

## A quantum logical and geometrical approach to the study of improper mixtures

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We study improper mixtures from a quantum logical and geometrical point of view. Taking into account the fact that improper mixtures do not admit an ignorance interpretation and must be considered as states in their own right, we do not follow the standard approach which considers improper mixtures as measures over the algebra of projections. Instead of it, we use the convex set of states in order to construct a new lattice whose atoms are all physical states: pure states and improper mixtures. This is done in order to overcome one of the problems which appear in the standard quantum logical formalism, namely, that for a subsystem of a larger system in an entangled state, the conjunction of all actual properties of the subsystem does not yield its actual state. In fact, its state is an improper mixture and cannot be represented in the von Neumann lattice as a minimal property which determines all other properties as is the case for pure states or classical systems. The new lattice also contains all propositions of the von Neumann lattice. We argue that this extension expresses in an algebraic form the fact that—alike the classical case—quantum interactions produce nontrivial correlations between the systems. Finally, we study the maps which can be defined between the extended lattice of a compound system and the lattices of its subsystems. © 2010 American Institute of Physics. [doi:10.1063/1.3429619]

### I. INTRODUCTION

Nonseparability of the states of quantum systems is considered with continuously growing interest in relation to quantum information theory. In fact, today entanglement is regarded not only as a feature that gives rise to interesting foundational questions. It is considered also as a powerful resource for quantum information processing. In this paper we pose the problem of studying nonseparability with algebraic and geometrical tools related to quantum logic (QL).

The algebraic approach to the formalization of quantum mechanics was initiated by Birkhoff and von Neumann,<sup>1</sup> who gave it the name of “quantum logic.” Although an algebraic structure, for historical reasons it has conserved its name. QL was developed mainly by Mackey,<sup>2</sup> Jauch,<sup>3</sup> Piron,<sup>4</sup> Kalmbach,<sup>5,6</sup> Varadarajan,<sup>7,8</sup> Greechie,<sup>9</sup> Gudder,<sup>10</sup> Giuntini,<sup>11</sup> Pták and Pulmannova,<sup>12</sup> Beltrametti and Cassinelli,<sup>13</sup> among others. For a complete bibliography, see, for example, Refs. 14 and 15. The Geneva school of QL extended this research to analysis of compound systems. The first results were obtained by Aerts and Daubechies<sup>16,17</sup> and Randall and Foulis.<sup>18</sup>

In the tradition of the quantum logical research, a property of (or a proposition about) a quantum system is related to a closed subspace of the Hilbert space  $\mathcal{H}$  of its (pure) states or, analogously, to the projector operator onto that subspace. Moreover, each projector is associated with a dichotomic question about the actuality of the property (Ref. 19, p. 247). A physical

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magnitude  $\mathcal{M}$  is represented by an operator  $M$  acting over the state space. For bounded self-adjoint operators, conditions for the existence of the spectral decomposition  $M = \sum_i a_i P_i = \sum_i a_i |a_i\rangle\langle a_i|$  are satisfied (along this work we will restrict the study to the finite dimensional case). The real numbers  $a_i$  are related to the outcomes of measurements of the magnitude  $\mathcal{M}$  and projectors  $|a_i\rangle\langle a_i|$  to the mentioned properties. The physical properties of the system are organized in the lattice of closed subspaces  $\mathcal{L}(\mathcal{H})$  that, for the finite dimensional case, is a modular lattice.<sup>20</sup> In this frame, the pure state of the system is represented by the meet (i.e., the lattice infimum) of all actual properties, more on this below. A comprehensive description of QL in present terminology may be found in Ref. 21.

Mixed states represented by density operators had a secondary role in the classical treatise by von Neumann because they did not add new conceptual features to pure states. In fact, in his book, mixtures meant “statistical mixtures” of pure states (Ref. 19, p. 328), which are known in the literature as “proper mixtures” (Ref. 22, Chap. 6). They usually represent the states of realistic physical systems whose preparation is not well described by pure states.

Today we know that the restriction to pure states and their mixtures is unduly because there are also “improper mixtures” and they do not admit an ignorance interpretation.<sup>22–27</sup> This fact is an expression of one of the main features of quantum systems, namely, nonseparability. Improper mixtures are now considered as states on their own right, and they appear, for example, in processes such as measurements on some degrees of freedom of the system, and also when considering one system in a set of interacting systems. In fact, in each (nontrivial) case in which a part of the system is considered, we have to deal with improper mixtures. [Also for statistical mixtures the ignorance interpretation becomes untenable in cases of nonunique decomposability of the density operator (Ref. 13, Chap. 2).]

In the standard formulation of QL, mixtures as well as pure states are included as measures over the lattice of projections (Ref. 28, Chap. 3), that is, a state  $s$  is a function,

$$s: \mathcal{L}(\mathcal{H}) \rightarrow [0; 1],$$

such that,

- (1)  $s(\mathbf{0})=0$  ( $\mathbf{0}$  is the null subspace);
- (2) for any pairwise orthogonal family of projections  $P_j$ ,  $s(\sum_j P_j) = \sum_j s(P_j)$ .

In a similar way, in classical mechanics statistical distributions are represented as measures over the phase space. But while pure states can be put in a bijective correspondence to the atoms of  $\mathcal{L}(\mathcal{H})$ , this is not the case for mixtures of neither kind. On the contrary, the standard formulation of QL treats improper mixtures in an analogous way as classical statistical distributions. But improper mixtures have a very different physical content because they do not admit an ignorance interpretation. After a brief review of the problem of quantum nonseparability in Sec. II, we turn in Sec. III to the reasons why this difference leads to a dead end when compound systems are considered from the standard quantum logical point of view. We also discuss that the physical necessity to consider mixtures indicates that the algebraic structure of the properties of compound systems should be studied in a frame that takes into account the fact that density operators are states in their own right. We show in Sec. IV that a frame with these characteristics can be built by enlarging the scope of standard QL. We do this by constructing a lattice based on the convex set of density operators which incorporates improper mixtures as atoms of the lattice. Then, in Sec. V we study the relationship between this lattice and the lattices of its subsystems and show how our construction overcomes the problem posed in Sec. III. Finally, we draw our conclusions in Sec. VI.

## II. QUANTUM NONSEPARABILITY

We briefly review here the main arguments and results of the analysis of nonseparability and relate them to the frame of quantum logical research for the sake of completeness. We start by analyzing classical compound systems in order to illustrate their differences with the quantum case.

## A. Classical systems

When considering in classical mechanics two systems  $S_1$  and  $S_2$  and their own state spaces  $\Gamma_1$  and  $\Gamma_2$  (or, analogously, two parts of a single system), the state space  $\Gamma$  of the composite system is the Cartesian product  $\Gamma = \Gamma_1 \times \Gamma_2$  of the phase spaces of the individual systems, independently of the kind of interaction between both of them. The physical intuition behind this fact is that, no matter how they interact, every interesting magnitude corresponding to the parts and the whole may be written in terms of the points in phase space.

In the logical approach, classical properties are associated with subsets of the phase space, precisely with the subsets consisting of the points corresponding to those states such that, when being in them, one may say that the system has the mentioned property. Thus, subsets of  $\Gamma$  are good representatives of the properties of a classical system. The power set  $\wp(\Gamma)$  of  $\Gamma$ , partially ordered by set inclusion  $\subseteq$  (the implication) and equipped with set intersection  $\cap$  as the meet operation, sets union  $\cup$  as the join operation and relative complement  $'$  as the complement operation gives rise to a complete Boolean lattice  $\langle \wp(\Gamma), \cap, \cup, ', \mathbf{0}, \mathbf{1} \rangle$ , where  $\mathbf{0}$  is the empty set  $\emptyset$  and  $\mathbf{1}$  is the total space  $\Gamma$ . According to the standard interpretation, partial order and lattice operations may be put in correspondence with the connectives *and*, *or*, *not*, and the *material implication* of classical logic.

In this frame, the points  $(p, q) \in \Gamma$  (pure states of a classical system) represent pieces of information that are maximal and logically complete. They are maximal because they represent the maximum of information about the system that cannot be consistently extended [any desired magnitude is a function of  $(p, q)$ ] and complete in the sense that they semantically decide any property.<sup>14</sup> Statistical mixtures are represented by measurable functions,

$$\sigma: \Gamma \rightarrow [0; 1],$$

such that

$$\int_{\Gamma} \sigma(p, q) d^3p d^3q = 1.$$

We point out that statistical mixtures are not fundamental objects in classical mechanics, in the sense that they admit an ignorance interpretation. They appear as a state of affairs in which the observer cannot access to an information which lies objectively in the system. Although the physical status of quantum improper mixtures is very different, they are treated in a similar way as classical mixtures by standard QL. We discuss in Sec. III how this misleading treatment leads to problems.

