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Exact solution for a two-phase Stefan problem with variable latent heat and a convective boundary condition at the fixed face

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Abstract. Recently, in Tarzia (Thermal Sci 21A:1–11, 2017) for the classical two-phase Lamé–Clapeyron–Stefan problem an equivalence between the temperature and convective boundary conditions at the fixed face under a certain restriction was obtained. Motivated by this article we study the two-phase Stefan problem for a semi-infinite material with a latent heat defined as a power function of the position and a convective boundary condition at the fixed face. An exact solution is constructed using Kummer functions in case that an inequality for the convective transfer coefficient is satisfied generalizing recent works for the corresponding one-phase free boundary problem. We also consider the limit to our problem when that coefficient goes to infinity obtaining a new free boundary problem, which has been recently studied in Zhou et al. (J Eng Math 2017. https://doi.org/10.1007/s10665-017-9921-y).

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List of symbols

d_{l}, d_{s}	Diffusivity coefficient (m^2/s)
h_0	Coefficient that characterizes the heat transfer in condition (1.6) [$\mathrm{kg}/(^\circ\mathrm{C}\mathrm{s}^{5/2})$]
k_{l},k_{s}	Thermal conductivity [$\mathrm{W}/(\mathrm{m}\ ^\circ\mathrm{C})$]
s	Position of the free front (m)
t	Time (s)
T_∞	Coefficient that characterizes the bulk temperature in condition (1.6) [$^\circ\mathrm{C}/\mathrm{s}^{\alpha/2}$]
T_i	Coefficient that characterizes the initial temperature of the material in condition (1.7), [$^\circ\mathrm{C}/\mathrm{m}^\alpha$]
x	Spatial coordinate (m)

Greek symbols

α	Power of the position that characterizes the latent heat per unit volume (dimensionless)
γ	Coefficient that characterizes the latent heat per unit volume [$\mathrm{kg}/(\mathrm{s}^2\mathrm{m}^{\alpha+1})$]
ν	Coefficient that characterizes the free interface (dimensionless)
η	Similarity variable in expression (2.1) (dimensionless)
Ψ_{l}, Ψ_{s}	Temperature ($^\circ$C).

Subscripts

l	Liquid phase
s	Solid phase

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1. Introduction

The study of heat transfer problems with phase change such as melting and freezing has attracted growing attention in the last decades due to their wide range of engineering and industrial applications. Stefan problems can be modelled as basic phase-change processes where the location of the interface is a priori unknown. They arise in a broad variety of fields like melting, freezing, drying, friction, lubrication, combustion, finance, molecular diffusion, metallurgy and crystal growth. Due to their importance, they have been largely studied since the last century [2,5–8,10,15,19]. For an account of the theory, we refer the reader to [20].

In the classical formulation of Stefan problems, there are many assumptions on the physical factors involved that are taken into account in order to simplify the description of the process. The latent heat, which is the energy required to accomplish the phase change, is usually considered constant. However in many practical problems a constant latent heat may be not appropriate, being necessary to assume a variable one. The physical bases of this particular assumption can be found in the movement of a shoreline [22], in the ocean delta deformation [9] or in the cooling body of a magma [12].

In [13], sufficient conditions for the existence and uniqueness of solution of a one-phase Stefan problem taking a latent heat as a general function of the position were found. In [22], as well as in [16] an exact solution was found for a one-phase and two-phase Stefan problem, respectively, considering the latent heat as a linear function of the position. [24] generalized [22] by considering the one-phase Stefan problem with the latent heat as a power function of the position with an integer exponent. Recently, in [25] the latter problem was studied assuming a real non-negative exponent. The explicit solution for two different problems defined according to the boundary conditions considered was presented: temperature and flux.

Boundary conditions imposed at a surface of a body in order to have a well-posed mathematical problem can be specified in terms of temperature or energy flow. One of the most realistic boundary conditions is the convective one, in which the heat flux depends not only on the ambient conditions but also on the temperature of the surface itself. In [18], the relationship between a classical two-phase Stefan problem considering temperature and convective boundary condition at the fixed face $x = 0$ was studied. In [4], a nonlinear one-phase Stefan problem with a convective boundary condition in Storm's materials was studied.

Motivated by [18] and [25], in [3] we studied the one-phase Stefan problem considering a variable latent heat and a convective boundary condition at the fixed face $x = 0$. In the present paper, we are going to analyse the existence and uniqueness of solution of a two-phase Stefan problem, considering an homogeneous semi-infinite material, with a latent heat as a power function of the position and a convective boundary condition at the fixed face $x = 0$. This problem can be formulated in the following way: find the temperatures $\Psi_{\mathrm{l}}(x,t)$, $\Psi_{\mathrm{s}}(x,t)$ and the moving melt interface $s(t)$ such that:

$$\Psi_{lt}(x,t) = d_{\mathrm{l}}\Psi_{\mathrm{l}xx}(x,t), \qquad 0 < x < s(t), \quad t > 0, \tag{1.1}$$

$$\Psi_{\mathrm{s}t}(x,t) = d_{\mathrm{s}}\Psi_{\mathrm{s}xx}(x,t), \qquad x > s(t), \quad t > 0, \tag{1.2}$$

$$s(0) = 0, \tag{1.3}$$

$$\Psi_{\mathrm{l}}(s(t),t) = \Psi_{\mathrm{s}}(s(t),t) = 0, \qquad t > 0, \tag{1.4}$$

$$k_{\mathrm{s}}\Psi_{\mathrm{s}x}(s(t),t) - k_{\mathrm{l}}\Psi_{\mathrm{l}x}(s(t),t) = \gamma s(t)^{\alpha}\dot{s}(t), \qquad t > 0, \tag{1.5}$$

$$k_{\mathrm{l}}\Psi_{\mathrm{l}x}(0,t) = h_0 t^{-1/2}\left[\Psi_{\mathrm{l}}(0,t) - T_{\infty}t^{\alpha/2}\right] \qquad t > 0, \tag{1.6}$$

$$\Psi_{\mathrm{s}}(x,0) = -T_i x^{\alpha}, \qquad x > 0. \tag{1.7}$$

where the liquid (solid) phase is represented by the subscript l (s), Ψ is the temperature, d is the diffusion coefficient, γx^{α} is the variable latent heat per unity of volume, $-T_i x^{\alpha}$ is the depth-varying initial temperatures and the phase-transition temperature is zero. Condition (1.6) represents the convective boundary condition at the fixed face $x = 0$. T_{∞} is the bulk temperature at a large distance from the fixed face $x = 0$, and h_0 is the coefficient that characterizes the heat transfer at the fixed face. Moreover, $\dot{s}(t)$ represents the velocity of the phase-change interface. We will work under the assumption that $\gamma, T_i, T_\infty, h_0 > 0$ which corresponds to the melting case.

