Han's Conjecture and Hochschild Homology for Null-Square Projective Algebras

CLAUDE CIBILS, MARÍA JULIA REDONDO & ANDREA SOLOTAR

ABSTRACT. Let $\mathcal{H}$ be the class of algebras verifying Han's conjecture. In this paper we analyse two types of algebras with the aim of providing an inductive step towards the proof of this conjecture. First, we show that if an algebra Λ is triangular with respect to a system of non-necessarily primitive idempotents, and if the algebras at the idempotents belong to $\mathcal{H}$, then Λ is in $\mathcal{H}$. Second, we consider a 2×2 matrix algebra, with two algebras on the diagonal, two projective bimodules in the corners, and zero corner products. They are not triangular with respect to the system of the two diagonal idempotents. However, the analogous result holds: namely, if both algebras on the diagonal belong to $\mathcal{H}$, then the algebra itself is in $\mathcal{H}$.

1. INTRODUCTION

In this paper, a *smooth* finite-dimensional algebra is a finite-dimensional algebra over a field k of finite global dimension. The word "smooth" originated in commutative algebra and is convenient for brevity. Observe that in [11], for finite-dimensional algebras, "smooth" corresponds to algebras of global dimension at most one, that is, hereditary or semisimple algebras.

In 2006, Y. Han conjectured in [13] that a finite dimensional algebra whose Hochschild homology vanishes in large enough degrees is smooth. In the same paper he proved the conjecture for monomial algebras, while in [4] P. A. Bergh and D. Madsen proved it in characteristic zero for graded finite-dimensional local algebras, Kozsul algebras, and graded cellular algebras. Recently, the same authors showed in [6] that trivial extensions of selfinjective algebras, local algebras, and graded algebras have infinite Hochschild homology, a result which confirms Han's

conjecture for these algebras. In the 90s, the work of the Buenos Aires Cyclic Homology Group [8], and of L. Avramov and M. Vigué-Poirrier [1] provided the result for finitely generated commutative algebras.

In relation with Han's conjecture, lower bounds are obtained in [5] for the dimension of the Hochschild homology groups of fiber products of algebras, trivial extensions, path algebras of quivers containing loops, and quantum complete intersections. Note that P. A. Bergh and K. Erdmann proved in [2] that quantum complete intersections—not at a root of unity—satisfy Han's conjecture. In [19] A. Solotar and M. Vigué-Poirrier proved Han's conjecture for a generalization of quantum complete intersections and for a family of algebras which are in a sense opposite to these. Moreover, in [18], A. Solotar, M. Suárez-Alvarez, and Q. Vivas considered quantum generalized Weyl algebras and proved Han's conjecture for these algebras (out of a few exceptional cases).

In this paper we consider null-square algebras over a field k, that is, algebras Λ of the form

$$\begin{pmatrix} A & N \\ M & B \end{pmatrix}$$

where A and B are k-algebras, M and N are bimodules, and the product is given by matrix multiplication subject to $MN = 0 = NM$. For these algebras, $I = M \oplus N$ is a two-sided ideal verifying $I^2 = 0$ and $C = A \times B$ is a subalgebra. Actually, $\Lambda = C \oplus I$, that is, Λ is a cleft singular extension (see [16, p. 284]).

Hochschild homology is a functor HH_* from k-algebras to graded vector spaces. Thus, for a null-square algebra, $HH_*(C)$ is a direct summand of $HH_*(\Lambda)$. Moreover, note that $HH_*(C) = HH_*(A) \oplus HH_*(B)$.

In relation to Han's conjecture, this paper treats two opposite cases: one corresponds to quivers without cycles, while in the other case the quiver contains cycles. Both of them aim to provide an inductive step towards proving the conjecture. In Section 2, we consider algebras which are E-triangular; that is, they do not have oriented cycles with respect to a complete system E of non-necessarily primitive orthogonal idempotents—for brevity we call such a set E a "system." In Sections 3 and 4, on the contrary, we study a case where there is an oriented cycle. In this last case our analysis requires the involved bimodules to be projective.

A null-square algebra with $N = 0$ will be called a corner algebra. For these algebras $HH_*(\Lambda) = HH_*(C)$, by a direct computation that we briefly recall in Section 2 (see also [15] or [10]). Moreover, we show that if a corner algebra is finite dimensional, with A and B smooth, then the corner algebra is also smooth. This leads to our first result: namely, corner algebras built on the class of algebras $\mathcal{H}$ verifying Han's conjecture also belong to $\mathcal{H}$. Note that no extra assumption on M is required in the foregoing.

Based on the previous results, we go further. We associate with a system E of a k-algebra Λ its Peirce E-quiver: the set of vertices is E, and for $x \neq y$ elements of E, there is an arrow from x to y if $y\Lambda x \neq 0$. If the Peirce E-quiver has no oriented cycles then Λ is called E-triangular. For instance, the Peirce E-quiver of a corner

algebra with respect to the system E given by the two diagonal idempotents is an arrow if $M \neq 0$. We show that if Λ is E-triangular, then there is a decomposition $HH_*(\Lambda) = \bigoplus_{x \in E} HH_*(x\Lambda x)$. Moreover, for a finite-dimensional E-triangular algebra Λ such that $x\Lambda x$ is smooth for all $x \in E$, the algebra is also smooth. We infer that finite-dimensional E-triangular algebras built on the class $\mathcal{H}$ also belong to $\mathcal{H}$, without requiring additional assumptions on the bimodules $y\Lambda x$.

In Section 3, we consider null-square algebras Λ with non-zero bimodules M and N: in other words, the Peirce E-quiver with respect to the two diagonal idempotents is $\cdot \rightleftarrows \cdot$. If M and N are projective bimodules, Λ is called a null-square projective algebra. We provide a long exact sequence computing $HH_*(\Lambda)$, which is associated to the short exact sequence obtained from the product map $\Lambda \otimes_C \Lambda \to \Lambda$. We obtain a projective resolution of the kernel $K_C^1(\Lambda)$ of this map, which enables us to compute $\mathrm{Tor}_*^{\Lambda - \Lambda}(K_C^1(\Lambda), \Lambda)$ through invariants or coinvariants of a natural action of cyclic groups C_m on the zero-degree Hochschild homology of tensor powers $N \otimes_B M$: that is, $H_0(A, (N \otimes_B M)^{\otimes_A m})^{C_m}$ and $H_0(A, (N \otimes_B M)^{\otimes_A m})_{C_m}$. We thus obtain the long exact sequence of Theorem 3.15.