When considering two systems, it is meaningful to organize the whole set of their properties in the corresponding (Boolean) lattice built up as the Cartesian product of the individual lattices. Informally one may say that each factor lattice corresponds to the properties of each physical system. More precisely, in the category of lattices as objects and lattice morphisms as arrows, the Cartesian product of lattices is the categorial product. This category is *Ens*, and the Cartesian product is the categorial product in *Ens*.

## B. Quantum systems

The quantum case is completely different. When two or more systems are considered together, the state space of their pure states is taken to be the tensor product of their Hilbert spaces. Given the Hilbert state spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  as representatives of two systems, the pure states of the compound system are given by rays in the tensor product space  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ . But it is not true—as a naive classical analogy would suggest—that any pure state of the compound system factorizes after the interaction in pure states of the subsystems, and that they evolve with their own Hamiltonian operators.<sup>23,29</sup> The mathematics behind the persistence of entanglement is the lack of a product of lattices and even posets.<sup>30–32</sup> A product of structures is available for weaker structures

(Ref. 15, Chap. 4) but those structures, although mathematically very valuable and promising, have a less direct relation with the standard formalism of quantum mechanics.

In the standard quantum logical approach, properties (or propositions regarding the quantum system) are in correspondence with closed subspaces of Hilbert space  $\mathcal{H}$ . The set of subspaces  $\mathcal{C}(\mathcal{H})$  with the partial order defined by set inclusion  $\subseteq$ , intersection of subspaces  $\cap$  as the lattice meet, closed linear span of subspaces  $\oplus$  as the lattice join, and orthocomplementation  $\neg$  as lattice complement gives rise in the finite dimensional case to a modular lattice  $\mathcal{L}(\mathcal{H}) = \langle \mathcal{C}(\mathcal{H}), \cap, \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle$ , where  $\mathbf{0}$  is the empty set  $\emptyset$  and  $\mathbf{1}$  is the total space  $\mathcal{H}$ . We will refer to this lattice as  $\mathcal{L}_{vN}$ , the “von Neumann lattice.”

When trying to repeat the classical procedure of taking the tensor product of the lattices of the properties of two systems to obtain the lattice of the properties of the composite the procedure fails.<sup>18,33</sup> Mathematically, this is the expression of the fact that the category of Hilbert lattices as objects and lattice morphisms as arrows has not a categorical product because of the failure of orthocomplementation. This problem is studied, for example, in Refs. 10 and 17. Attempts to vary the conditions that define the product of lattices have been made,<sup>34,35</sup> but in all cases it results that the Hilbert lattice factorizes only in the case in which one of the factors is a Boolean lattice or when systems have never interacted. For a complete review, see Ref. 32.

Let us briefly recall the defining properties of the tensor product of a finite collection of vector spaces in order to discuss the main features that make the difference with the classical case. Let us first define (following Ref. 36)  $\otimes \mathcal{H}_i$  as the unique vector space which satisfies the following properties.

- (1) For each family  $\{|x_i\rangle\}$ ,  $|x_i\rangle \in \mathcal{H}_i$ , there exists an element  $\otimes_i |x_i\rangle \in \otimes_i \mathcal{H}_i$  depending multilinearly on the  $\{|x_i\rangle\}$ . All vectors in  $\otimes_i \mathcal{H}_i$  are finite linear combinations of such elements.
- (2) (Universal property) For each multilinear mapping  $\pi$  of the product of the  $\mathcal{H}_i$  into a vector space  $Y$ , there exists a unique linear map  $\varphi: \otimes_i \mathcal{H}_i \rightarrow Y$ , such that

$$\varphi(\otimes_i |x_i\rangle) = \pi(\{|x_i\rangle\})$$

for all  $|x_i\rangle \in \mathcal{H}_i$ .

- (3) (Associativity) For each partition  $\cup_k I_k$  of  $\{1, \dots, n\}$  there exists a unique isomorphism from  $\otimes_i \mathcal{H}_i$  onto  $\otimes_k (\otimes_{i \in I_k} \mathcal{H}_i)$  transforming  $\otimes_i |x_i\rangle$  into  $\otimes_k (\otimes_{i \in I_k} |x_i\rangle)$ .

When the spaces  $\mathcal{H}_i$  are Hilbert spaces, it is possible to define an inner product on  $\otimes \mathcal{H}_i$  by extending the following definition by linearity:

$$(\otimes_i |x_i\rangle, \otimes_i |y_i\rangle) = \prod_i (|x_i\rangle, |y_i\rangle).$$

Note that as we are using Dirac notation, we may write  $\langle x_i | y_i \rangle$  instead of  $(|x_i\rangle, |y_i\rangle)$ . The completion of  $\otimes \mathcal{H}_i$  in the associated norm is the tensor product of the Hilbert spaces  $\otimes_i \mathcal{H}_i$ . Thus, we see that the tensor product of Hilbert spaces is in essence a multilinear extension of the direct product. From a physical point of view, it is for this reason that the state of the joint system contains much more information than “the sum” of the information contained in the states of its parts.

This feature of quantum systems may be regarded as a strange fact when using classical reasoning, but it not strange at all in a landscape where the superposition principle holds. Given two systems  $S_1$  and  $S_2$ , if we prepare them independently in states  $|a\rangle$  and  $|b\rangle$ , respectively, then we would have something like the direct product of the states of both systems  $|a\rangle \times |b\rangle$  for the state of the joint system. We could perform also different preparations and obtain  $|a'\rangle \times |b'\rangle$ . Then, if there are no superselection rules, and according to the superposition principle, it is quite natural to suppose that it is at least in principle possible to prepare the superposition state of the form  $\alpha|a\rangle \otimes |b\rangle + \beta|a'\rangle \otimes |b'\rangle$ , and so, we need  $\otimes$  instead of  $\times$ . This last state is not a product of the states of the parties. It is for this reason that the product in quantum mechanics has to be the multilinear extension of the direct product.

Let us now briefly review the standard relationship between the states of the joint system and the states of the subsystems. If  $\{|x_k^{(i)}\rangle\}$  is an orthonormal basis for  $\mathcal{H}_i$ , then

$$\otimes_{i=1}^n |x_{k_i}^{(i)}\rangle$$

forms a basis of  $\otimes_i \mathcal{H}_i$ . Let us focus for simplicity on the case of two systems,  $S_1$  and  $S_2$ . If  $\{|x_i^{(1)}\rangle\}$  and  $\{|x_j^{(2)}\rangle\}$  are the corresponding orthonormal basis of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively, then  $\{|x_i^{(1)}\rangle \otimes |x_j^{(2)}\rangle\}$  is an orthonormal basis for  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . A general (pure) state of the composite system can be written as

$$\rho = |\psi\rangle\langle\psi|,$$

where  $|\psi\rangle = \sum_{i,j} \alpha_{ij} |x_i^{(1)}\rangle \otimes |x_j^{(2)}\rangle$ . Moreover, if  $M$  represents an observable, its mean value  $\langle M \rangle$  is given by

$$\text{tr}(\rho M) = \langle M \rangle.$$

When observables of the form  $O_1 \otimes \mathbf{1}_2$  and  $\mathbf{1}_1 \otimes O_2$  (with  $\mathbf{1}_1$  and  $\mathbf{1}_2$  the identity operators over  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively) are considered, then partial state operators  $\rho_1$  and  $\rho_2$  can be defined for systems  $S_1$  and  $S_2$ . The relation between  $\rho$ ,  $\rho_1$ , and  $\rho_2$  is given by

$$\rho_1 = \text{tr}_2(\rho), \quad \rho_2 = \text{tr}_1(\rho),$$

where  $\text{tr}_i$  stands for the partial trace over the  $i$  degrees of freedom. It can be shown that

$$\text{tr}_1(\rho_1 O_1 \otimes \mathbf{1}_2) = \langle O_1 \rangle$$

and that a similar equation holds for  $S_2$ . Operators of the form  $O_1 \otimes \mathbf{1}_2$  and  $\mathbf{1}_1 \otimes O_2$  represent magnitudes related to  $S_1$  and  $S_2$ , respectively. When  $S$  is in a product state  $|\varphi_1\rangle \otimes |\varphi_2\rangle$ , the mean value of the product operator  $O_1 \otimes O_2$  will yield

$$\text{tr}(|\varphi_1\rangle \otimes |\varphi_2\rangle \langle\varphi_1| \otimes \langle\varphi_2| O_1 \otimes O_2) = \langle O_1 \rangle \langle O_2 \rangle,$$

reproducing statistical independence. But, as is well known, this is not the general case.

As we pointed out above,  $\rho_1$  and  $\rho_2$  do not accept an ignorance interpretation.<sup>22-27</sup> Moreover, the state of the whole system  $\rho = |\psi\rangle\langle\psi|$  carries the information about the correlations between  $S_1$  and  $S_2$ . The fact that  $\rho_1$  and  $\rho_2$  are not pure states is an expression of the nontriviality of these correlations that are stronger and of a different kind than those of the classical case. This radical difference expresses itself also in the violation of Bell inequalities by quantum systems.<sup>37</sup> These facts suggest that mixtures have to be considered as states in their own right and be given a place in the algebraic approach to the study of quantum properties.