In Sect. 2, we will quickly review fundamental results that allow us to apply the similarity transformation technique to our problem. We will analyse the fusion of a semi-infinite material which is initially at the solid phase, where a convective condition is imposed at the fixed boundary $x = 0$ and where the latent heat is considered as a power function of the position with power α. In Sect. 3, we will provide an explicit solution of a similarity type of problems (1.1)–(1.7) under certain conditions on the data, proving in addition its uniqueness in case that α is a positive non-integer exponent. We will study the particular case when α is a non-negative integer, recovering for $\alpha = 0$ the results obtained by [18]. Finally Sect. 4 will show that the solution to our problem converges to the solution of a different free boundary problem with a prescribed temperature at $x = 0$ when the coefficient $h_0 \to +\infty$ which has been recently studied in [23].

The main contribution of this paper is to generalize the work that has been done in [18,25] and [3], by obtaining the explicit solution of a one-dimensional two-phase Stefan problem for a semi-infinite material where a variable latent heat and a convective boundary condition at the fixed face are considered, as well as to obtain the results given in [23] when the coefficient that characterizes the convective boundary condition goes to infinity.

2. Explicit solution with latent heat depending on the position and a convective boundary condition at $x = 0$

In this section, the explicit solution of the problem governed by (1.1)–(1.7) will be found. The proof will be split into two subsections. The first one results from the work of Zhou and Xia in [25] and corresponds to the case when α is positive and non-integer. The second one is correlated with the case when α is a non-negative integer, based on [24].

2.1. Case when α is a positive non-integer exponent

The following lemma has already been developed by Zhou-Xia in [25] and constitutes the base on which we will find solutions for the differential heat Eqs. (1.1)–(1.2).

Lemma 2.1. 1. *Let*

$$\Psi(x,t) = t^{\alpha/2} f(\eta), \quad \textit{with} \quad \eta = \frac{x}{2\sqrt{dt}} \tag{2.1}$$

then $\Psi = \Psi(x,t)$ is a solution of the heat equation $\Psi_t(x,t) = d\Psi_{xx}(x,t)$, with $d > 0$ if and only if $f = f(\eta)$ satisfies the following ordinary differential equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}(\eta) + 2\eta \frac{\mathrm{d}f}{\mathrm{d}\eta}(\eta) - 2\alpha f(\eta) = 0. \tag{2.2}$$

2. *An equivalent formulation for Eq. (2.2), introducing the new variable $z = -\eta^2$ is:*

$$z\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}(z) + \left(\frac{1}{2} - z\right)\frac{\mathrm{d}f}{\mathrm{d}z}(z) + \frac{\alpha}{2} f(z) = 0. \tag{2.3}$$

3. *The general solution of the ordinary differential Eq. (2.3), called Kummer's equation, is given by:*

$$f(z) = \widehat{c_{11}} M\left(-\frac{\alpha}{2}, \frac{1}{2}, z\right) + \widehat{c_{21}} U\left(-\frac{\alpha}{2}, \frac{1}{2}, z\right). \tag{2.4}$$

where $\widehat{c_{11}}$ *and* $\widehat{c_{21}}$ *are arbitrary constants and* $M(a,b,z)$ *and* $U(a,b,z)$ *are the Kummer functions defined by:*

$$M(a,b,z)=\sum_{s=0}^{\infty}\frac{(a)_s}{(b)_s s!}z^s,\ \text{where } b \text{ cannot be a non-positive integer,} \tag{2.5}$$

$$U(a,b,z)=\tfrac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a,b,z)+\tfrac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}M(a-b+1,2-b,z). \tag{2.6}$$

where $(a)_s$ *is the Pochhammer symbol:*

$$(a)_s=a(a+1)(a+2)\dots(a+s-1),\qquad (a)_0=1 \tag{2.7}$$

Proof. See [25]. □

Remark 2.2. All the properties of Kummer's functions to be used in the following arguments can be found in "Appendix A".

Remark 2.3. Taking into account Eq. (2.4) and definition (2.6), we can rewrite the general solution of the ordinary differential Eq. (2.3) as:

$$f(z)=\overline{c_{11}}M\left(-\frac{\alpha}{2},\frac{1}{2},z\right)+\overline{c_{21}}z^{1/2}M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},z\right), \tag{2.8}$$

where $\overline{c_{11}}$ and $\overline{c_{21}}$ are arbitrary constants.

Remark 2.4. Taking into account Lemma 2.1 and Remark 2.3, we can assure that $\Psi(x,t)=t^{\alpha/2}f(\eta)$ satisfies the heat equation $\Psi_t(x,t)=d\Psi_{xx}(x,t)$ if and only if it is defined as:

$$\Psi(x,t)=t^{\alpha/2}\left[c_{11}M\left(-\frac{\alpha}{2},\frac{1}{2},-\eta^2\right)+c_{21}\eta M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-\eta^2\right)\right], \tag{2.9}$$

with $\eta=\frac{x}{2\sqrt{dt}}$ and where c_{11} and c_{21} are arbitrary constants (not necessarily real).

Our main outcome is given by the following theorem, which constitutes a generalization to the two-phase case of [3]. This theorem ensures the existence and uniqueness of solution of problems (1.1)–(1.7) under a restriction for the convective coefficient, providing in addition to the explicit solution.

