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THE GRASSMANN MANIFOLD OF A HILBERT SPACE

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ABSTRACT. The present paper surveys the geometric properties of the Grassmann manifold $Gr(\mathscr{H})$ of an infinite dimensional complex Hilbert space $\mathscr{H}$. $Gr(\mathscr{H})$ is viewed as a set of operators, identifying each closed subspace $\mathscr{S} \subset \mathscr{H}$ with the orthogonal projection $P_{\mathscr{S}}$ onto $\mathscr{S}$. Most of the results surveyed here were stated by G. Corach, H. Porta and L. Recht: submanifold structure, homogeneous reductive structure, local minimality of geodesics. Some recent results concerning the existence and uniqueness of a geodesic joining two given projections, which were obtained by the present author, are also presented.

1. INTRODUCTION

This is a survey article, examining the geometric structure the the Grassmann manifold $Gr(\mathscr{H})$ of an (eventually) infinite dimensional complex Hilbert space $\mathscr{H}$,

$$Gr(\mathscr{H}) = \{\text{closed linear subspaces of } \mathscr{H}\}.$$

We adopt the point of view that we learned from G. Corach, H. Porta and L. Recht, whose works on this subject are cited below, which consists of identifying a closed subspace $\mathscr{S} \subset \mathscr{H}$ with the orthogonal projection $P_{\mathscr{S}}$ onto $\mathscr{S}$. Thus the Grassmann manifold becomes a submanifold of the Banach space $\mathscr{B}(\mathscr{H})$ of bounded linear operators in $\mathscr{H}$. There is a natural metric in $\mathscr{B}(\mathscr{H})$, the usual (spectral) norm. Most of the results cited here belong to the mentioned authors ([14, 15, 5]). The few contributions of the present author concern the problems of existence and uniqueness of geodesics joining two given subspaces / projections ([2, 3]).

As said above, to a closed subspace $\mathscr{S}$ corresponds an orthogonal projection $P_{\mathscr{S}} \in \mathscr{B}(\mathscr{H})$ (charaterized by the conditions $(P_{\mathscr{S}})^2 = P_{\mathscr{S}}$, $P^*_{\mathscr{S}} = P_{\mathscr{S}}$ and $R(P_{\mathscr{S}}) = \mathscr{S}$). We shall use this correspondence to identify

$$Gr(\mathscr{H}) = \{P \in \mathscr{B}(\mathscr{H}) : P \text{ is an orthogonal projection}\}.$$

One advantage of this viewpoint is that it enables one to regard the $Gr(\mathscr{H})$ as a subset of a Banach space (namely $\mathscr{B}(\mathscr{H})$). Most of the facts concerning this viewpoint were presented in [15, 14, 5].

The main tool to study the geometry of $Gr(\mathscr{H})$ is the action of the unitary group $\mathscr{U}(\mathscr{H})$ on this set. Recall that $U \in \mathscr{U}(\mathscr{H})$ if $U^*U = UU^* = 1$. The action is given by

$$U \cdot P = UPU^*, \qquad U \in \mathscr{U}(\mathscr{H}), \quad P \in Gr(\mathscr{H}).$$

In terms of subspaces, if $P = P_{\mathscr{S}}$, then $U \cdot P$ corresponds to the subspace $U(\mathscr{S})$.

We shall denote by $P^{\perp} = 1 - P$ (note that if $P = P_{\mathscr{S}}$, then $1 - P = P_{\mathscr{S}^{\perp}}$).

This action is locally transitive: if $P, Q \in Gr(\mathscr{H})$ verify $\|P - Q\| < 1$, then there exists $U \in \mathscr{U}(\mathscr{H})$ such that $U \cdot P = Q$. This fact is well known (for instance [10, 9, 7, 5, 4]); we include an elementary proof:

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Lemma 1.1. *Let $P,Q \in Gr(\mathcal{H})$ such that $\|P-Q\|<1$. Then there exists $U=U(P,Q)\in \mathcal{U}(\mathcal{H})$ such that*

$$UPU^*=Q.$$

Proof. We claim that $S=S(P,Q)=QP+Q^\perp P^\perp$ is an invertible operator. Indeed,

$$S^*S=(PQ+P^\perp Q^\perp)(QP+Q^\perp P^\perp)=PQP+P^\perp Q^\perp P^\perp.$$

Then, using that $1=P+P^\perp$,

$$1-S^*S=P-PQP+P^\perp-P^\perp Q^\perp P^\perp.$$

The first term $P-PQP$ can be regarded as a difference of bounded operators in the space $R(P)$. Note that PQP is invertible there:

$$\|P-PQP\|=\|P(P-Q)P\|\le\|P\|\|P-Q\|\|Q\|\le\|P-Q\|<1.$$

It is a well known fact that an operator whose distance to the identity operator is less than 1 is invertible (in this case P is the identity operator of $R(P)$). Similarly, $P^\perp Q^\perp P^\perp$ is invertible in $R(P^\perp)=R(P)^\perp$ (note that $\|P^\perp-Q^\perp\|=\|Q-P\|$). Then S^*S is invertible in $R(P)\oplus R(P^\perp)=\mathcal{H}$. Analogously, SS^* is invertible in $\mathcal{H}$.

Note that

$$SP=(QP+Q^\perp P^\perp)P=QP=Q(QP+Q^\perp P^\perp)=QS.$$

Then $S^*Q=(QS)^*=(SP)^*=PS^*$, and thus S^*S commutes with P,

$$S^*SP=S^*QS=PS^*S.$$

Consider the polar decomposition $S=U|S|$ (here $|S|=(S^*S)^{1/2}$). Since S is invertible, U is unitary, and by the above computation, $|S|$ commutes with P. Then

$$UP=S|S|^{-1}P=SP|S|^{-1}=QS|S|^{-1}=QU,$$

i.e. $UPU^*=Q$. □

In the above proof, note that U is a smooth formula in terms of the operators P and Q. One easy consequence of the above result, is that the connected components of $Gr(\mathcal{H})$ are the orbits of the action by the unitary group. Note the fact that for any pair P,Q of projections, one always has $\|P-Q\|\le 1$. And if $\|P-Q\|<1$ they are conjugate by a unitary operator, and thus lie in the same connected component. Here we use the known fact that $\mathcal{U}(\mathcal{H})$ is connected (in fact it is contractible if $\dim\mathcal{H}=\infty$ [12]).

We shall denote by $\mathcal{B}_h(\mathcal{H})\subset\mathcal{B}(\mathcal{H})$ the (real, complemented) subspace of selfadjoint (or Hermitian) operators in $\mathcal{H}$.

