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ORIGINAL ARTICLE

The Sitnikov problem for several primary bodies configurations

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Abstract In this paper we address an $n+1$-body gravitational problem governed by the Newton's laws, where n primary bodies orbit on a plane Π and an additional massless particle moves on the perpendicular line to Π passing through the center of mass of the primary bodies. We find a condition for the described configuration to be possible. In the case when the primaries are in a rigid motion, we classify all the motions of the massless particle. We study the situation when the massless particle has a periodic motion with the same minimal period as the primary bodies. We show that this fact is related to the existence of a certain pyramidal central configuration.

Keywords Sitnikov · n-Body · Periodic solutions · Central configurations

1 Introduction

In this paper we study the following restricted Newtonian $n+1$-body problem P (see Fig. 1):

P_1 We have n primary bodies of masses $m_1, \ldots, m_n$ and an additional massless particle.
P_2 The primary bodies are in a homographic motion (see Llibre et al. 2015, Section 2.9). This motion is carried out in a plane Π.

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P_3 The massless particle is moving on a perpendicular line to Π passing through the center of mass of the primary bodies.

Problems like the one presented above have been extensively discussed in the literature. Sitnikov (1960) considered the problem of two bodies in a Keplerian elliptic motion and a massless particle moving on the perpendicular line to the orbital plane passing through the center of mass. Sitnikov obtained deep results about the existence of solutions, for small $e > 0$, with a chaotic behavior [see Moser 1973, III(5)]. Periodic solutions for a Sitnikov configuration were considered in Corbera and Llibre (2000, 2002), Llibre and Ortega (2008) and Pustyl'nikov (1990).

Generalized circular Sitnikov problems, i.e., when there are $n \geq 3$ primaries in a relative equilibrium motion, were addressed more recently. Soulis et al. (2008) studied the existence, linear stability and bifurcations for a problem similar to P. They considered a Lagrangian equilateral triangle configuration for the primary bodies, which were supposed to have the same mass $m_1 = m_2 = m_3$. Bountis and Papadakis (2009) extended the results of Soulis et al. (2008) to n primaries ($n \geq 3$) in a polygonal equal masses configuration. Later, Pandey and Ahmad (2013) generalized the analysis started in Soulis et al. (2008) to the case with oblate primaries. Li et al. (2013) studied a special type of the restricted circular $n + 1$-body problem with equal masses for the primaries in a regular polygonal configuration. Periodic solutions for generalized Sitnikov problems with primaries performing no rigid motions were studied in Pustyl'nikov (1990) and Rivera (2013). We emphasize that in Bountis and Papadakis (2009), Li et al. (2013), Pandey and Ahmad (2013), Pustyl'nikov (1990), Rivera (2013) and Soulis et al. (2008), it is assumed that the primary bodies are in the vertices of a regular polygon. As far as we know, the first non-polygonal configuration of primary bodies was considered in Marchesin and Vidal (2013) where Marchesin and Vidal studied the problem P for a rigid motion of primaries in a rhomboidal configuration. Bakker and Simmons (2015) studied escape regions for the massless particle in a similar problem to P where the primaries perform certain type of periodic orbits including non homographic motions.

In the present paper, after introducing preliminary facts in Sect. 2, we obtain in Sect. 3 necessary and sufficient conditions on the configuration of the primary bodies in order to the z-axis be invariant for the flow associated with the equations of motion of the massless particle. For this type of configurations, that we call *admissible*, the Sitnikov problem has sense. The conclusions of Sect. 3 are obtained basically by elementary linear algebra arguments. We consider that the main contribution of Sect. 3 is to expand the variety of problems of Sitnikov type. In Sect. 4, we find all admissible configurations for $n \leq 4$ primaries. The

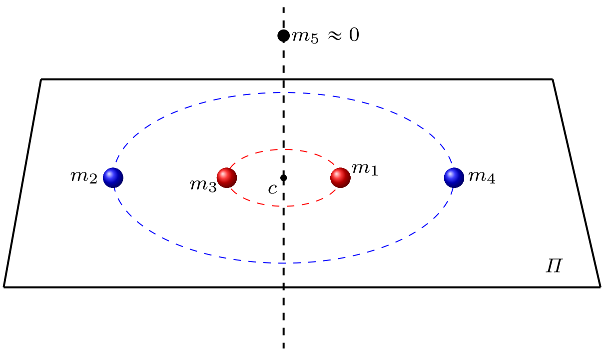

Fig. 1 Five-body problem with primaries in a collinear configuration

Perpendicular Bisector Theorem of Moeckel (see Moeckel 1990) is an important help to solve this question. In Sect. 5 we describe all possible motions of the massless particle when the primaries are in a relative equilibrium (or rigid) motion. In this direction, we observe that only escape (both parabolic and hyperbolic) and periodic motions are possible. We also give in Theorem 5 a formula expressing the period of solutions by means of integrals. We prove in Corollary 1 that the complete $n+1$-body system has infinite number of periodic solutions. We solve some problems raised in Sect. 5 by two alternative techniques: (1) elementary arguments, by using energy conservation (Arnold 1989, Ch. 2) and (2) variational techniques inspired in Li et al. (2013), Ferrario and Terracini (2004) and Zhao and Zhang (2015). In Sect. 6 we discuss the situation when the entire system has a solution with the same period as the motion of primaries. We call it *synchronous solution*. Surprisingly, the existence of synchronous solutions is related to the existence of certain pyramidal central configurations (for the definition of this concept see Fayçal 1996, 1995; Ouyang et al. 2004). Finally, in the last section, we study certain non-admissible configurations which provide some particular solutions of problem P.

In this paper, we generalize and extend some previously obtained results. For example, the results in Sect. 5, obtained for admissible configurations, generalize some results in Marchesin and Vidal (2013) established for rhomboidal configurations. In Sect. 6 we prove that there exist synchronous solutions for primaries in a regular polygonal equal mass configuration if and only if $2 \leq n \leq 472$. The sufficiency of this fact was established in Li et al. (2013).

2 Preliminaries

We start considering n mass points, $n > 2$, of masses $m_1, \ldots, m_n$ moving in a Euclidean three-dimensional space according to Newton's laws of motion. We assume that $x_1(t), \ldots, x_n(t)$ are the coordinates of the bodies in some inertial Cartesian coordinate system. We can suppose, without any loss of generality, that the center of mass $C := \sum_j m_j x_j / M$ ($M := \sum_j m_j$) is fixed at the origin ($C = 0$).