In Section 4 we focus on basic finite-dimensional algebras A and B over a perfect field. After choosing a complete system of primitive orthogonal idempotents for each algebra, the projective bimodules M and N are given explicitly as direct sums of indecomposable projective bimodules. We first prove that if the invariants are zero (i.e, $H_0(A, (N \otimes_B M)^{\otimes_A m})^{C_m} = 0$), then the space itself is zero. We infer that if the null-square projective algebra has zero homology in large degrees, then the 0-homology of any tensor power of $N \otimes_A M$ vanishes. Hence, the long exact sequence obtained before provides $HH_*(\Lambda) = HH_*(A \times B)$. We prove that the tensor powers of $N \otimes_A M$ and of $M \otimes_B N$ vanish in large enough degrees. Observe that $H_0(A, (N \otimes_B M)^{\otimes_A *})$ is related to 2-truncated cycles: namely, cycles in the Gabriel quiver of a basic algebra in which the product of any two consecutive arrows is zero, as considered in [3] in order to guarantee that Hochschild homology is infinite dimensional.

Another important result that we obtain in this section is the following Theorem 4.7. For a perfect field k, let Λ be a finite-dimensional null-square projective k-algebra, where A and B are smooth. Assuming the bimodules verify $(N \otimes_B M)^{\otimes_A *} = 0$ for large enough exponents, the algebra Λ is also smooth. The proof relies on the construction of an explicit projective resolution obtained through successive cones of the identity.

One of the main results of this paper follows: a finite-dimensional null-square projective algebra built on the class of basic algebras in $\mathcal{H}$ also belongs to $\mathcal{H}$.

In the last section we give a presentation by quiver and relations of a null-square projective algebra, starting from the same type of presentations of A and B. This is useful for producing examples where our results apply.

2. HAN'S CONJECTURE FOR CORNER AND E-TRIANGULAR ALGEBRAS

In this section we first consider null-square algebras and their category of representations. Next, we will study corner algebras which are particular cases of null-square algebras, in relation with Han's conjecture. The results that we obtain in this section for corner (and then for triangular algebras) do not require a projectivity hypothesis on the bimodules considered in the definition of a null-square algebra below.

Definition 2.1. Let k be a field and let A and B be k-algebras. Let M and N be, respectively, a $B - A$-bimodule and an $A - B$-bimodule. The corresponding *null-square algebra* is

$$\begin{pmatrix} A & N \\ M & B \end{pmatrix}$$

where the product is given by matrix multiplication using the products of A and B, the bimodule structures of M and N, and setting $mn = 0$ and $nm = 0$ for all $m \in M$ and $n \in N$.

Remark 2.2. A *square algebra* is an algebra

$$\begin{pmatrix} A & N \\ M & B \end{pmatrix}$$

as before, with two bimodule maps $\alpha : N \otimes_B M \to A$ and $\beta : M \otimes_A N \to B$ verifying the obvious "associativity" conditions that ensure the associativity of the corresponding matrix product on Λ. A null-square algebra is a square algebra where $\alpha = 0 = \beta$. Observe that in [7], R. O. Buchweitz studies square algebras which are called "(generalised) Morita context" or "pre-equivalence," and focuses on the case where α or β are surjective.

Example 2.3. Let Λ be a k-algebra with a decomposition $\Lambda = P \oplus Q$ as a right Λ-module. Then, Λ is a square algebra of the form

$$\begin{pmatrix} \mathrm{End}_\Lambda P & \mathrm{Hom}_\Lambda(Q,P) \\ \mathrm{Hom}_\Lambda(P,Q) & \mathrm{End}_\Lambda Q \end{pmatrix}.$$

If for all $f \in \mathrm{Hom}_\Lambda(P,Q)$ and for all $g \in \mathrm{Hom}_\Lambda(Q,P)$ the compositions gf and fg are zero, the algebra is null-square.

Remark 2.4. Any square algebra Λ is obtained as above by considering the right module decomposition:

$$\begin{pmatrix} A & N \\ M & B \end{pmatrix} = \begin{pmatrix} A & N \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ M & B \end{pmatrix}.$$

Recall that a *cleft singular extension algebra* (see [16, p. 284]) is an algebra Λ with a decomposition $\Lambda = C \oplus I$, where C is a subalgebra and I is a two-sided ideal

of Λ verifying $I^2 = 0$. A null-square algebra $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ is an instance of a cleft singular extension with $C = A \times B$ and $I = M \oplus N$. Indeed, I is a two-sided ideal precisely because $MN = NM = 0$.

We will next consider systems of idempotents of an arbitrary algebra in order to recall the representation theory of a null-square algebra.

Definition 2.5. Let Λ be a k-algebra. A *system* of Λ is a finite set E of non-zero orthogonal idempotents which is complete, that is, $\sum_{x \in E} x = 1$. The system is trivial if $E = \{1\}$.

Observe that in the above definition we do not require the idempotents to be primitive. With a system E of a k-algebra Λ we associate a k-category $C_{\Lambda,E}$ as follows: its objects are the elements of E, while the vector space $_y(C_{\Lambda,E})_x$ of morphisms from x to y is $y\Lambda x$. The composition is provided by the product of Λ. Of course, Λ is recovered as the direct sum of all the morphisms spaces of $C_{\Lambda,E}$, endowed with the matrix product. It is well known and easy to prove that the k-categories of left Λ-modules, and of k-functors from $C_{\Lambda,E}$ to k-vector spaces, are isomorphic.

Let now C be a small k-category, with set of objects C_0. Notice that a k-functor $\mathcal{M}$ from C to k-vector spaces is given by a family of vector spaces $\{_x\mathcal{M}\}_{x \in C_0}$ and a collection of linear maps

$$_yC_x \otimes {_x\mathcal{M}} \xrightarrow{ _ym_x } {_y\mathcal{M}}$$

such that, for any objects x, y, and z, the following diagram commutes:

$$
\begin{array}{ccc}
_zC_y \otimes {_yC_x} \otimes {_x\mathcal{M}} & \xrightarrow{1 \otimes {_ym_x}} & _zC_y \otimes {_y\mathcal{M}} \\
\downarrow{\scriptstyle c \otimes 1} & & \downarrow{\scriptstyle _zm_y} \\
_zC_x \otimes {_x\mathcal{M}} & \xrightarrow[{\scriptstyle _zm_x}]{} & _z\mathcal{M}.
\end{array}
$$

Next, we define a k-category, which will be isomorphic to the category of left modules over a square algebra.

Definition 2.6. Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a square algebra. The objects of the linear category $S(\Lambda)$ are $X \underset{\nu}{\overset{\mu}{\rightleftarrows}} Y$, where X is an A-module, Y is a B-module, $X \overset{\mu}{\rightarrow} Y$ stands for a map of B-modules $\mu : M \otimes_A X \to Y$, and analogously $X \underset{\nu}{\rightarrow} Y$ stands for a map of A-modules $\nu : N \otimes_B Y \to X$ which verify

$$(2.1) \qquad \nu(1_N \otimes \mu) = \alpha \otimes 1_X \quad \text{and} \quad \mu(1_M \otimes \nu) = \beta \otimes 1_Y.$$

Note that we identify the vector spaces $A \otimes_A X$ and X through the canonical isomorphism, as well as $Y \otimes_B B$ and Y.