### C. The convex set of states of a quantum system

From the analysis of Sec. II B it becomes clear that for a complete description that includes compound systems it is not sufficient to consider only pure states, but we have to consider also mixtures. The standard way of doing this is by representing the states of the system by positive, Hermitian, and trace one operators (also called “density matrices”). The set of all density matrixes forms a convex set (of states), which we will denote by  $\mathcal{C}$ ,

$$\mathcal{C} := \{\rho \in \mathcal{A} | \text{tr}(\rho) = 1, \rho \geq 0\}.$$

As usual, physical observables  $\mathcal{M}$  are represented by elements  $M$  of  $\mathcal{A}$ , the  $\mathbb{R}$ -vector space of Hermitian operators acting on  $\mathcal{H}$ ,

$$\mathcal{A} := \{M \in B(\mathcal{H}) | M = M^\dagger\},$$

$B(\mathcal{H})$  stands for the algebra of bounded operators in  $\mathcal{H}$ . The mean value of the observable represented by the operator  $M$  when the system is in a state  $\rho$  is given by  $\langle M \rangle = \text{tr}(\rho M)$ .

The set  $P$  of pure states can be defined as

$$P := \{\rho \in \mathcal{C} | \rho^2 = \rho\}.$$

This set is in correspondence with the rays of  $\mathcal{H}$  by the usual association (using Dirac notation)  $[|\psi\rangle] \mapsto |\psi\rangle\langle\psi|$  between the elements of the projective space of  $\mathcal{H}$  and the class defined by the normalized vector  $|\psi\rangle$  ( $|\varphi\rangle \sim |\psi\rangle \leftrightarrow |\varphi\rangle = \lambda|\psi\rangle, \lambda \neq 0$ ).  $\mathcal{C}$  is a convex set inside the hyperplane  $\{\rho \in \mathcal{A} | \text{tr}(\rho) = 1\}$ . If  $\dim_{\mathbb{C}}(\mathcal{H}) = n < \infty$ , we have an  $\mathbb{R}$ -linear isomorphism  $B(\mathcal{H}) \cong M_n(\mathbb{R}) \times M_n(\mathbb{R})$ , then

$$\mathcal{A} \cong \{(R, I) \in M_n(\mathbb{R}) \times M_n(\mathbb{R}) | R^t = R, I^t = -I\} = S_n(\mathbb{R}) \times \wedge_n(\mathbb{R}),$$

$$\mathcal{A} \cap \{\text{tr}(\rho) = 1\} \cong \{(R, I) \in S_n(\mathbb{R}) \times \wedge_n(\mathbb{R}) | \text{tr}(R) = 1\}.$$

So the convex set  $\mathcal{C}$  lies inside an  $\mathbb{R}$ -algebraic variety of dimension,

$$\dim_{\mathbb{R}}(\{\rho \in \mathcal{A} | \text{tr}(\rho) = 1\}) = n^2 - 1.$$

When a system  $S$  composed of subsystems  $S_1$  and  $S_2$  is considered, the state of  $S$  cannot be decomposed, in general, in a product state  $\rho = \rho_1 \otimes \rho_2$ , as said before. *Separable states* are those states of  $S$  which can be written as a convex combination of product states,<sup>38</sup>

$$\rho_{\text{Sep}} = \sum_k \lambda_k \rho_k^{(1)} \otimes \rho_k^{(2)},$$

where  $\rho_k^{(1)} \in \mathcal{C}_1$  and  $\rho_k^{(2)} \in \mathcal{C}_2$ ,  $\sum_k \lambda_k = 1$ , and  $\lambda_k \geq 0$ . It is easy to see that this expression may be written as

$$\rho_{\text{Sep}} = \sum_{i,j} \lambda_{ij} \rho_i^{(1)} \otimes \rho_j^{(2)},$$

with  $\sum_{i,j} \lambda_{ij} = 1$  and  $\lambda_{ij} \geq 0$ . We will denote  $\mathcal{S}(\mathcal{H})$  the (convex) set of separable states. As said above, it is a remarkable fact that there are many states in  $\mathcal{C}$  which are nonseparable. If the state is nonseparable, it is said to be *entangled*. The estimation of the volume of  $\mathcal{S}(\mathcal{H})$  is of great interest (see, for example, Refs. 39–41).

### III. THE PROBLEM OF THE STATES OF THE SUBSYSTEMS IN QL

In the quantum logical approach, there is a bijective correspondence between the states of the system and the atoms of the lattice  $\mathcal{L}_{vN}$  of its properties: the atoms of  $\mathcal{L}_{vN}$  are the pure states. The relationship between pure states  $\rho_{\text{pure}} = |\psi\rangle\langle\psi|$  of the quantum system and its actual properties  $p$  is given by

$$\langle |\psi\rangle \rangle = \wedge \{p \in \mathcal{L}_{vN} | p \text{ is actual}\},$$

and an equivalent relation holds for the classical case. This is an expected fact because in  $\mathcal{L}_{vN}$  states are the most elemental properties of the system, up from which all other properties are inferred. We claim that any reasonable definition of state must satisfy this property. Furthermore, the representatives of states must be atoms of the lattice, in order to grant that no other nontrivial property be more elementary. But pure states form, in general, a quite small subset of the border of  $\mathcal{C}$  (the atoms of  $\mathcal{L}_{vN}$  are in one to one correspondence with this subset): pure states are in a  $2(N-1)$ -dimensional subset of the  $(N^2-2)$ -dimensional boundary of  $\mathcal{C}$ . Moreover, so all nonpure

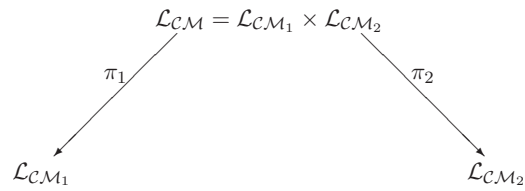


FIG. 1. In the classical case, we can go from the state of the system to the states of the subsystems using the set-theoretical projections  $\pi_1$  and  $\pi_2$ .

states are excluded from  $\mathcal{L}_{vN}$ . Or in the best case, they have a different status, when considered (as in the classical case) as measures over the lattice of projections.

Let us emphasize that a remarkable problem appears in standard QL, linked to the status that it gives to improper mixtures (see Ref. 43, for more discussion on this problem and a proposal for its solution different than the one presented here). Suppose that  $S_1$  and  $S_2$  are subsystems of a larger system  $S$  which is in a pure entangled state  $|\psi\rangle$ . Then we may ask which the states of its subsystems are. If we make the conjunction of all actual properties for, say  $S_1$ , we will no longer obtain an atom of  $\mathcal{L}_{vN_1}$ .<sup>33</sup> Instead of it, we will obtain a property which corresponds, in the nontrivial case, to a subspace of dimension strictly greater than 1 and does not correspond to the state of the subsystem. In fact, the state of the subsystem is the (improper) mixture given by the partial trace  $\text{tr}_2(|\psi\rangle\langle\psi|)$ . Thus, there is no way to obtain the actual physical state of  $S_1$  using the  $\wedge$  operation of  $\mathcal{L}_{vN_1}$ , as it would be reasonable according to the definition of state as minimal property out of which all other properties are inferred.

To put things graphically, consider Figs. 1 and 2. For the classical case, there exist set-theoretical projections  $\pi_1$  and  $\pi_2$  from  $\mathcal{L}_{CM}$  to  $\mathcal{L}_{CM_1}$  and  $\mathcal{L}_{CM_2}$  which relate the states of the system  $S$  and the states of the subsystems  $S_1$  and  $S_2$ . In the quantum case (Fig. 2), we do not have arrows which map states of  $\mathcal{L}_{vN}$  into states of  $\mathcal{L}_{vN_i}$  ( $i=1,2$ ) simply because nonpure states are not properly included in the property lattice. Thus, the “?” arrows of Fig. 2 are missing.

In spite of the fact that mixtures are also considered in classical mechanics, they pose there no fundamental problem. This is so because classical mixtures represent a lack of information that is—at least in principle—available. On the contrary, according to the orthodox interpretation of *QM*, information encoded in (improper) mixtures is all that exists, there is no further information available: there is no ignorance interpretation of improper mixtures. But the orthodox quantum logical approach puts in different level pure states and mixtures (the lattice of properties and a measure over it) as is done in the classical case. In the classical case this works, for pure states of the whole system and its subsystems can be properly linked as Fig. 1 shows. But we cannot do the same in the quantum case because subsystems are rarely found in pure states.