Theorem 2.5. *If the coefficient* h_0 *satisfies the inequality:*

$$h_0>\frac{2^{\alpha}\Gamma\left(\frac{\alpha}{2}+1\right)k_{\mathrm{s}}T_i d_{\mathrm{s}}^{(\alpha-1)/2}}{T_\infty\sqrt{\pi}} \tag{2.10}$$

then there exists an instantaneous fusion process and the free boundary problems (1.1)–(1.7) *has a unique solution of a similarity type given by:*

$$s(t)=2\nu\sqrt{d_l t}, \tag{2.11}$$

$$\Psi_l(x,t)=t^{\alpha/2}\left[E_l M\left(-\frac{\alpha}{2},\frac{1}{2},-\eta_l^2\right)+F_l\eta_l M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-\eta_l^2\right)\right], \tag{2.12}$$

$$\Psi_{\mathrm{s}}(x,t)=t^{\alpha/2}\left[E_s M\left(-\frac{\alpha}{2},\frac{1}{2},-\eta_s^2\right)+F_s\eta_s M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-\eta_s^2\right)\right], \tag{2.13}$$

where $\eta_l=\frac{x}{2\sqrt{d_l t}}$, $\eta_s=\frac{x}{2\sqrt{d_{\mathrm{s}} t}}$ *and the constants* E_l, F_l, E_s *and* F_s *are given by:*

$$E_l=\frac{-\nu M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-\nu^2\right)}{M\left(-\frac{\alpha}{2},\frac{1}{2},-\nu^2\right)}F_l, \tag{2.14}$$

$$F_l = \frac{-h_0 T_\infty 2\sqrt{d_l} M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right)}{\left[k_l M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right) + 2\sqrt{d_l} h_0 \nu M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\right)\right]}, \tag{2.15}$$

$$E_s = \frac{-\nu\omega M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\omega^2\right)}{M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\omega^2\right)} F_s, \qquad \textit{with} \quad \omega = \sqrt{d_l/d_s}, \tag{2.16}$$

$$F_s = \frac{-T_i 2^{\alpha+1} d_s^{\alpha/2} M\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, \nu^2\omega^2\right)}{U\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, \nu^2\omega^2\right)}. \tag{2.17}$$

and the dimensionless coefficient ν is the unique positive solution of the following equation:

$$-\frac{k_s T_i d_s^{(\alpha-1)/2}}{\gamma d_l^{(\alpha+1)/2}} f_1(x) + \frac{h_0 T_\infty}{\gamma 2^\alpha d_l^{(\alpha+1)/2}} f_2(x) = x^{\alpha+1}, \qquad x > 0. \tag{2.18}$$

in which functions f_1 and f_2 are defined by:

$$f_1(x) = \frac{1}{U\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, x^2\omega^2\right)}, \qquad x > 0, \tag{2.19}$$

$$f_2(x) = \frac{1}{\left[M\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, x^2\right) + 2\frac{\sqrt{d_l} h_0}{k_l} x M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, x^2\right)\right]}, \ x > 0. \tag{2.20}$$

Proof. The general solution of Eqs. (1.1)–(1.2) based on Kummer functions is given by Lemma 2.1 and Remark 2.4:

$$\Psi_l(x,t) = t^{\alpha/2}\left[E_l M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta_l^2\right) + F_l \eta_l M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta_l^2\right)\right], \tag{2.21}$$

$$\Psi_s(x,t) = t^{\alpha/2}\left[E_s M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta_s^2\right) + F_s \eta_s M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta_s^2\right)\right], \tag{2.22}$$

where $\eta_l = \frac{x}{2\sqrt{d_l t}}$, $\eta_s = \frac{x}{2\sqrt{d_s t}}$, and E_l, F_l, E_s and F_s are coefficients that must be determined.

Furthermore, condition (1.4) together with (2.21) implies that the free boundary should take the following form:

$$s(t) = 2\nu\sqrt{d_l t}. \tag{2.23}$$

where ν is a constant that also has to be computed.

Using the derivation formulas for the Kummer functions (A.4)–(A.5) presented in "Appendix A", it is deduced that:

$$\begin{aligned} \Psi_{lx}(x,t) = \frac{t^{(\alpha-1)/2}}{\sqrt{d_l}} \Bigg[& E_l \alpha \eta_l M\left(-\frac{\alpha}{2} + 1, \frac{3}{2}, -\eta_l^2\right) \\ & + \frac{F_l}{2} M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, -\eta_l^2\right)\Bigg], \end{aligned} \tag{2.24}$$

$$\Psi_{sx}(x,t) = \frac{t^{(\alpha-1)/2}}{\sqrt{d_s}} \left[E_s \alpha \eta_s M\left(-\frac{\alpha}{2} + 1, \frac{3}{2}, -\eta_s^2\right)\right.$$

$$+\frac{F_s}{2}M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2},-\eta_s^2\right)\Bigg]. \tag{2.25}$$

From Eq. (1.4), we have:

$$t^{\alpha/2}\left[E_l M\left(-\frac{\alpha}{2},\frac{1}{2},-\nu^2\right)+F_l\nu M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-\nu^2\right)\right]=0. \tag{2.26}$$

Isolating E_l, we obtain (2.14).

On the other hand, using (2.21) and (2.24), condition (1.6) becomes

$$k_{\rm l}\frac{F_l}{2\sqrt{d_{\rm l}}}=h_0\left[E_l-T_\infty\right], \tag{2.27}$$

and replacing E_l given by (2.14) into (2.27) we get that F_l is given by (2.15).

Condition (1.4), $\Psi_{\rm s}(s(t),t)=0$ implies:

$$E_s M\left(-\frac{\alpha}{2},\frac{1}{2},-\nu^2\omega^2\right)+F_s\nu\omega M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-\nu^2\omega^2\right)=0 \quad \text{where } \omega=\sqrt{\tfrac{d_{\rm l}}{d_{\rm s}}}, \tag{2.28}$$

leading us to define E_s by (2.16).