A morphism in $S(\Lambda)$ from $X \overset{\mu}{\underset{\nu}{\rightleftharpoons}} Y$ to $X' \overset{\mu'}{\underset{\nu'}{\rightleftharpoons}} Y'$ is a couple (φ, ψ) where $\varphi : X \to X'$ is a morphism of A-modules, $\psi : Y \to Y'$ is a morphism of B-modules such that the following diagrams commute:

$$
\begin{array}{ccc}
M \otimes_A X & \overset{\mu}{\longrightarrow} & Y \\
{\scriptstyle 1 \otimes \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\
M \otimes_A X' & \underset{\mu'}{\longrightarrow} & Y'
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \overset{\nu}{\longleftarrow} & N \otimes_B Y \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle 1 \otimes \psi} \\
X' & \underset{\nu'}{\longleftarrow} & N \otimes_B Y'
\end{array}
$$

Proposition 2.7. *Let Λ be a square algebra. The category of left Λ-modules is isomorphic to $S(\Lambda)$.*

Proof. Consider the complete set of orthogonal idempotents $E = \{e, 1 - e\}$ of Λ, where $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The result is an immediate consequence of the previous observations. ∎

In what follows, the categories of the above proposition will be identified. Note that for a null-square algebra, the equalities (2.1) become

$$\nu(1_N \otimes \mu) = 0 \quad \text{and} \quad \mu(1_M \otimes \nu) = 0.$$

Lemma 2.8. *Let Λ be a square algebra and let P be a projective A-module. The Λ-module $(P \overset{1}{\underset{\alpha}{\rightleftharpoons}} M \otimes_A P)$ is projective.*

Proof. Let Λ_1 be the $\Lambda - A$-bimodule given by the first column of Λ, that is, $\Lambda_1 = \left(\begin{smallmatrix} A & 0 \\ M & 0 \end{smallmatrix}\right) = (A \overset{1}{\underset{\alpha}{\rightleftharpoons}} M)$. Note that $\Lambda_1 = \Lambda e$ is a projective Λ-module. Moreover, if X is an A-module, $\Lambda_1 \otimes_A X = (X \overset{1}{\underset{\alpha \otimes 1_X}{\rightleftharpoons}} M \otimes_A X)$. Since $\Lambda_1 \otimes_A A$ is isomorphic to Λ_1, we infer that $\Lambda_1 \otimes_A P$ is a projective Λ-module. ∎

The analogous result holds for B-modules.
From now on we focus on null-square algebras.

Proposition 2.9. *Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a null-square algebra where A, B, M, and N are finite dimensional. A simple Λ-module is isomorphic to*

$$S \rightleftharpoons 0 \quad \text{or} \quad 0 \rightleftharpoons T$$

where S and T are simple A and B-modules, respectively.

Proof. We assert that the Jacobson radical of Λ is $\left(\begin{smallmatrix} \mathrm{rad}\,A & N \\ M & \mathrm{rad}\,B \end{smallmatrix}\right)$ where $\mathrm{rad}\,A$ and $\mathrm{rad}\,B$ are the Jacobson radicals of A and B. Indeed, this vector space is a nilpotent two-sided ideal, and the quotient of Λ by it is semisimple. ∎

Definition 2.10. A *corner algebra* Λ is a square algebra with $N = 0$. In this case, the objects of $S(\Lambda)$ are denoted by $X \overset{\mu}{\to} Y$.

In this section we consider Han's conjecture for corner algebras first, and secondly for E-triangular algebras which will be defined below. We emphasize that for corner algebras we do not make any hypothesis on the projectivity of M. First, we recall the following result.

Proposition 2.11 ([12, Proposition 10, p. 86]). *Let A and B be smooth finite-dimensional k-algebras. Then, the k-algebra $A \otimes B$ is smooth.*

Theorem 2.12. *Let $\Lambda = \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix} \right)$ be a corner finite-dimensional algebra, where M is a $B - A$-bimodule. If A and B are smooth, then Λ is smooth.*

Proof. It is well known that if a finite-dimensional algebra A is smooth, the same holds for A^{op}. By the previous proposition, $B \otimes A^{\mathrm{op}}$ is smooth.

Let $0 \to Q_q \to \cdots \to Q_1 \to Q_0 \to M \to 0$ be a finite resolution of M by projective $B - A$-bimodules.

First, let $S \to 0$ be a simple Λ-module where S is a simple A-module. Let $0 \to P_p \to \cdots \to P_1 \to P_0 \to S \to 0$ be a resolution of S by projective A-modules. Observe that the sequence of Λ-modules obtained by tensoring the previous resolution by Λ_1,

$$0 \to (P_p \overset{1}{\to} M \otimes_A P_p) \to \cdots \to (P_0 \overset{1}{\to} M \otimes_A P_0) \to (S \to 0) \to 0,$$

is not exact in general unless M is a projective A-module. Instead, we consider the double complex obtained by tensoring both resolutions over A:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Q_1 \otimes_A P_2 & \longrightarrow & Q_1 \otimes_A P_1 & \longrightarrow & Q_1 \otimes_A P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Q_0 \otimes_A P_2 & \longrightarrow & Q_0 \otimes_A P_1 & \longrightarrow & Q_0 \otimes_A P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M \otimes_A P_2 & \longrightarrow & M \otimes_A P_1 & \longrightarrow & M \otimes_A P_0 & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The total complex of this double complex is exact, since each column is obtained by tensoring an exact complex by a projective module. Hence, we obtain a finite

exact sequence of Λ-modules:

$$
\begin{array}{ccccccc}
\vdots & & & & \vdots & & \\
P_2 & \rightarrow & M \otimes_A P_2 & \oplus & Q_0 \otimes_A P_1 & \oplus & Q_1 \otimes_A P_0. \\
\downarrow & & \downarrow & & \downarrow & & \\
P_1 & \rightarrow & M \otimes_A P_1 & \oplus & Q_0 \otimes_A P_0 & & \\
\downarrow & & \downarrow & & & & \\
P_0 & \rightarrow & M \otimes_A P_0 & & & & \\
\downarrow & & \downarrow & & & & \\
S & \rightarrow & 0 & & & & \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}
$$

We assert that this is a projective resolution of $S \rightarrow 0$. Indeed, the i-th module is

$$(P_i \rightarrow M \otimes P_i) \oplus (0 \rightarrow Q_0 \otimes_A P_{i-1}) \oplus \cdots \oplus (0 \rightarrow Q_{i-1} \otimes_A P_0).$$

The first summand $\Lambda_1 \otimes_A P_i$ is projective by Remark 2.8. For the other summands, we first notice that if Q is a projective $B - A$-bimodule and X is any A-module, $Q \otimes_A X$ is a projective B-module. Moreover, for a corner algebra Λ, if W is a projective left B-module, then the left Λ-module $0 \rightarrow W$ is projective.

Second, let T be a simple B-module and let $0 \rightarrow T$ the corresponding simple Λ-module. Let $R_\bullet \rightarrow T$ be a finite B-projective resolution of T; then, we have that $(0 \rightarrow R_\bullet) \rightarrow (0 \rightarrow T)$ is a finite resolution of $0 \rightarrow T$ by projective Λ-modules. $\quad\square$

Now, we will define E-triangular algebras with respect to a chosen system E. We define first a quiver inferred from the Peirce decomposition $\Lambda = \bigoplus_{x,y \in E} y\Lambda x$.