Initially we suppose that the bodies are in a *planar homographic motion* on the plane Π (see Llibre et al. 2015), where Π is the plane determined by the first two coordinate axes. Concretely, we are assuming that

$$x_j(t) = r(t) Q(\theta(t)) q_j, \tag{1}$$

where

$$Q(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $q_j \in \Pi$, $j = 1, \ldots, n$ are vectors in a planar *central configuration* (CC) in Π. We recall the following definition of this concept (see Llibre et al. 2015).

Definition 1 Let $q = (q_1, \ldots, q_n)$ be an n-tuple of positions in $\mathbb{R}^3$ and let $m = (m_1, \ldots, m_n)$ be a vector of masses. We say that (q, m) is a central configuration if there exists $\lambda \in \mathbb{R}$ such that

$$\nabla_j U(q_1, \ldots, q_n) + \lambda m_j q_j = 0, \quad j = 1, \ldots, n, \tag{2}$$

where

$$U(q_1, \ldots, q_n) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}, \tag{3}$$

$r_{ij} = |q_i - q_j|$ and ∇_j denotes the 3-dimensional partial gradient with respect to q_j.

From Llibre et al. (2015, Eq. (2.16)), the functions $r(t)$ and $\theta(t)$ solve the two-dimensional Kepler problem in polar coordinates, which is

$$\begin{aligned} \ddot{r}(t) - r(t)\dot{\theta}(t)^2 &= -\frac{\lambda}{r(t)^2} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)^2\dot{\theta}(t)\right] &= 0. \end{aligned} \tag{4}$$

It may be the case that the solutions of (4) are defined only on a proper subset of $\mathbb{R}$. We denote by $\mathcal{O}$ the domain of the solutions r and θ. In the particular case of *rigid motion*, we have $\mathcal{O} = \mathbb{R}$, $r(t) \equiv 1$ and $\theta(t) = \sqrt{\lambda}t + \theta(0)$. In this case the primary bodies perform a periodic motion with minimal period $T := 2\pi/\sqrt{\lambda}$.

Let $x_0(t)$ be the position of the massless particle. According to the Newtonian equations of motion, x_0 satisfies

$$\ddot{x}_0 = \sum_{i=1}^{n} \frac{m_i(x_i - x_0)}{|x_i - x_0|^3} =: f(t, x_0). \tag{5}$$

In the previous equation, we assume that we know the positions of the primaries. Therefore, this equation plus the initial conditions determine the position of the particle completely.

3 Admissible configurations

Henceforth, we denote by L the coordinate z axis.

A necessary and sufficient condition for that L be invariant under the flow associated with the non-autonomous system (5) is $f(t, L) \subset L$ for all $t \in \mathcal{O}$, i.e., L is f-*invariant* for every $t \in \mathcal{O}$. This fact follows by applying (Brezis 1970, Th. 1) to the first-order autonomous system

$$\begin{cases} \frac{\mathrm{d}s}{\mathrm{d}t} &= 1 \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= v \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= f(s, x) \end{cases}$$

which is equivalent to Eq. (5). In addition, the following observations must be taken into account: i) the autonomous vector field $F(s, x, v) = (1, v, f(s, x))$ satisfies $F(\mathcal{O} \times L \times L) \subset \mathcal{O} \times L \times L$ if and only if $f(t, L) \subset L$ for all $t \in \mathcal{O}$ and ii) if $A \subset \mathbb{R}^d$ is a subspace, $x \in A$ and $v \in \mathbb{R}^d$ then $d(x + hv, A)/h \to 0$, when $h \to 0$ if and only if $v \in A$. In the last assertion d denotes the distance function.

Definition 2 We say that a central configuration (q, m) is *admissible* if and only if

1. $q_i \neq 0$, for $i = 1, \ldots, n$.

2. For any $r > 0$, if the set

$$F_r := \{i : |q_i| = r\}$$

is non-empty, then

$$\sum_{i \in F_r} m_i q_i = 0, \tag{6}$$

i.e., every maximal set of bodies which are equidistant from the origin has a center of mass equal to 0.

Remark 1 In the previous definition, we introduced the condition $q_i \neq 0$ in order to avoid collisions between the primaries and the particle.

Theorem 1 *L is f-invariant for every $t \in \mathcal{O}$ if and only if (q, m) is admissible.*

For the proof of the previous theorem we need the following result.

Lemma 1 *For $c > 0$ we define the function $y_c(t) := (c+t)^{-3/2}$. If $0 < t_1 < t_2 < \ldots < t_k$ then the functions $y_j(t) := y_{t_j}(t)$ are linearly independent on each open interval $I \subset \mathbb{R}^+$.*

Proof It is sufficient to prove that the Wronskian

$$W := W(y_1, \ldots, y_k)(t) = \det \begin{pmatrix} y_1 & \cdots & y_k \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} & \cdots & \frac{\mathrm{d}y_k}{\mathrm{d}t} \\ \vdots & \ddots & \vdots \\ \frac{\mathrm{d}^{k-1}y_1}{\mathrm{d}t^{k-1}} & \cdots & \frac{d^{k-1}y_k}{\mathrm{d}t^{k-1}} \end{pmatrix}$$

is not null on I.

Using induction, it is easy to show that

$$\frac{\mathrm{d}^i y_c}{\mathrm{d}t^i} = \beta_i y_c^{\frac{2i+3}{3}}, \quad \text{for some } \beta_i \neq 0, \text{ and for all } i = 1, \ldots. \tag{7}$$

Fix any $t \in I$. Then, according to (7) and writing $\lambda_j := (t + t_j)^{-1}$, we have

$$\begin{aligned} W(t) &= \det \begin{pmatrix} \lambda_1^{3/2} & \lambda_2^{3/2} & \cdots & \lambda_k^{3/2} \\ \beta_1 \lambda_1^{5/2} & \beta_1 \lambda_2^{5/2} & \cdots & \beta_1 \lambda_k^{5/2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k-1}\lambda_1^{k+1/2} & \beta_{k-1}\lambda_2^{k+1/2} & \cdots & \beta_{k-1}\lambda_k^{k+1/2} \end{pmatrix} \\ &= \beta_1 \beta_2 \cdots \beta_{k-1} \lambda_1^{3/2} \lambda_2^{3/2} \cdots \lambda_k^{3/2} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix} \\ &= \beta_1 \beta_2 \cdots \beta_{k-1} \lambda_1^{3/2} \lambda_2^{3/2} \cdots \lambda_k^{3/2} \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i), \end{aligned}$$

where the last equality follows from the well known Vandermonde determinant identity. Therefore, $W \neq 0$ if and only if $\lambda_i \neq \lambda_j$, $i \neq j$, which in turn is equivalent to $t_i \neq t_j$, $i \neq j$. □

Proof (*Proof of Theorem* 1) The condition $f(t, L) \subset L$ for all $t \in \mathcal{O}$ is equivalent to

$$\sum_{i=1}^{n} \frac{m_i r(t) Q(\theta(t)) q_i}{\left(r(t)^2 |q_i|^2 + z^2\right)^{3/2}} = 0 \in \mathbb{R}^2, \tag{8}$$

for every $t \in \mathcal{O}$ and $z \in \mathbb{R}$.