In view of condition (1.7), it is necessary to compute $\Psi_{\rm s}(x,0)$, given by the expression:

$$\begin{aligned}\Psi_{\rm s}(x,0)=\lim_{t\to 0}\Psi_{\rm s}(x,t)&=E_s\left[\lim_{t\to 0}t^{\alpha/2}M\left(-\tfrac{\alpha}{2},\tfrac{1}{2},-\eta_s^2\right)\right]\\ &+F_s\left[\lim_{t\to 0}t^{\alpha/2}\eta_s M\left(-\tfrac{\alpha}{2}+\tfrac{1}{2},\tfrac{3}{2},-\eta_s^2\right)\right]\end{aligned} \tag{2.29}$$

Taking into account formula (A.9) from "Appendix A", we obtain:

$$\begin{aligned}M\left(-\frac{\alpha}{2},\frac{1}{2},-\eta_s^2\right)=&\left[\frac{\sqrt{\pi}}{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)}e^{-\frac{\alpha}{2}\pi i}U\left(-\frac{\alpha}{2},\frac{1}{2},-\eta_s^2\right)+\right.\\ &\left.+\frac{\sqrt{\pi}}{\Gamma\left(-\frac{\alpha}{2}\right)}e^{-\frac{(\alpha+1)}{2}\pi i}e^{-\eta_s^2}U\left(\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2},\eta_s^2\right)\right].\end{aligned} \tag{2.30}$$

and

$$\begin{aligned}M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-\eta_s^2\right)=&\left[\frac{\sqrt{\pi}}{2\Gamma\left(\frac{\alpha}{2}+1\right)}e^{(-\frac{\alpha}{2}+\frac{1}{2})\pi i}U\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-\eta_s^2\right)+\right.\\ &\left.+\frac{\sqrt{\pi}}{2\Gamma\left(-\frac{\alpha}{2}+\frac{1}{2}\right)}e^{-(\frac{\alpha}{2}+1)\pi i}e^{-\eta_s^2}U\left(\frac{\alpha}{2}+1,\frac{3}{2},\eta_s^2\right)\right].\end{aligned} \tag{2.31}$$

We can observe that if α is a non-negative even integer then $\Gamma\left(-\frac{\alpha}{2}\right)$ is not defined, and so (2.30) is not valid. In the same way if α is a non-negative odd integer, then $\Gamma\left(-\frac{\alpha}{2}+\frac{1}{2}\right)$ is neither defined and (2.31) cannot be applied. From this fact, we restrict α to be positive and non-integer.

Considering (2.30) and (2.31) and applying (A.7), we obtain the following limits:

$$\lim_{t\to 0}\left[t^{\alpha/2}M\left(-\frac{\alpha}{2},\frac{1}{2},-\eta_s^2\right)\right]=\frac{\sqrt{\pi}}{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)}\frac{x^\alpha}{2^\alpha d_{\rm s}^{\alpha/2}}. \tag{2.32}$$

and

$$\lim_{t\to 0}t^{\alpha/2}\eta_s M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-\eta_s^2\right)=\frac{\sqrt{\pi}}{\Gamma\left(\frac{\alpha}{2}+1\right)}\frac{x^\alpha}{2^{\alpha+1}d_{\rm s}^{\alpha/2}}, \tag{2.33}$$

Combining (2.29), (2.32) and (2.33), we deduce that:

$$\Psi_s(x,0) = E_s \frac{\sqrt{\pi}}{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)} \frac{x^\alpha}{(4d_s)^{\alpha/2}} + F_s \frac{\sqrt{\pi}}{2\Gamma\left(\frac{\alpha}{2}+1\right)} \frac{x^\alpha}{(4d_s)^{\alpha/2}} \tag{2.34}$$

Considering the initial temperature given by (1.7), and replacing E_s by (2.16) in (2.34), it is obtained:

$$-\nu\omega \frac{M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-\nu^2\omega^2\right)}{M\left(-\frac{\alpha}{2},\frac{1}{2},-\nu^2\omega^2\right)} \frac{\sqrt{\pi}}{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)} F_s + \frac{\sqrt{\pi}}{2\Gamma\left(\frac{\alpha}{2}+1\right)} F_s = -T_i(4d_s)^{\alpha/2} \tag{2.35}$$

Then we can determine F_s using the definition of the U-Kummer function and identity (A.10) presented in “Appendix A” arriving to definition (2.17).

Until now we have obtained E_l, F_l, E_s and F_s as functions of ν, arriving to expressions (2.14)–(2.17).

Finally, it remains to take into account the Stefan condition (1.5) from which we will deduce an equation that must be satisfied by the unknown coefficient ν that characterized the free boundary. Substituting Eqs. (2.14)–(2.17), (2.24)–(2.25) into (1.5) and applying formula (A.11), it can be obtained that ν must satisfy the following equation:

$$\frac{k_l h_0 T_\infty}{\left[k_l M\left(\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2},x^2\right) + 2\sqrt{d_l} h_0 x M\left(\frac{\alpha}{2}+1,\frac{3}{2},x^2\right)\right]} + \\ - \frac{k_s T_i 2^\alpha d_s^{(\alpha-1)/2}}{U\left(\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2},x^2\omega^2\right)} = \gamma 2^\alpha x^{\alpha+1} d_l^{(\alpha+1)/2}, \qquad x>0. \tag{2.36}$$

that can be rewritten, arriving to the result that ν must be a solution of Eq. (2.18).

Our proof is going to be completed by showing that there exists a unique solution ν for Eq. (2.36) (i.e. (2.18)). With this purpose, we will study the behaviour of the functions f_1 and f_2.

On the one hand, due to the derivation formula (A.6), and its integral representation (A.8) we can assure that f_1 is an increasing function of x. It follows immediately that the first term of the left-hand side of Eq. (2.18) decreases from $\Delta_1 = -\frac{k_s T_i d_s^{(\alpha-1)/2}}{\gamma d_l^{(\alpha+1)/2}} \frac{\Gamma(\alpha/2+1)}{\sqrt{\pi}}$ to $-\infty$ when x increases from 0 to $+\infty$.

On the other hand, taking into account Eqs. (A.4) and (A.5) we arrive to the conclusion that f_2 is a decreasing function of x. Therefore, the second term of the left-hand side of Eq. (2.18) decreases from $\Delta_2 = \frac{h_0 T_\infty}{\gamma 2^\alpha d_l^{(\alpha+1)/2}}$ to 0 when x increases from 0 to $+\infty$.