2. P CO-DIAGONAL OPERATORS

Following ideas in [14] and [5], we shall base our study of the geometry of $Gr(\mathcal{H})$ on the following decomposition. Fix $P_0\in Gr(\mathcal{H})$. Then operators A in $\mathcal{B}(\mathcal{H})$ can be written as 2×2 block matrices:

$$A=\begin{pmatrix} P_0AP_0 & P_0AP_0^\perp \\ P_0^\perp AP_0 & P_0^\perp AP_0^\perp \end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix},$$

where $a_{11}=P_0AP_0$ is regarded as an operator in $\mathcal{B}(R(P_0))$, $a_{12}=P_0AP_0^\perp$ as an operator in $\mathcal{B}(N(P_0),R(P_0))$, and so on. Then, based on P_0, $\mathcal{B}(\mathcal{H})$ can be decomposed as

$$A=\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22}\end{pmatrix}+\begin{pmatrix} 0 & a_{12} \\ a_{21} & 0\end{pmatrix}=A_d+A_c,$$

where A_d will be called the P_0-diagonal part of A, and A_c the P_0-co-diagonal part of A. Note that A_d commutes with P_0.

Let us denote by $\mathscr{D}_{P_0}$ and $\mathscr{C}_{P_0}$ the (closed, complemented) subpaces of P_0-diagonal and P_0-co-diagonal selfadjoint operators. Clearly $\mathscr{D}_{P_0} \oplus \mathscr{C}_{P_0} = \mathscr{B}_h(\mathscr{H})$.

It shall be sometimes useful to consider the symmetry induced by a projection. Namely, projections are in one to one correspondence with symmetries $\varepsilon \in \mathscr{B}(\mathscr{H})$: $\varepsilon^* = \varepsilon^{-1} = \varepsilon$. The correspondence is given by

$$P \longleftrightarrow \varepsilon_P = 2P - 1.$$

Note that ε_P equals 1 in $R(P)$ and -1 in $N(P)$.

Returning to the diagonal/co-diagonal decomposition, it is apparent that the P_0-diagonal part A_d commutes with ε_{P_0} and the P_0-co-diagonal part A_c anti-commutes with ε_{P_0} ($A_c\varepsilon_{P_0} = -\varepsilon_{P_0}A_c$).

Fix $P_0 \in \mathscr{P}$, and consider the following map

$$\varphi : \mathscr{B}_h(\mathscr{H}) \to \mathscr{B}_h(\mathscr{H}), \;\; \varphi(X) = X_d + e^{\tilde{X}_c} P_0 e^{-\tilde{X}_c},$$

where

$$\tilde{X}_c = \begin{pmatrix} 0 & -x_{12} \\ x_{12}^* & 0 \end{pmatrix}, \quad \text{if } X_c = \begin{pmatrix} 0 & x_{12} \\ x_{12}^* & 0 \end{pmatrix}.$$

$X_c^* = X_c$ implies $\tilde{X}_c^* = -\tilde{X}_c$ and thus $e^{\tilde{X}_c}$ is a unitary operator. This implies that $\varphi(X)$ is indeed selfadjoint.

Also note that $X_c \mapsto \tilde{X}_c$ is a linear isometric isomorphism between P_0-co-diagonal selfadjoint operators, and P_0-co-diagonal *anti*-selfadjoint operators (an operator B is anti-selfadjoint if $B^* = -B$).

Lemma 2.1. $d\varphi_0 = 1_{\mathscr{B}_h(\mathscr{H})}$.

Proof. Let $X \in \mathscr{B}_h(\mathscr{H})$ and $X(t)$ a smooth path in $\mathscr{B}_h(\mathscr{H})$ with $X(0) = 0$, $\dot{X}(0) = X$. For instance, $X(t) = tX$. Then

$$d\varphi_0(X) = \frac{d}{dt}|_{t=0}(X_d(t) + e^{\tilde{X}_c(t)} P_0 e^{-\tilde{X}_c(t)}) = X_d + \tilde{X}_c P_0 - P_0 \tilde{X}_c.$$

Note that

$$\begin{aligned} \tilde{X}_c P_0 - P_0 \tilde{X}_c &= \begin{pmatrix} 0 & -x_{12} \\ x_{12}^* & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -x_{12} \\ x_{12}^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x_{12} \\ x_{12}^* & 0 \end{pmatrix} = X_c. \end{aligned}$$

Then $d\varphi_0(X) = X_d + X_c = X$. □

By the inverse mapping theorem, it follows that φ is a local diffeomorphism between neighbourhoods of 0 and $\varphi(0) = P_0$ in $\mathscr{B}_h(\mathscr{H})$: there exist open subsets $\mathscr{V}$, $\mathscr{W}$ in $\mathscr{B}_h(\mathscr{H})$, such that $0 \in \mathscr{V}$, $P_0 \in \mathscr{W}$, and

$$\varphi : \mathscr{V} \to \mathscr{W}$$

is a C^∞ diffeomorphism.

Proposition 2.2. *$Gr(\mathscr{H})$ is a C^∞ differentiable complemented submanifold of $\mathscr{B}(\mathscr{H})$. For any fixed $P_0 \in Gr(\mathscr{H})$, the map*

$$\pi_{P_0} : \mathscr{U}(\mathscr{H}) \to Gr(\mathscr{H})_{P_0}, \quad \pi_{P_0}(U) = UP_0U^*$$

is a C^∞ submersion. Here $Gr(\mathcal{H})_{P_0}$ denotes the connected component of P_0 in $Gr(\mathcal{H})$, which coincides with the (unitary) orbit of P_0:

$$Gr(\mathcal{H})_{P_0} = \{UP_0U^* : U \in \mathcal{U}(\mathcal{H})\}.$$

Proof. The local diffeomorphism $\varphi : \mathcal{V} \to \mathcal{W}$ maps $\mathcal{V} \cap \mathcal{C}_{P_0}$ onto $\mathcal{W} \cap Gr(\mathcal{H})$. This provides a local chart for $Gr(\mathcal{H})$ near P_0. Since P_0 is arbitary, one obtains a chart for every point in $Gr(\mathcal{H})$. The Lemma in the previous section shows that elements in different orbits / connected components lie at distance at least 1. On the other hand, any given pair of projections lie at distance less than or equal to 1, thus elements in different components lie at distance 1.

