morgan.*
3. *If g is a quality type then, $\mathcal{G}$ is De Morgan if and only if $\mathcal{S}$ is.*

PROOF. To prove the first item notice that $\mathrm{Sub}_{\mathcal{G}}(g^*A) \simeq \mathrm{Sub}_{\mathcal{S}}(A)$ by A4.6.6 in [1]. So $\mathcal{S}$ is De Morgan.

To prove the second item we use that g_* preserves the subobject classifier (again A4.6.6 in [1]). So, as $\mathcal{S}$ is De Morgan by hypothesis, 2 is a retract of $g_*\Omega$ in $\mathcal{S}$. Then 2 is a retract of $g^*(g_*\Omega)$ in $\mathcal{G}$. As Ω is Leibniz, $g^*(g_*\Omega)$ is a retract of Ω in $\mathcal{G}$. Composing the retracts we obtain that 2 is a retract of Ω in $\mathcal{G}$. That is, $\mathcal{G}$ is De Morgan.

If g is a quality type then every object is Leibniz by Lemma 1.1 so the third item follows from the other two. ∎

Let us say that an object X in a topos is *De Morgan* if and only if the Heyting algebra of subobjects of X is actually a De Morgan algebra. It is well-known that a topos is De Morgan if and only if every object in it is De Morgan.

5.4. PROPOSITION. *Let $p : \mathcal{E} \to \mathcal{S}$ be local and hyperconnected. Then $\mathcal{L}$ is De Morgan if and only if $\mathcal{S}$ is. Moreover, in this case, Leibniz objects are De Morgan.*

PROOF. By Theorem 4.3, $q : \mathcal{L} \to \mathcal{S}$ is a quality type so we may apply Lemma 5.3 to conclude that $\mathcal{L}$ is De Morgan if and only if $\mathcal{S}$ is. In this case, every object X in $\mathcal{L}$ is De Morgan, so s^*X is De Morgan because s is hyperconnected. ∎

6. Essentiality of the canonical intensive quality

Let $p : \mathcal{E} \to \mathcal{S}$ be local and essential. We continue with the notation in earlier sections and we also let σ and τ be the unit and counit of $p_! \dashv p^*$.

6.1. LEMMA. *The two composites below*

$$p_* \xrightarrow{\tau^{-1}_{p_*}} p_! p^* p_* \xrightarrow{p_! \beta} p_! \qquad p_* \xrightarrow{p_* \sigma} p_* p^* p_! \xrightarrow{\alpha^{-1}_{p_!}} p_!$$

are equal. The resulting transformation is denoted

$$\theta : p_* \to p_!$$

and the following diagrams

$$\begin{tikzcd} p^*p_* \arrow[r, "\beta"] \arrow[dr, "p^*\theta"'] & 1_{\mathcal{E}} \arrow[d, "\sigma"] \\ & p^*p_! \end{tikzcd} \qquad \begin{tikzcd} p_*p^* \arrow[r, "\alpha^{-1}"] \arrow[dr, "\theta_{p^*}"'] & 1_{\mathcal{S}} \arrow[d, "\tau^{-1}"] \\ & p_!p^* \end{tikzcd}$$

commute.

PROOF. Analogous to Lemma 1.1. See [3, 2]. ∎

Lemma 1.2 above and Lemma 2.3 in [2] imply that, p is hyperconnected if and only if $\phi : p^* \to p^!$ is monic if and only if $\theta : p_* \to p_!$ is epic.

Assume for the rest of the section that the local and essential $p : \mathcal{E} \to \mathcal{S}$ is also hyperconnected.

The following result confirms that the canonical intensive quality of p coincides with that described in [3, Theorem 2].

6.2. LEMMA. *For any X in $\mathcal{E}$, X is Leibniz if and only if $\theta : p_*X \to p_!X$ is an isomorphism.*

PROOF. If $\theta : p_*X \to p_!X$ is an isomorphism then, using Lemma 6.1, it is easy to check that the transposition $X \to p^*(p_*X)$ of $\theta^{-1} : p_!X \to p_*X$ is a retract of $\beta : p^*(p_*X) \to X$.

On the other hand, if $r : X \to p^*(p_*X)$ is a retract of the discrete core $\beta : p^*(p_*X) \to X$ then $p_*r : p_*X \to p_*(p^*(p_*X))$ is an isomorphism. So, in the following commutative square,

$$\begin{tikzcd}
p_*X \arrow[r, "p_*r"] \arrow[d, "\theta"'] & p_*(p^*(p_*X)) \arrow[d, "\theta_{p^*}"] \\
p_!X \arrow[r, "p_!r"'] & p_!(p^*(p_*X))
\end{tikzcd}$$

the top map is an iso. Also, the right vertical map is an iso by Lemma 6.1, so the left vertical map is monic. ■

The next construction is borrowed from the proof of [3, Theorem 2]. Let X in $\mathcal{E}$ and define $\overline{X}$ by declaring that the following square

$$\begin{tikzcd}
p^*(p_*X) \arrow[r, "p^*\theta_X"] \arrow[d, "\beta_X"'] & p^*(p_!X) \arrow[d, "t_1"] \\
X \arrow[r, "t_0"'] & \overline{X}
\end{tikzcd}$$

is a pushout in $\mathcal{E}$.

6.3. LEMMA. *The object $\overline{X}$ is Leibniz.*

PROOF. As p_* preserves pushouts the following square

$$\begin{tikzcd}
p_*p^*p_*X \arrow[r, "p_*p^*\theta_X"] \arrow[d, "p_*\beta_X"'] & p_*p^*p_!X \arrow[d, "p_*t_1"] \\
p_*X \arrow[r, "p_*t_0"'] & p_*\overline{X}
\end{tikzcd}$$

is a pushout. Since $p_*\beta_X$ is an iso, we conclude that p_*t_1 is an iso. On the other hand, Lemma 6.1 tells us that the exterior of the diagram

$$\begin{tikzcd}
p^*p_*X \arrow[r, "p^*\theta_X"] \arrow[d, "\beta_X"'] & p^*p_!X \arrow[d, "t_1"] \arrow[ddr, bend left, "1_{p^*p_!X}"] & \\
X \arrow[r, "t_0"'] \arrow[drr, bend right, "\sigma_X"'] & \overline{X} \arrow[dr, dashed, "t"] & \\
& & p^*p_!X
\end{tikzcd}$$

commutes. So there exists a unique $t : \overline{X} \to p^*p_!X$ such that the above diagram commutes. Then p_*t is a retraction of p_*t_1, and this latter map is an iso, thus p_*t is an iso. The retraction of $\beta_{\overline{X}}$ is the composite $\overline{X} \xrightarrow[t]{} p^*p_!X \xrightarrow[p^*\alpha_{p_!X}]{} p^*p_*p^*p_!X \xrightarrow[p^*p_*t_1]{} p^*p_*\overline{X}$. ■