Let $D = \{|q_i| : i = 1, \ldots, n\}$. Suppose that $D = \{s_1, \ldots, s_k\}$, with $s_i \neq s_j$ for $i \neq j$. Therefore $\{1, \ldots, n\} = F_{s_1} \cup \cdots \cup F_{s_k}$. Then, multiplying Eq. (8) by $r(t)^2 Q^{-1}(\theta(t))$ and writing $\zeta = (z/r(t))^2$ we have that (8) is equivalent to

$$\sum_{j=1}^{k} \left\{ \frac{1}{(s_j^2 + \zeta)^{3/2}} \sum_{i \in F_{s_j}} m_i q_i \right\} = 0.$$

According to Lemma 1, the last equation is equivalent to (6). □

4 Admissible configurations for $n \leq 4$

In this section, we find all admissible configurations with $n \leq 4$. Since the center of mass is an excluded position, an admissible configuration satisfies

$$\# F_r \neq 1. \tag{9}$$

It is a trivial fact that the configuration of two point masses m_1 and m_2 is admissible if and only if $m_1 = m_2$.

From (9), a three-body admissible configuration consists of equidistant bodies from the origin. Therefore, it must be the Lagrangian equilateral triangle. Now, by Eq. (6) and an elementary geometrical reasoning, we have $m_1 = m_2 = m_3$.

The case $n = 4$ is more interesting. We include Definition 3 and Theorem 2, which were introduced for the first time in Moeckel (1990), for the reader's convenience.

Definition 3 Let q be a planar configuration. For each pair i, j, the line containing q_i and q_j together with its perpendicular bisector form axes which divide the plane into four quadrants. The union of the first and third quadrants is an hourglass-shaped region which will be called a 'cone'; similarly, the second and fourth quadrants together form another cone. The phrase 'open cone' refers to a cone minus the axes.

Theorem 2 (Perpendicular Bisector Theorem) *Let (q, m) be a planar central configuration and let q_i and q_j be any two of its points. Then if one of the two open cones determined by the line through q_i and q_j and its perpendicular bisector contains points of the configuration, so does the other one.*

Next, we characterize all the four-body admissible configurations.

Theorem 3 *Let (q, m) be a four-body central configuration. Then (q, m) is admissible if and only if $q_i \neq 0$ and for a suitable enumeration of bodies, $q_1 = -q_3$, $q_2 = -q_4$, $m_1 = m_3$, $m_2 = m_4$, and (q, m) is of some of the following mutually exclusive types:*

CCcl. collinear,
CCr. a rhombus with $r_{13} < r_{24}$ and $m_1 > m_2$,
CCs. a square with four equal masses.

Remark 2 In Shoaib and Faye (2011), central configurations of type CCcl were studied, while CCr configurations were treated in Long and Sun (2002) and Perez-Chavela and Santoprete (2007).

Proof From (9) we have to consider two cases.

Case 1 $m_1 \geq m_2$, $|q_1| \neq |q_2|$, $|q_1| = |q_3|$ and $|q_2| = |q_4|$. Now (6) implies that $m_1 = m_3$, $m_2 = m_4$, $q_1 = -q_3$ and $q_2 = -q_4$. We divide the plane into two open cones C_i, $i = 1, 2$, by means of a line $\boldsymbol{P}$ joining q_1 and q_3 together with its perpendicular bisector M. From Theorem 2, if q_2 is in C_1, then q_4 is in C_2, and vice versa. This is a contradiction with the fact that $q_2 = -q_4$. Then $q_2, q_4 \in P$ or $q_2, q_4 \in M$, i.e., q is collinear or a rhombus with equal masses in opposite vertices. In the first case, (q, m) is of CCcl type. In the second case, if $m_1 > m_2$, was proved in Long and Sun (2002, Eqs. (3.44) and (3.45)) that $r_{13} < r_{24}$. Hence (q, m) is of CCr type. From Perez-Chavela and Santoprete (2007, Corollary 2) if $m_1 = m_2$ then the configuration is a square which is a contradiction with the fact that $|q_1| \neq |q_2|$.

Case 2 $|q_1| = |q_2| = |q_3| = |q_4|$. In this situation, in Hampton (2005) it was proved that the configuration is the equal mass square. □

5 Massless particle motion

In this section and in Sect. 6, we will suppose that the primary bodies are in a T-periodic rigid motion associated with an admissible CC (q, m), i.e $r(t) \equiv 1$ and according to the remark that follows Eq. (4), $\theta(t) = \sqrt{\lambda} t$ (w.l.o.g we assume that $\theta(0) = 0$). As to the particle, we suppose that it is moving on L, i.e., $x_0(t) = (0, 0, z(t))$. From Theorem 1, x_0 is a solution of (5), if and only if $z(t)$ is a solution of the autonomous equation

$$\ddot{z} = -\sum_{i=1}^{n} \frac{m_i z}{(s_i^2 + z^2)^{3/2}}, \tag{10}$$

where $s_i = |q_i|$.

We will analyze all possible motions for the massless particle x_0. In particular, we shall see that every motion is either periodic or an escape trajectory. We shall find that there exist T_0-periodic solutions for all T_0 in an interval $(\sigma(q, m), +\infty)$. This fact implies that there exists an infinite quantity of periodic solutions for the entire $n + 1$-body system.

The second-order Eq. (10) is conservative, and therefore its solutions conserve the energy

$$E(z, v) := \frac{|v|^2}{2} - \sum_{i=1}^{n} \frac{m_i}{\left(s_i^2 + z^2\right)^{\frac{1}{2}}}, \tag{11}$$

i.e., $E(z(t), \dot{z}(t))$ is constant.

Following Arnold et al. (2007) (see also Marchesin and Vidal 2013), we introduce the next concepts.

Definition 4 (Chazy 1922) Let $z(t)$ be a solution of (10) such that $\lim_{t\to\infty} z(t) = \infty$. Then $z(t)$ is called:

– hyperbolic when there exists $\lim_{t\to\infty} \dot{z}(t)$ and it is not null,
– parabolic if $\lim_{t\to\infty} \dot{z}(t) = 0$.