Definition 2.13. Let Λ be a k-algebra and let E be a system of Λ. The *Peirce E-quiver* Q_E has set of vertices E; for x and y different elements of E, there is an arrow from x to y in case $y\Lambda x \neq 0$. Note that Q_E contains no loops.

Definition 2.14. An algebra Λ is *E-triangular* with respect to a non-trivial system E if Q_E has no oriented cycles.

Remark 2.15. In case $|E| = 2$, the Peirce E-quiver of an E-triangular algebra is an arrow, and the algebra is a corner algebra. Observe that a finite-dimensional algebra which is E-triangular with respect to a system E may have oriented cycles in its Gabriel quiver.

Lemma 2.16. *Let Λ be a k-algebra which is E-triangular. There exists a system F of two idempotents such that Λ is a corner algebra.*

Proof. The Peirce E-quiver has no oriented cycles, is finite and has at least two vertices. Then, there exists a source vertex e, that is, a vertex with no arrows ending at it. The idempotent $f = \sum_{x \neq e} x$ is not zero. Since $e \Lambda f = 0$, the algebra Λ is a corner algebra with respect to the system $F = \{e, f\}$.   □

Corollary 2.17. *Let Λ be a finite-dimensional k-algebra which is E-triangular with respect to a system E. If $x \Lambda x$ is smooth for every $x \in E$, then Λ is smooth.*

Proof. We proceed by induction on the number of vertices. Let e be a source vertex of Q_E, let $f = \sum_{x \neq e} x = 1 - e$, and let F be the system $\{e, f\}$.

Let $E' = E \setminus \{e\}$, which is a system of the algebra $f \Lambda f$. The E'-quiver of $f \Lambda f$ has no oriented cycles since $y(f \Lambda f)x = y \Lambda x$ for every $x, y \in E'$. By hypothesis, the algebras $x(f \Lambda f)x = x \Lambda x$ are smooth for every $x \in E'$. By induction, $f \Lambda f$ is smooth. Theorem 2.12 provides the result since $e \Lambda e$ is smooth and Λ is a corner algebra with respect to F, as in the proof of the previous lemma.   □

By definition, the Hochschild homology vector spaces of a k-algebra Λ with coefficients in a Λ-bimodule Z are

$$H_*(\Lambda, Z) = \mathrm{Tor}_*^{\Lambda \otimes \Lambda^{\mathrm{op}}}(\Lambda, Z),$$

where the latter is also denoted by $\mathrm{Tor}_*^{\Lambda-\Lambda}(\Lambda, Z)$.

Next, we recall the computation of the Hochschild homology of a corner algebra (see, e.g., [10, 15]). The following well known result will be required; we provide a sketch of its proof for the convenience of the reader.

Lemma 2.18. *Let Λ be a k-algebra, let D be a separable subalgebra of Λ, let Z be a Λ-bimodule, and let $Z_D = Z / \langle dz - zd \mid z \in Z, d \in D \rangle$. The homology of the complex*

$$\cdots \xrightarrow{b} Z \otimes_{D-D} (\Lambda \otimes_D \Lambda \otimes_D \Lambda) \xrightarrow{b} Z \otimes_{D-D} (\Lambda \otimes_D \Lambda) \xrightarrow{b} Z \otimes_{D-D} \Lambda \xrightarrow{b} Z_D \to 0$$

is $H_*(\Lambda, Z)$, where $\otimes_{D-D}$ stands for $\otimes_{D \otimes D^{\mathrm{op}}}$, and where the maps b are given by the usual formulas for computing Hochschild homology:

$$b(x_0 \otimes x_1 \otimes \cdots \otimes x_n) = x_0 x_1 \otimes x_2 \otimes \cdots \otimes x_n$$
$$+ \sum_0^{n-1} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n$$
$$+ (-1)^n x_n x_0 \otimes x_2 \otimes \cdots \otimes x_{n-1}.$$

Proof. Consider the complex with differential d defined by the usual formulas for the canonical resolution of Λ over the ground field

$$\cdots \xrightarrow{d} \Lambda \otimes_D \Lambda \otimes_D \Lambda \xrightarrow{d} \Lambda \otimes_D \Lambda \xrightarrow{d} \Lambda \to 0.$$

The map s given by $s(x_1 \otimes \cdots \otimes x_n) = 1 \otimes x_1 \otimes \cdots \otimes x_n$ is well defined and verifies $ds + sd = 1$; this proves that the complex is acyclic. Since D is separable, $D \otimes D^{\mathrm{op}}$ is also separable, and any D-bimodule is projective. Consequently, the acyclic complex above is a projective resolution of Λ by projective Λ-bimodules. The statement of the lemma is obtained by applying the functor $Z \otimes_{D-D}-$ to this resolution and observing that $Z \otimes_{\Lambda-\Lambda} (\Lambda \otimes_D X \otimes_D \Lambda)$ is canonically isomorphic to $Z \otimes_{D-D} X$ for any D-bimodule X.   □

Theorem 2.19 ([10, 15]). *Let* $\Lambda = \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix} \right)$ *be a corner algebra, where A and B are k-algebras and M is a $B - A$-bimodule. There is a decomposition*

$$HH_*(\Lambda) = HH_*(A) \oplus HH_*(B).$$

Proof. Let e be the idempotent

$$\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix},$$

and let

$$f = 1 - e = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}.$$

Let $D = \left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix} \right) = ke \times kf$; note that D is a separable subalgebra of Λ. We assert that the complex of the previous lemma is actually the direct sum of the complexes that compute $HH_*(A)$ and $HH_*(B)$. Indeed, notice that the D-bimodule decomposition $\Lambda = A \oplus B \oplus M$ provides a direct sum decomposition

$$\Lambda \otimes_{D-D} (\Lambda \otimes_D \cdots \otimes_D \Lambda) = (A \otimes \cdots \otimes A) \oplus (B \otimes \cdots \otimes B)$$

since

$$0 = M \otimes_{D-D} B = M \otimes_{D-D} A = M \otimes_{D-D} M = B \otimes_{D-D} M = A \otimes_{D-D} M$$

and $A \otimes_{D-D} A = A \otimes A$ while $B \otimes_{D-D} B = B \otimes B$. Observe that in degree 0 we obtain $\Lambda \otimes_{D-D} D = A \oplus B$.   □

Corollary 2.20. *For any k-algebra Λ which is E-triangular with respect to a system E, there is a decomposition*

$$HH_*(\Lambda) = \bigoplus_{x \in E} HH_*(x\Lambda x).$$

Proof. The idea of the proof is similar to the proof of Corollary 2.17. It follows by induction once a source vertex of Q_E is chosen.   □

Next, we turn to Han's conjecture that we recall: if A is a finite-dimensional algebra over a field such that $HH_n(A) = 0$ for n large enough, then A is smooth.