6.4. Proposition. *For every X the map $t_0 : X \to \overline{X}$ is universal from X to $s^* : \mathcal{L} \to \mathcal{E}$.*

Proof. Let X be an object in $\mathcal{E}$. Lemma 6.3 tells us that $\overline{X}$ is Leibniz. To prove that it is universal from X to $s^* : \mathcal{L} \to \mathcal{E}$ let L be a Leibniz object and $l : X \to L$ be a map in $\mathcal{E}$. It is enough to prove that there exists a unique map $l' : p^*(p_!X) \to L$ such that the diagram below

$$\begin{tikzcd}
p^*(p_*X) \arrow[r, "p^*\theta_X"] \arrow[d, "\beta_X"'] & p^*(p_!X) \arrow[d, "l'"] \\
X \arrow[r, "l"'] & L
\end{tikzcd}$$

commutes. Uniqueness follows because $p^*\theta$ is epic. For existence just take

$$p^*(p_!X) \xrightarrow{p^*(p_!l)} p^*(p_!L) \xrightarrow{p^*\theta_L^{-1}} p^*(p_*L) \xrightarrow{\beta_L} L$$

and confirm that the square above commutes. ■

We denote the left adjoint by $s_! : \mathcal{E} \to \mathcal{L}$.

To summarize, we state the following improvement of Theorem 2 in [3].

6.5. Theorem. *If $p : \mathcal{E} \to \mathcal{S}$ is an essential, hyperconnected and local geometric morphism then the intensive part of its canonical intensive/quality-type factorization is also essential.*

Proof. This is the combination of Theorem 4.3 and Proposition 6.4. ■

Let us also mention a relevant fact related to $s_! : \mathcal{E} \to \mathcal{L}$. Intuitively, $s_!$ forces cohesion between points into self-cohesion. It does not omit any cohesion. So, roughly speaking, if $s_! X$ is discrete then X should be discrete.

6.6. COROLLARY. *For any object X in $\mathcal{E}$, X is discrete if and only if $s^*(s_! X)$ is discrete.*

PROOF. Let us recall that the following diagram

$$\begin{tikzcd}
p^*(p_* X) \arrow[r, "p^*\theta"] \arrow[d, "\beta"'] & p^*(p_! X) \arrow[d, "t_1"] \\
X \arrow[r, "t_0"'] & s^*(s_! X)
\end{tikzcd}$$

is a pushout by the definition of $s^*(s_! X)$. (Notice that the square is also a pullback because the left vertical map is monic.)

Assume first that X is discrete, so the left map in the square is an isomorphism. Then so is the right map, which means that $s^*(s_! X)$ is in the image of $p^* : \mathcal{S} \to \mathcal{E}$.

Recall from the proof of Lemma 6.3 that $p_* t_1$ is an isomorphism and notice that since the following diagram commutes

$$\begin{tikzcd}
p^* p_* p^* p_! X \arrow[r, "\beta"] \arrow[d, "p^* p_* t_1"'] & p^* p_! X \arrow[d, "t_1"] \\
p^* p_* s^* s_! X \arrow[r, "\beta"'] & s^* s_! X
\end{tikzcd}$$

and the span is made of isos, $\beta : p^* p_* s^* s_! X \to s^* s_! X$ is an iso if and only of t_1 is an iso. So, if we assume that $s^* s_! X$ is discrete then t_1 is an isomorphism and so X is discrete because the first square is a pullback. ∎

Let us call an object X in $\mathcal{E}$ *boolean* if every subobject of X has a complement.

6.7. PROPOSITION. *Boolean objects in $\mathcal{E}$ are discrete. So, if $\mathcal{E}$ is Boolean as a category, then p is an equivalence.*

PROOF. If X is Boolean in $\mathcal{E}$ then we can take the complement $c : C \to X$ of the discrete core $p^*(p_* X) \to X$ and, clearly, X is discrete if and only if C is initial. As $p_* : \mathcal{E} \to \mathcal{S}$ preserves coproducts, the following diagram

$$p_*(p^*(p_* X)) \xrightarrow{p_*\beta} p_* X \xleftarrow{p_* c} p_* C$$

is a coproduct in $\mathcal{S}$. Since the left leg is an iso, $p_* C$ is initial and so C is initial because, as p is hyperconnected essential, p_* reflects 0 (see [11, Lemma 3.1]). ∎

The intuition is appealing: a topos 'of spaces' is unlikely to be Boolean because if you take the complement of a point then you are likely to miss the 'infinitesimals around the point'.

6.8. COROLLARY. *If $\mathcal{L}$ is Boolean then $q : \mathcal{L} \to \mathcal{S}$ is an equivalence (so p is an equivalence).*

PROOF. The geometric morphism $q : \mathcal{L} \to \mathcal{S}$ is pre-cohesive so we may apply Proposition 6.7 to conclude that q is an equivalence. Then p is an equivalence by Corollary 5.2. ∎

One may wonder if the hyperconnected essential $s : \mathcal{E} \to \mathcal{L}$ is pre-cohesive. The answer, in general, is 'no', as the following examples show. First, let us consider the question of whether $s : \mathcal{E} \to \mathcal{L}$ may be local.

6.9. EXAMPLE. [s_* does not preserve coequalizers in general] Consider the pre-cohesive topos $p : \widehat{\Delta_1} \to \mathbf{Set}$. Let A be the (representable) reflexive graph consisting of a single arrow between two different nodes. The object A has two points $\bot, \top : 1 \to A$. Their coequalizer $A \to D$ collapses the two nodes into one, and the arrow into a loop. On the other hand, $s_*A \to s_*D = D$ is not epic, because $s_*A = 1 + 1$.

Even is $s : \mathcal{E} \to \mathcal{L}$ is not local in general, one may also wonder if $s_!$ preserves finite products.

6.10. EXAMPLE. [$s_!$ does not preserve products in general] Consider again the pre-cohesive topos $p : \widehat{\Delta_1} \to \mathbf{Set}$ and the single arrow A. The graph $A \times A$ has 5 non-identity arrows and hence, so does $s_!(A \times A)$. On the other hand, $s_!A \times s_!A$ has three non-trivial arrows. Therefore, the canonical $s_!(A \times A) \to s_!A \times s_!A$ cannot be an iso.

7. The case of pre-cohesive presheaf toposes

In this section we discuss the canonical intensive quality of a pre-cohesive presheaf topos. Fix a small category $\mathcal{C}$ with terminal object and such that every object has a point, so that the canonical geometric morphism $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is local, hyperconnected and essential (and, in fact, pre-cohesive).

7.1. LEMMA. *Let $s : \widehat{\mathcal{C}} \to \mathcal{L}$ be the canonical intensive quality. Then, for any X in $\widehat{\mathcal{C}}$, the counit $s^*(s_*X) \to X$ may be identified with*

$$(s^*(s_*X))C = \{x \in XC \mid \text{for all } a, b : 1 \to C, x \cdot a = x \cdot b\} \to XC$$

for every $C \in \mathcal{C}$.