The same Lemma provides an explicit formula for the a unitary operator $U = U(P,Q)$ (which is C^∞ in both variables P and Q), which implements the conjugation $UPU^* = Q$. Thus $\Sigma_{P_0} = U(P_0, \cdot\,)$ provides a C^∞ local cross section for π_{P_0} defined on a neighbourhood of P_0 in $Gr(\mathcal{H})$, namely, $\{Q \in Gr(\mathcal{H}) : \|Q - P_0\| < 1\}$. One obtains local cross sections near other points P of $Gr(\mathcal{H})_{P_0}$ by translating this one. Explicitly: if $P = U_0P_0U_0^*$ for some $U_0 \in \mathcal{U}(\mathcal{H})$, put

$$\Sigma_P(Q) = U_0\, \Sigma_{P_0}(U_0^*QU_0)$$

defined on the open set

$$\{Q \in Gr(\mathcal{H})_{P_0} : \|U_0^*QU_0 - P_0\| = \|Q - P\| < 1\}.$$

It follows that π_{P_0} has C^∞ local cross sections defined on neighbourhoods of any point of $Gr(\mathcal{H})_{P_0}$. Then it is a submersion. □

Tangent vectors are co-diagonal:

Remark 2.3. The tangent space $(TGr(\mathcal{H}))_{P_0}$ is $\mathcal{C}_{P_0}$. Indeed, let $p(t)$, $t \in I$, be a smooth curve in $Gr(\mathcal{H})$, with $p(0) = P_0$ and $\dot{p}(0) = X$. Clearly, X is selfadjoint. Differentiating the identity $p(t) = p(t)^2$ at $t = 0$ we obtain

$$X = XP_0 + P_0X.$$

Thus $P_0XP_0 = P_0XP_0 + P_0XP_0$, which implies $P_0XP_0 = 0$, and

$$P_0^\perp XP_0^\perp = P_0^\perp(XP_0 + P_0X)P_0^\perp = 0.$$

That is, X is co-diagonal. Note also the useful relations: if $X \in (TGr(\mathcal{H}))_{P_0}$,

$$[X,P_0] = XP_0 - P_0X = \tilde{X}\ ,\ \ [\tilde{X},P_0] = X.$$

Conversely, if $X \in \mathcal{C}_{P_0}$, then $\tilde{X}$ is anti-selfadjoint, and thus $e^{t\tilde{X}}$ is a one-parameter group of unitaries, in particular it is a smooth curve of unitaries. Thus

$$p(t) = e^{t\tilde{X}}P_0e^{-t\tilde{X}} \in Gr(\mathcal{H})$$

satisfies $p(0) = P_0$ and

$$\dot{p}(0) = \tilde{X}P_0 - P_0\tilde{X} = [\tilde{X},P_0] = X,$$

i.e. $X \in (TGr(\mathcal{H}))_{P_0}$.

Let us show how the diagonal / co-diagonal decomposition provides also a natural linear connection for $Gr(\mathcal{H})$. We shall make it here explicit, though it is a particular case of what in classical differential geometry is a reductive structure for a homogeneous space. Again, we follow ideas in [14], [15] and [5].

For further use, note the following straightforward facts about co-diagonal operators:

Remark 2.4.

- $X \in \mathcal{C}_P$ if and only if $VXV^* \in \mathcal{C}_{VPV^*}$, for any $V \in \mathcal{U}(\mathcal{H})$.

- If $X \in \mathscr{C}_P$, $X = XP + PX$.

Definition 2.5. *As above, let $p(t)$, $t \in I = [0,1]$ be a smooth curve in $Gr(\mathscr{H})$. A smooth curve $U(t)$, $t \in I$ of unitary operators is a co-diagonal lifting (or in classical terms, a horizontal lifting) por p, if*

$$U(t)p(t_0)U(t)^* = p(t) \quad \text{and} \quad iU^*(t)\dot{U}(t) \in \mathscr{C}_{p(t)}.$$

If one requires that $U(t_0) = 1$, then the curve U is unique. Existence and uniqueness of such liftings follow from the next result.

Lemma 2.6. *The co-diagonal lifting satisfying $U(t_0) = 1$, is charaterized as the unique solution of the following linear differential equation:*

$$\begin{cases} \dot{U} = [\dot{p}, p]U \\ U(t_0) = 1 \end{cases} \tag{1}$$

Proof. Pick first a co-diagonal lifting $U = U(t)$: $Up(t_0)U^* = p$ and $pU^*\dot{U}p = P^\perp U^*\dot{U}p^\perp = 0$. Denote $P_0 = p(t_0)$. The first observation in the above remark implies that $U^*\dot{U} \in \mathscr{C}_{P_0}$. We need to show that U satisfies the equation (1), or equivalently,

$$\dot{U}U^* = [\dot{p}, p].$$

Differentiating $UP_0U^* = p$, one obtains $\dot{U}P_0U^* + UP_0\dot{U}^* = \dot{p}$. Then

$$[\dot{p}, p] = [\dot{U}P_0U^* + UP_0\dot{U}^*, UP_0U^*] = \dot{U}P_0U^* + UP_0\dot{U}^*P_0UP_0U^* - UP_0U^*\dot{U}P_0U^* - UP_0\dot{U}^*.$$

Note that differentiating $UU^* = 1$ one obtains $U\dot{U}^* = -\dot{U}U^*$. Then the second term in the above sum vanishes

$$-UP_0U^*\dot{U}P_0U^* = 0,$$

because $U^*\dot{U}$ is P_0-co-diagonal. The third term equals 0 for the same reason. Thus U satisfies equation (1) if and only if

$$\dot{U}U^* = \dot{U}P_0U^* - UP_0\dot{U}^*,$$

or equivalently (multiplying by U^* on the left and by U on the right)

$$U^*\dot{U} = U^*\dot{U}P_0 - P_0\dot{U}^*U = U^*\dot{U}P_0 + P_0U^*\dot{U},$$

which holds, by the second item in the above remark, because $U^*\dot{U}$ is P_0-co-diagonal.

Conversely, suppose that U is a solution of the equation $\dot{U} = [\dot{p}, p]U$ with $U(t_0) = 1$. First note that $[\dot{p}, p]$, being the commutant of two selfadjoint operators, is anti-selfadjoint. It is a general fact that an operator valued linear equation with initial data a unitary operator (in this case the identity operator 1), with anti-selfadjoint Hamiltonian, has a solution which consists of unitary operators. On the other hand,

$$\dot{U}U^* = [\dot{p}, p]$$

is clearly p-co-diagonal. Let us show that U is a lifting of p, or equivalently, that

$$U^*pU = P_0.$$

Differentiating U^*pU one obtains

$$\dot{U}^*pU + U^*\dot{p}U + U^*p\dot{U} = U^*(U\dot{U}^*p + \dot{p} + p\dot{U}U^*)U.$$

The term inside the parenthesis equals

$$\begin{aligned} -\dot{U}U^*p + p\dot{U}U^* + \dot{p} &= -[\dot{p}, p]p + p[\dot{p}, p] + \dot{p} \\ &= -\dot{p}p + p\dot{p}p + p\dot{p}p - p\dot{p} + \dot{p}. \end{aligned}$$

Recall that tangent vectors are co-diagonal, thus $p\dot{p}p = 0$, and the above expression equals

$$-\dot{p}p - p\dot{p} + \dot{p} = 0.$$

Thus U^*pU is constant, and equals P_0 at $t = t_0$, which completes the proof. $\square$

With the aid of the co-diagonal lifting we define the parallel transport of tangent vectors in $Gr(\mathscr{H})$.