The following theorem characterizes all the possible motions for the massless particle.

Theorem 4 *We assume that* (q, m) *is an admissible configuration and the primaries are in a rigid motion. Every solution of* (10) *is of some of the following types:*

1. *Hyperbolic, when* $E > 0$,
2. *Parabolic, when* $E = 0$,
3. *Periodic, when* $E_{\min} := -\sum_{i=1}^{n} \frac{m_i}{s_i} < E < 0$,
4. *Equilibrium solution when* $E = E_{\min}$.

Proof We follow a standard argument for Hamiltonian systems (see Arnold 1989).

We consider the level sets $S(E) = \{(z, v) : E(z, v) = E\}$ on the phase space (z, v). An elementary analysis shows that

- If $E \geq 0$ then $S(E)$ is the union of two bounded graphs. They are symmetric with respect to the z-axis, each of which is contained in some semiplane $v > 0$ or $v < 0$. The v-positive branch is the graph of a function $v(E, z)$, which is decreasing with respect to $|z|$. Moreover, $\lim_{|z|\to\infty} v(E, z) = \sqrt{2E}$.
- For every $E \geq E_{\min}$, the energy curve $S(E)$ cuts the v-axis at the value $\pm(2E + 2\sum_{i=1}^{n} m_i s_i^{-1})^{\frac{1}{2}}$.
- If $E_{\min} < E < 0$ then $S(E)$ is a simple closed curve, symmetric with respect to the z and v axes.
- An energy curve cuts the z-axis, only in the case that $E < 0$, at the point $\pm z_E$, where z_E is the only positive solution of $-\sum_{i=1}^{n} m_i (s_i^2 + z_E^2)^{-\frac{1}{2}} = E$.

In Fig. 1, we show the phase portrait for a rhomboidal configuration with masses $m_1 = m_3 = 1$ and $m_2 = m_4 = 0.5$.

The function $\varphi(t) = (z(t), \dot{z}(t))$ solves the system $\dot{\varphi}(t) = F(\varphi(t))$, where $F(z, v) = (v, -\sum_{i=1}^{n} m_i z(s_i^2 + z^2)^{-3/2})$. It is easy to show that the vector field F has a bounded Jacobian DF. Therefore $F(z, v)$ is a global Lipschitz function on $\mathbb{R}^2$. This fact and Betounes (2009, Th. B.1) imply that the trajectories $t \mapsto (z(t), \dot{z}(t))$ are defined for every time. On the other hand, since $\dot{z} = v$, the motion along trajectories is in clockwise direction. The only fixed point of F is $(z, v) = (0, 0)$. Therefore, the level surfaces $S(E)$, with $E \neq E_{\min}$, do not contain stationary points. Then the $\lim_{t\to\infty} \varphi(t)$ does not exist. As a consequence, the map $t \mapsto \varphi(t)$ fills completely one connected component of its corresponding energy curve (Fig. 2).

We observe that any solution z crosses the v-axis. On the other hand, if $E \geq 0$ and $v(E, 0) > 0$ ($v(E, 0) < 0$) then $z(t)$ is increasing (decreasing) with respect to t. If $z(t)$ remained bounded when $t \to +\infty$, then there would be the limit $\zeta_\infty := \lim_{t\to\infty} z(t)$. This would imply that $(\zeta_\infty, 0)$ would be a fixed point of F, which is a contradiction. As a consequence, if $E \geq 0$ then $|z(t)| \to \infty$ when $t \to +\infty$. Moreover $\lim_{t\to+\infty} \dot{z}(t) = \pm\sqrt{2E}$. From this fact, we conclude that the trajectory is hyperbolic when $E > 0$ and it is parabolic in the case when $E = 0$.

In the case when $E_{\min} < E < 0$, we have that the trajectory is contained in a closed curve; therefore, it is a periodic orbit.

Finally, if $E = E_{\min}$ we clearly have that $z(t) \equiv 0$. □

Theorem 5 *We denote by* $T_0(E)$ *the minimal period for a solution of* (10) *with* $E_{\min} < E < 0$. *Then*

1.

$$T_0(E) = 2^{3/2} \int_0^{z_E} \left(E + \sum_{i=1}^{n} m_i (s_i^2 + z^2)^{-\frac{1}{2}} \right)^{-\frac{1}{2}} \mathrm{d}z, \tag{12}$$

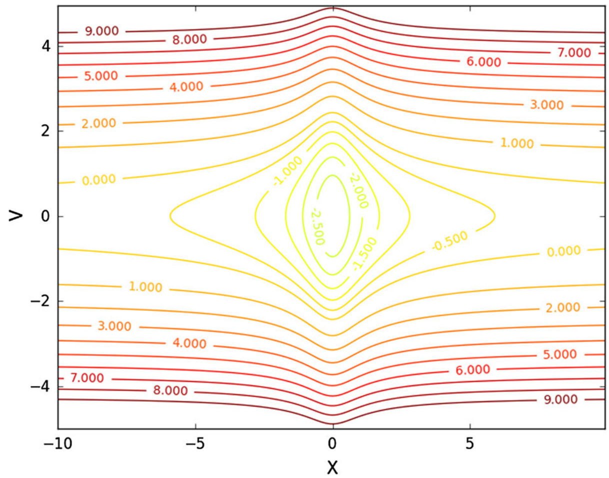

Fig. 2 Energy level for a rhomboidal configuration with masses $m_1 = m_3 = 1$ and $m_2 = m_4 = 0.5$

where z_E is the only positive solution of $-\sum_{i=1}^n m_i(s_i^2 + z_E^2)^{-\frac{1}{2}} = E$,

2. $T_0(E)$ *is an increasing function.*
3. $T_0\left((E_{\min}, 0)\right) = (T_{\min}, +\infty)$, *where* $T_{\min} = 2\pi \left(\sum_{i=1}^n \frac{m_i}{s_i^3}\right)^{-1/2}$.