PROOF. Follows from the explicit construction of s_* in Section 4. ∎

Lemma 7.1 shows that we can easily calculate the 'Leibniz core' of a presheaf, but this gives no immediate information on the nature of $\mathcal{L}$. We show below that $\mathcal{L}$ is a presheaf topos and make some related observations.

For any $f, g : C' \to C$ we write $f \sim g$ if both maps are constant (in the sense that they factor through the terminal object). Write $\equiv$ for the congruence on $\mathcal{C}$ generated by $\sim$. Let $\mathcal{D} = \mathcal{C}/\equiv$ and $r : \mathcal{C} \to (\mathcal{C}/\equiv) = \mathcal{D}$ be the resulting quotient. This functor induces a hyperconnected geometric morphism $r : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$, and $\widehat{\mathcal{D}}$ is identified via r^* with the full subcategory of $\widehat{\mathcal{C}}$ that consists of the functors that respect this congruence. (See [1, A4.6.9].)

7.2. Lemma. *For any X in $\widehat{\mathcal{C}}$, the following are equivalent:*

1. *The counit $r^*(r_* X) \to X$ is an iso.*
2. *For every $h, h' : 1 \to C$ in $\mathcal{C}$ and $x \in XC$, $x \cdot h = x \cdot h'$.*

Proof. For any $f, g : C' \to C$ define $f \cong g$ if $f = g$ or $f \sim g$. We want to show that $\cong$ is the relation $\equiv$ generated by $\sim$. It suffices to show that $\cong$ is a congruence. Consider a not necessarily commutative diagram

$$\begin{tikzcd} C_0 \arrow[r, "a"] & C' \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \arrow[r, "b"] & C_1 \end{tikzcd}$$

with $f \cong g$. If $f = g$ then clearly $bfa = bga$. If f and g are constant then so are bfa and bga and hence $bfa \cong bga$. Therefore $\cong$ is a congruence and its simple description implies that it is the least one containing $\sim$. ∎

7.3. Proposition. *The canonical intensive quality of the pre-cohesive geometric morphism $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is $r : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}/\equiv}$.*

Proof. By 7.1 and 7.2 we obtain that $\mathcal{L}$ and $\widehat{\mathcal{C}/\equiv}$ can be identified with the subcategory of those functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ that satisfy the condition: for every $h, h' : 1 \to C$ in $\mathcal{C}$ and $x \in XC$, $x \cdot h = x \cdot h'$. ∎

Applying Proposition 7.3 to Δ_1 one obtains the concrete description of the canonical intensive quality of the topos of reflexive graphs discussed in [4]. Similarly for the case dealt with in [3]. For an arbitrary small $\mathcal{C}$ the quotient $\mathcal{C}/\equiv$ may not be so easy to visualize. On the other hand, a little experience with the simpler pre-cohesive toposes in algebraic geometry suggests the possibility of more informative descriptions of the Leibniz cores of certain objects. We first argue abstractly.

Following [9, Definition 5.5], a map $f : B \to C$ in $\mathcal{C}$ is *pseudo-constant* if $fa = fb$ for every $a, b : 1 \to B$. A *little figure* (of C) is a map with codomain C whose domain has exactly one point. (Of course, every little figure is pseudo-constant.) The category $\mathcal{C}$ is said to have *enough little figures* if every pseudo-constant $B \to C$ factors through a little figure of C.

7.4. Lemma. *If $\mathcal{C}$ has enough little figures then, for every C in $\mathcal{C}$, the Leibniz core of $\mathcal{C}(\_, C)$ is the sieve generated by the little figures of C.*

Proof. Lemma 7.1 implies that the Leibniz core of $\mathcal{C}(\_, C)$ is the sieve of pseudo-constants. ∎

Let $\mathcal{C}_! \to \mathcal{C}$ be the full subcategory of objects with exactly one point. If $\mathcal{C}$ has enough little figures, [9, Lemma 5.9] shows that $\widehat{\mathcal{C}_!} \to \widehat{\mathcal{C}}$ is the largest subtopos $f : \mathcal{F} \to \widehat{\mathcal{C}}$ of $\widehat{\mathcal{C}}$ such that pf is a quality type. So Lemma 7.4 suggests interesting connection between the canonical intensive quality (of p) and level ϵ (of p) to be studied elsewhere.

There is a further aspect that appears in practice but was not treated in [9]. Namely, the possibility that every little figure $B \to C$ factors as $B \to B' \to C$ with $B \to B'$ epic and $B' \to B$ a monic little figure. In this case, it follows from Lemma 7.4 that the Leibniz core of $\mathcal{C}(\_, C)$ is the sieve generated by the monic little figures of C. In the next section we discuss a conspicuous concrete example.

8. The case of the complex Gaeta topos

Let **Ring** be the category of rings, let $\mathbb{C}$ be the field of complex numbers. Let $\mathcal{C}$ be the opposite of the category of finitely generated $\mathbb{C}$-algebras with exactly two idempotents. It is well-known, that $\widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive and classifies $\mathbb{C}$-algebras with exactly two idempotents. (It is also well-known that $\widehat{\mathcal{C}}$ is the Gaeta topos of $\mathbb{C}$.) It follows from [9, Theorem 7.7] that $\mathcal{C}$ has enough little figures.

As in the previous section we denote by $\mathcal{C}_! \to \mathcal{C}$ the full subcategory of the objects that have exactly one point. The opposite of the category $\mathcal{C}_!$ is equivalent to the category of *Weil algebras* (over $\mathbb{C}$), that is, the local and finitely generated $\mathbb{C}$-algebras. See [9, Lemma 6.2] and references therein.

8.1. Lemma. *If $L \to D$ is an epimorphism in $\mathcal{C}$ and L is in $\mathcal{C}_!$ then so is D.*

Proof. We prove a slightly more general dual statement: Any finitely generated subalgebra of a Weil algebra is Weil. (This result is mentioned in passing in Weil's classical paper.) Let W be a Weil algebra and let $\gamma : W \to \mathbb{C}$ be the quotient by the nilradical. For any $w \in W$, $w = \gamma w + (w - \gamma w)$ and $\gamma(w - \gamma w) = 0$ trivially. Hence, every $w \in W$ is of the form $w = \gamma w + n$ for some nilpotent n.

Let $f : A \to W$ be monic in $\mathbb{C}/\mathbf{Ring}$. For every $a \in A$, $fa = \gamma(fa) + n$ for some nilpotent $n \in W$. Trivially, $f(a - \gamma(fa)) = fa - \gamma(fa) = n$ so, as f is injective, $a - \gamma(fa)$ is nilpotent. Then, $a = \gamma(fa) + (a - \gamma(fa))$ is the sum of an invertible and a nilpotent and is therefore invertible. To summarize, f reflects invertible elements.