Definition 2.7. *Given $X \in (TGr(\mathscr{H}))_{P_0}$ and $p(t)$, $t \in [0,1]$, a smooth curve in $Gr(\mathscr{H})$ with $p(0) = P_0$, the parallel transport of X along p is defined as*

$$U(t)XU^*(t),$$

where U is the solution of equation (1), $\dot{U} = [\dot{p}, p]U$, $U(0) = 1$.

Note that by the remark above, $X \in (TGr(\mathscr{H}))_{P_0} = \mathscr{C}_{P_0}$ implies that $UXU^* \in \mathscr{C}_{UP_0U^*} = (TGr(\mathscr{H}))_p$.

This notion of parallel transport induces a covariant derivative.

Definition 2.8. *If $X(t)$ is a smooth field of tangent vectors along $p(t)$ (i.e. $X(t)$ is a smooth curve of selfadjoint operators with $X(t) \in \mathscr{C}_{p(t)}$), define*

$$\frac{DX}{dt} = U\{\frac{d}{dt}(U^*XU)_{t=0}\}U^*, \tag{2}$$

where U is the solution of equation (1): $\dot{U} = [\dot{p}, p]U$, $U(0) = 1$.

Note that U^*XU is a smooth curve in $\mathscr{C}_{P_0}$, thus its derivative is an element in $\mathscr{C}_{P_0}$, and therefore $\frac{DX}{dt} \in \mathscr{C}_p = (TGr(\mathscr{H}))_p$.

The main data of the linear connection, for instance its torsion and curvature tensors, can be computed (as in classical differential geometry of reductive homogeneous spaces [11], or [13] for the specific framework on homogeneous spaces of operators). We shall focus on the geodesics. The equation of a geodesic δ (with co-diagonal lifting Ω, $\Omega(t_0) = 1$) of this connection is

$$\frac{D\dot{\delta}}{dt}\Big|_{t=t_0} = 0.$$

In our case,

$$\Omega\{\frac{d}{dt}(\Omega^*\dot{\delta}\Omega)\}_{t=t_0}\Omega^* = 0,$$

Explicitly,

$$\begin{aligned} 0 &= \Omega\dot{\Omega}^*\dot{\delta} + \ddot{\delta} + \dot{\delta}\dot{\Omega}\Omega^* = -\dot{\Omega}\Omega^*\dot{\delta} + \ddot{\delta} + \dot{\delta}\dot{\Omega}\Omega^* \\ &= -[\dot{\delta}, \delta]\dot{\delta} + \ddot{\delta} + \dot{\delta}[\dot{\delta}, \delta]. \end{aligned}$$

Thus we arrive at

$$\ddot{\delta} = [[\dot{\delta}, \delta], \dot{\delta}]. \tag{3}$$

We know from the general theory of homogeneous reductive spaces that the horizontal (or co-diagonal) liftings of geodesics are exponentials with co-diagonal exponents:

Proposition 2.9. *Let $P_0 \in Gr(\mathscr{H})$ and $X_0 \in (TGr(\mathscr{H}))_{P_0}$. The unique geodesic δ with $\delta(0) = P_0$ and $\dot{\delta}(0) = X_0$ is given by*

$$\delta(t) = e^{t\tilde{X}_0}P_0e^{-t\tilde{X}_0}.$$

Proof. Apparently $\delta(0) = P_0$ and

$$\dot{\delta} = \tilde{X}_0\delta - \delta\tilde{X}_0 = [\tilde{X}_0, \delta] = e^{t\tilde{X}_0}[\tilde{X}_0, P_0]e^{-t\tilde{X}_0}$$

so that

$$\dot{\delta}(0) = [\tilde{X}_0, P_0] = X_0.$$

Also

$$\ddot{\delta} = [\tilde{X}_0, [\tilde{X}_0, \delta]] = e^{t\tilde{X}_0}[\tilde{X}_0, [\tilde{X}_0, P_0]]e^{-t\tilde{X}_0}.$$

Note that, since $\tilde{X}_0$ is P_0-co-diagonal, $\tilde{X}_0^2$ commutes with P_0,

$$[\tilde{X}_0, [\tilde{X}_0, P_0]] = 2\tilde{X}_0^2 P_0 - 2\tilde{X}_0 P_0 \tilde{X}_0.$$

On the other hand,

$$[[\dot{\delta}, \delta], \dot{\delta}] = e^{t\tilde{X}_0}[[\tilde{X}_0, P_0], [\tilde{X}_0, P_0]]e^{-t\tilde{X}_0}.$$

Again, that $\tilde{X}_0$ is P_0-co-diagonal, implies that $P_0\tilde{X}_0P_0 = 0$, and thus a straightforward computation shows that

$$[[\tilde{X}_0, P_0], [\tilde{X}_0, P_0]] = 2P_0\tilde{X}_0^2P_0 - 2\tilde{X}_0P_0\tilde{X}_0 = 2\tilde{X}_0^2P_0 - 2\tilde{X}_0P_0\tilde{X}_0.$$

□

3. CONDITIONS FOR THE EXISTENCE OF A GEODESIC JOINING TWO GIVEN PROJECTIONS

The problem of whether two given projections P and Q lie in the same connected component of $Gr(\mathcal{H})$ (or equivalently, whether there exists a unitary operator U such that $UPU^* = Q$, shortly: whether P and Q are unitarily equivalent) is solved computing the numbers

$$\dim N(P) = n(P), \quad \dim R(P) = r(P).$$

Namely, P and Q lie in the same connected component if and only if $n(P) = n(Q)$ and $r(P) = r(Q)$.

The problem of whether they can be joined by a geodesic, and in this case whether the geodesic is unique (up to reparametrization) requires another study.

P. Halmos [9], J. Dixmier [8], and C. Davis [7], among others, suggest that to understand the finer geometric relative properties of P and Q, one needs to consider the following subspaces:

$$(R(P) \cap R(Q)) \oplus (N(P) \cap N(Q)) \oplus (R(P) \cap N(Q)) \oplus (N(P) \cap R(Q)) \oplus \mathcal{H}_0,$$

where $\mathcal{H}_0$ is the orthogonal complement of the sum of the first three subspaces. We denote these subspaces, respectively,

$$\mathcal{H}_{11} \oplus \mathcal{H}_{00} \oplus \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_0.$$

It can be proved that $\mathcal{H}_{11}$, $\mathcal{H}_{00}$ and $\mathcal{H}' = \mathcal{H}_{10} \oplus \mathcal{H}_{01}$ reduce P and Q simultaneously, which implies that their orthogonal complement $\mathcal{H}_0$ also does. $\mathcal{H}_0$ is usually called the *generic part* of P and Q.

It will be useful to refer these subspaces to the operator $A = P - Q$:

$$N(A) = \mathcal{H}_{00} \oplus \mathcal{H}_{11}, \quad N(A-1) = \mathcal{H}_{10} \quad \text{and} \quad N(A+1) = \mathcal{H}_{01}.$$

In $\mathcal{H}_{11}$, P and Q act as the indentity, in $\mathcal{H}_{00}$ they are both trivial. Therefore interesting phenomena occur in $\mathcal{H}'$ and in the generic part $\mathcal{H}_0$. We shall denote by P', Q' the restrictions of P and Q to $\mathcal{H}'$, and by P_0, Q_0 their restrictions to $\mathcal{H}_0$.