Theorem 2.21. *Finite-dimensional corner k-algebras built on the class of k-algebras $\mathcal{H}$ verifying Han's conjecture also belong to $\mathcal{H}$.*

Proof. Let $\Lambda = \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix} \right)$ be a finite-dimensional corner algebra, and suppose $HH_*(\Lambda) = 0$ for large enough degrees. Theorem 2.19 shows that the same holds for A and B. Since A and B belong to $\mathcal{H}$, they are smooth. By Theorem 2.12, Λ is smooth. □

Corollary 2.22. *Let Λ be a finite-dimensional k-algebra which is E-triangular with respect to a system E of Λ. If for every $x \in E$ the algebras $x\Lambda x$ belong to $\mathcal{H}$, then Λ belongs to $\mathcal{H}$.*

Proof. The proof follows from Corollaries 2.17 and 2.20. □

Remark 2.23. Let Λ be a smooth finite-dimensional algebra such that $\Lambda/\operatorname{rad}\Lambda$ is a product of copies of the ground field k, and which admits a Wedderburm decomposition $\Lambda = D \oplus \operatorname{rad}\Lambda$ where D is a subalgebra of Λ. Note that if k is perfect, a Wedderburm decomposition always exists. If Λ is smooth, it is proven by B. Keller in [14, 2.5] that there is a K-theoretical equivalence between Λ and D. In particular, the cyclic homologies of these algebras are isomorphic, as well as the Hochschild homologies because of the Connes long exact sequences relating cyclic and Hochschild homologies, for Λ and $\Lambda/\operatorname{rad}\Lambda$ (see, e.g., [20]). Consequently, the Hochschild homology of Λ is concentrated in degree zero. In this situation, it follows from Han's conjecture that if the Hochschild homology vanishes in large enough degrees, then it actually vanishes in all positive degrees.

We observe that in the situation of Corollary 2.22, the result that we have proven agrees with the previous observation. Indeed, we have shown using Corollary 2.20 that Hochschild homology is the direct sum of the Hochschild homologies at the idempotents of the system.

3. HOCHSCHILD HOMOLOGY OF NULL-SQUARE PROJECTIVE ALGEBRAS

In this section we consider a *null-square projective algebra* Λ, that is, a null-square algebra $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix} \right)$ where M and N are projective $B - A$ and $A - B$-bimodules, respectively; we recall that $MN = NM = 0$. We will provide a long exact sequence which computes $HH_*(\Lambda)$.

First, we consider a cleft extension algebra $\Lambda = C \oplus I$, where C is a subalgebra and I is a two-sided ideal (see [16, p. 284]). Let

$$K_C^1(\Lambda) = \operatorname{Ker}(\Lambda \otimes_C \Lambda \xrightarrow{d} \Lambda),$$

where d is given by the product of Λ. In case I is projective as a C-bimodule we will provide a resolution of $K_C^1(\Lambda)$ by projective Λ-bimodules. This resolution specialized to a null-square projective algebra will allow us to compute $\mathrm{Tor}_*^{\Lambda-\Lambda}(K_C^1(\Lambda),\Lambda)$. The mentioned long exact sequence will be obtained as the Tor exact sequence associated with the short exact sequence of Λ-bimodules

$$(3.1) \qquad 0 \longrightarrow K_C^1(\Lambda) \longrightarrow \Lambda \otimes_C \Lambda \longrightarrow \Lambda \longrightarrow 0.$$

Remark 3.1. This short exact sequence splits as a sequence of C-bimodules, but it does not split as a sequence of Λ-bimodules.

Lemma 3.2. *Let* $\Lambda = C \oplus I$ *be a cleft extension algebra. The complex*

$$\cdots \xrightarrow{d} \Lambda \otimes_C I \otimes_C I \otimes_C \Lambda \xrightarrow{d} \Lambda \otimes_C I \otimes_C \Lambda \xrightarrow{d} \Lambda \otimes_C \Lambda \xrightarrow{d} \Lambda \longrightarrow 0$$

is acyclic, with differentials for $n \geq 3$

$$d(\ell_1 \otimes x_2 \otimes \cdots \otimes x_{n-1} \otimes \ell_n) =$$

$$= \ell_1 x_2 \otimes x_3 \otimes \ldots x_{n-1} \otimes \ell_n + \sum_{2}^{n-2} (-1)^{i+1} \ell_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes \ell_n$$

$$+ (-1)^n \ell_1 \otimes x_2 \otimes \cdots \otimes x_{n-1} \ell_n,$$

and, for $n = 2$, *the product of the algebra is denoted by* d *as before.*

Proof. Let $\ell \in \Lambda$ and let $\ell = \ell_C + \ell_I$ be its decomposition in $C \oplus I$. Let s be the map given as follows:

$$s(\ell_1 \otimes x_2 \otimes \cdots \otimes x_{n-1} \otimes \ell_n) = 1 \otimes (\ell_1)_I \otimes x_2 \otimes \cdots \otimes x_{n-1} \otimes \ell_n.$$

It is straightforward to check that s is well defined with respect to the tensor products over C. The verification that s is a homotopy contraction is not completely trivial; we illustrate this by checking the property in degree two:

$$ds(\ell \otimes x \otimes \ell') = \ell_I \otimes x \otimes \ell' - 1 \otimes \ell_I x \otimes \ell' + 1 \otimes \ell_I \otimes x\ell',$$

$$sd(\ell \otimes x \otimes \ell') = 1 \otimes (\ell x)_I \otimes \ell' - 1 \otimes \ell_I \otimes x\ell'.$$

Note that $(\ell x)_I = \ell x = \ell_C x + \ell_I x$. Hence,

$$(ds + sd)(\ell \otimes x \otimes \ell')$$

$$= \ell_I \otimes x \otimes \ell' - 1 \otimes \ell_I x \otimes \ell' + 1 \otimes (\ell x)_I \otimes \ell'$$

$$= \ell_I \otimes x \otimes \ell' - 1 \otimes \ell_I x \otimes \ell' + 1 \otimes \ell_C x \otimes \ell' + 1 \otimes \ell_I x \otimes \ell'$$

$$= \ell_I \otimes x \otimes \ell' + 1 \otimes \ell_C x \otimes \ell' = \ell_I \otimes x \otimes \ell' + \ell_C \otimes x \otimes \ell'$$

$$= (\ell_I + \ell_C) \otimes x \otimes \ell' = \ell \otimes x \otimes \ell'. \qquad \square$$

Proposition 3.3. *Let* $\Lambda = C \oplus I$ *be a cleft extension algebra and suppose* I *is a projective* C*-bimodule. The following is a resolution of* $K_C^1(\Lambda)$ *by projective* Λ*-bimodules:*

$$\cdots \xrightarrow{d} \Lambda \otimes_C I \otimes_C I \otimes_C \Lambda \xrightarrow{d} \Lambda \otimes_C I \otimes_C \Lambda \xrightarrow{d} K_C^1(\Lambda) \longrightarrow 0.$$

Proof. The complex is acyclic by the previous result. We claim that if P and Q are projective C-bimodules, then $P \otimes_C Q$ is also a projective C-bimodule. Indeed, $(C \otimes C) \otimes_C (C \otimes C)$ is a projective bimodule and the result follows. Consequently, $I \otimes_C \cdots \otimes_C I$ is a projective C-bimodule. Moreover, if P is a projective C-bimodule it is clear that $\Lambda \otimes_C P \otimes_C \Lambda$ is a projective Λ-bimodule. $\qquad\square$

Let Λ be a k-algebra and Z be a Λ-bimodule. We recall

$$H_0(\Lambda, Z) = \Lambda \otimes_{\Lambda \otimes \Lambda^{\mathrm{op}}} Z = \Lambda \otimes_{\Lambda - \Lambda} = Z / \langle \lambda z - z\lambda \rangle,$$

where $\langle \lambda z - z\lambda \rangle$ is the vector subspace of Z generated by the set $\{\lambda z - z\lambda\}$ for all $\lambda \in \Lambda$ and $z \in Z$.