Proof Let $E_{\min} < E < 0$ and let $z(t)$ be the only solution with $z(0) = 0$, $\dot{z}(0) > 0$ and energy equals to E. Therefore $z(t)$ is $T_0(E)$-periodic. As a consequence of the symmetries of the equation, we have that $z(T_0(E)/4) = z_E$. Then, taking account of (11), we have

$$\begin{aligned}\frac{T_0}{4} &= \frac{1}{\sqrt{2}} \int_0^{T_0/4} \left(E + \sum_{i=1}^n m_i(s_i^2 + z^2)^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \dot{z}\mathrm{d}t \\ &= \frac{1}{\sqrt{2}} \int_0^{z_E} \left(E + \sum_{i=1}^n m_i(s_i^2 + z^2)^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \mathrm{d}z,\end{aligned}$$

and we have proved item *1*. In order to prove item *2*, we note that

$$\begin{aligned}2^{-3/2} T_0(E) &= \int_0^{z_E} \left(\sum_{i=1}^n m_i \left((s_i^2 + z^2)^{-\frac{1}{2}} - (s_i^2 + z_E^2)^{-\frac{1}{2}}\right)\right)^{-\frac{1}{2}} \mathrm{d}z \\ &= \int_0^{z_E} \left(z_E^2 - z^2\right)^{-\frac{1}{2}} f(z, z_E)\mathrm{d}z \\ &= \int_0^1 \left(1 - u^2\right)^{-\frac{1}{2}} f(z_E u, z_E)\mathrm{d}u,\end{aligned}$$

where

$$f(z, z_E) = \left(\sum_{i=1}^n m_i \left\{(s_i^2 + z^2)(s_i^2 + z_E^2)\right\}^{-\frac{1}{2}} \left\{(s_i^2 + z^2)^{\frac{1}{2}} + (s_i^2 + z_E^2)^{\frac{1}{2}}\right\}^{-1}\right)^{-\frac{1}{2}}.$$

We point out that $f(z_E u, z_E)$ is an increasing function with respect to z_E for any fixed $u \in [0, 1]$. This assertion implies item 2.

On the other hand,

$$\lim_{z_E \to 0} f(z_E u, z_E) = \left(\sum_{i=1}^{n} \frac{m_i}{2s_i^3} \right)^{-\frac{1}{2}} \quad \text{and} \quad \lim_{z_E \to +\infty} f(z_E u, z_E) = +\infty.$$

Thus, from the dominated convergence theorem and monotone convergence theorem, we have

$$\lim_{E \to E_{\min}} T_0 = \lim_{z_E \to 0} T_0 = 2\pi \left(\sum_{i=1}^{n} \frac{m_i}{s_i^3} \right)^{-\frac{1}{2}} \quad \text{and} \quad \lim_{E \to 0} T_0 = \lim_{z_E \to +\infty} T_0 = +\infty.$$

Finally, since $T_0 = T_0(z_E)$ is continuous and increasing with respect to z_E, we conclude the statement of item 3. □

Remark 3 It is possible to use the classical theory of Hamiltonian systems (see Arnold 1989) to derive the formula (12) (see Acinas et al. 2014 for this approach in a related problem).

Remark 4 Let us show a second proof of item 3 of Theorem 5.

The inequality $T_0 > T_{\min}$ is a consequence of a comparison of Sturm's theorem applied to equations $\ddot{z} + h(z)z = 0$, where $h(z) = \sum_{i=1}^{n} m_i \left(s_i^2 + z^2\right)^{-3/2}$, and $\ddot{z} + \left(\sum_{i=1}^{n} m_i s_i^{-3}\right) z = 0$. This proves that $T_0\left((E_{\min}, 0)\right) \subset (T_{\min}, +\infty)$.

For the reverse inclusion, we follow the arguments of Zhao and Zhang (2015) and Li et al. (2013) based on variational principles.

Let $T_0 > T_{\min}$. We consider the action integral

$$\mathcal{I}(z) = \int_0^{T_0} \frac{1}{2}|\dot{z}|^2 + \sum_{i=1}^{n} \frac{m_i}{\sqrt{s_i^2 + z^2}} \mathrm{d}t.$$

Then T_0-periodic solutions of (10) are critical points of $\mathcal{I}$ in the space $H^1(\mathbb{T}, \mathbb{R})$ of the functions which are absolutely continuous, T_0-periodic with $\dot{z} \in L^2(\mathbb{T}, \mathbb{R})$ and being $\mathbb{T} = \mathbb{R}/T_0\mathbb{Z}$ (see Mawhin and Willem 1989, Cor. 1.1). We prove the existence of critical points by means of the direct method of calculus of variations, i.e., we will prove that $\mathcal{I}$ has a minimum. The functional $\mathcal{I}$ is not coercive in $H^1(\mathbb{T}, \mathbb{R})$. This deficiency is overcome with symmetry techniques (see Ferrario and Terracini 2004). The group $\mathbb{Z}_2$ acts on $H^1(\mathbb{T}, \mathbb{R})$ according to the following assignments $(\bar{0} \cdot z)(t) = z(t)$ and $(\bar{1} \cdot z)(t) = -z(t + \frac{T_0}{2})$. The symmetry involved in the previous definition is called *Italian Symmetry* (see Meyer et al. 2009, p. 327). The functional $\mathcal{I}$ is $\mathbb{Z}_2$-invariant, i.e., $\mathcal{I}(g \cdot z) = \mathcal{I}(z)$. We define the space of all $\mathbb{Z}_2$-symmetric functions

$$\Lambda(\mathbb{T}, \mathbb{R}) := \left\{ z \in H^1(\mathbb{T}, \mathbb{R}) | \forall g \in \mathbb{Z}_2 : z = g \cdot z \right\}.$$

The functional $\mathcal{I}$ restricted to Λ is coercive. This fact follows from an obvious adaptation of Proposition 4.1 of Ferrario and Terracini (2004). We note that $F(z) := \sum_{i=1}^{n} m_i (s_i^2 + z^2)^{-\frac{1}{2}}$ satisfies the condition (A) in Mawhin and Willem (1989), p. 12, then $\mathcal{I}$ is continuously differentiable and weakly lower semicontinuous on $H^1(\mathbb{T}, \mathbb{R})$ (see Mawhin and Willem 1989, p. 13). Therefore $\mathcal{I}$ has a minimum z_0 in $\Lambda(\mathbb{T}, \mathbb{R})$. Then by the Palais' principle of symmetric criticality, z_0 is a critical point of $\mathcal{I}$ in $H^1(\mathbb{T}, \mathbb{R})$ (see Ferrario and Terracini 2004; Palais 1979).