An important result obtained by the mentioned authors (see for instance [9], which is the form we employ here) is that in the generic part, P_0 and Q_0 are unitarily equivalent, and unitarily equivalent to their orthogonal complements $1-P_0$, $1-Q_0$. And more important for us, these equivalences are implemented in the following manner:

There exists a unitary isomorphism between $\mathscr{H}_0$ and a product Hilbert space $\mathscr{K}\times\mathscr{K}$, and an operator $0\le X\le \pi/2$ in $\mathscr{K}$, such that P_0 and Q_0 are carried to (the operator matrices in $\mathscr{K}\times\mathscr{K}$):

$$P_0=\begin{pmatrix} 1_{\mathscr{K}} & 0 \\ 0 & 0\end{pmatrix} \quad \text{and} \quad Q_0=\begin{pmatrix} C^2 & CS \\ CS & S^2\end{pmatrix},$$

where $C=\cos(X)$ and $S=\sin(X)$ have trivial nullspace.

In the space $\mathscr{H}'$, as matrices in terms of the decomposition $\mathscr{H}'=\mathscr{H}_{10}\oplus\mathscr{H}_{01}$, the projections P' and Q' are given by:

$$P'=\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} \quad \text{and} \quad Q'=\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}.$$

Apparently P' and Q' are unitarily equivalent (in $\mathscr{H}'$) if and only if

$$\dim \mathscr{H}_{10}=\dim \mathscr{H}_{01}.$$

Therefore, if these numbers coincide, the whole projections P and Q are unitarily equivalent. However, as remarked in the first paragraph of this section, P and Q can be unitarily equivalent without the coincidence of these dimensions (the unitary operator implementing the equivalence need not be reduced by $\mathscr{H}'$). For instance, pick $Q\le P$, both with infinite rank and nullity.

Coming back to

$$P_0=\begin{pmatrix} 1_{\mathscr{K}} & 0 \\ 0 & 0\end{pmatrix} \quad \text{and} \quad Q_0=\begin{pmatrix} C^2 & CS \\ CS & S^2\end{pmatrix},$$

a straightforward computation shows that the unitary operator implementing the equivalence of these projections can be chosen to be

$$U=\begin{pmatrix} C & -S \\ S & C\end{pmatrix}=\exp(i\begin{pmatrix} 0 & iX \\ -iX & 0\end{pmatrix}).$$

Note that the exponent

$$Z_0=\begin{pmatrix} 0 & iX \\ -iX & 0\end{pmatrix}$$

is (if we bring it back to $\mathscr{H}_0$ with the unitary isomorphism between $\mathscr{H}_0$ and $\mathscr{K}\times\mathscr{K}$) selfadjoint and P_0-co-diagonal. That is, in the Grassmannian of $\mathscr{H}_0$ there is a geodesic curve joining P_0 and Q_0,

$$\delta_0(t)=e^{itZ_0}P_0e^{-itZ_0},$$

which additionally satisfies $\|Z_0\|\le\pi/2$, with equality in some cases.

Let us explicitly describe how in $\mathscr{H}'=\mathscr{H}_{10}\oplus\mathscr{H}_{01}$, if both subspaces have the same dimension, P' and Q' are unitarily equivalent, by means of a unitary operator which is the exponential of a P'-co-diagonal exponent:

Let $W:\mathscr{H}_{10}\to\mathscr{H}_{01}$ be a unitary isomorphism, and put $U':\mathscr{H}'\to\mathscr{H}'$,

$$U'(\xi,\eta)=(W^*\eta,-W\xi).$$

A straightforward computation shows that

$$U'P'U'^*=Q'.$$

Moreover, if we put $Z'=-i\frac{\pi}{2}U$, one has that Z' is selfadjoint, P'-co-diagonal, and satisfies that

$$e^{iZ'}=U'.$$

In other words, the geodesic $\delta'(t) = e^{itZ'}P'e^{-itZ'}$ joins P' and Q' (inside $\mathcal{H}'$). Note that $\|Z'\| = \pi/2$. Thus we have proved the following partial result:

Proposition 3.1. *Let P, Q be two projetions such that*

$$\dim R(P) \cap N(Q) = \dim R(Q) \cap N(P).$$

Then there exists a geodesic $\delta(t) = e^{itZ}Pe^{-itZ}$ of $Gr(\mathcal{H})$, such that $\delta(0) = P$ and $\delta(1) = Q$. The exponent can be chosen so that $\|Z\| \leq \pi/2$.

Conversely, suppose now that there exists a selfadjoint operator Z, which is P-codiagonal (in $\mathcal{H}$) and such that

$$e^{iZ}Pe^{-iZ} = Q.$$

Pick $\xi \in \mathcal{H}_{10}$, i.e. $P\xi = \xi$ and $Q\xi = 0$.

The formula above implies that $e^{iZ}(R(P)) = R(Q)$, so that $e^{iZ}\xi \in R(Q)$. Let us show that also $e^{iZ}\xi \in N(P)$. The same formula also means that $Pe^{-iZ} = e^{-iZ}Q$. The fact that Z is P-codiagonal, means that Z anti-commutes with the symmetry $2P-1$. It follows that

$$(2P-1)e^{iZ}\xi = e^{-iZ}(2P-1)\xi = e^{-iZ}\xi,$$

and then

$$Pe^{iZ}\xi = P(2P-1)e^{iZ}\xi = Pe^{-iZ}\xi = e^{-iZ}Q\xi = 0.$$

That is,

$$e^{iZ}(\mathcal{H}_{10}) \subset \mathcal{H}_{01}.$$

Similarly (or by the symmetry of the argument, reasoning with the orthocomplements $P^{\perp}$ and $Q^{\perp}$), one obtains

$$e^{iZ}(\mathcal{H}_{01}) \subset \mathcal{H}_{10}.$$

In particular, $\dim \mathcal{H}_{10} = \dim \mathcal{H}_{01}$

We may summarize our computations in the following

Theorem 3.2. *Let P and Q be projections. There is a geodesic of $Gr(\mathcal{H})$ which joins them if and only if*

$$\dim R(P) \cap N(Q) = \dim R(Q) \cap N(P).$$

Note that, in particular, this equality implies that the projections are unitarily equivalent.

4. Uniqueness of geodesics joining two given projections

Let us focus now on the problem of uniqueness (or multiplicity) of geodesics in $Gr(\mathcal{H})$ joining two given projections.

In the previous section we saw that in the case that $\dim R(P) \cap N(Q) = \dim R(Q) \cap N(P)$ **any** unitary isomorphism $W : R(P) \cap N(Q) \to R(Q) \cap N(P)$ induces a geodesic between P' and Q'. It follows that if

$$\dim R(P) \cap N(Q) = \dim R(Q) \cap N(P) \neq 0,$$

there are infinitely many geodesics of $Gr(\mathcal{H})$ joining P and Q.