In consequence we can assure that the left-hand side of (2.18) decreases from $\Delta_1+\Delta_2$ to $-\infty$ when x increases from 0 to $+\infty$.

As the right-hand side of (2.18) is an increasing function of x that goes from 0 to $+\infty$, we claim that Eq. (2.18) has a unique solution if and only if it is satisfied the following condition:

$$\Delta_1+\Delta_2>0 \tag{2.37}$$

which is equivalent to (2.10). □

Remark 2.6. An inequality of type (2.10) in order to obtain an instantaneous phase-change process was given firstly in [21]; see also [14].

Corollary 2.7. *If the coefficient h_0 satisfies the following inequality:*

$$0 < h_0 \leq \frac{2^\alpha \Gamma\left(\frac{\alpha}{2}+1\right) k_{\mathrm{s}} T_i d_{\mathrm{s}}^{(\alpha-1)/2}}{T_\infty \sqrt{\pi}} \tag{2.38}$$

then the free boundary problems (1.1)–(1.7) reduce to a classical heat transfer problem for the initial solid phase governed by:

$$\Psi_{\mathrm{s}t}(x,t) = d_{\mathrm{s}} \Psi_{\mathrm{s}xx}(x,t), \qquad x>0, \quad t>0, \tag{2.39}$$

$$k_{\mathrm{s}} \Psi_{\mathrm{s}x}(0,t) = h_0 t^{-1/2} \left[\Psi_{\mathrm{s}}(0,t) - T_\infty t^{\alpha/2}\right], \qquad t>0, \tag{2.40}$$

$$\Psi_{\mathrm{s}}(x,0) = -T_i x^\alpha, \qquad x>0, \tag{2.41}$$

whose explicit solution is given by:

$$\Psi_{\mathrm{s}}(x,t) = t^{\alpha/2}\left[E_s M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta_s^2\right) + F_s \eta_s M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\eta_s^2\right)\right], \tag{2.42}$$

where $\eta_s = x/\sqrt{4 d_{\mathrm{s}} t}$ and :

$$E_s = \frac{-T_i d_{\mathrm{s}}^{\alpha/2} k_{\mathrm{s}} \Gamma(\alpha+1) + \Gamma\left(\frac{\alpha+1}{2}\right) h_0 \sqrt{d_{\mathrm{s}}} T_\infty}{\left[k_{\mathrm{s}} \Gamma\left(\frac{\alpha}{2}+1\right) + h_0 \sqrt{d_{\mathrm{s}}} \Gamma\left(\frac{\alpha+1}{2}\right)\right]}, \tag{2.43}$$

$$F_s = \frac{2\sqrt{d_{\mathrm{s}}} h_0 (E_s - T_\infty)}{k_{\mathrm{s}}}. \tag{2.44}$$

Proof. From Lemma 2.1 and Remark 2.4, we have that the temperature is given by:

$$\Psi_{\mathrm{s}}(x,t) = t^{\alpha/2}\left[E_s M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta_s^2\right) + F_s \eta_s M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}. -\eta_s^2\right)\right] \tag{2.45}$$

where $\eta_s = \dfrac{x}{\sqrt{4 d_{\mathrm{s}} t}}$ and E_s and F_s are coefficients that must be determined.

Taking into account conditions (2.40)–(2.41), coefficients E_s and F_s are obtained in an analogous way as in the proof of Theorem 2.5. □

2.2. Case when α is a non-negative integer

This section is intended to present the exact solution of problems (1.1)–(1.7) in the particular case that α is a non-negative integer. Using formulas (A.12)–(A.13) from "Appendix A", it can be proved the following assertion.

Lemma 2.8. *Consider problems (1.1)–(1.7), where $\alpha = n \in \mathbb{N}_0$. If the coefficient h_0 satisfies the inequality:*

$$h_0 > \frac{2^n \Gamma\left(\frac{n}{2}+1\right) k_{\mathrm{s}} T_i d_{\mathrm{s}}^{(n-1)/2}}{T_\infty \sqrt{\pi}} \tag{2.46}$$

then the explicit solution of this problem is given by:

$$s(t) = 2\nu\sqrt{d_{\mathrm{l}} t}, \tag{2.47}$$

$$\Psi_{\mathrm{l}}(x,t) = -\frac{t^{n/2} 2^n h_0 T_\infty \sqrt{d_{\mathrm{l}}} \Gamma\left(\frac{n}{2}+\frac{1}{2}\right) \Gamma\left(\frac{n}{2}+1\right) \left[F_n(\eta_l) E_n(\nu) - F_n(\nu) E_n(\eta_l)\right]}{\left[k_{\mathrm{l}} \Gamma\left(\frac{n}{2}+1\right) E_n(\nu) + \sqrt{d_{\mathrm{l}}} h_0 \Gamma\left(\frac{n}{2}+\frac{1}{2}\right) F_n(\nu)\right]}, \tag{2.48}$$

$$\Psi_{\rm s}(x,t)=t^{n/2}2^nT_id_{\rm s}^{n/2}\Gamma(n+1)\left[\frac{E_n(\eta_s)F_n(\nu\omega)-E_n(\nu\omega)F_n(\eta_s)}{E_n(\nu\omega)-F_n(\nu\omega)}\right], \tag{2.49}$$

where $\eta_l=\frac{x}{2\sqrt{d_{\rm l}t}}$, $\eta_s=\frac{x}{2\sqrt{d_{\rm s}t}}$, $\omega=\sqrt{\frac{d_{\rm l}}{d_{\rm s}}}$ *and* ν *is the unique solution of the following equation:*

$$\frac{h_0T_\infty}{\gamma 2^n d_{\rm l}^{(n+1)/2}}\frac{1}{\left[e^{x^2}2^n\Gamma\left(\frac{n}{2}+1\right)E_n(x)+\frac{2^n\sqrt{d_{\rm l}}h_0}{k_{\rm l}}e^{x^2}\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)F_n(x)\right]}+$$
$$-\frac{k_{\rm s}T_id_{\rm s}^{(n-1)/2}}{\gamma d_{\rm l}^{(n+1)/2}}\frac{1}{2^ne^{x^2\omega^2}\sqrt{\pi}\left(E_n(x\omega)-F_n(x\omega)\right)}=x^{n+1},\qquad x>0. \tag{2.50}$$

Proof. Inequality (2.46), functions (2.47)–(2.49) and Eq. (2.50) can be deduced following the same reasoning used in the demonstration of Theorem 2.5 by using the relationship between the Kummer functions and the family of the repeated integrals of the complementary error function given by (A.12) and (A.13).