All of this motivates our search of algebraic structures which contain mixtures in such a way that they may be given an equal treatment as the one given to pure states. We will show that this is possible and that such structures may be defined in a natural manner, extending (in a sense explained below)  $\mathcal{L}_{vN}$  so to be compatible with the physics of compounded quantum systems. Precisely, in Sec. IV we construct a lattice  $\mathcal{L}$  that has all physical states as its atoms and whose

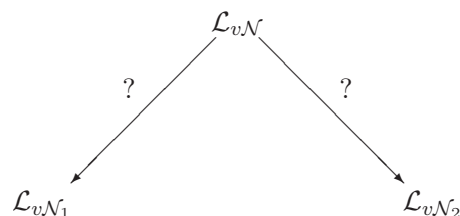


FIG. 2. We cannot apply partial traces in order to go down from  $\mathcal{L}_{vN}$  to  $\mathcal{L}_{vN_1}$ , and  $\mathcal{L}_{vN_2}$ .

meet operation over all actual properties of a system gives the actual physical state of that system. It also includes  $\mathcal{L}_{vN}$  set theoretically, so we are able to reobtain all well known results of single isolated systems.

$\mathcal{L}$  is constructed in such a way that there exist projection functions which map *all* states (atoms) of the structure corresponding to the whole system  $S$  to the corresponding states (atoms) of its subsystems  $S_1$  and  $S_2$ . This assignation rule is compatible with the physics of the problem, i.e., it is constructed using partial traces, which are the natural functions which map states of the larger system with the states of its subsystems. Improper mixtures are put in correspondence with atoms of  $\mathcal{L}$ , granting that they are the most elementary properties.

There is another important feature of  $\mathcal{L}$ . In  $\mathcal{L}_{vN}$  from two given pure states, say  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , a new state  $\alpha|\psi_1\rangle + \beta|\psi_2\rangle$  may be constructed; we have at hand the superposition principle. The  $\vee_{\mathcal{L}_{vN}}$  operation of the von Neumann lattice is directly linked to the superposition principle: starting with two rays, the  $\vee_{\mathcal{L}_{vN}}$  operation yields the closed subspace formed by all linear combinations of the generators of the rays. But there is another operation available, namely, we can mix states, we can perform a “mixing operation” to get  $p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$ . There is no place for such a thing in  $\mathcal{L}_{vN}$ , but it may be performed in  $\mathcal{L}$ . The  $\vee_{\mathcal{L}}$  operation reflects the fact that we can mix states, playing an analogous role to that of  $\vee_{\mathcal{L}_{vN}}$  in relation to the superposition principle.

Let us mention before presenting  $\mathcal{L}$  that there exists a trivial example of a lattice that fulfills the requirement that its atoms are improper mixtures, namely, the set of all subsets of  $\mathcal{C}$ , which we call  $\mathcal{P}(\mathcal{C})$ . If we use set intersection as the meet operation and set union as the join operation, this structure is a Boolean lattice. But this lattice is not of interest because its disjunction is not connected with the mixing operation mentioned above, alike  $\vee_{\mathcal{L}}$ . Its boolean structure hides the radical differences between classical and quantum mechanics. But this trivial example shows that our construction below may be one among a family of possible lattices which overcome the problem of  $\mathcal{L}_{vN}$  presented in this section.

#### IV. THE LATTICE OF DENSITY OPERATORS

In order to construct a lattice for density operators, let us consider the pair  $G(\mathcal{A}) := (\mathcal{A}, \text{tr})$ , where  $\mathcal{A}$  is the  $\mathbb{R}$ -vector space of operators over  $\mathcal{H}$  and  $\text{tr}$  is the usual trace operator on  $B(\mathcal{H})$ , which induces the scalar product  $\langle A, B \rangle = \text{tr}(A \cdot B)$  ( $\dim(\mathcal{H}) < \infty$ ). The restriction to  $\mathcal{A}$  of  $\text{tr}$  makes  $\mathcal{A}$  into an  $\mathbb{R}$ -Euclidean vector space. With the standard  $\vee$ ,  $\wedge$ , and  $\neg$  operations,  $G(\mathcal{A})$  is a modular, orthocomplemented, atomic, and complete lattice (not distributive, hence not a Boolean algebra).

Let  $\mathcal{L}_o$  be the set of subspaces,

$$\mathcal{L}_o := \{L = S \cap \mathcal{C} | S \in G(\mathcal{A})\}.$$

There are a lot of subspaces  $S, S_i \in G(\mathcal{A})$ , such that  $S \cap \mathcal{C} = S_i \cap \mathcal{C}$ , so for each  $L \in \mathcal{L}_o$  we may choose the subspace with the least dimension  $[S]$  as the representative element,

$$[S] := \min\{\dim_{\mathbb{R}}(S) | L = S \cap \mathcal{C}, S \in G(\mathcal{A})\}.$$

Let  $[S] = L$ , being  $S \in G(\mathcal{A})$  an element of the class  $L$ , then

$$S \cap \mathcal{C} \subseteq \langle S \cap \mathcal{C} \rangle_{\mathbb{R}} \subseteq S \Rightarrow S \cap \mathcal{C} \cap \mathcal{C} \subseteq \langle S \cap \mathcal{C} \rangle_{\mathbb{R}} \cap \mathcal{C} \subseteq S \cap \mathcal{C} \Rightarrow \langle S \cap \mathcal{C} \rangle \cap \mathcal{C} = S \cap \mathcal{C}.$$

So  $\langle S \cap \mathcal{C} \rangle$  and  $S$  are in the same class  $L$ . Note that  $\langle S \cap \mathcal{C} \rangle \subseteq S$  and if  $S$  is the subspace with the least dimension, then  $\langle S \cap \mathcal{C} \rangle = S$ . Also note that the representative with least dimension is unique because if we choose  $S'$  such that  $S' \cap \mathcal{C} = S \cap \mathcal{C}$ , then

$$S = \langle S \cap \mathcal{C} \rangle = \langle S' \cap \mathcal{C} \rangle = S'.$$

Finally, the representative of a class  $L$  that we choose is the unique  $\mathbb{R}$ -subspace  $S \subseteq \mathcal{A}$ , such that

$$S = \langle S \cap \mathcal{C} \rangle_{\mathbb{R}}.$$



We call it the *good representative*. It is important to remark that in the case of infinite dimensional Hilbert spaces we cannot define good representatives in such a way.

Let us now define  $\vee$ ,  $\wedge$ , and  $\neg$  operations in  $\mathcal{L}_o$  as

$$(S \cap \mathcal{C}) \wedge (T \cap \mathcal{C}) = \langle S \cap \mathcal{C} \rangle \cap \langle T \cap \mathcal{C} \rangle \cap \mathcal{C},$$

$$(S \cap \mathcal{C}) \vee (T \cap \mathcal{C}) = (\langle S \cap \mathcal{C} \rangle + \langle T \cap \mathcal{C} \rangle) \cap \mathcal{C},$$

$$\neg(S \cap \mathcal{C}) = \langle S \cap \mathcal{C} \rangle^\perp \cap \mathcal{C}.$$

They are well defined for every element of the classes  $[S]$  and  $[T]$ . It is easy to see that  $\mathcal{L} = \langle \mathcal{L}_o, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$  is a complete lattice, with  $\mathbf{0} = \emptyset$  represented by the class of  $G(\mathcal{A})$  whose elements are disjoint with  $\mathcal{C}$  and  $\mathbf{1} = \mathcal{C}$ , represented by the class of  $\mathcal{A}$ . It is an atomic lattice: the atoms of  $\mathcal{L}$  are given by the intersection of rays in  $G(\mathcal{A})$  and  $\mathcal{C}$ . They are the sets  $\{\rho\}$ , with  $\rho$  a density operator.

It is important to notice that with respect to the  $\neg$  operation,  $\mathcal{L}$  is not an orthocomplemented lattice—alike  $\mathcal{L}_{oN}$ —because if we take  $L = \{(1/N)\mathbf{1}\} = \langle (1/N)\mathbf{1} \rangle \cap \mathcal{C}$ , then

$$\neg(\neg L) = \neg\left(\left\langle \frac{1}{N}\mathbf{1} \right\rangle^\perp \cap \mathcal{C}\right) = \neg\emptyset = \mathcal{C} \neq L.$$

On the other hand it is easy to show that noncontradiction holds,

$$L \wedge \neg L = \mathbf{0},$$

and also contraposition,

$$L_1 \leq L_2 \Rightarrow \neg L_2 \leq \neg L_1.$$

*Proposition 4.1:* If  $\dim(\mathcal{H}) < \infty$ ,  $\mathcal{L}$  is a modular lattice.

*Proof:* To prove the modular equality,

$$[S] \leq [R] \Rightarrow [S] \vee ([T] \wedge [R]) = ([S] \vee [T]) \wedge [R],$$

the key point is that

$$[S] \leq [R] \Leftrightarrow S \cap \mathcal{C} \subseteq R \cap \mathcal{C} \Rightarrow S = \langle S \cap \mathcal{C} \rangle \subseteq \langle R \cap \mathcal{C} \rangle = R.$$

So, using  $S \subseteq R$ , is easy to see that  $(S + (T \cap R)) \cap \mathcal{C} = ((S + T) \cap R) \cap \mathcal{C}$ . ■

Furthermore, we can prove the following.