Let $\mathfrak{m} \subseteq A$ be the restriction of the nilradical of W. If $x \in A$ but $x \notin \mathfrak{m}$ then $fx \in W$ is not nilpotent, so it is invertible and hence x is invertible. In other words, the non-invertibles of A form a prime ideal. Then A is local. ∎

In $\mathcal{C}$, every map factors as an epic followed by a regular monomorphism. So, if $L \to C$ is a little figure of C in $\mathcal{C}$ then we can factor it and, by Lemma 8.1, the regular-monic part is also a little figure of C. So, by the comments at the end of Section 7, the Leibniz cores of representables are generated by the (regular-)monic little figures of the representing object.

8.2. Proposition. *In the Gaeta topos of* $\mathbb{C}$*, the Leibniz core of the generic model, i.e. the representable by* $\mathbb{C}[x]$*, is dually generated by the quotients* $\mathbb{C}[x] \to W$ *with Weil codomain.*

Of course, there is an analogue for any finite set of generators.

9. The case of pre-cohesive Grothendieck toposes

Given a geometric morphism $g : \mathcal{F} \to \mathcal{E}$ between pre-cohesive toposes (over some base), it is natural to wonder if g restricts to a geometric morphism $\mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{E})$ between the respective quality types. We will see that this question is relevant for the more 'concrete' problem of calculating sites for codomains of intensive qualities. In any case, we give here a sufficient condition for g to restrict as above. This sufficient condition is formulated using *pieces-preserving* geometric morphisms as introduced in [8]. We assume some familiarity with that paper.

9.1. Proposition. *Let*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{S} & \end{tikzcd}$$

be a diagram of toposes and geometric morphisms, where

(i) $g_* f_* = p_*$;

(ii) f *and* p *are connected, essential and hyperconnected.*

(iii) p_* *inverts the unit of* g.

1. *If* g_* *preserves* $\mathcal{S}$*-indexed coproducts, then* g^* *preserves and reflects Leibniz objects.*

2. *If* g *preserves pieces, then* g_* *preserves and reflects Leibniz objects. Thus, if furthermore* f *and* p *are local,* g *restricts to a geometric morphism* $h : \mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{E})$ *over* $\mathcal{S}$.

3. *If, moreover,* $g : \mathcal{F} \to \mathcal{E}$ *is a subtopos, then the top-right composite in the commutative diagram below (in the category of toposes and geometric morphisms)*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "g"'] & \mathcal{L}(\mathcal{F}) \arrow[d, "h"] \\ \mathcal{E} \arrow[r, "s"'] & \mathcal{L}(\mathcal{E}) \end{tikzcd}$$

is the surjection/inclusion factorization of the left-bottom composite.

PROOF. We stick to the notation of Section 6 of [8]. Assume first that g_* preserves $\mathcal{S}$-indexed coproducts. We next prove that the diagram

$$\begin{tikzcd}
p_*g_*g^* \arrow[r, equal] & f_*g^* \arrow[r, "\overline{\theta}_{g^*}"] & f_!g^* \arrow[d, "\varrho"] \\
p_* \arrow[u, "p_*\nu"] \arrow[rr, "\theta"'] & & p_!
\end{tikzcd}$$

commutes. Indeed, the diagram

$$\begin{tikzcd}
f_*g^* \arrow[r, "\overline{\theta}_{g^*}"] \arrow[dr, "f_*g^*\sigma"] \arrow[d, equal] & f_!g^* \arrow[r, "f_!g^*\sigma"] & f_!g^*p^*p_! \arrow[rr, equal] & & f_!f^*p_! \arrow[dd, "\overline{\tau}_{p_!}"] \\
p_*g_*g^* & f_*g^*p^*p_! \arrow[ur, "\overline{\theta}_{g^*p^*p_!}"'] \arrow[r, equal] & f_*f^*p_! \arrow[urr, "\overline{\theta}_{f^*p_!}"'] \arrow[drr, bend left=10, "\overline{\alpha}^{-1}_{p_!}"] \arrow[d, equal] & & \\
p_* \arrow[u, "p_*\nu"] \arrow[r, "p_*\sigma"] \arrow[rrrr, bend right=20, "\theta"'] & p_*p^*p_! \arrow[r, "p_*\nu_{p^*p_!}"] \arrow[rr, equal, bend right=10] & p_*g_*g^*p^*p_! \arrow[r, "(p_*\nu_{p^*p_!})^{-1}"] & p_*p^*p_! \arrow[r, "\alpha^{-1}_{p_!}"] & p_!
\end{tikzcd}$$

commutes. Since g_* preserves $\mathcal{S}$-indexed coproducts, ϱ is an iso (Lemma 6.4 in [8]), and $p_*\nu$ is assumed to be an iso. Thus, for $Y \in \mathcal{E}$, θ_Y is an iso if and only if $\overline{\theta}_{g^*Y}$ is an iso. It follows that $g^* : \mathcal{E} \to \mathcal{F}$ restricts to a functor $\mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{F})$.

Assume now that g preserves pieces. Let $X \in \mathcal{L}(F)$. Then in the commutative diagram

$$\begin{tikzcd}
p_*g_*X \arrow[r, "\theta_{g_*X}"] \arrow[d, equal] & p_!g_*X \arrow[d, "\lambda_X"] \\
f_*X \arrow[r, "\overline{\theta}_X"'] & f_!X
\end{tikzcd}$$

of Proposition 6.12 in [8], applied to X, we know that λ is an iso. Thus $\overline{\theta}_X$ is an iso if and only if θ_{g_*X} is an iso. Clearly, when f and p are local, $\mathcal{L}(\mathcal{E})$ and $\mathcal{L}(\mathcal{F})$ are toposes by Theorem 6.5, and the restriction $g^* : \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{F})$ preserves finite limits, so g restricts to a geometric morphism $h : \mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{E})$.

Finally, assume that $g : \mathcal{F} \to \mathcal{E}$ is a subtopos. The restriction result shows that the diagram on the left below commutes

$$\begin{tikzcd}
\mathcal{L}(\mathcal{E}) \arrow[r, "h^*"] \arrow[d, "s^*"'] & \mathcal{L}(\mathcal{F}) \arrow[d, "t^*"] \\
\mathcal{E} \arrow[r, "g^*"'] & \mathcal{F}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{F} \arrow[r, "g"] \arrow[d, "t"'] & \mathcal{E} \arrow[d, "s"] \\
\mathcal{L}(\mathcal{F}) \arrow[r, "h"'] & \mathcal{L}(\mathcal{E})
\end{tikzcd}$$

where $t : \mathcal{F} \to \mathcal{L}(\mathcal{F})$ is the canonical intensive quality of $\mathcal{F}$ and h^* is the restriction of g^*. It follows that the diagram on the right above commutes in the category of toposes; but h_* is the restriction of the full and faithful g_* so it is also full and faithful. That is, $h : \mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{E})$ is a subtopos. Since t is a surjection, the result follows. ∎

For example, for $n \geq 1$, we have that

$$\begin{tikzcd} \widehat{\Delta_n} \arrow[r, "g"] \arrow[dr, "f"'] & \widehat{\Delta} \arrow[d, "p"] \\ & \mathbf{Set} \end{tikzcd}$$

(where $g^* : \widehat{\Delta} \to \widehat{\Delta_n}$ is truncation of simplicial sets at level n) satisfies the conditions (i)-(iii) of Proposition 9.1. Furthermore g preserves pieces (so, in particular, g_* preserves $\mathbf{Set}$-indexed coproducts). We thus obtain a geometric morphism $\mathcal{L}(\widehat{\Delta_n}) \to \mathcal{L}(\widehat{\Delta})$. This of course could have been done directly since Leibniz on both categories means that every connected component has exactly one point.