Let us note that in order to follow the arguments of Theorem 2.5 we must show that the limits given by (2.32) and (2.33) remain true in case that α is a non-negative integer. But this can be easily proved due to the formula presented by Tao in [17]:

$$\lim_{t\to 0}\ t^{n/2}E_n\left(\eta_s\right)=\lim_{t\to 0}\ t^{n/2}F_n\left(\eta_s\right)=\frac{x^n}{\Gamma\left(n+1\right)2^nd_{\rm s}^{n/2}}. \tag{2.51}$$

and due to the Legendre duplication formula for the Gamma function [1]:

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x). \tag{2.52}$$

□

Remark 2.9. Considering $n=0$ and taking into account that $E_0(z)=1$ and $F_0(z)=erf(z)$, condition (2.46) and functions (2.47)–(2.49) reduce to:

$$h_0>\frac{k_{\rm s}T_i}{T_\infty\sqrt{\pi d_{\rm s}}} \tag{2.53}$$

$$s(t)=2\nu\sqrt{d_{\rm l}t}, \tag{2.54}$$

$$\Psi_{\rm l}(x,t)=\frac{h_0T_\infty\sqrt{\pi d_{\rm l}}}{k_{\rm l}}\frac{\left[erf(\nu)-erf\left(\frac{x}{2\sqrt{d_{\rm l}t}}\right)\right]}{\left[1+\frac{\sqrt{\pi d_{\rm l}}h_0}{k_{\rm l}}erf(\nu)\right]}, \tag{2.55}$$

$$\Psi_{\rm s}(x,t)=-T_i\left[1-\frac{erfc\left(\frac{x}{2\sqrt{d_{\rm s}t}}\right)}{erf(\nu\omega)}\right], \tag{2.56}$$

where ν is the unique solution of the following equation:

$$-\frac{k_{\rm s}T_i}{\gamma\sqrt{\pi d_{\rm l}d_{\rm s}}}\frac{e^{-x^2\omega^2}}{erfc(x\omega)}+\frac{h_0T_\infty}{\gamma\sqrt{d_{\rm l}}}\frac{e^{-x^2}}{\left[1+\frac{\sqrt{\pi d_{\rm l}}h_0erf(x)}{k_{\rm l}}\right]}=x,\qquad x>0. \tag{2.57}$$

This formulas are in agreement with the explicit solution of the problem presented by Tarzia in [18] which is in contrast to our problem corresponds to a solidification process.

Remark 2.10. The results of Remark 2.9 in the one-phase case with a convective boundary condition are also recovered in [3].

3. Limit behaviour when $h_0 \to +\infty$

In this section, we are going to study the limit behaviour of the solution of the problem governed by Eqs. (1.1)–(1.7) when the coefficient h_0 that characterizes the heat transfer in the convective condition (1.6) tends to infinity. The main reason for doing this analysis is due to the fact that the convective heat input:

$$k_{\mathrm{l}}\Psi_{l\,x}(0,t) = h_0 t^{-1/2}[\Psi_{\mathrm{l}}(0,t) - T_\infty t^{\alpha/2}], \tag{3.1}$$

constitutes a generalization of the Dirichlet condition in the sense that if we take the limit when $h_0 \to \infty$ in (3.1) we must obtain $\Psi_{\mathrm{l}}(0,t) = T_\infty t^{\alpha/2}$. Therefore, we will prove that the solution to our problem in which we consider a convective condition at the fixed face $x = 0$ converges to the solution of a problem with a temperature condition at the fixed face.

Bearing in mind that the solution to problems (1.1)–(1.7), it means that the free boundary and the temperatures in the solid and the liquid phases depend on h_0, and we will rename them as:

$$\begin{cases} s_{h_0}(t) & : \text{free boundary given by (2.11)}, \\ \nu_{h_0} & : \text{unique solution of Eq. (2.18)}, \\ \Psi_{lh_0}(t) & : \text{liquid temperature given by (2.12)}, \\ \Psi_{sh_0}(t) & : \text{liquid temperature given by (2.13)}. \end{cases}$$

Theorem 3.1. *Let us consider the problem given by conditions (1.1)–(1.7), where the solutions s_{h_0}, $\Psi_{\mathrm{l}h_0}$, $\Psi_{\mathrm{s}h_0}$ and ν_{h_0} are defined by (2.11), (2.12), (2.13) and (2.18), respectively. If we take the limit when $h_0 \to \infty$, we obtain that s_{h_0}, $\Psi_{\mathrm{l}h_0}$, $\Psi_{\mathrm{s}h_0}$ and ν_{h_0} converge to s_∞, $\Psi_{l\infty}$, $\Psi_{s\infty}$ and ν_∞, respectively, which corresponds to the solution of the following problem:*

$$\Psi_{l\infty t}(x,t) = d_{\mathrm{l}}\Psi_{l\infty xx}(x,t), \qquad 0 < x < s_\infty(t), \quad t > 0, \tag{3.2}$$
$$\Psi_{s\infty t}(x,t) = d_{\mathrm{s}}\Psi_{s\infty xx}(x,t), \qquad x > s_\infty(t), \quad t > 0, \tag{3.3}$$
$$s_\infty(0) = 0, \tag{3.4}$$
$$\Psi_{l\infty}(s_\infty(t),t) = \Psi_{s\infty}(s_\infty(t),t) = 0, \qquad t > 0, \tag{3.5}$$
$$k_{\mathrm{s}}\Psi_{s\infty x}(s_\infty(t),t) - k_{\mathrm{l}}\Psi_{l\infty x}(s_\infty(t),t) = \gamma s_\infty(t)^\alpha \dot{s}_\infty(t), \qquad t > 0, \tag{3.6}$$
$$\Psi_{l\infty}(0,t) = T_\infty t^{\alpha/2} \quad t > 0, \tag{3.7}$$
$$\Psi_{s\infty}(x,0) = -T_i x^\alpha, \qquad x > 0 \quad . \tag{3.8}$$

with $s_\infty = 2\nu_\infty\sqrt{d_{\mathrm{l}}t}$ and where a temperature $T_\infty t^{\alpha/2}$ is prescribed at the fixed face $x = 0$.