Let Λ be an algebra and let C be a subalgebra. Let U be a C-bimodule and let $\Lambda \otimes_C U \otimes_C \Lambda$ be the induced Λ-bimodule. The next result gives a decomposition of the Hochschild homology in degree zero of a cleft algebra $\Lambda = C \oplus I$ with coefficients in an induced bimodule. We provide a proof for further use.

Proposition 3.4. *Let* $\Lambda = C \oplus I$ *be a cleft algebra and let* U *be a* C*-bimodule. Then,*

$$H_0(\Lambda, \Lambda \otimes_C U \otimes_C \Lambda) = H_0(C, U) \oplus H_0(C, I \otimes_c U).$$

Proof. The mutual inverse isomorphisms are given by

$$a \otimes u \otimes b \mapsto (ba)_C u + (ba)_I \otimes u,$$
$$u + x \otimes v \mapsto 1 \otimes u \otimes 1 + x \otimes v \otimes 1. \qquad\square$$

We will use next the previous result for $U = I^{\otimes_c n}$. Let $I(n) = H_0(C, I^{\otimes_c n})$.

Corollary 3.5. *Let* $\Lambda = C \oplus I$ *be a cleft algebra. There is a decomposition*

$$H_0(\Lambda, \ \Lambda \otimes_C I^{\otimes_c n} \otimes_C \Lambda) = I(n) \oplus I(n+1).$$

Proposition 3.6. *Let* $\Lambda = C \oplus I$ *be a cleft algebra where* I *is a projective* C*-bimodule. The vector spaces* $\mathrm{Tor}_*^{\Lambda - \Lambda}(K_C^1(\Lambda), \Lambda)$ *are the homology spaces of the complex*

$$\cdots \xrightarrow{b} I(n) \oplus I(n+1) \xrightarrow{b} I(n-1) \oplus I(n) \xrightarrow{b} \cdots$$

$$\cdots \xrightarrow{b} I(2) \oplus I(3) \xrightarrow{b} I(1) \oplus I(2) \longrightarrow 0,$$

where
$$b : I(n) \oplus I(n+1) \to I(n-1) \oplus I(n)$$

is as follows:

- *If* $z_1 \otimes \cdots \otimes z_n \in I(n)$, *then*

  $$b(z_1 \otimes \cdots \otimes z_n)$$
  $$= z_1 \otimes \cdots \otimes z_n + \sum_{1}^{n-1} (-1)^i z_1 \otimes \cdots \otimes z_i z_{i+1} \otimes \cdots \otimes z_n$$
  $$+ (-1)^n z_n \otimes z_1 \otimes \cdots \otimes z_{n-1},$$

  where the first and the last terms belong to $I(n)$ *and the middle sum belongs to* $I(n-1)$.

- *If* $z_0 \otimes \cdots \otimes z_n \in I(n+1)$, *then*

  $$b(z_0 \otimes \cdots \otimes z_n)$$
  $$= z_0 z_1 \otimes \cdots \otimes z_n + \sum_{0}^{n-1} (-1)^i z_0 \otimes \cdots \otimes z_i z_{i+1} \otimes \cdots \otimes z_n$$
  $$+ (-1)^n z_n z_0 \otimes \cdots \otimes z_{n-1},$$

  which belongs to $I(n)$.

Proof. The formulas are obtained by applying the functor $H_0(\Lambda, -)$ to the projective resolution of $K_C^1(\Lambda)$ of Proposition 3.3, and by translating the differentials to the present setting through the isomorphisms provided in Proposition 3.4. □

Lemma 3.7. *Let A and B be k-algebras, let $C = A \times B$, and let I be a C-bimodule of the form $I = M \oplus N$ where M is a $B - A$-bimodule and N is a $A - B$-bimodule. For n odd, $I(n) = 0$.*

Proof. First, we notice that $M \otimes_C M = 0 = N \otimes_C N$, since, for instance, $m \otimes m' = m(1_A, 0) \otimes m' = m \otimes (1_A, 0)m' = m \otimes 0 = 0$.

Moreover, $N \otimes_C M = N \otimes_B M$ and $M \otimes_C N = M \otimes_A N$.

Consequently,

$$I^{\otimes_C n} = (\cdots \otimes_A N \otimes_B M \otimes_A N \otimes_B M) \oplus (\cdots \otimes_B M \otimes_A N \otimes_B M \otimes_A N)$$

with n tensorands in each summand. In particular, for n odd we have

$$I^{\otimes_C n} = (M \otimes_A \cdots \otimes_B M \otimes_A N \otimes_B M) \oplus (N \otimes_B \cdots \otimes_A N \otimes_B M \otimes_A N),$$

and we assert that $H_0(C, I^{\otimes_C n}) = 0$. Indeed, $(1_A, 0)x = 0$ for every $x \in M$, while $x(1_A, 0) = x$, and $(0, 1_B)y = 0$, while $y(0, 1_B) = y$ for every $y \in N$. □

Lemma 3.8. *In the same situation as in the previous lemma, for* $n = 2m$,

$$I^{\otimes_C n} = (N \otimes_B M) \otimes_A \cdots \otimes_A (N \otimes_B M)$$
$$\oplus (M \otimes_A N) \otimes_B \cdots \otimes_B (M \otimes_A N)$$
$$= (N \otimes_B M)^{\otimes_A m} \oplus (M \otimes_A N)^{\otimes_B m}.$$

Corollary 3.9. *Let A and B be k-algebras, let $C = A \times B$, and let I be a C-bimodule of the form $I = M \oplus N$ where M is a $B - A$-bimodule and where N is a $A - B$-bimodule. The following decomposition holds:*

$$I(2m) = H_0(A, (N \otimes_B M)^{\otimes_A m}) \oplus H_0(B, (M \otimes_A N)^{\otimes_B m}).$$

Definition 3.10. *Let $C_m = \langle t \mid t^m = 1 \rangle$ be a cyclic group of order m. The kC_m-module structures of*

$$H_0(B, (M \otimes_A N)^{\otimes_B m}) \quad \text{and} \quad H_0(A, (N \otimes_B M)^{\otimes_A m})$$

are given by the following action of t by cyclic permutation:

$$t(x_m \otimes y_m \otimes \cdots x_2 \otimes y_2 \otimes x_1 \otimes y_1) = x_1 \otimes y_1 \otimes x_m \otimes y_m \otimes \cdots x_2 \otimes y_2,$$
$$t(y_m \otimes x_m \otimes \cdots y_2 \otimes x_2 \otimes y_1 \otimes x_1) = y_1 \otimes x_1 \otimes y_m \otimes x_m \otimes \cdots y_2 \otimes x_2.$$

Note that the above actions are not well defined either on $M \otimes_A N$ or on $N \otimes_B M$; on the other hand, they are well defined on the 0-degree homology of these bimodules.