Fix now a site $(\mathcal{C}, J)$. We recall the following definition (see [2] for instance).

9.2. Definition. The site $(\mathcal{C}, J)$ is *locally connected* if each J-covering sieve on C is connected as a full subcategory of $\mathcal{C}/C$. If furthermore $\mathcal{C}$ has a terminal object, then we say that $(\mathcal{C}, J)$ is *connected and locally connected.*

Assume from now on that $\mathcal{C}$ has a terminal object and that every object has a point. Assume also that J is a coverage making $(\mathcal{C}, J)$ into a (connected and) locally connected site. It follows from [2] that both $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ and $f : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Set}$ are pre-cohesive. We thus have the following situation

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}, J) \arrow[rr, "g"] \arrow[dr, "f"'] & & \widehat{\mathcal{C}} \arrow[dl, "p"] \\ & \mathbf{Set} & \end{tikzcd}$$

with both f and p pre-cohesive and g the canonical geometric morphism. So conditions (i) and (ii) of Proposition 9.1 are satisfied. We observe that (iii) is also satisfied.

9.3. Lemma. *The functor p_* inverts the unit of $g^* \dashv g_* : \mathbf{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$.*

Proof. Let $P \in \widehat{\mathcal{C}}$. Then the unit of the adjunction is the composite

$$P \to P^+ \to (P^+)^+$$

where P^+ is Grothendieck's $+$-construction. Thus it will be sufficient to show that the map $p_*(P \to P^+)$ is an isomorphism. We know that p_* is evaluation at 1. Thus we have to check that

$$P1 \to \varinjlim_{S \in J(1)} \widehat{\mathcal{C}}(S, P)$$

is an iso. Let $S \in J(1)$. Since $(\mathcal{C}, J)$ is locally connected, S is non-empty. So take $(f : C \to 1) \in S$. Now, taking a point $1 \to C$, we see that $1_1 : 1 \to 1$ is an element of S, so the only element of $J(1)$ is the maximal sieve. Thus $\varinjlim_{S \in J(1)} \widehat{\mathcal{C}}(S, P) = \widehat{\mathcal{C}}(\mathcal{C}(-,1), P) \cong P(1)$. ∎

We may now prove the main result of the section.

9.4. Theorem. *Let $(\mathcal{C}, J)$ be a connected and locally connected site such that every object in $\mathcal{C}$ has a point. Let $\mathcal{C}/\equiv$ be the category that results from identifying all the points, and let $r : \mathcal{C} \to \mathcal{C}/\equiv$ be the quotient functor. If r_+J is the largest topology on $\mathcal{C}/\equiv$ such that r reflects covers, then*

$$\mathcal{L}(\mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Sh}(\mathcal{C}/\equiv, r_+J).$$

Proof. Lemma 7.7 in [8] implies that the geometric morphism $g : \mathbf{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ is pieces preserving so, by Lemma 9.3, we may apply Proposition 9.1 to conclude that the canonical intensive quality $\mathbf{Sh}(\mathcal{C}, J) \to \mathcal{L}(\mathbf{Sh}(\mathcal{C}, J))$ is the surjective part of the factorization of the composite

$$\mathbf{Sh}(\mathcal{C}, J) \longrightarrow \widehat{\mathcal{C}} \xrightarrow{s} \mathcal{L}(\widehat{\mathcal{C}})$$

in the category of toposes. Proposition 7.3 shows that s may be identified with the geometric morphism $r : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}/\equiv}$ induced by $r : \mathcal{C} \to \mathcal{C}/\equiv$. By Lemma C2.3.19 in [1], r_+J is the coverage on $\mathcal{C}/\equiv$ corresponding to the image of the composite

$$\mathbf{Sh}(\mathcal{C}, J) \longrightarrow \widehat{\mathcal{C}} \xrightarrow{r} \widehat{\mathcal{C}/\equiv}$$

so the result follows. ∎

Since it relates to the applicability of Theorem 9.4, it is important to recall, also from Lemma C2.3.19 in [1], that r_+J has a very simple description. Indeed, a sieve S on C in the category $\mathcal{C}/\equiv$ is in $(r_+J)C$ if and only if the sieve

$$\{g : \operatorname{dom} g \to C \text{ in } \mathcal{C} | r(g) \in S\}$$

is in JC.

10. The case of a cohesive Grothendieck topos

Recall from [10] the example where the category is $\mathbf{L}$ of intervals and linear functions between them, with the topology given by a base K formed of all the families of inclusions from a partition of an interval. Recall that the canonical $\mathbf{Sh}(L, K) \to \mathbf{Set}$ is cohesive (see Section 10 in loc. cit.). Based on these ideas, [13] determines a whole family of monoids of continuous endomorphisms of the unit interval that produce models of cohesion. Since the argument that follows is basically the same in both cases, we deal here with the latter one.

Thus, in this section $\mathbf{M}$ denotes the category of closed real intervals induced by a suitable submonoid M of the monoid of continuous endomorphisms of the unit interval $[0,1]$, such that every function in M is unilateral, with the Grothendieck topology K of finite intervals, as in [13], and we produce a subcanonical site of definition for the Leibniz category $\mathcal{L}(\mathbf{Sh}(\mathbf{M},K))$ associated with the cohesive geometric morphism $\mathbf{Sh}(\mathbf{M},K) \to \mathbf{Set}$.

Recall that a continuous endomorphism f of the unit interval is *unilateral* if for any $t \in [0,1]$ there is an $\varepsilon > 0$ such that the function $f - f(t)$ does not change sign in the interval $[t, t+\varepsilon] \cap [0,1]$ and does not change sign in the interval $[t-\varepsilon, t] \cap [0,1]$.

Given a submonoid M of the monoid of continuous endomorphisms of the unit interval $[0,1]$ that contains all the linear endomorphisms, [13] shows that the following defines a subcanonical site: the category $\mathbf{M}$ has as objects all the real closed intervals $[a,b]$ ($a, b \in \mathbb{R}$, $a \leq b$), and as morphisms all the piecewise M-functions (a function $f : [a,b] \to [c,d]$ is an *M-function* if it can be written in the form

$$[a,b] \longrightarrow [0,1] \xrightarrow{m} [0,1] \longrightarrow [c,d],$$

where the arrows on both extremes are linear functions, and $m \in M$). A continuous function $f : [a,b] \to [c,d]$ is a *piecewise M-function* if there is a partition $a = r_0 < \cdots < r_n = b$ such that the restriction of f to every $[r_i, r_{i+1}]$ is an M-function.