Proof. On the one hand, if we consider the problem governed by Eqs. (3.2)–(3.8), we can obtain by the following similar arguments of the proof of Theorem 2.5 that the solution is given by:

$$s_\infty(t) = 2\nu_\infty\sqrt{d_{\mathrm{l}}t}, \tag{3.9}$$
$$\Psi_{l\infty}(x,t) = t^{\alpha/2}\left[E_{l\infty}M\left(-\tfrac{\alpha}{2},\tfrac{1}{2},-\nu_\infty^2\right) + F_{l\infty}\nu_\infty M\left(-\tfrac{\alpha}{2}+\tfrac{1}{2},\tfrac{3}{2},-\nu_\infty^2\right)\right], \tag{3.10}$$
$$\Psi_{s\infty}(x,t) = t^{\alpha/2}\left[E_{s\infty}M\left(-\tfrac{\alpha}{2},\tfrac{1}{2},-\nu_\infty^2\right) + F_{s\infty}\nu_\infty M\left(-\tfrac{\alpha}{2}+\tfrac{1}{2},\tfrac{3}{2},-\nu_\infty^2\right)\right], \tag{3.11}$$

where

$$E_{l\infty} = T_\infty, \tag{3.12}$$

$$F_{l\infty} = -\frac{T_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu_\infty^2\right)}{\nu_\infty M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\nu_\infty^2\right)}, \tag{3.13}$$

$$E_{s\infty} = -\frac{\nu_\infty \omega M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\nu_\infty^2\omega^2\right)}{M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu_\infty^2\omega^2\right)} F_{s\infty}, \tag{3.14}$$

$$F_{s\infty} = -\frac{T_i 2^{\alpha+1} d_{\mathrm{s}}^{\alpha/2} M\left(\frac{\alpha}{2}+\frac{1}{2}, \frac{1}{2}, \nu_\infty^2\omega^2\right)}{U\left(\frac{\alpha}{2}+\frac{1}{2}, \frac{1}{2}, \nu_\infty^2\omega^2\right)}, \tag{3.15}$$

with $\omega = \sqrt{\frac{d_{\mathrm{l}}}{d_{\mathrm{s}}}}$ and where ν_∞ is the unique solution of equation:

$$\frac{k_{\mathrm{l}} T_\infty}{2^{\alpha+1} d_{\mathrm{l}}^{(\alpha/2+1)}\gamma} f_3(x) - \frac{k_{\mathrm{s}} T_i d_{\mathrm{s}}^{(\alpha-1)/2}}{\gamma d_{\mathrm{l}}^{(\alpha+1)/2}} f_1(x) = x^{\alpha+1}, \qquad x > 0, \tag{3.16}$$

with

$$f_3(x) = \frac{1}{xM\left(\frac{\alpha}{2}+1, \frac{3}{2}, x^2\right)}, \qquad x > 0 \tag{3.17}$$

and $f_1(x)$ defined in (2.19).

The proof that ν_∞ is the unique solution of (3.16) derive from analysing the growth of functions f_1 and f_3. On the one hand, we have seen in the proof of Theorem (2.5) that f_1 is an increasing function that satisfies $f_1(0) = \frac{\Gamma(\alpha/2+1)}{\sqrt{\pi}}$ and $f_1(+\infty) = +\infty$. On the other hand, taking into account the derivation formula (A.4) we can easily prove that $f_3(x)$ is a decreasing function that verifies $f_3(0) = +\infty$ and $f_3(+\infty) = 0$. Thus we obtain that the left-hand side of Eq. (3.16) is a decreasing function that goes from $+\infty$ to $-\infty$ when x goes from 0 to $+\infty$. As the right-hand side of Eq. (3.16) is an increasing function that increases from 0 to $+\infty$, we can assure that (3.16) has a unique positive solution. We remark here that the solution of problems (3.2)–(3.8) was obtained in [23] by using results for a heat flux condition from an argument not so clear for us, and for this reason we have proved it with details.

Once we have calculated the solution of problems (3.2)–(3.8), let us show that the solution of problems (1.1)–(1.7) converges to it when $h_0 \to +\infty$. We know that ν_{h_0}, which is the parameter that characterizes the free front in (1.1)–(1.7), is the unique solution of (2.18). Taking limit in (2.18), we obtain:

$$\begin{aligned}
&\lim_{h_0\to+\infty}\left[-\frac{k_{\mathrm{s}} T_i d_{\mathrm{s}}^{(\alpha-1)/2}}{\gamma d_{\mathrm{l}}^{(\alpha+1)/2}} \frac{1}{U\left(\frac{\alpha}{2}+\frac{1}{2}, \frac{1}{2}, x^2\omega^2\right)}\right] + \\
&\quad + \lim_{h_0\to+\infty}\left[\frac{h_0 T_\infty}{\gamma 2^\alpha d_{\mathrm{l}}^{(\alpha+1)/2}} \frac{1}{\left[M\left(\frac{\alpha}{2}+\frac{1}{2}, \frac{1}{2}, x^2\right) + 2\frac{\sqrt{d_{\mathrm{l}}} h_0}{k_{\mathrm{l}}} xM\left(\frac{\alpha}{2}+1, \frac{3}{2}, x^2\right)\right]}\right] = \\
&= -\frac{k_{\mathrm{s}} T_i d_{\mathrm{s}}^{(\alpha-1)/2}}{\gamma d_{\mathrm{l}}^{(\alpha+1)/2}} \frac{1}{U\left(\frac{\alpha}{2}+\frac{1}{2}, \frac{1}{2}, x^2\omega^2\right)} + \frac{k_{\mathrm{l}} T_\infty}{\gamma 2^{\alpha+1} d_{\mathrm{l}}^{(\alpha/2+1)}} \frac{1}{xM\left(\frac{\alpha}{2}+1, \frac{3}{2}, x^2\right)} = \\
&= -\frac{k_{\mathrm{s}} T_i d_{\mathrm{s}}^{(\alpha-1)/2}}{\gamma d_{\mathrm{l}}^{(\alpha+1)/2}} f_1(x) + \frac{k_{\mathrm{l}} T_\infty}{\gamma 2^{\alpha+1} d_{\mathrm{l}}^{(\alpha/2+1)}} f_3(x).
\end{aligned} \tag{3.18}$$