Remark 4.1. If Z is the (selfadjoint, P-co-diagonal) exponent of a geodesic joining P and Q, reparametrizing this curve in order to reverse its path ($t \longleftrightarrow 1-t$) one concludes that Z is also Q-co-diagonal. In particular,

$$Z(N(P)) \subset R(P), \quad Z(R(P)) \subset N(P), \quad Z(N(Q)) \subset R(Q), \quad Z(R(Q)) \subset N(Q).$$

Then

$$Z(N(P) \cap N(Q)) \subset R(P) \cap R(Q) \quad \text{and} \quad Z(R(P) \cap R(Q)) \subset N(P) \cap N(Q),$$

as well as

$$Z(R(P)\cap N(Q))\subset N(P)\cap R(Q) \quad \text{and} \quad Z(N(P)\cap R(Q))\subset R(P)\cap N(Q).$$

In other words, any exponent Z of a geodesic joining P and Q is reduced by the (**three**) subspace decomposition

$$\mathscr{H}=\mathscr{H}''\oplus\mathscr{H}'\oplus\mathscr{H}_0,$$

where

$$\mathscr{H}''=(N(P)\cap N(Q))\oplus(R(P)\cap R(Q))=\mathscr{H}_{00}\oplus\mathscr{H}_{11},$$

and

$$\mathscr{H}'=(R(P)\cap N(Q))\oplus(N(P)\cap R(P))=\mathscr{H}_{10}\oplus\mathscr{H}_{01}.$$

This implies that any geodesic of $Gr(\mathscr{H})$ between P and Q induces three geodesics:

- Between
$$1\oplus 0 \ \text{ and } \ 1\oplus 0 \ \text{ in } \ \mathscr{H}'',$$
- between
$$P'=1\oplus 0 \ \text{ and } \ Q'=0\oplus 1 \ \text{ in } \ \mathscr{H}',$$
- and between
$$P_0 \text{ and } Q_0 \ \text{ in } \ \mathscr{H}_0.$$

The first geodesic is reduced to a point. This fact seems apparent, but needs a proof (it could be a loop inside $Gr(\mathscr{H}'')$).

Proposition 4.2. *Let Z_1 be a selfadjoint operator in $(N(P)\cap N(Q))\oplus(R(P)\cap R(Q))$, co-diagonal with respect $1\oplus 0$, with $\|Z_1\|\leq\pi/2$, such that*

$$e^{iZ_1}(1\oplus 0)e^{-iZ_1}=1\oplus 0.$$

Then $Z_1=0$.

Proof. Let us write the projections as symmetries,

$$e^{2iZ_1}(1\oplus -1)=1\oplus -1. \tag{4}$$

The operator $2iZ_1$, in matrix form in terms of $(N(P)\cap N(Q))\oplus(R(P)\cap R(Q))$, is

$$2iZ_1=\begin{pmatrix} 0 & a \\ -a^* & 0 \end{pmatrix}$$

with $\|a\|\leq\pi$. The exponential of this matrix can be computed

$$e^{2iZ_1}=\begin{pmatrix} \cos(|a^*|) & -a\ \mathrm{sinc}(|a|) \\ -a^*\ \mathrm{sinc}(|a^*|) & \cos(|a|) \end{pmatrix},$$

where $\mathrm{sinc}(t)=\frac{\sin(t)}{t}$ is the cardinal sine. In particular, the formula (4) above implies that

$$\cos(|a|)=1.$$

Since $0\leq|a|\leq\pi$, this implies $|a|=0$, i.e. $Z_1=0$. □

In the second part $\mathscr{H}'=(R(P)\cap N(Q))\oplus(N(P)\cap R(P))$, as seen previously, there might be infinitely many geodesics (granted that $\dim\mathscr{H}_{10}=\dim\mathscr{H}_{01}\neq 0$).

In order to examine what happens in the third part $\mathscr{H}_0$, the generic part, we shall use a result by Chandler Davis ([7, Thm. 6.1]):

The following condition is necessary and sufficient in order that a selfadjoint operator A be equal to the difference of two orthogonal projections:

$-1 \leq A \leq 1$ and in the generic part $\mathscr{H}_0$ of A (in terms of A: $\mathscr{H}_0 = (N(A) \oplus N(A-1) \oplus N(A+1)^{\perp})$ there is a symmetry V ($V^* = V^{-1} = V$) such that

$$VA_0 = -A_0 V,$$

where A_0 denotes the part of A which acts in $\mathscr{H}_0$.

In this case, Davis shows, to every V there corresponds a unique pair P_V, Q_V of projections such that $A_0 = P_V - Q_V$.

Apparently, only A_0 needs a decomposition. In $N(A)$ and $N(A-1) \oplus N(A+1)$, A is naturally written as a difference of projections: $0 + P_{N(A-1)} - P_{N(A+1)}$.

Davis gives explicit formulas for P_V and Q_V:

$$P_V = \frac{1}{2}\{1 + A_0 + V(1-A_0^2)^{1/2}\}, \qquad Q_V = \frac{1}{2}\{1 - A_0 + V(1-A_0^2)^{1/2}\}.$$

It can be easily shown how the projections determine V:

$$P_V + Q_V - 1 = V(1-A_0^2)^{1/2} = (1-A_0^2)^{1/2}V,$$

knowing that $(1-A_0^2)^{1/2}$ has trivial nullspace, because

$$N((1-A_0^2)^{1/2}) = N(1-A_0^2) = N(A_0-1) \oplus N(A_0+1) = 0,$$

one obtains that

$$V = (1-A_0^2)^{-1/2}(P_V + Q_V - 1).$$

What Davis' theorem means to our problem is explained in the following result:

Lemma 4.3. *Let P_0 and Q_0 be projections in the generic part $\mathscr{H}_0$, $A_0 = P_0 - Q_0$ and let V be the unique symmetry given by Davis' theorem which verifies*

$$P_V = P_0, \quad Q_V = Q_0.$$

Let Z_0 be a selfadjoint operator, $\|Z_0\| \leq \pi/2$, which is P_0-co-diagonal, such that $e^{iZ_0}P_0e^{-iZ_0} = Q_0$. Then

$$V = e^{iZ_0}(2P_0 - 1).$$

Proof. Indeed, recalling that

$$P_0 = \begin{pmatrix} 1_{\mathscr{K}} & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \quad \text{and} \quad e^{iZ_0} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix},$$

one has that (using $C^2 + S^2 = 1$)

$$A_0^2 = \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix}, \quad (1-A_0^2)^{-1/2} = \begin{pmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix}$$

and then

$$V = (1-A_0^2)^{-1/2}(P_0 + Q_0 - 1) = \begin{pmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} \begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix} = \begin{pmatrix} C & S \\ S & -C \end{pmatrix},$$

which coincides with

$$e^{iZ_0}(2P_0 - 1) = \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

□

In consequence, e^{iZ_0} is determined by P_0 and Q_0.