A basis K of a Grothendieck topology on $\mathbf{M}$ is given by finite partitions: $K([a,a]) = \{1_{[a,b]}\}$ for any $a \in \mathbf{R}$, and for $a < b$ in $\mathbb{R}$, $K([a,b])$ consists of those families

$$\{[r_{j-1}, r_j] \hookrightarrow [a,b] | 1 \leq j \leq n\}$$

determined by partitions $a = r_0 < \cdots < r_n = b$ of $[a,b]$. Then $(\mathbf{M}, K)$ is a subcanonical site and $\mathbf{Sh}(\mathbf{M},K)$ is a cohesive topos over $\mathbf{Set}$ (see Theorem 2.8 in [13]).

According to Theorem 9.4 we have that a site of definition for $\mathcal{L}(\mathbf{Sh}(\mathbf{M},K))$ is given by $(\mathbf{M}/\equiv, r_+K)$, where $r : \mathbf{M} \to \mathbf{M}/\equiv$ is the canonical functor. So we could use Giraud's theorem (see the appendix of [7]) to obtain a subcanonical site of definition for $\mathcal{L}(\mathbf{Sh}(\mathbf{M},K))$ by taking the full subcategory of $\mathbf{Sh}(\mathbf{M}/\equiv, r_+K)$ whose objects are sheafifications of the presheaves represented by objects in $\mathbf{M}/\equiv$, and give this category the topology that consists of those families that are epimorphic in $\mathbf{Sh}(\mathbf{M}/\equiv, r_+K)$. The process does not produce an explicit description of the site, but we next observe the following two facts. On the one hand, if $F : \mathbf{M}/\equiv \to \mathbf{Sh}(\mathbf{M}/\equiv, r_+K)$ denotes Yoneda followed by sheafification, then the topology given by epimorphic families coincides with the topology $F_+(r_+K)$, and on the other, every representable functor is separated for this topology. One can then take the category whose objects are the objects of $\mathbf{M}/\equiv$, and as arrows from $[a,b]$ to $[c,d]$, the set

$$\mathbf{Sh}(\mathbf{M}/\equiv, r_+K)([a,b]^+, [c,d]^+),$$

where $[a,b]^+ := (\mathbf{M}/\!\equiv(-,[a,b]))^+$ is Grothendieck's plus construction applied to the presheaf represented by $[a,b]$. But now

$$\mathbf{Sh}(\mathbf{M}/\!\equiv, r_+K)([a,b]^+,[c,d]^+) \simeq [c,d]^+([a,b]),$$

so it is clear that what one must understand is compatible families of arrows. Since we have identified all the points in $\mathbf{M}/\!\equiv$ and the coverings are determined by partitions, the condition of being compatible trivializes.

This last observation suggests a more incisive direct description of essentially the same subcanonical site for $\mathcal{L}(\mathbf{Sh}(\mathbf{M},K))$ for a suitable monoid M as follows.

We first require that the monoid M satisfy the following two conditions:

1. Every $m \in M$ is either constant or strictly monotone.

2. For every pair of M-functions $f : [a,b] \to [c,d]$, $g : [a',b'] \to [c,d]$ with $a < a' < b < b'$ and $f|_{[a',b]} = g|_{[a',b]} : [a',b] \to [c,d]$, the unique continuous function $h : [a,b'] \to [c,d]$ that restricts to f on $[a,b]$ and to g on $[a',b']$ is also an M-function.

There may be milder conditions on M for which the following reasoning works, but we still have plenty of monoids M that satisfy these conditions: for instance the monoid M of functions that are either constant or strictly monotone.

Consider the following category **OI**.

The objects of **OI** are open real intervals (a,b) with $a \leq b$ in $\mathbb{R}$ (so $\varnothing \in \mathbf{OI}$).

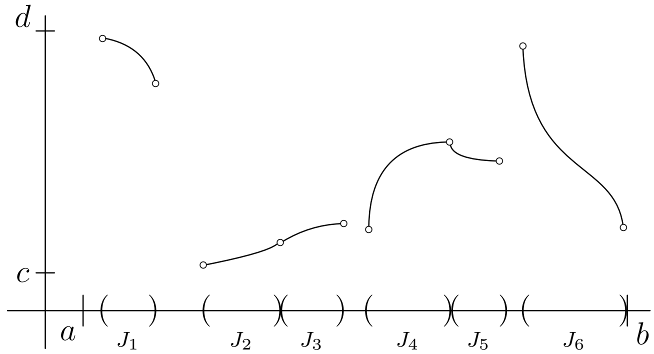
The morphisms $f : (a,b) \to (c,d)$ in **OI** are partial functions that satisfy the following conditions:

a) the set of points where f is defined is of the form $\bigcup_{i=1}^m I_i$, with $m \in \mathbb{N}$, where $\{I_i\}_{i=1}^m$ is a disjoint family of non-empty open subintervals of (a,b), and

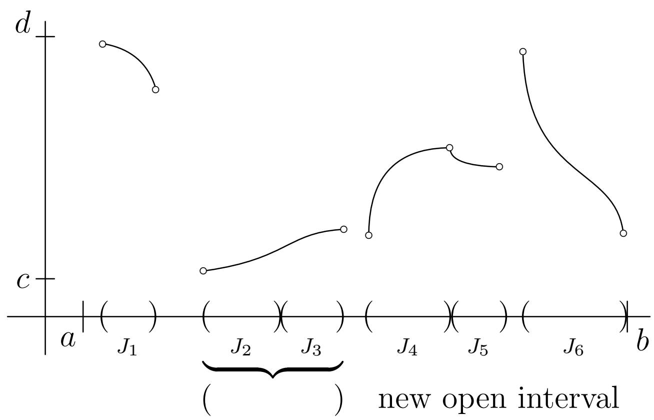
b) for every $i \in \{1,\ldots,m\}$, $f|_{I_i} : I_i \to (c,d)$ is a restriction of a strictly monotone M-function $\overline{I_i} \to [c,d]$ where $\overline{I_i}$ is the closure of I_i.

Composition in **OI** is defined as the usual composition of partial functions. One must of course show that such a composite satisfies the conditions expressed above, a task we leave to the reader.