*Proposition 4.2:* There is a one to one correspondence between the states of the system and the atoms of  $\mathcal{L}$ .

*Proof:* For every  $\rho \in \mathcal{C}$ , we have that  $\langle \rho \rangle \cap \mathcal{C} = \{\rho\}$ . This is so because the only positive matrix of trace one that is a multiple of  $\rho$  is  $\rho$  itself. Then,  $\{\rho\}$  is an element of  $\mathcal{L}$ . Suppose that there exists  $L$  such that  $\mathbf{0} \leq L \leq \{\rho\}$ . If  $L \neq \mathbf{0}$ , we can write  $L = S \cap \mathcal{C}$ , with  $S$  being the good representative for the class of  $L$ .  $L \leq \{\rho\}$  implies that  $S \subseteq \langle \rho \rangle$  and thus  $S = \langle \rho \rangle$ , so it follows that  $L = \{\rho\}$ . Conversely, if  $L$  is an atom of  $\mathcal{L}$ , take  $\rho \in L$ . Define  $L' = \langle \rho \rangle \cap \mathcal{C} = \{\rho\}$ . It is clear that  $L' \subseteq L$  and, given that  $L' \neq \mathbf{0}$ , we have  $L' = L$ . ■

The last proposition shows that we can represent the states of subsystems of a larger system as elements of the lattice  $\mathcal{L}$  giving them a similar status as pure states, something impossible in the standard formalism of QL and one of the desiderata in searching a structure to deal with composite systems. It is a well established fact<sup>42</sup> that there is a lattice isomorphism between the complemented and complete lattice of faces of the convex set  $\mathcal{C}$  and  $\mathcal{L}_{oN}$ . As desired,  $\mathcal{L}_{oN}$  is included in  $\mathcal{L}$  guaranteeing to represent all the good features of standard QL in the new algebra. This is a nontrivial result, and it is ensured by the following proposition and its corollary.

*Proposition 4.3:* Every face of  $\mathcal{C}$  is an element of  $\mathcal{L}$ .

*Proof:* Let  $F \subseteq \mathcal{C}$  be a face. Then there exists a  $\mathbb{R}$ -hyperplane  $H$  inside  $\{\rho \in \mathcal{A} | \text{tr}(\rho) = 1\}$ , such that  $F = H \cap \mathcal{C}$ .

Given that  $H = \{l = \alpha\}$  with  $\alpha \in \mathbb{R}$  and  $l$  an  $\mathbb{R}$ -linear form on  $\mathcal{A}$ , we have that

$$F = H \cap \mathcal{C} = H \cap \mathcal{C} \cap \{\text{tr} = 1\} = \{l = \alpha, \text{tr} = 1\} \cap \mathcal{C} = \{l = \alpha \text{tr}, \text{tr} = 1\} \cap \mathcal{C} = \{l = \alpha \text{tr}\} \cap \mathcal{C} \cap \{\text{tr} = 1\} = \{l - \alpha \text{tr} = 0\} \cap \mathcal{C}.$$

So  $\{\rho \in \mathcal{A} | l(\rho) - \alpha \text{tr}(\rho) = 0\} \in G(\mathcal{A})$ , and then  $F \in \mathcal{L}$ . ■

So, we can naturally embed  $\mathcal{L}_{vN}$  in  $\mathcal{L}$  as a poset.

*Corollary 4.4:* The complemented and complete lattice of faces of the convex set  $\mathcal{C}$  is a subposet of  $\mathcal{L}$ .

*Proof:* We have already seen that  $\mathcal{L}_{vN} \subseteq \mathcal{L}$  as sets. Moreover, it is easy to see that if  $F_1 \leq F_2$  in  $\mathcal{L}_{vN}$  then  $F_1 \leq F_2$  in  $\mathcal{L}$ . This is so because both orders are set theory inclusions. ■

The previous corollary shows that  $\mathcal{L}$  and  $\mathcal{L}_{vN}$  are closely connected. Let us analyze the relationship between the operations of the two lattices in order to characterize this connection. We recall that the meet of two faces is their intersection and the join is the smallest face containing both. In  $\mathcal{L}_{vN}$ , the meet of two subspaces is their intersection and the join is their closed linear span.

$\wedge$ :  $F_1, F_2 \in \mathcal{L}_{vN}$ , then  $F_1 \wedge F_2$  in  $\mathcal{L}_{vN}$  is the same as in  $\mathcal{L}$ . So the inclusion  $\mathcal{L}_{vN} \subseteq \mathcal{L}$  preserves the  $\wedge$ -operation.

$\vee$ : In general, it does not preserve the  $\vee$ -operation. The relation between the two operations is

$$F_1 \vee_{\mathcal{L}} F_2 \leq F_1 \vee_{\mathcal{L}_{vN}} F_2,$$

$$F_1 \leq F_2 \Rightarrow F_1 \vee_{\mathcal{L}} F_2 = F_1 \vee_{\mathcal{L}_{vN}} F_2 = F_2.$$

For example, if the convex set  $\mathcal{C}$  is a rectangle and  $F_1$  and  $F_2$  are two opposite vertices then, the face-join of them is the whole rectangle, and the  $\mathcal{L}$ -join is the diagonal joining them.

$\neg$ : In any lattice,  $x$  is a complement to  $y$  if  $x \vee y = 1$  and  $x \wedge y = 0$ . In general, the lattice of faces of a convex set is complemented, but in the case of  $\mathcal{C}$  it is orthocomplemented, that is, it has a distinguished complemented face for every face  $F \subseteq \mathcal{C}$ . Given that  $\mathcal{L}_{vN} \cong \mathcal{P}(\mathcal{H})$ , the lattice of projectors in  $\mathcal{H}$ , the  $\neg$ -operation in  $\mathcal{L}_{vN}$  is that induced from  $\mathcal{P}(\mathcal{H})$ . If  $F \subseteq \mathcal{C}$  is a face, there exists a unique projector  $P \in \mathcal{A}$ , such that

$$F = \{\rho \in \mathcal{C} | \text{tr}(P\rho) = 0\} = \{\rho \in \mathcal{C} | \rho \perp P\} \Rightarrow \neg_{\mathcal{L}_{vN}} F = \{\rho \in \mathcal{C} | \rho \perp 1 - P\}.$$

It is easy (using eigenvalues) to see that it is well defined and that  $\neg F$  is again a face. Given that  $F \in \mathcal{L}$ , it has a good representative  $F = [S]$ . Then

$$\neg_{\mathcal{L}} F = S^\perp \cap \mathcal{C}.$$

Using this, we can prove that  $\neg_{\mathcal{L}} F \leq \neg_{\mathcal{L}_{vN}} F$  because

$$v \in \neg_{\mathcal{L}} F \text{ then } v \perp \rho, \rho \in F$$

and, in particular,

$$v \perp 1 - P \text{ then } v \in \neg_{\mathcal{L}_{vN}} F.$$

## A. Quantum interactions enlarge the lattice of properties

The results presented in Sec. IV show that  $\mathcal{L}$  is a quite natural extension of  $\mathcal{L}_{vN}$  and satisfies that improper mixtures are in a bijective correspondence with the atoms of the lattice. This feature allows this lattice to avoid the problems which appear in the standard formulation of QL posed in Ref. 43 and also discussed in Sec. III of this work. In the new lattice, the conjunction of all actual

properties yields the physical state of the system because all states are in correspondence with atoms, which are minimal elements. From the physical point of view the necessity of an extension becomes clear from the comparison between classical and quantum compound systems. When we have a single classical system, its properties are faithfully represented by the subsets of its phase space. When another classical system is added and the compound system is considered, no enrichment of the state space of the former system is needed in order to describe its properties, *even* in the presence of interactions. No matter which the interaction may be, the Cartesian product of the individual phase spaces gives all is needed to represent the compound system, and the same stands for the property lattices. But the situation is quite different in quantum mechanics. This is so because if we add a new quantum system to the first one, pure states are no longer faithful in order to describe subsystems. Interactions produce nontrivial correlations, which are reflected in the presence of entangled states and violation of Bell inequalities. These nontrivial correlations are behind the fact that within the standard quantum logical approach, the conjunction of all actual properties does not yield the physical state of the subsystem. Thus, besides their own properties, we need information about the nontrivial *correlations* that each subsystem has with other subsystems—for example, a system with the environment—that may be regarded as new elements in the structure of properties and cannot be described otherwise. For this reason an enlargement of the lattice of properties is needed to represent improper mixtures by atoms  $\{\rho\}$  in  $\mathcal{L}$ . We will come back to this point in Sec. V B, where we study the projections from the lattice of the compound system onto the lattices of the subsystems.