We provide two isomorphisms between these kC_m-modules that will be used in the proof of the next result:

$$H_0(A, (N \otimes_B M)^{\otimes_A m}) \overset{\sigma}{\longrightarrow} H_0(B, (M \otimes_A N)^{\otimes_B m})$$
$$y_m \otimes x_m \otimes \cdots \otimes y_1 \otimes x_1 \mapsto x_1 \otimes y_m \otimes x_m \otimes \cdots \otimes y_1,$$

$$H_0(B, (M \otimes_A N)^{\otimes_B m}) \overset{\tau}{\longrightarrow} H_0(A, (N \otimes_B M)^{\otimes_A m})$$
$$x_m \otimes y_m \otimes \cdots \otimes x_1 \otimes y_1 \mapsto y_1 \otimes x_m \otimes y_m \otimes \cdots \otimes x_1.$$

Notice that the compositions $\sigma\tau$ and $\tau\sigma$ are the actions of t on the corresponding vector spaces.

Finally, we recall that for a group G and a kG-module H, the invariants (or fixed points) of the action are $H^G = \{x \in H \mid sx = x \text{ for all } s \in G\}$. The coinvariants are $H_G = H/\langle sx - x \rangle$ where $\langle sx - x \rangle$ is the vector subspace of H generated by the elements of the form $sx - x$ for all $s \in G$ and $x \in H$. If G is finite and the characteristic of the field does not divide its order, then H_G and H^G are canonically isomorphic through the action of $(1/|G|) \sum_{s \in G} s$.

Theorem 3.11. *Let* $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ *be a null-square projective algebra, and let* $I = M \oplus N$. *For* $m \geq 0$,

$$\mathrm{Tor}_{2m+1}^{\Lambda-\Lambda}(K_C^1(\Lambda), \Lambda) = H_0\Big(B, (M \otimes_A N)^{\otimes_B m+1}\Big)^{C_{m+1}}$$

and

$$\mathrm{Tor}_{2m}^{\Lambda-\Lambda}(K_C^1(\Lambda), \Lambda) = H_0\Big(B, (M \otimes_A N)^{\otimes_B m+1}\Big)_{C_{m+1}}.$$

Proof. We recall that for a null-square projective algebra $MN = 0 = NM$, and hence $I^2 = 0$. Moreover, $I(n) = 0$ for n odd, by Lemma 3.7. Consequently, the complex of Proposition 3.6 reduces to

$$\cdots \xrightarrow{b} I(6) \xrightarrow{0} I(4) \xrightarrow{b} I(4) \xrightarrow{0} I(2) \xrightarrow{b} I(2) \to 0,$$

where for $n = 2m$,

$$b(z_1 \otimes \cdots \otimes z_n) = z_1 \otimes \cdots \otimes z_n + z_n \otimes z_1 \otimes \cdots \otimes z_{n-1}.$$

Furthermore, the matrix of

$$b : H_0(A, (N \otimes_B M)^{\otimes_A m}) \oplus H_0(B, (M \otimes_A N)^{\otimes_B m})$$
$$\longrightarrow H_0(A, (N \otimes_B M)^{\otimes_A m}) \oplus H_0(B, (M \otimes_A N)^{\otimes_B m})$$

with respect to the decomposition of Proposition 3.9 is $\left(\begin{smallmatrix} 1 & \tau \\ \sigma & 1 \end{smallmatrix}\right)$. Moreover,

$$\begin{aligned}
\mathrm{Ker}\, b &= \{(u, v) \mid u + \tau(v) = 0 = \sigma(u) + v\} \\
&= \{(u, -\sigma u) \mid u = \tau \sigma u\} \\
&= \{u \mid tu = u\} \\
&= H_0(A, (N \otimes_B M)^{\otimes_A m})^{C_m}.
\end{aligned}$$

In order to compute $\mathrm{Coker}\, b$, note that $(u, v) = -(\tau v, \sigma u)$ holds in $\mathrm{Coker}\, b$. Hence, $(u, 0) = (0, -\sigma u) = (\tau \sigma(u), 0)$. This shows that the map

$$H_0(A, (N \otimes_B M)^{\otimes_A m})_{C_m} \to \mathrm{Coker}\, b$$

given by $u \mapsto (u, 0)$ is well defined. Its inverse is given by $(u, v) \mapsto u - \tau(v)$. Hence, $\mathrm{Coker}\, b = H_0(A, (N \otimes_B M)^{\otimes_A m})_{C_m}$. $\qquad\square$

Towards describing the long exact sequence mentioned above, we consider now some tools of homological algebra to compute $\mathrm{Tor}_*^{\Lambda-\Lambda}(\Lambda \otimes_C \Lambda, \Lambda)$. The next result will be used for a null-square projective algebra $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ and for the inclusion of algebras $C \otimes C^{\mathrm{op}} \subset \Lambda \otimes \Lambda^{\mathrm{op}}$, where $C = A \times B$.

Lemma 3.12. *Let $F \subset D$ be an inclusion of k-algebras, and suppose D is projective as a left F-module. Let U be a right F-module and let $U \uparrow^D = U \otimes_F D$ be the induced right module. Let Z be a left D-module and let $_F \downarrow Z$ be the left F-module obtained by restricting the action to F. Then, $\mathrm{Tor}_*^D(U \uparrow^D, Z) = \mathrm{Tor}_*^F(U, {}_F \downarrow Z)$.*

Proof. The lefthand side functor in the variable Z is characterised by its universal property:

- $\mathrm{Tor}_0^D(U \uparrow^D, Z) = U \uparrow^D \otimes_D Z = U \otimes_F Z$.
- $\mathrm{Tor}_0^D(U \uparrow^D, Z) = 0$ if Z is projective.
- A short exact sequence of D-modules provides a long exact sequence.

It is clear that the righthand side functor in the variable Z verifies the same properties. Note that the second property is fulfilled precisely because we assume $_F \downarrow D$ is projective. $\qquad\square$

Lemma 3.13. *Let Λ be a null-square projective algebra, and let $C = A \times B$. The C-bimodule $\Lambda \otimes \Lambda$ is projective.*

Proof. Note first that by hypothesis M is a projective $B - A$-bimodule. It becomes a C-bimodule by extending the actions by zero; then, M is a projective C-bimodule. The same holds for N; then, $I = M \oplus N$ is a projective C-bimodule.