Given $f : (a,b) \to (c,d)$ in **OI**, we define the *M-saturation* $\overline{f} : (a,b) \to (c,d)$ of f as follows. The partial function $\overline{f}$ is defined at a point $t \in (a,b)$ if there exist $x < t < y$ such that f is defined on $[x,t)$ and on $(t,y]$ and $f|_{[x,t)\cup(t,y]}$ can be extended to an M-function $l : [x,y] \to [c,d]$ (and observe that l turns out to be strictly monotone); in such a case we define $\overline{f}(t) = l(t)$. It is not hard to see that $\overline{f}$ is a morphism in **OI**, that the set of points where $\overline{f}$ is defined is the union of the set where f is defined and a set with a finite number of points, and that for any $t \in (a,b)$, $\overline{f}(t) = f(t)$ when the latter is defined. We have below an example (of the graph) of an f in **OI**:



and (the graph) of its M-saturation $\overline{f}$:



(assuming of course that the right end t of J_2 is the left end of J_3, and that we can fill in the point "$f(u)$" in such a way that the resulting function on $J_2 \cup \{t\} \cup J_3$ is an M-function).

10.1. Lemma. *For any $f : (a,b) \to (c,d)$ and $g : (c,d) \to (u,v)$ in* **OI**, *we have that*

$$\overline{gf} = \overline{\overline{g}f} = \overline{g\overline{f}}.$$

Proof. We show first that $\overline{gf} = \overline{\overline{g}f}$.

Let $t \in (a,b)$ be such that $\overline{gf}$ is defined at t. There exist $x, y \in \mathbb{R}$ such that $x < t < y$, gf is defined on $[x,t) \cup (t,y]$, and an M-function $l : [x,y] \to [u,v]$ that agrees with gf on $(x,t) \cup (t,y)$. Then it is clear that $\overline{g}f$ satisfies these same conditions. Thus $\overline{gf}(t) = \overline{\overline{g}f}(t)$.

Assume now that $\overline{\overline{g}f}$ is defined at $t \in (a,b)$. There are $x < t < y$ such that $\overline{g}f$ is defined on $[x,t) \cup (t,y]$ and an M-function $l : [x,y] \to [u,v]$ that agrees with $\overline{g}f$ on $[x,t) \cup (t,y]$. This means that f is defined on $[x,t) \cup (t,y]$, and for every $s \in [x,t) \cup (t,y]$, $\overline{g}$ is defined on $f(s)$. Observe that $f(x,t)$ is an open subinterval of (c,d), and that the number of points at which $\overline{g}$ is defined but g is not, is finite. Thus, increasing x if necessary, we may assume that g is defined on $f(x,t)$. We similarly may assume that g is defined on $f(t,y)$.

In this way gf is defined on $[x,t) \cup (t,y]$ and agrees with $l : [x,y] \to [u,v]$ there. Thus $\overline{gf}$ is defined at t and $\overline{gf}(t) = \overline{\overline{g}f}(t)$.

This completes the proof that $\overline{gf} = \overline{\overline{g}f}$. The proof that $\overline{gf} = \overline{g\overline{f}}$ is similar, and is left to the reader. ∎

The lemma allows us to define the following category **OIs**. The objects of **OIs** are the same as those of **OI**. A morphism $f : (a,b) \to (c,d)$ in **OIs** is a morphism $f : (a,b) \to (c,d)$ in **OI** that is M-saturated (i.e. $\overline{f} = f$). The composition of $f : (a,b) \to (c,d)$ and $g : (c,d) \to (u,v)$ in **OIs** is given by the formula

$$\overline{gf} : (a,b) \to (u,v).$$

The fact that this defines a category follows from the observation that the identities in **OI** are M-saturated, and the following calculation

$$\overline{\overline{hg}f} = \overline{hgf} = \overline{h\overline{gf}}$$

for composable $f, g, h \in$ **OIs**.

Now we can consider any piecewise M-function as a morphism in **OIs**: for $f : [a,b] \to [c,d]$ with $a < b$, take the minimal partition $a = r_0 < \cdots < r_n = b$ such that every $f|_{[r_i, r_{i+1}]}$ is an M-function; take as domain of the morphism in **OIs** those (r_i, r_{i+1}) at which f is not constant, and define the partial function as f in those intervals. For $a = b$ there is only one possible function to consider. This clearly defines a functor $F : \mathbf{M} \to$ **OIs** where $F([a,b]) = (a,b)$ on objects. The next lemma gives us a more concrete description of the topology F_+K.

10.2. Lemma. *$F_+K(\varnothing)$ consists only of the maximal sieve, and for $a < b$, $R \in F_+K((a,b))$ if and only if there is a partition $a = r_0 < \cdots < r_k = b$ of $[a,b]$ such that for every $i \in \{1, \ldots, k\}$, the inclusion $(r_{i-1}, r_i) \rightarrowtail (a,b)$ belongs to R.*

Proof. Let $S \in F_+K(\varnothing)$. Since $\varnothing$ is a zero object in **OIs**, to show that S is the maximal sieve it suffices to show that S is non-empty. But since $1_\varnothing : F1 \to \varnothing$ and $\varnothing \notin K1$, this is clear. Take now $a < b$ real numbers. Let $R \in F_+K((a,b))$. Since we have $1_{(a,b)} : F([a,b]) \to (a,b)$, then we have that

$$\{g : X \to [a,b] | F(g) \in 1^*_{(a,b)} R = R\} \in K([a,b]).$$

There is, therefore, a partition $0 = s_0 < \cdots < s_k = 1$ of $[a,b]$ such that for every $i \in \{1, \ldots, k\}$, the inclusion $f_i : [s_{i-1}, s_i] \to [a,b]$ is such that $F(f_i) \in R$. But $F(f_i)$ is of course the inclusion $(s_{i-1}, s_i) \rightarrowtail (a,b)$.

Assume now that a sieve R on (a,b) satisfies the condition given on the statement of the lemma. We must show that for every $f : FX \to (a,b)$ in **OIs** we have that

$$\{g : Y \to X | F(g) \in f^* R\} \in K(X).$$

For $X = 1$ this is clear since R is non-empty. So assume that $X = [c,d]$ with $c < d$. So now $f : (c,d) \to (a,b)$ is an M-saturated partial function, where the set of points at which f is defined is of the form $\bigcup_{j=1}^{m} I_j$, where the union is disjoint, for every j, I_j is an open subinterval of (c,d), and $f|_{I_j} : I_j \to (a,b)$ is a strictly monotone M-function. This means that there is a partition $c = s_0 < \cdots < s_m = d$ such that for every $j \in \{1,\ldots,m\}$, $f|_{(s_{j-1},s_j)} : (s_{j-1},s_j) \to (a,b)$ is either the empty partial function or it is a strictly monotone function defined everywhere on (s_{j-1},s_j). In the first case, if $\iota_j : [s_{j-1},s_j] \rightarrowtail [c,d]$ denotes the inclusion, we have that $f \circ F(\iota_j)$ is the empty partial function, and thus it belongs to R. In the second case consider the partition of $[s_{j-1},s_j]$ given by $\{s_{j-1},s_j\} \cup f^{-1}(\{r_0,\ldots,r_k\})$. If $u < v$ are two consecutive points in this partition, then the inclusion $\iota : [u,v] \rightarrowtail [c,d]$ is such that $f \circ F(\iota)$ factors through some inclusion $(r_{i-1},r_i) \rightarrowtail (a,b)$, and thus it belongs to R. We conclude that $R \in F_+K((a,b))$. ∎

10.3. Proposition. *The site* $(\mathbf{OIs}, F_+K)$ *is subcanonical and the geometric morphism*

$$\mathbf{Sh}(\mathbf{M},K) \to \mathbf{Sh}(\mathbf{OIs},F_+K)$$

induced by F is the canonical intensive quality of the cohesive $\mathbf{Sh}(\mathbf{M},K) \to \mathbf{Set}$.