We use the second variation $\delta^2\mathcal{I}$ in order to show that $z_0 \not\equiv 0$. It is well known (see Jost and Li-Jost 1998, Th. 1.3.1) that if z_0 is a minimum of $\mathcal{I}$ on $H^1(\mathbb{T},\mathbb{R})$ then $\delta^2\mathcal{I}(z_0,\varphi) \geq 0$ for all $\varphi \in H^1(\mathbb{T},\mathbb{R})$. In our case,

$$\delta^2\mathcal{I}(0,\varphi) = \int_0^{T_0} |\dot{\varphi}|^2 - \sum_{i=1}^{n} \frac{m_i}{s_i^3}\varphi^2 \mathrm{d}t$$

(see Jost and Li-Jost 1998, Eq. 1.3.6). In particular, if $\varphi(t) = \sin(2\pi t/T_0)$ it follows from $T_0 > T_{\min}$ that

$$\delta^2\mathcal{I}(0,\varphi) = \left(\frac{4\pi^2}{T_0^2} - \sum_{i=1}^{n} \frac{m_i}{s_i^3}\right)\frac{T_0}{2} < 0. \tag{13}$$

It is sufficient to guarantee that $z_0 \equiv 0$ is not a minimum. $\square$

This second proof, unlike the first one, does not prove that T_0 is the minimum period of z_0. It could happen that z_0 had period T_0/m, with natural $m \in \mathbb{N}$. Because of the Italian symmetry this m should be odd.

Corollary 1 *The complete $n+1$-body system has an infinite quantity of periodic solutions.*

Proof We recall that T denotes the minimal period of the primaries. Let l/m be a positive rational number with $Tl/m > T_{\min}$. Then, there exists a solution of the entire system with period lT. $\square$

6 Synchronous solutions and pyramidal CC

If Eq. (10) has a T-periodic solution, we say that the solution is *synchronous*. In Li et al. (2013) the problem of existence of synchronous solutions for n equal mass primary bodies in a regular polygon configuration was studied.

In this section we establish a relation between the existence of synchronous solutions and the concept of pyramidal central configuration (see Fayçal 1996, 1995; Ouyang et al. 2004).

Definition 5 A central configuration of $n+1$ mass point $q_0,\ldots,q_n$ in $\mathbb{R}^3$ is called a pyramidal central configuration (PCC) if and only if n points, we say $q_1,\ldots,q_n$, are in some plane Π and $q_0 \notin \Pi$.

The following lemma was proved in Ouyang et al. (2004) (see also Fayçal 1995).

Lemma 2 (Ouyang et al. 2004, Lemma 2.1) *Let $q_0,\ldots,q_n$ be a PCC such that m_0 is off the plane containing $m_1,\ldots,m_n$. If $m_0 > 0$ then m_0 is equidistant from $m_1,\ldots,m_n$.*

We remark that the condition $m_0 > 0$ is important in the previous lemma. In the examples below, we will show two PCC with $m_0 = 0$ which do not satisfy the conclusion of Lemma 2.

Proposition 1 *We assume that $q = q_1,\ldots,q_n$ is an admissible configuration and that the primaries are in a rigid motion. Then, there is a synchronous solution if and only if there exists $c \in \mathbb{R}$ such that the points $(0,0,c), q_1,\ldots,q_n$ associated with the masses $0, m_1,\ldots,m_n$ form a PCC.*

Proof We start assuming that there exist a synchronous solution. As a consequence of Theorem 5(3) and the fact that $T^2 = 4\pi^2/\lambda$, we get

$$\lambda < \sum_{i=1}^{n} \frac{m_i}{s_i^3}. \tag{14}$$

Since $\sum_{i=1}^{n} m_i \left(s_i^2 + c^2\right)^{-3/2} \to 0$, when $c \to +\infty$, there exists $c \in \mathbb{R}$ such that $\sum_{i=1}^{n} m_i \left(s_i^2 + c^2\right)^{-3/2} = \lambda$. Therefore

$$-\sum_{i=1}^{n} \frac{m_i c}{\left(s_i^2 + c^2\right)^{3/2}} = -\lambda c. \tag{15}$$

As $q_1, \ldots, q_n$ is an admissible configuration, then

$$\sum_{i=1}^{n} \frac{m_i q_i}{\left(s_i^2 + c^2\right)^{3/2}} = (0, 0). \tag{16}$$

Equations (15), (16) and the fact that $q_1, \ldots, q_n$ is a CC with constant λ, complete the proof. The proof of the reciprocal statement follows in a direct way. □

Corollary 2 *We assume that* (q, m) *is an admissible configuration and the primaries are in a rigid motion. Then, there is a synchronous solution if and only if*

$$\sum_{i<j} \frac{m_i m_j}{r_{ij}} < \left(\sum_{i=1}^{n} \frac{m_i}{s_i^3}\right)\left(\sum_{i=1}^{n} m_i s_i^2\right). \tag{17}$$

Proof The result is a consequence of (14) and the fact that $T^2 = 4\pi^2 \sum_{i=1}^{n} m_i s_i^2 / U$ (see Llibre et al. 2015, p. 109). □

Remark 5 Let (q, m) be an admissible CC with constant $\lambda > 0$ satisfying (17) and let r, μ be positive numbers. Then $(rq, \mu m)$ is a CC with constant $\lambda\mu r^3$, and (17) remains unchanged. In virtue of the previous observation, we can assume that any length and any mass take any desired value. Equation (10) has a synchronous solution if and only if the same equation with $(rq, \mu m)$ instead of (q, m) has a synchronous solution.

The sufficiency of the condition $n \leq 472$ in the following corollary was proved in Li et al. (2013).

Corollary 3 *We suppose that* (q, m) *is the equal masses regular polygon configuration (this is an admissible CC). Then, there exists a synchronous solution if and only if* $2 \leq n \leq 472$.

Proof In this case $s_1 = s_2 = \cdots = s_n =: r$ and $m_1 = m_2 = \cdots = m_n =: M$. Then, from the law of cosines, we obtain

$$\sum_{i<j} \frac{m_i m_j}{r_{ij}} = \frac{nM^2}{4r} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)}.$$

Therefore, the condition (17) is equivalent to

$$\frac{1}{n} \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{j\pi}{n}\right)} < 4. \tag{18}$$

This inequality was also derived by Li et al. (2013), where the authors proved (performing computer calculations) that inequality (18) holds true for $2 \leq n \leq 472$. Let us prove that any other n does not satisfy (18).