## V. THE RELATIONSHIP BETWEEN $\mathcal{L}$ AND $\mathcal{L}_i$

Given two systems with Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , we can construct the lattices  $\mathcal{L}_1$  and  $\mathcal{L}_2$  according to the procedure of Sec. IV. We can also construct  $\mathcal{L}$ , the lattice associated with the product space  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . In this section we examine their mutual relations. We study some special maps between these lattices and their properties, in order to get an insight in the characterization of compound quantum systems.

### A. Separable states (going up)

We start defining the map,

$$\Psi: \mathcal{L}_1 \times \mathcal{L}_2 \rightarrow \mathcal{L},$$

$$(S_1 \cap \mathcal{C}_1, S_2 \cap \mathcal{C}_2) \rightarrow S \cap \mathcal{C},$$

$$\text{where } S = (\langle S_1 \cap \mathcal{C}_1 \rangle \otimes \langle S_2 \cap \mathcal{C}_2 \rangle).$$

In terms of good representatives,  $\Psi([S_1], [S_2]) = [S_1 \otimes S_2]$ . We can prove the following.

*Proposition 5.1:* Fixing  $[U] \in \mathcal{L}_2$  then  $\mathcal{L}_1$  is isomorphic (as complete lattice) to  $\mathcal{L}_1 \times [U] \subseteq \mathcal{L}$ . The same is true for  $\mathcal{L}_2$  and an arbitrary element of  $\mathcal{L}_1$ .

*Proof:* Let us prove it for  $\mathcal{L}_1$ . Let  $([S], [U]) \in \mathcal{L}_1 \times [U]$  with  $S$  a good representative for  $[S]$  and  $U$  for  $[U]$ . When we apply  $\Psi$  we obtain the proposition  $[S \otimes U] \in \mathcal{L}$ , then, we can consider the image under  $\Psi$  of  $\mathcal{L}_1 \times [U] \subseteq \mathcal{L}_1 \times \mathcal{L}_2$ ,

$$\Psi(\mathcal{L}_1 \times [U]) = \{[S \otimes U] \text{ where } S \text{ is a good representative for } [S] \in \mathcal{L}_1\}.$$

From this characterization it is easy to see that  $\Psi$  is injective. If  $[S \otimes U] = [T \otimes U]$  ( $S$  and  $T$  are good representatives), taking partial traces (more in Sec. V B) then  $[S] = [T]$ .

Moreover,  $\Psi(-, [U])$  is a lattice morphism: let  $[S \otimes U], [T \otimes U] \in \mathcal{L}$  with  $S$  and  $T$  good representatives of  $[S], [T] \in \mathcal{L}_1$ . The key observation is that  $S \otimes U$  and  $T \otimes U$  are also good representatives (taking partial traces). Then we have

$$[S \otimes U] \wedge [T \otimes U] = [(S \cap T) \otimes U] = \Psi([S] \wedge [T], [U]),$$

$$[S \otimes U] \vee [T \otimes U] = [(S \oplus T) \otimes U] = \Psi([S] \vee [T], [U]).$$

This ensures that  $\mathcal{L}_1$  is a sublattice of  $\mathcal{L}$ . The same is true for  $\mathcal{L}_2$ . ■

Notice that we can use an arbitrary atom  $\rho_2 \in \mathcal{C}_2$  instead of some  $[U] \in \mathcal{L}_2$  and that the application  $\Psi$  restricted to  $\mathcal{L}_1 \times \rho_2$  does not preserve the  $\neg$ -operation. This is so because

$$\Psi(\neg[S], \rho_2) = [S^\perp \otimes \rho_2] \subset [(S \otimes \rho_2)^\perp] = \neg[S \otimes \rho_2] = \neg\Psi([S], \rho_2).$$

The inclusion holds because if  $\rho \in [S^\perp \otimes \rho_2]$ , then  $\rho = (\sum \lambda_i \rho_i) \otimes \rho_2 = \sum \lambda_i \rho_i \otimes \rho_2$ , with  $\rho_i \in S^\perp$ . It is clear that all the  $\rho_i \otimes \rho_2$  are orthogonal to  $S \otimes \rho_2$ , and then  $\rho \in (S \otimes \rho_2)^\perp$ . In general, the inclusion is strict because we can have elements of the form  $\rho_1 \otimes \rho'_2$ , with  $\rho_1 \in S^\perp$  and  $\rho'_2 \neq \rho_2$ . Then,  $\rho_1 \otimes \rho_2 \in S \otimes \rho_2$ , but  $\rho_1 \otimes \rho_2 \notin S^\perp \otimes \rho_2$ . This has a clear physical meaning: in fact, when the system  $S_1$  is isolated, its lattice of properties  $\mathcal{L}_1$  is equivalent to  $\mathcal{L}_1 \times \rho_2$ . But when we add system  $S_2$  we can, for example, prepare the systems independently, in such a way that the state after preparation is  $\rho_1 \otimes \rho'_2$  with  $\rho_1 \in S^\perp$  and  $\rho'_2$  an arbitrary state of  $S_2$ . Then, we see that there is much more freedom in the space of all states.

Let us study now the image of  $\Psi$ . First, we note that given  $L_1 \in \mathcal{L}_1$  and  $L_2 \in \mathcal{L}_2$ , we can define the following convex tensor product:

$$L_1 \tilde{\otimes} L_2 := \left\{ \sum \lambda_{ij} \rho_i^1 \otimes \rho_j^2 \mid \rho_i^1 \in L_1, \rho_j^2 \in L_2, \sum \lambda_{ij} = 1 \text{ and } \lambda_{ij} \geq 0 \right\}.$$

This product is formed by all possible convex combinations of tensor products of elements of  $L_1$  and elements of  $L_2$ , and it is again a convex set.

*Proposition 5.2:*  $L_1 \tilde{\otimes} L_2 \subseteq \Psi(L_1, L_2)$ .

*Proof:* If  $\rho \in L_1 \tilde{\otimes} L_2$ , then  $\rho = \sum \lambda_{ij} \rho_i^1 \otimes \rho_j^2$ , with  $\rho_i^1 \in L_1$ ,  $\rho_j^2 \in L_2$ ,  $\sum \lambda_{ij} = 1$ , and  $\lambda_{ij} \geq 0$ . For each  $i, j$ ,  $\rho_i^1 \otimes \rho_j^2$  is again a positive trace one operator and so belongs to  $\mathcal{C}$ . It belongs to  $\langle L_1 \rangle \otimes \langle L_2 \rangle$  because of the definition of tensor product. Then, it belongs to  $\Psi(L_1, L_2)$ . As  $\mathcal{C}$  is convex, then  $\rho \in \mathcal{C}$  because it is a convex combination of elements in  $\mathcal{C}$ . It is a linear combination of elements of  $\langle L_1 \rangle \otimes \langle L_2 \rangle$  also, and so it belongs to it. This proves that  $\rho \in \Psi(L_1, L_2)$ . ■

We can also prove the following.

*Proposition 5.3:* If  $L \in \text{Im}(\Psi)$ , then  $L \cap \mathcal{S}(\mathcal{H}) \neq \emptyset$ .

*Proof:*  $L \in \text{Im}(\Psi)$  implies that there exist  $L_1$  and  $L_2$  such that  $L = \Psi(L_1, L_2)$ . By definition  $\Psi(L_1, L_2) = (S_1 \otimes S_2) \cap \mathcal{C}$ , with  $L_1 = S_1 \cap \mathcal{C}_1$  and  $L_2 = S_2 \cap \mathcal{C}_2$ . Let  $\rho_1 \in L_1$  and  $\rho_2 \in L_2$ . Then,  $\rho_1 \otimes \rho_2 \in L$ . But we have also that  $\rho_1 \otimes \rho_2 \in \mathcal{S}(\mathcal{H})$ , and then  $L \cap \mathcal{S}(\mathcal{H}) \neq \emptyset$ . ■

From the last proposition it follows that  $\text{Im}(\Psi) \subset \mathcal{L}$  because if we take a nonseparable state  $\rho \in \mathcal{C}$ , then  $\rho \in \mathcal{L}$ , but  $\rho \cap \mathcal{S}(\mathcal{H}) = \emptyset$ , and so, it cannot belong to  $\text{Im}(\Psi)$ . Note that, in general,  $L_1 \tilde{\otimes} L_2$  is not an element of  $\mathcal{L}$ .

Let us compute  $\mathcal{C}_1 \tilde{\otimes} \mathcal{C}_2$ . Remember that  $\mathcal{C}_1 = [\mathcal{A}_1] \in \mathcal{L}_1$  and  $\mathcal{C}_2 = [\mathcal{A}_2] \in \mathcal{L}_2$ ,

$$\mathcal{C}_1 \tilde{\otimes} \mathcal{C}_2 = \left\{ \sum \lambda_{ij} \rho_i^1 \otimes \rho_j^2 \mid \rho_i^1 \in \mathcal{C}_1, \rho_j^2 \in \mathcal{C}_2, \sum \lambda_{ij} = 1 \text{ and } \lambda_{ij} \geq 0 \right\}.$$

So, using the definition of  $\mathcal{S}(\mathcal{H})$ , the set of all separable states, we have

$$\mathcal{S}(\mathcal{H}) = \mathcal{C}_1 \tilde{\otimes} \mathcal{C}_2.$$

We know that  $\mathcal{C}_1 \tilde{\otimes} \mathcal{C}_2 \subset \mathcal{C}$ . But it does not necessarily belong to  $\mathcal{L}$ . We can prove also the following propositions.