Moreover, since $\|Z_0\| \le \pi/2 < \pi$, this implies that Z_0 is determined P_0 and Q_0. Indeed, the exponential of the unitary group, $Z \mapsto e^{iZ}$ is one to one between $\{Z = Z^*, \|Z\| < \pi\}$ and $\{U \text{ unitary}, \|U-1\| < 2\}$, by a straightforward use of the spectral theorem.

Therefore we have

Theorem 4.4. *Let P and Q be projections which can be joined by a geodesic δ of $Gr(\mathcal{H})$. This geodesic determines a geodesic δ_0 between the generic parts P_0 and Q_0, which is unique (satisfying that $\|Z_0\| \le \pi/2$).*

In particular, if P and Q are in generic position ($\mathcal{H} = \mathcal{H}_0$), there is a unique geodesic of $Gr(\mathcal{H})$ (with exponent Z of norm less than or equal to $\pi/2$) which joins them.

In terms of the question originating this section:

Theorem 4.5. *There is a unique geodesic joining P and Q in $Gr(\mathcal{H})$ if and only if*

$$R(P) \cap N(Q) = N(P) \cap R(Q) = 0.$$

Remark 4.6. If $\mathcal{H}$ is finite dimensional, or if $P-Q$ is a trace class operator, one has that

$$\mathrm{Tr}(P-Q) = \dim N(A-1) - \dim N(A+1).$$

This fact was stated by several authors (Effros [6], Amrein-Sinha [1], Avron, Seiler and Simon [4]), and has a direct proof using Davis' theorem of 1954 [7]. Indeed, if A_0 is the generic part of $A = P-Q$, A_0 is also a trace class operator, which anti-commutes with a symmetry, therefore it has null trace:

$$\mathrm{Tr}(A_0) = \mathrm{Tr}(VA_0V) = -\mathrm{Tr}(A_0).$$

In the spectral resolution of A, taking trace, only two projections remain: $\mathrm{Tr}(P_{N(A-1)}) - \mathrm{Tr}(P_{N(A+1)})$, which equals the differences of dimensions above. One obtains in this fashion a proof that in this context P and Q can be connected by a geodesic of $Gr(\mathcal{H})$ if and only if $\mathrm{Tr}(P-Q) = 0$. In the finite dimensional case, this equality means that $r(P) = r(Q)$ and thus also $n(P) = n(Q)$, i.e. P and Q lie in the same connected component of $Gr(\mathcal{H})$. The fact, that this condition implies that P and Q can be joined by a geodesic, is a consequence of the Theorem of Hopf-Rinow, due to the fact that the connected components of $Gr(\mathcal{H})$ are compact if $\dim \mathcal{H} < \infty$.

5. Finsler metric in $Gr(\mathcal{H})$

Since $Gr(\mathcal{H})$ is a complemented submanifold of $\mathcal{B}(\mathcal{H})$, it is natural to endow its tangents spaces $(TGr(\mathcal{H}))_P = \mathcal{C}_P \subset \mathcal{B}(\mathcal{H})$ with the operator norm of $\mathcal{B}(\mathcal{H})$. We shall call the metric thus obtained a Finsler metric, though it is not a Finsler metric in the usual sense of the term: for instance, the norm of a smooth tangent field is a (continuous) non differentiable map.

We shall expose the remarkable result by Porta and Recht [14], that geodesics of the linear connection are locally minimal. To do this, it will be useful to consider the metric of the unitary group $\mathcal{U}(\mathcal{H})$, induced by endowing the tangent spaces of $\mathcal{U}(\mathcal{H})$ also with the usual operator norm. Let us state in the next remark the basic facts on the geometry of $\mathcal{U}(\mathcal{H})$.

Remark 5.1. The unitary group $\mathcal{U}(\mathcal{H})$ is a submanifold of $\mathcal{B}(\mathcal{H})$. One can parametrize unitaries with selfadjoint operators by means of the exponential map:

$$\exp : \{X \in \mathcal{B}_h(\mathcal{H}) : \|X\| < \pi\} \mapsto \{U \in \mathcal{U}(\mathcal{H}) : \|U-1\| < 2\}, \quad \exp(X) = e^{iX}$$

which is one to one. Then $\mathscr{U}(\mathscr{H})$ acquires a differentiable structure (translating this local chart for $1 \in \mathscr{U}(\mathscr{H})$ using the left action of $\mathscr{U}(\mathscr{H})$ on itself). Therefore

$$(T\mathscr{U}(\mathscr{H}))_1 = i\mathscr{B}_h(\mathscr{H}) \quad \text{and} \quad (T\mathscr{U}(\mathscr{H}))_U = iU \cdot \mathscr{B}_h(\mathscr{H}).$$

As said above, we endow the tangent spaces with the usual operator norm. This norm is unitary invariant: $\|UX\| = \|XU\| = \|X\|$. With this metric, one parameter groups $t \mapsto \exp(tX)$ are minimal curves for time $|t| \leq \pi/\|X\|$.

Let us state and prove this fact, which is well known:

Proposition 5.2. *Let $U \in \mathscr{U}(\mathscr{H})$ and $X_0 \in \mathscr{B}_h(\mathscr{H})$ with $\|X_0\| \leq \pi$. Then the smooth curve*

$$\mu(t) = Ue^{itX_0}$$

has minimal length along its path, for all $t \in [-1,1]$. Any pair of unitaries $U,V \in \mathscr{U}(\mathscr{H})$ can be joined by such a curve.

Proof. Note that the length of μ in the interval $[t_1,t_2]$ is $(t_2 - t_1)\|X_0\|$. Suppose first that the selfadjoint operator X_0 has a norming eigenvector. This means that there exists a unit vector $\xi \in \mathscr{H}$ such that $X_0\xi = \lambda\xi$, where $\lambda = \pm\|X_0\|$. Consider the following smooth map:

$$\rho : \mathscr{U}(\mathscr{H}) \to \mathbb{S}_{\mathscr{H}} = \{\eta \in \mathscr{H} : \|\eta\| = 1\}, \quad \rho(U) = U\xi.$$

Clearly the differential of this map is

$$d\rho_U : iU\mathscr{B}_h(\mathscr{H}) \to (T\mathbb{S}_{\mathscr{H}})_{U\xi}, \quad \mathscr{D}\rho_U(UX) = iUX\xi.$$