Using that $1/\sin(x)$ is a convex function on $[0, \pi]$ and the composite trapezoid rule (see Kincaid and Cheney 1991), we have

$$\int_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi} \frac{1}{\sin(x)}\mathrm{d}x \leq \frac{\pi}{2n}\left\{\frac{1}{\sin\left(\frac{\pi}{n}\right)} + \frac{1}{\sin\left(\frac{n-1}{n}\pi\right)} + 2\sum_{j=2}^{n-2}\frac{1}{\sin\left(j\frac{\pi}{n}\right)}\right\}$$
$$= \frac{\pi}{n}\sum_{j=1}^{n-2}\frac{1}{\sin\left(j\frac{\pi}{n}\right)}.$$

Hence

$$\frac{1}{n}\sum_{j=1}^{n-1}\frac{1}{\sin\left(\frac{j\pi}{n}\right)} \geq \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\frac{\pi(n-1)}{n}}\frac{1}{\sin(x)}\mathrm{d}x + \frac{1}{n\sin\left(\frac{n-1}{n}\pi\right)}$$
$$= \frac{1}{2\pi}\log\left(\frac{1-\cos(x)}{1+\cos(x)}\right)\Bigg|_{\frac{\pi}{n}}^{\frac{n-1}{n}\pi} + \frac{1}{n\sin\left(\frac{\pi}{n}\right)}$$
$$= \frac{1}{\pi}\left\{\log\left(\frac{1+\cos(\frac{\pi}{n})}{1-\cos(\frac{\pi}{n})}\right) + \frac{\pi/n}{\sin\left(\frac{\pi}{n}\right)}\right\}$$
$$=: f\left(\frac{\pi}{n}\right).$$

It is easy to see that $f(x)$ is a decreasing function on $(0, \pi/2)$. Moreover $f(\pi/842) \approx 4.0006 > 4$. Thus, if $n \geq 842$ then n does not satisfy inequality (18). The validity of the inequality (18), for $n \leq 841$ can be easily checked using computer. This gives the result that the inequality holds only for $n \leq 472$. $\square$

Our next goal is to verify that condition (17) is satisfied for all admissible CC of three-body or four-body. Since (17) holds for an equilateral triangle and square configurations of equal masses bodies, it only rests to prove, in virtue of Theorem 3, the following result.

Theorem 6 *The central configurations CCcl and CCr satisfy condition* (17).

Proof Let us start by analyzing the central configuration CCr. From Remark 5, we can suppose without loss of generality that $q_1 = -q_3 = (0, y)$ for $0 < y < 1$, $q_2 = -q_4 = (1, 0)$. The condition (17) becomes

$$\frac{m_1^2}{2y} + \frac{4m_1m_2}{\sqrt{1+y^2}} + \frac{m_2^2}{2} < \left(\frac{2m_1}{y^3} + 2m_2\right)\left(2m_1y^2 + 2m_2\right).$$

As $m_1^2/(2y) < 4m_1^2/y$, $m_2^2/2 < 4m_2^2$ and $4m_1m_2/\sqrt{1+y^2} < 4m_1m_2/y^3$ (since $y < 1$), we have that the inequality holds.

Now we consider the central configuration CCl. From Remark 5 again, we can suppose that $q_1 = -q_3 = 1$, $q_2 = -q_4 = x$ with $0 < x < 1$, and $m_1 = m_3 = \mu$, $m_2 = m_4 = 1 - \mu$, with $0 < \mu < 1$. Then, inequality (17) becomes

$$\frac{2\mu(1-\mu)}{1-x}+\frac{2\mu(1-\mu)}{1+x}+\frac{\mu^2}{2}+\frac{(1-\mu)^2}{2x}<4\mu^2+4\mu(1-\mu)x^2$$
$$+\frac{4\mu(1-\mu)}{x^3}+\frac{4(1-\mu)^2}{x}.$$

As $\mu^2/2 < 4\mu^2$ and $(1-\mu)^2/(2x) < 4(1-\mu)^2/x$, it is sufficient to show that

$$\frac{2\mu(1-\mu)}{1-x}+\frac{2\mu(1-\mu)}{1+x}<\frac{4\mu(1-\mu)}{x^3},$$

and, this is equivalent to see that

$$\frac{x^3}{1-x^2}<1. \tag{19}$$

The values of x involved in the inequality above are such that the configuration of positions $(-1, -x, x, 1)$ and masses $(\mu, 1-\mu, 1-\mu, \mu)$ is central. It was shown in Moulton (1910) that given a mass μ there is only one value of x satisfying this condition (see also Shoaib and Faye 2011). Consequently, we can define $x(\mu)$ as such value of x. We note that $h(x) = x^3/(1-x^2)$ is an increasing function with respect to $x \in (0, 1)$ and $h(x) < 1$ for $x \in (0, 3/4)$. Hence, if we could prove that $x(\mu)$ is a decreasing function and

$$\lim_{\mu\to 0} x(\mu) < 3/4, \tag{20}$$

we would have justified (19).

Let us first prove that $x(\mu)$ is a decreasing function. Eliminating λ from Eqs. (2) and replacing q_j and m_j by their expressions in x and μ, we get

$$\frac{\mu}{4}-\frac{\mu}{x\,(x+1)^2}+\frac{\mu}{x\,(-x+1)^2}+\frac{-\mu+1}{(x+1)^2}+\frac{-\mu+1}{(-x+1)^2}-\frac{1}{x^3}\left(-\frac{\mu}{4}+\frac{1}{4}\right)=0,$$

which is equivalent to

$$\mu=-\frac{8x^5-x^4+8x^3+2x^2-1}{(x-1)\,(x+1)\left(x^5-9x^3+x^2-1\right)}.$$

Therefore

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}=\frac{x^2\left(16x^9-3x^8+32x^7+12x^6-304x^5-2x^4+44x^2-51\right)}{(x-1)^2\,(x+1)^2\left(x^5-9x^3+x^2-1\right)^2}.$$

Since $44x^2 < 51$ and $16x^9 + 32x^7 + 12x^6 < 304x^5$ for $x \in (0, 1)$, then $\mathrm{d}\mu/\mathrm{d}x < 0$ on the interval $(0, 1)$. Which, in turn, implies that x is decreasing with respect to μ.

Let us see now that (20) holds. When μ goes to 0, $x(\mu)$ converges to the only solution on the interval $(0, 1)$ of equation $8x(0)^5-x(0)^4+8x(0)^3+2x(0)^2-1 = 0$. Then, $8x(0)^3-1 < 0$ which implies that $x(0) < 3/4$ as we wanted to prove. □

Remark 6 As a consequence of the previous results, there exist five-body $PCC's$ with $m_1, \ldots, m_4$ in a CCcl or CCr configuration and the mass $m_0 = 0$ is in the perpendicular line to the plane containing $m_1, \ldots, m_4$ and passing by the center of mass. These are examples of $PCC's$ which do not verify the conclusion of Lemma 2.

Corollary 4 *For all admissible CC of three-body or four-body, the problem P has a synchronous solution.*

7 Non-admissible central configurations

The following result shows when a non-admissible CC has a solution of the problem P.