That means that $\lim_{h_0\to+\infty} \nu_{h_0}$ must be a solution of Eq. (3.16) which has a unique solution ν_∞, so we can conclude that $\lim_{h_0\to+\infty} \nu_{h_0} = \nu_\infty$.

Subsequently by simple algebraic calculations, we obtain:

$$\lim_{h_0\to+\infty} s_{h_0}(t) = s_\infty(t), \tag{3.19}$$

$$\lim_{h_0\to+\infty} \Psi_{lh_0}(x,t) = \Psi_{l\infty}(x,t), \tag{3.20}$$

$$\lim_{h_0\to+\infty} \Psi_{sh_0}(x,t) = \Psi_{s\infty}(x,t). \tag{3.21}$$

□

4. Conclusions

In this article, a closed analytical solution of a similarity type has been obtained for a one-dimensional two-phase Stefan problem in a semi-infinite material using Kummer functions. The novel feature in the problem studied concerns a variable latent heat that depends on the position as well as a convective boundary condition at the fixed face $x = 0$ of the material. Assuming a latent heat defined as a power function of the position allows the generalization of some previous theoretical results. We have also generalized the classical two-phase Stefan problem with constant latent heat and a convective boundary condition [18] and the one-phase Stefan problem with latent heat depending on the position and a convective boundary condition at the fixed face $x = 0$ [3].

Furthermore, we have shown that when h_0 increases, the solution of problems (1.1)–(1.7) converges to the solution of a different free boundary problems (3.2)–(3.8) where a temperature condition at the fixed face is considered instead of a convective one [23].

The key contribution of this paper has been to prove the existence and uniqueness of the explicit solution of problems (1.1)–(1.7) when a restriction on the data is satisfied. We have presented the exact solution which is worth finding not only to understand better the process involved but also to verify the accuracy of numerical methods that solve Stefan problems.

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Appendix A.

This appendix presents a review of some of the significant mathematical results of the Kummer functions which are used in the main body of the paper.

Definition of Kummer functions

Kummer functions are defined by

$$M(a,b,z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_s s!} z^s, \qquad \text{with } b \text{ non-positive integer}, \tag{A.1}$$

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}M(a-b+1,2-b,z) \tag{A.2}$$

where $(a)_s$ is the Pochhammer symbol:

$$(a)_s = a(a+1)(a+2)\dots(a+s-1), \qquad (a)_0 = 1 \tag{A.3}$$

and $\Gamma(\cdot)$ is the Gamma function. In order that U is well defined it is necessary that a and $a-b+1$ be non-positive integers.

Differentiation formulas

From [11], we have

$$\frac{\mathrm{d}}{\mathrm{d}z}M(a,b,z) = \frac{a}{b}M(a+1,b+1,z) \tag{A.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{b-1}M(a,b,z)\right) = (b-1)z^{b-2}M(a,b-1,z) \tag{A.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}U(a,b,z) = -aU(a+1,b+1,z) \tag{A.6}$$

Connection Formulas

From [11] and [25], we know that

- Relationship with the generalized hypergeometric function

$$U(a,b,z) \sim z^{-a}, \quad z \to \infty, |z| \le \frac{3}{2}\pi - \delta \quad \text{where } \delta \text{ is an arbitrary small positive constant.} \tag{A.7}$$

- Integral Representation of U

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}\mathrm{d}t \qquad \text{with } \mathrm{Re}(a) > 0 \text{ and } |\mathrm{ph}(z)| < \frac{\pi}{2} \tag{A.8}$$

- Relationship between U and M

$$\frac{1}{\Gamma(b)}M(a,b,z) = \frac{e^{a\pi i}}{\Gamma(b-a)}U(a,b,z) + \frac{e^{-(b-a)\pi i}}{\Gamma(a)}e^{z}U(b-a,b,e^{-\pi i}z) \tag{A.9}$$

- Relationship with the exponential function

$$M(a,b,z) = e^{z}M(b-a,b,-z) \tag{A.10}$$

$$e^{-z^2} = -2\alpha z^2 M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{3}{2},-z^2\right)M\left(-\frac{\alpha}{2}+1,\frac{3}{2},-z^2\right) + M\left(-\frac{\alpha}{2},\frac{1}{2},-z^2\right)M\left(-\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2},-z^2\right) \tag{A.11}$$

  where α is real and non-negative.

- Relationship with the family of the repeated integrals of the complementary error function

$$M\left(-\frac{n}{2},\frac{1}{2},-z^2\right) = 2^n\Gamma\left(\frac{n}{2}+1\right)E_n(z) \tag{A.12}$$

$$zM\left(-\frac{n}{2}+\frac{1}{2},\frac{3}{2},-z^2\right) = 2^{n-1}\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)F_n(z) \tag{A.13}$$

where n is an integer, E_n and F_n are defined by

$$E_n(z) = \left[i^n erfc(z) + i^n erfc(-z)\right]/2 \tag{A.14}$$

$$F_n(z) = \left[i^n erfc(-z) + i^n erfc(z)\right]/2 \tag{A.15}$$

in which

$$i^0 erfc(x) = erfc(x) \tag{A.16}$$

$$i^n erfc(x) = \int_x^{+\infty} i^{n-1} erfc(t)\mathrm{d}t \tag{A.17}$$

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