*Proposition 5.4:* Let  $L \in \text{Im}(\Psi)$  and  $\rho \in L$ . Then,  $\rho$  is a linear combination of product states.

*Proof:* Let  $L \in \text{Im}(\Psi)$ . Then, there exist  $L_1 \in \mathcal{L}_1$  and  $L_2 \in \mathcal{L}_2$  such that  $\Psi(L_1, L_2) = L$ . If  $L_1 = S_1 \cap \mathcal{C}_1$  and  $L_2 = S_2 \cap \mathcal{C}_2$ , with  $S_1$  and  $S_2$  good representatives, we have

$$L = (S_1 \otimes S_2) \cap \mathcal{C} \Rightarrow \rho = \sum_{i,j} \lambda_{ij} \rho_i^1 \otimes \rho_j^2.$$

■

*Proposition 5.5:* Let  $\rho = \rho_1 \otimes \rho_2$ , with  $\rho_1 \in \mathcal{C}_1$  and  $\rho_2 \in \mathcal{C}_2$ . Then  $\{\rho\} = \Psi(\{\rho_1\}, \{\rho_2\})$  with  $\{\rho_1\} \in \mathcal{L}_1$ ,  $\{\rho_2\} \in \mathcal{L}_2$ , and  $\{\rho\} \in \mathcal{L}$ .

*Proof:* We already know that the atoms are elements of the lattices,

$$\Psi(\{\rho_1\}, \{\rho_2\}) = (\langle \rho_1 \rangle \otimes \langle \rho_2 \rangle) \cap \mathcal{C} = \langle \rho_1 \otimes \rho_2 \rangle \cap \mathcal{C} = \{\rho_1 \otimes \rho_2\} = \{\rho\}.$$

*Proposition 5.6:* Let  $\rho \in \mathcal{S}(\mathcal{H})$ , the set of separable states. Then, there exist  $L \in \mathcal{L}$ ,  $L_1 \in \mathcal{L}_1$ , and  $L_2 \in \mathcal{L}_2$ , such that  $\rho \in L$  and  $L = \Psi(L_1, L_2)$ .

*Proof:* If  $\rho \in \mathcal{S}(\mathcal{H})$ , then  $\rho = \sum_{ij} \lambda_{ij} \rho_i^1 \otimes \rho_j^2$ , with  $\sum_{ij} \lambda_{ij} = 1$  and  $\lambda_{ij} \geq 0$ . Consider the subspaces,

$$S_1 = \langle \rho_1^1, \rho_2^1, \dots, \rho_k^1 \rangle, \quad S_2 = \langle \rho_1^2, \rho_2^2, \dots, \rho_l^2 \rangle.$$

Take  $L_1 = S_1 \cap \mathcal{C}_1$  and  $L_2 = S_2 \cap \mathcal{C}_2$ . Let us observe first that  $\langle S_1 \cap \mathcal{C}_1 \rangle \subseteq S_1$ . We have  $\rho_i^1 \in \mathcal{C}_1$  and so,  $\langle S_1 \cap \mathcal{C}_1 \rangle = S_1$  because  $S_1$  is generated by the set  $\rho_i^1$ . We also have that  $\langle S_2 \cap \mathcal{C}_2 \rangle = S_2$ . Now we can compute

$$\Psi(L_1, L_2) = (\langle S_1 \cap \mathcal{C}_1 \rangle \otimes \langle S_2 \cap \mathcal{C}_2 \rangle) \cap \mathcal{C} = (S_1 \otimes S_2) \cap \mathcal{C}.$$

But the set  $\{\rho_i^1 \otimes \rho_j^2\}$  generates  $S_1 \otimes S_2$ , and then  $(S_1 \otimes S_2) \cap \mathcal{C}$  is formed by all the possible convex combinations of  $\{\rho_i^1 \otimes \rho_j^2\}$ . This proves that  $\rho \in L$ . ■

The above propositions show that  $\text{Im}(\Psi)$  encodes information related to separable states. As a general state in  $\mathcal{S}$  is nonseparable, we obtain that  $\text{Im}(\Psi)$  is not equal to  $\mathcal{L}$ . This is a reasonable result. If we interpret  $\mathcal{L}_1$  and  $\mathcal{L}_2$  as encoding all the information that is available for  $S_1$  and  $S_2$  expressed via observables of the subsystems separately, it will never be possible to reconstruct from it alone all the information about the correlations between  $S_1$  and  $S_2$ , which is encoded in  $\mathcal{L}$ . This information is available only in observables of the whole system  $\mathcal{S}$ . From  $\text{Im}(\Psi)$  it is possible to recover information about separated states only. As said above, the tensor product contains more information than that of its parties, and this is directly linked to the nonexistence of a satisfactory theory of tensor products of orthomodular posets and lattices compatible with physics.

## B. Projections onto $\mathcal{L}_1$ and $\mathcal{L}_2$ (going down)

There are other maps of interest. If the whole system is in a state  $\rho$ , using partial traces we can define states for the subsystem  $\rho_1 = \text{tr}_2(\rho)$  and similarly for  $\rho_2$ . Then, we can consider the maps,

$$\text{tr}_i: \mathcal{C} \rightarrow \mathcal{C}_i | \rho \rightarrow \text{tr}_i(\rho),$$

from which we can construct the induced projections,

$$\tau_i: \mathcal{L} \rightarrow \mathcal{L}_i | S \cap \mathcal{C} \rightarrow \text{tr}_i(\langle S \cap \mathcal{C} \rangle) \cap \mathcal{C}_i.$$

In terms of good representatives  $\tau_i([S]) = [\text{tr}_i(S)]$ . Then we can define the product map,

$$\tau: \mathcal{L} \rightarrow \mathcal{L}_1 \times \mathcal{L}_2 | L \rightarrow (\tau_1(L), \tau_2(L)).$$

We can prove the following about the image of  $\tau_i$ .

*Proposition 5.7:* The functions  $\tau_i$  are surjective and preserve the  $\vee$ -operation. They are not injective.

*Proof:* Take  $L_1 \in \mathcal{L}_1$ . Choose an arbitrary element of  $\mathcal{C}_2$ , say  $\rho_2$ . Now consider the following element of  $\mathcal{L}$ :

$$L = \langle L_1 \otimes \rho_2 \rangle \cap \mathcal{C}.$$

It is clear that  $\tau_1(L) = L_1$  because if  $\rho_1 \in L_1$ , then  $\text{tr}(\rho_1 \otimes \rho_2) = \rho_1$ . So,  $\tau_1$  is surjective. On the other hand, the arbitrariness of  $\rho_2$  implies that it is not injective. An analogous argument follows for  $\tau_2$ .

Let us see that  $\tau_i$  preserves the  $\vee$ -operation,

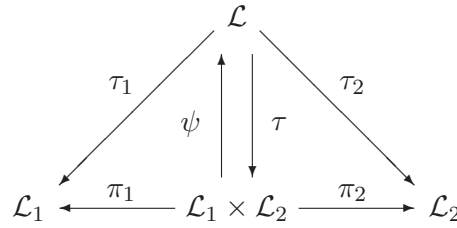


FIG. 3. The different maps between  $\mathcal{L}_1$ ,  $\mathcal{L}_2$ ,  $\mathcal{L}_1 \times \mathcal{L}_2$ , and  $\mathcal{L}$ .  $\pi_1$  and  $\pi_2$  are the canonical projections.

$$\tau_i([S] \vee [T]) = \tau_i([S \oplus T]) = [\text{tr}_i(S \oplus T)] = [\text{tr}_i(S) \oplus \text{tr}_i(T)] = [\text{tr}_i(S)] \vee [\text{tr}_i(T)] = \tau_i([S]) \vee \tau_i([T]).$$

Let us now consider the  $\wedge$ -operation. Let us compute

$$\tau_i([S] \wedge [T]) = \tau_i([S \cap T]) = [\text{tr}_i(S \cap T)] \subseteq [\text{tr}_i(S) \cap \text{tr}_i(T)] = [\text{tr}_i(S)] \wedge [\text{tr}_i(T)] = \tau_i([S]) \wedge \tau_i([T]).$$

It is easy to see that  $\text{tr}_i(S \cap T) \subseteq \text{tr}_i(S) \cap \text{tr}_i(T)$ . This is because if  $\rho \in \text{tr}_i(S \cap T)$ , then  $\rho = \text{tr}_i(\sigma)$ , with  $\sigma \in S$  and  $\sigma \in T$ . This means that  $\rho \in \text{tr}_i(S) \cap \text{tr}_i(T)$ , and so we have the inclusion of classes. But these sets are not equal, in general, as the following example shows. Take  $\{\rho_1 \otimes \rho_2\} \in \mathcal{L}$  and  $\{\rho_1 \otimes \rho'_2\} \in \mathcal{L}$ , with  $\rho' \neq \rho$ . It is clear that  $\{\rho_1 \otimes \rho_2\} \wedge \{\rho_1 \otimes \rho'_2\} = \mathbf{0}$  and so,  $\tau_1(\{\rho_1 \otimes \rho_2\}) \wedge \tau_1(\{\rho_1 \otimes \rho'_2\}) = \mathbf{0}$ . On the other hand,  $\tau_1(\{\rho_1 \otimes \rho_2\}) = \{\rho_1\} = \tau_1(\{\rho_1 \otimes \rho'_2\})$ , and so,  $\tau_1(\{\rho_1 \otimes \rho_2\}) \wedge \tau_1(\{\rho_1 \otimes \rho'_2\}) = \{\rho_1\}$ . A similar fact holds for the  $\neg$ -operation.