Theorem 7 *We suppose that (q, m) is a non-admissible CC with $q_i \neq 0$ and that the primaries are in a homographic motion, i.e., Eq. (1) is satisfied. Assume that the massless particle is moving on the z-axis with position vector $x_0(t) = (0, 0, z(t))$. Then, one and only one of the following statements is satisfied:*

1. *The massless particle is in a stationary motion and*

$$\sum_{i=1}^{n} \frac{m_i q_i}{s_i^3} = 0, \tag{21}$$

i.e., the positions $0, q_1, \ldots, q_n$ and the masses $0, m_1, \ldots, m_n$ are in a CC.

2. *The $n+1$-body system is in a homothetic motion, i.e., $Q(\theta(t))$ in Eq. (1) is the identity matrix and $z(t) = cr(t)$, for some constant c. Moreover, the configuration $q_0, \ldots, q_n$ is a PCC, where $q_0 = (0, 0, c)$ and $m_0 = 0$.*

Proof We recall the definition of the function f and line L from Sect. 3.

The fact that the massless particle is moving on L is equivalent to the condition $f(t, x_0(t)) \in L$ for all $t \in \mathcal{O}$, which is equivalent to the equality

$$\sum_{i=1}^{n} \frac{m_i r(t) Q(\theta(t)) q_i}{\left(r(t)^2 |q_i|^2 + z(t)^2\right)^{3/2}} = 0, \tag{22}$$

for every $t \in \mathcal{O}$.

With the same notation and reasoning as in the proof of Theorem 1, we prove that

$$\sum_{j=1}^{k} \left\{ \frac{1}{(s_j^2 + (z(t)/r(t))^2)^{3/2}} \sum_{i \in F_j} m_i q_i \right\} = 0. \tag{23}$$

If $z(t)/r(t)$ would be a non-constant function then the previous equation and Lemma 1 would imply that q is admissible, which is a contradiction. Hence, there exists $c \in \mathbb{R}$ such that $z(t) = cr(t)$. Now, we have two cases.

Case 1 $c = 0$. Then $z \equiv 0$ and (21) follows from (22).

Case 2 $c \neq 0$. From Eq. (10), the Kepler equations (4) and the fact that $z(t) = cr(t)$, we have

$$-\frac{1}{r(t)^2} \sum_{i=1}^{n} \frac{m_i}{(s_i^2 + c^2)^{3/2}} = -\frac{\lambda}{r(t)^2} + r(t)\dot{\theta}(t)^2. \tag{24}$$

The second equality in (4) implies Kepler's second law, i.e., there exists $d \in \mathbb{R}$ such that $r^2\dot{\theta} \equiv d$. Replacing $\dot{\theta}$ in Eq. (24) and multiplying by $r(t)^3$, we obtain

$$-r(t) \left(\sum_{i=1}^{n} \frac{m_i}{(s_i^2 + c^2)^{3/2}} - \lambda \right) = d^2. \tag{25}$$

Therefore, if $d \neq 0$ then $\dot{r}(t) \equiv 0$, and this implies $\dot{z}(t) \equiv 0$. As $z(t)$ is a constant function and it solves Eq. (10), then $z(t) \equiv 0$. Hence we are in case 1 again. Consequently we suppose $d = 0$. Therefore $\theta(t)$ is a constant function and the motion is homothetic. From (23) and (25), we deduce that in this new situation Eqs. (15) and (16) hold. This fact, as in the proof of Proposition 1, implies the desired result. □

Example 1 We present an example of a $3+1$-body system satisfying the situation described in item 1 of Theorem 7, i.e., (q, m) is a non-admissible CC and $z(t) \equiv 0$. For this purpose, it is sufficient to find a four-body CC with a zero mass body located in the center of mass.

We start with an Euler's collinear central configuration formed by three primary bodies of masses $m_1 = 4-\mu$, $m_2 = 2+\mu$ and $m_3 = 1$, where $0 < \mu < 1$, and positions, with respect to a convenient 1-dimensional coordinate system, given by $q_1 = 0$, $q_2 = 1$ and $q_3 = 1+r$. It is known (see Moeckel 2014) that r is the only positive solution of

$$p(r, \mu) := 6r^5 + (16-\mu)\, r^4 + (14-2\mu)\, r^3 - (\mu+5)\, r^2 - (2\mu+7)\, r - \mu - 3 = 0.$$

Since $p(0, \mu) = -\mu-3$ and $p(1, \mu) = -7\mu+21$, then $r = r(\mu) \in (0, 1)$, for all $0 < \mu < 1$.

Therefore, as the center of mass $C = C(\mu)$ is equal to $(\mu+r+3)/7$, we obtain $C \in (0, 1)$.

We consider a massless particle with coordinate x. The acceleration resulting from the action of the gravitational field is equal to

$$f(x) = -\frac{4-\mu}{x^2} + \frac{\mu+2}{(-x+1)^2} + \frac{1}{(r-x+1)^2}.$$

Note that the right-hand side of the previous equation is an increasing function that tends to $-\infty$ when x goes to 0, and tends to $+\infty$ when x goes to 1, so there is a unique point $\bar{x} = \bar{x}(\mu) \in (0, 1)$ such that the equality $f(\bar{x}) = 0$ holds. This point is an equilibrium for the gravitational field generated for the primaries.

Let us see that there exists $\mu \in (0, 1)$ such that $C(\mu) = \bar{x}$, i.e., $f(C) = 0$. For this purpose, since C is a continuous function with respect to μ, it is sufficient to show that f changes its sign on $(0, 1)$. The function $f(x)$ can be written as

$$f(x) = \frac{g(x)}{h(x)},$$

where $h(x) = x^2\,(x-1)^2\,(r-x+1)^2$. Note that $h(x) > 0$ for all $x \in (0, 1)$. If we consider $\mu = 0$ and compute $g(C)$, we have

$$g(C) = \frac{r^4}{2401} + \frac{1514r^3}{2401} + \frac{2245r^2}{2401} + \frac{1110r}{2401} + \frac{333}{2401} > 0.$$

On the other hand, if $\mu = 1$ then

$$g(C) = -\frac{71r^4}{2401} + \frac{1486r^3}{2401} + \frac{401r^2}{2401} - \frac{1480r}{2401} - \frac{592}{2401} < 0,$$

because $0 < r < 1$.

Remark 7 The following question is posed. Is there some non-admissible central configuration (q, m) such that the $n+1$-body system perform the motion described in Theorem 7(2)?

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