Proof. We begin by showing that the site $(\mathbf{OIs}, F_+K)$ is subcanonical.

Clearly $\mathbf{OIs}(-,\varnothing) = 1$, so it is a sheaf.

Let $a < b$ be real numbers. We must show that $\mathbf{OIs}(-,(a,b))$ is a sheaf. So let c,d be real numbers with $c \leq d$ and let $S \in F_+K((c,d))$. By Lemma 10.2, S is generated by a family of the form

$$\{\iota_i : (r_{i-1},r_i) \rightarrowtail (c,d) | c = r_0 < \cdots < r_k = d\}.$$

A compatible family for S is now just a family

$$\{f_i : (r_{i-1},r_i) \rightarrowtail (a,b) | i \in \{1,\ldots,k\}\},$$

(compatibility is automatic). If we put all these f_i together, we obtain a partial function $f : (c,d) \to (a,b)$. The amalgam of the given family is $\overline{f}$. Uniqueness of the amalgam follows from the fact that $r_0 < \cdots < r_k$ is a partition of (c,d).

Now, the functor $F : \mathbf{M} \to \mathbf{OIs}$ clearly identifies all the points in $\mathbf{M}$. Thus there is a unique functor $G : \mathbf{M}/{\equiv} \to \mathbf{OIs}$ such that

$$\begin{tikzcd} \mathbf{M} \arrow[r, "r"] \arrow[dr, "F"'] & \mathbf{M}/{\equiv} \arrow[d, "G"] \\ & \mathbf{OIs} \end{tikzcd}$$

commutes. Observe that $F_+K = G_+(r_+(K))$ (use the description of $(\ )_+$ in terms of images given in the proof C2.3.19(ii)). The functor $G : \mathbf{M}/{\equiv} \to \mathbf{OIs}$ is faithful. In fact, G induces an isomorphism of categories between $\mathbf{M}/{\equiv}$ and the subcategory $\mathcal{D}$ of $\mathbf{OIs}$ whose objects are of the form (a,b); and whose arrows are all the arrows that begin or end at $\varnothing$, and those arrows $f : (c,d) \to (a,b)$ (with $c < d$ and $a < b$) that are defined everywhere, plus the empty partial function $(c,d) \to (a,b)$. Once we transfer the coverage

r_+K from $\mathbf{M}/\equiv$ to $\mathcal{D}$, we observe that this coverage on $\mathcal{D}$ coincides with $(F_+K)_{\mathcal{D}}$ (see C2.2.2). $\mathcal{D}$ is clearly a F_+K-dense subcategory of **OIs** (see C2.2.1). So an application of the Comparison Lemma (see C2.2.3) produces the equivalence with the canonical intensive quality. ∎

It follows that the functor induced on sheaves by R is essentially the geometric morphism given by the Leibniz category $\mathbf{Sh}(\mathbf{L}_p, H_+K) \to \mathcal{L}(\mathbf{Sh}(\mathbf{L}_p, H_+K))$. Thus we know that there is an extra left adjoint. One advantage of having subcanonical sites is that this extra left adjoint has a simple form.

10.4. Proposition. *The geometric morphism* $s : \mathbf{Sh}(\mathbf{M}, K) \to \mathbf{Sh}(\mathbf{OIs}, F_+K)$ *induced by* $r : \mathbf{M} \to \mathbf{OIs}$ *is the canonical intensive quality of the cohesive* $\mathbf{Sh}(\mathbf{M}, K) \to \mathbf{Set}$*, and the diagram*

$$\begin{array}{ccc} \widehat{\mathbf{M}} & \xrightarrow{r_!} & \widehat{\mathbf{OIs}} \\ \uparrow & & \uparrow \\ \mathbf{Sh}(\mathbf{M}, K) & \xrightarrow[s_!]{} & \mathbf{Sh}(\mathbf{OIs}, F_+K) \end{array}$$

commutes, where the vertical arrows are the corresponding inclusions.

Proof. We prove something a bit more general. Assume that we have a functor $F : \mathcal{C} \to \mathcal{D}$ and a topology K on $\mathcal{C}$. Thus we have a geometric morphism $\mathbf{Sh}(\mathcal{C}, K) \to \mathbf{Sh}(\mathcal{D}, F_+K)$ induced by a functor $F : \mathcal{C} \to \mathcal{D}$, where $(\mathcal{C}, J)$ is a site. Assume that composing with F, $F^* : \mathbf{Sh}(\mathcal{D}, F_+K) \to \mathbf{Sh}(\mathcal{C}, K)$, has a left adjoint $F_!$, and that both sites are subcanonical. Then $F_!$ preserves representables as the usual calculation

$$\begin{array}{c} F_!(\mathcal{C}(-, C)) \longrightarrow Y \\ \hline \mathcal{C}(-, C) \longrightarrow Y \cdot F \\ \hline Y(FC) \\ \hline \mathcal{D}(-, FC) \longrightarrow Y \end{array}$$

shows, for $C \in \mathcal{C}$, $Y \in \mathbf{Sh}(\mathcal{D}, K)$, using, of course, that both sites are subcanonical. This last fact gives us the following formula for $F_!$ since it preserves colimits, and every object in the domain is a canonical colimit of representables:

$$F_!(X) = \varinjlim_{\mathcal{C}(-,C) \to X} \mathcal{D}(-, FC),$$

for $X \in \mathbf{Sh}(\mathcal{C}, J)$. That is, the diagram

$$\begin{array}{ccc} \widehat{\mathcal{C}} & \xrightarrow{F_!} & \widehat{\mathcal{D}} \\ \uparrow & & \uparrow \\ \mathbf{Sh}(\mathcal{C}, J) & \xrightarrow[F_!]{} & \mathbf{Sh}(\mathcal{D}, F_+J) \end{array}$$

commutes, where the vertical arrows are inclusions. ∎

The fact that $s_!$ preserves representables allows us to obtain, for example, a very concrete expression of the "supercooling" of the unit interval $\mathbf{M}(-,[0,1])$, namely $\mathbf{OIs}(R(-),(0,1))$, and also of the corresponding counit $\mathbf{M}(-,[0,1]) \to s^* s_!(\mathbf{M}(-,[0,1]))$, namely

$$R : \mathbf{M}(-,[0,1]) \to \mathbf{OIs}(R(-),(0,1)).$$

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