Note that $d\rho_U$ is norm decreasing ($\mathbb{S}_{\mathscr{H}}$ is endowed with the Hilbert-Riemann metric consisting of the usual norm at very tangent space): $\|d\rho_U(UX)\| = \|UX\xi\| \leq \|UX\|$. It follows that if γ is a piecewise smooth curve in $\mathscr{U}(\mathscr{H})$, then

$$\ell(\rho(\gamma)) \leq \ell(\gamma),$$

where ℓ denotes the length functional of curves. On the other hand, note that for the special curve $\mu(t) = Ue^{itX_0}$, the length is preserved by ρ:

$$\|\frac{d}{dt}(\rho(\mu))\| = \|\frac{d}{dt}Ue^{itX_0}\xi\| = \|Ue^{itX_0}X_0\xi\| = \|X_0\xi\| = |\lambda| = \|X_0\| = \|\dot{\mu}\|.$$

Therefore $\ell(\mu) = \ell(\rho(\mu))$. Moreover, note that

$$\rho(\mu(t)) = Ue^{itX_0}\xi = U(e^{it\lambda}\xi) = e^{it\lambda}U\xi.$$

Since $|\lambda| \leq \pi$, the curve $\rho(\mu(t) = e^{it\lambda}U\xi$ is a minimal geodesic of the sphere $\mathbb{S}_{\mathscr{H}}$ for $t \in [-1,1]$. This implies that on any sub-interval $[t_1,t_2]$ of $[-1,1]$, $\rho(\mu)$ is shorter than any given curve of $\mathbb{S}_{\mathscr{H}}$ joining the same endpoints. In particular, if γ is a curve in $\mathscr{U}(\mathscr{H})$ joining $U_1 = \mu(t_1)$ and $U_2 = \mu(t_2)$, then

$$\ell(\mu|_{[t_1,t_2]}) = \ell(\rho(\mu)|_{[t_1,t_2]}) \leq \ell(\rho(\gamma)) \leq \ell(\gamma).$$

Suppose now that X_0 is an arbitrary selfadjoint operator. Then X_0 can be approximated by seladjoint operators X_n with norming eigenvectors, and $\|X_n\| \leq \|X_0\|$ (for instance, it can be approximated by selfadjoint operators of finite spectrum). Apparently it is sufficient to reason in the interval $[0,1]$ (the same argument holds on any sub-interval). Suppose that there exists a curve γ in $\mathscr{U}(\mathscr{H})$ joining the same endpoints as μ, such that

$$\ell(\gamma) < \ell(\mu) - \delta = \|X_0\| - \delta.$$

Pick X_n close to X_0, so that $0 \leq \|X_0\| - \|X_n\| < \delta/4$ and $\|Z_n\| < \delta/4$, where

$$Z_n = \log(e^{-iX_0}e^{iX_n}).$$

Here log is the (continuous) inverse of exp above. Consider the curve $\gamma\cdot\mu_n$ obtained by adjoining the curves γ, and $\mu_n(t)=Ue^{iX_0}e^{itZ_n}$, $t\in[0,1]$. Note that μ_n joins Ue^{iX_0} with

$$Ue^{iX_0}e^{iZ_n}=Ue^{iX_n},$$

with length $\ell(\mu_n)=\|Z_n\|$. Thus $\gamma\cdot\mu_n$ joins U with Ue^{iX_n} with length $\ell(\gamma\cdot\mu_n)=\ell(\gamma)+\ell(\mu_n)$. Note that $\ell(\gamma)<\|X_0\|-\delta<\|X_n\|-\frac{3}{4}\delta$ and $\ell(\mu_n)=\|Z_n\|<\delta/4$. Thus

$$\ell(\gamma\cdot\mu_n)<\|X_n\|-\delta/2,$$

which contradicts the previous paragraph.

Finally, it is well known that any unitary operator can be written in the form e^{iX_0} for some selfadjoint operator X_0 with $\|X_0\|\le\pi$. If $U_1,U_2\in\mathcal{U}(\mathcal{H})$, there exists such X_0 satisfying $U_1^*U_2=e^{iX_0}$. Then $\mu(t)=U_1e^{itX_0}$ is a minimal curve in $\mathcal{U}(\mathcal{H})$ joining $\mu(0)=U_1$ and $\mu(1)=U_1e^{iX_0}=U_2$ □

As said in the first section, there is a one to one correspondence between projections and symmetries,

$$P\longleftrightarrow\varepsilon_P=2P-1.$$

This correspondence allows one to regard $Gr(\mathcal{H})$ inside $\mathcal{U}(\mathcal{H})$. With respect to the Finsler metrics induced by the operator norm, this inclusion is an isometry multiplied by the factor 2. Thus we may compare lengths of curves of projections in the unitary group. The following statement was proved in [14], with a slightly different argument.

Theorem 5.3. *Let $P\in Gr(\mathcal{H})$, and let $Z=Z^*$ be P-codiagonal with $\|Z\|\le\pi/2$. Then the curve $\delta(t)=e^{itZ}Pe^{-itZ}$ is minimal along its path in $Gr(\mathcal{H})$, for $t\in[-1,1]$.*

Proof. As remarked in Section 1, that Z is P-codiagonal means that it anti-commutes with $\varepsilon_P=2P-1$: $Z\varepsilon_P=-\varepsilon_pZ$. Then

$$e^{itZ}\varepsilon_P=\varepsilon_Pe^{-itZ}.$$

Therefore, if we consider the geodesic δ inside $\mathcal{U}(\mathcal{H})$,

$$\varepsilon_{\delta(t)}=2e^{itZ}Pe^{-itZ}-1=e^{itZ}\varepsilon_Pe^{-itZ}=\varepsilon_Pe^{-2itZ}.$$

That is, ε_δ is a minimal curve in $\mathcal{U}(\mathcal{H})$ for $t\in[-1,1]$ (note that $\|2Z\|\le\pi$). Thus, if γ is a curve in $Gr(\mathcal{H})$ joining two endpoints in the path δ, say $\delta(t_1)$ and $\delta(t_2)$, then ε_γ is a curve in $\mathcal{U}(\mathcal{H})$, joining $\varepsilon_{\delta(t_1)}$ and $\varepsilon_{\delta(t_2)}$. By the above Proposition, it follows that

$$\ell(\varepsilon_\gamma)\ge\ell(\varepsilon_\delta|_{[t_1,t_2]}).$$

Since $\ell(\varepsilon_\gamma)=2\ell(\gamma)$ and $\ell(\varepsilon_\delta|_{[t_1,t_2]})=2\ell(\delta|_{[t_1,t_2]})$, the result follows. □

Remark 5.4. The proof above proves a stronger fact than stated, namely, that ε_δ is minimal among all curves of *unitaries* joining the same endpoints, and not merely among curves of symmetries.

Combining the above theorem with the results in section 3, one obtains:

Corollary 5.5. *Let $P,Q\in Gr(\mathcal{H})$. Then they can be joined by a minimal geodesic if and only if*

$$\dim R(P)\cap N(Q)=\dim N(P)\cap R(Q).$$

The length of a minimal geodesic is $\le\pi/2$. If this dimension above is non zero, then the length of a minimal geodesic equals $\pi/2$.

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