The lack of injectivity of the  $\tau_i$  may be physically recognized from the fact that the state of the whole system encodes information about correlations between its parts. It is again useful to make a comparison with the classical case in order to illustrate what is happening. The same as in classical mechanics, we have atoms in  $\mathcal{L}$  which are tensor products of atoms of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . But in contrast to classical mechanics, entangled states originate atoms of  $\mathcal{L}$  which cannot be expressed in such a way and thus, loosely using a topological language, we may say the fiber of the projection  $\tau_i$  is much bigger than that of its classical counterpart.

It is important to note that the projection function  $\tau$  cannot be properly defined within the frame of the traditional approach of QL because there was no place for improper mixtures in  $\mathcal{L}_{vN}$ , where they have to be defined as functions over the sublattices. On the contrary, mixtures are *elements* of the lattices  $\mathcal{L}$  and  $\mathcal{L}_i$ , and thus we can define the projections from the lattice of the whole system to the lattices of the subsystems mapping the states of  $S$  into the corresponding states of  $S_i$ . This enables a more natural approach when compound systems are considered from a quantum logical point of view.

It is interesting also to analyze the functions  $\Psi \circ \tau$  and  $\tau \circ \Psi$ .

*Proposition 5.8:*  $\tau \circ \Psi = Id$ .

*Proof:* Let us see it in terms of good representatives,

$$\tau_1(\Psi([S], [T])) = \tau_1([S \otimes T]) = [\text{tr}_1(S \otimes T)] = [S],$$

$$\tau_2(\Psi([S], [T])) = \tau_2([S \otimes T]) = [\text{tr}_2(S \otimes T)] = [T].$$

Then

$$\tau(\Psi([S], [T])) = ([S], [T]).$$

It is clear, from a physical point of view, that  $\Psi \circ \tau$  is not the identity function: when we take partial traces information is lost that cannot be recovered by making products of states. This can be summarized as “going down and then going up is not the same as going up and then going down” (another way to express quantum nonseparability). We show these maps in Fig. 3.

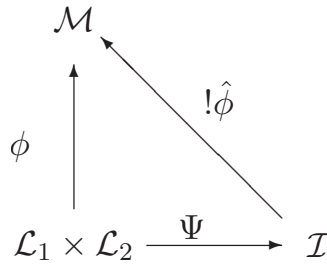


FIG. 4. This is a commutative diagram.  $(\mathcal{I}, \Psi)$  is the lattice tensor product (in categorical terms) of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ .

Let us finally make an observation about the image of  $\Psi$ . Consider the category of lattices as objects and lattice morphisms as arrows. A bimorphism is a morphism in each variable, and Proposition 5.1 ensures that  $\Psi$  is a bimorphism. Let us define  $\mathcal{I}$  as the lattice generated by  $\text{Im}(\Psi)$  inside  $\mathcal{L}$ . Then, the following relationship holds between  $\mathcal{I}$ ,  $\mathcal{L}_1$  and  $\mathcal{L}_2$  (see Fig. 4).

*Proposition 5.9:*  $(\mathcal{I}, \Psi)$  is the lattice tensor product (in categorical terms) of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . That is, it satisfies the following universal property: for every bimorphism of lattices  $\phi: \mathcal{L}_1 \times \mathcal{L}_2 \rightarrow \mathcal{M}$  there exists a unique  $\hat{\phi}: \mathcal{I} \rightarrow \mathcal{M}$  such that  $\hat{\phi}\Psi = \phi$ . Moreover, if  $(\mathcal{I}', \Psi')$  is another product then they are isomorphic by a unique isomorphism.

*Proof:* Let  $\phi: \mathcal{L}_1 \times \mathcal{L}_2 \rightarrow \mathcal{M}$  a bimorphism where  $\mathcal{M}$  is an arbitrary lattice. Given that  $\text{Im}(\Psi)$  lattice generates  $\mathcal{I}$  we can define  $\hat{\phi}$  over the elements of the form  $[S \otimes T]$ ,

$$\hat{\phi}([S \otimes T]) := \phi([S], [T]).$$

Note that it is unique by definition and  $\hat{\phi}\Psi = \phi$ .

The unicity of  $(\mathcal{I}, \Psi)$  follows from a standard categorical argument: Given that  $\Psi'$  is a bimorphism we have  $\hat{\Psi}'\Psi' = \Psi'$  because  $\Psi$  has the universal property. Given that  $\Psi'$  also has the universal property we have  $\text{Id}_{\mathcal{I}'}\Psi' = \Psi'$ . The same holds for  $\Psi$ , that is,  $\hat{\Psi}\Psi = \Psi$  and  $\text{Id}_{\mathcal{I}}\Psi = \Psi$ . Note that  $\hat{\Psi}', \hat{\Psi}, \text{Id}_{\mathcal{I}'}, \text{Id}_{\mathcal{I}}$  are all unique having this property. Given that  $\hat{\Psi}'\hat{\Psi}\Psi' = \Psi'$  and  $\hat{\Psi}\hat{\Psi}'\Psi = \Psi$  then we have

$$\hat{\Psi}\hat{\Psi}' = \text{Id}_{\mathcal{I}}, \quad \hat{\Psi}'\hat{\Psi} = \text{Id}_{\mathcal{I}'}. \quad \blacksquare$$

So  $\mathcal{I}$  and  $\mathcal{I}'$  are isomorphic by a unique isomorphism. ■

## VI. CONCLUSIONS

In this article we have shown that it is possible to construct a lattice theoretical framework which incorporates improper mixtures as atoms. This is done in order to overcome a problem of the standard QL formalism posed in Sec. III, namely, that the conjunction of all actual properties of the system does not yield the actual state of the system when compound systems are considered. We showed that this is directly linked with the fact that QL treats improper mixtures as measures over the projection lattice, in an analogous way as classical statistical distributions are measured over the phase space. But unlike classical mixtures, improper mixtures in quantum mechanics do not admit an ignorance interpretation, and this was at the origin of the problems posed in Sec. III. Our construction is a quite natural extension of the von Neumann lattice, and its properties and characteristics are consistent with the constraints imposed by quantum mechanics. More precisely, in the standard quantum logical approach, when the whole system is in a pure entangled state there are no elements available in the lattices of the subsystems to represent the states of the subsystems as elements of the lattice. This is expressed in the absence of projection functions which map the states of the lattice of the whole system to the states of the lattices of the subsystems which satisfy in turn to be compatible with the physical description. Unlike the standard approach, the projections

defined in the frame of the enlarged structure satisfy this condition. They are also the canonical ones in the sense that they are constructed using partial traces, in accordance with the quantum formalism. This was shown in Sec. V B.

Traditionally, the difference between classical and quantum lattices is said to be that the classical lattice is a Boolean lattice while von Neumann lattice is an orthomodular one. We claim that this is not the only difference, the other one—although not independent—being their behavior with respect to the coupling of two or more systems. The necessity of the enlargement of the von Neumann lattice in order to preserve the condition that the meet of actual properties defines the state of the system may be seen as an algebraic expression of the existence of entanglement. The approach presented here shows, in an algebraic fashion, the radical difference between quantum mechanics and classical mechanics when two systems interact. If the systems are classical, no nontrivial enlargement of the lattice is needed even in the presence of interactions. It is enough in order to describe all relevant physics about the subsystems. But the existence of entanglement in quantum mechanics forces an enlargement of the state space of pure states to the convex set  $\mathcal{C}$  to deal with the states of subsystems and thus the enlargement of  $\mathcal{L}_{vN}$ . A possible candidate to fulfill this task, namely, the lattice  $\mathcal{L}$ , has been presented in this work and the relations among  $\mathcal{L}$  and  $\mathcal{L}_i$  have been analyzed. We think that paying more attention to this kind of approaches would shed new light on the algebraic properties of quantum nonseparability.

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