Consider the C-bimodule decomposition

$$\Lambda \otimes \Lambda = (C \otimes C) \oplus (C \otimes I) \oplus (I \otimes C) \oplus (I \otimes I).$$

We assert that a projective C-bimodule is also projective as a left (or right) C-module. Indeed, the free rank-one C-bimodule $C \otimes C$ is free as a left (or right) C-module. This observation makes the proof of the assertion immediate. We infer that I is projective as a left and as a right C-module.

We record that if P is a projective left C-module and Q is a projective right C-module, the C-bimodule $P \otimes Q$ is a projective C-bimodule.

Consequently, the four terms of the above direct sum decomposition of the C-bimodule $\Lambda \otimes \Lambda$ are projective C-bimodules. $\qquad\square$

Theorem 3.14. *Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix} \right)$ be a null-square projective algebra, and let $C = A \times B$. There is a decomposition*

$$\mathrm{Tor}_*^{\Lambda - \Lambda}(\Lambda \otimes_C \Lambda, \Lambda) = HH_*(A) \oplus HH_*(B).$$

Proof. We consider the inclusion $C \otimes C^{\mathrm{op}} \subset \Lambda \otimes \Lambda^{\mathrm{op}}$. By Lemma 3.12 with $U = C$, we have the following:

$$\begin{aligned}
\mathrm{Tor}_*^{\Lambda - \Lambda}(\Lambda \otimes_C \Lambda, \Lambda) &= \mathrm{Tor}_*^{\Lambda - \Lambda}(\Lambda \otimes_C C \otimes_C \Lambda, \Lambda) \\
&= \mathrm{Tor}_*^{C - C}(C, {}_C \downarrow \Lambda \downarrow_C) \\
&= H_*(C, {}_C \downarrow \Lambda \downarrow_C) \\
&= HH_*(C) \oplus H_*(C, M) \oplus H_*(C, N).
\end{aligned}$$

We assert that $H_*(C, M) = H_*(C, N) = 0$. Indeed, let $P_\bullet \to A$ be a projective resolution of the A-bimodule A, and analogously for $Q_\bullet \to B$. Note that $P_\bullet \oplus Q_\bullet \to A \oplus B$ is a projective resolution of the C-bimodule C, where the C-bimodule structure of $P_\bullet$ is obtained by extending the action to B by zero, and analogously for $Q_\bullet$. The functor $M \otimes_{C-C}-$ applied to $P_\bullet \oplus Q_\bullet$ gives the zero complex by simple arguments already used in the proof of Lemma 3.7 and $H_*(C, M) = 0$. Analogously, $H_*(C, N) = 0$. Note that the assertion also follows from [9, p. 173].

To prove $HH_*(C) = HH_*(A) \oplus HH_*(B)$, observe that the summands $A \otimes_{C-C} Q_\bullet$ and $B \otimes_{C-C} P_\bullet$ of $C \otimes_{C-C} (P_\bullet \oplus Q_\bullet)$ are zero for analogous reasons. □

The previous results and the exact sequence (3.1) provide the following result.

Theorem 3.15. Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix} \right)$ be a null-square projective algebra. There is a long exact sequence as follows:

$$\cdots$$

$$H_0\left(A, (N \otimes_B M)^{\otimes_A m+1}\right)^{C_{m+1}} \to HH_{2m+1}(A) \oplus HH_{2m+1}(B)$$
$$\to HH_{2m+1}(\Lambda) \to$$
$$H_0\left(A, (N \otimes_B M)^{\otimes_A m+1}\right)_{C_{m+1}} \to HH_{2m}(A) \oplus HH_{2m}(B)$$
$$\to HH_{2m}(\Lambda) \to$$

$$\cdots$$

$$H_0\left(A, (N \otimes_B M)^{\otimes_A 3}\right)^{C_3} \to HH_5(A) \oplus HH_5(B) \to HH_5(\Lambda) \to$$
$$H_0\left(A, (N \otimes_B M)^{\otimes_A 3}\right)_{C_3} \to HH_4(A) \oplus HH_4(B) \to HH_4(\Lambda) \to$$
$$H_0\left(A, (N \otimes_B M)^{\otimes_A 2}\right)^{C_2} \to HH_3(A) \oplus HH_3(B) \to HH_3(\Lambda) \to$$
$$H_0\left(A, (N \otimes_B M)^{\otimes_A 2}\right)_{C_2} \to HH_2(A) \oplus HH_2(B) \to HH_2(\Lambda) \to$$
$$H_0\left(A, (N \otimes_B M)\right) \to HH_1(A) \oplus HH_1(B) \to HH_1(\Lambda) \to$$
$$H_0\left(A, (N \otimes_B M)\right) \to HH_0(A) \oplus HH_0(B) \to HH_0(\Lambda) \to 0.$$

Corollary 3.16. Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix} \right)$ be a null-square projective algebra. If we have $HH_n(\Lambda) = 0$ for n large enough, then

$$H_0\left(A, (N \otimes_B M)^{\otimes_A n}\right)_{C_n} = H_0\left(B, (M \otimes_A N)^{\otimes_B n}\right)^{C_n} = 0$$

for n large enough.

Proof. Hochschild homology is a functor from the category of algebras to the category of vector spaces. Let $\Lambda = C \oplus I$ where C is a subalgebra of Λ and I is a two-sided ideal. In other words, there is an algebra surjection $\Lambda \to C$ which splits in

the category of algebras, and so $HH_*(C)$ is a direct summand of $HH_*(\Lambda)$. Consequently, if $HH_n(\Lambda) = 0$ for n large enough, then the same holds for $HH_n(C)$. The long exact sequence of the previous theorem provides the result. $\qquad\square$

Remark 3.17. The morphisms induced by the inclusion $K_C^1(\Lambda) \to \Lambda \otimes_C \Lambda$ of the short exact sequence (3.1) are zero. Indeed, if $f : M \to M'$ and $g : N \to N'$ are C-bimodule morphisms, we associate functorially with them a morphism between the corresponding short exact sequences (3.1) for the corresponding algebras Λ and Λ'. This induces a functorial morphism between the corresponding long exact sequences of Theorem 3.15. In particular, for $M' = N' = 0$ we infer that the morphisms induced by the inclusion of (3.1) factor through zero, and so they are zero. Consequently, there are short exact sequences as follows for $m > 0$:

$$0 \to HH_{2m}(A) \oplus HH_{2m}(B) \to HH_{2m}(\Lambda)$$
$$\to H_0(A, (N \otimes_B M)^{\otimes_A m})_{C_m} \to 0$$
$$0 \to HH_{2m+1}(A) \oplus HH_{2m+1}(B) \to HH_{2m+1}(\Lambda)$$
$$\to H_0(A, (N \otimes_B M)^{\otimes_A m+1})_{C_{m+1}} \to 0.$$

For $m = 0$ we obtain that $HH_0(A) \oplus HH_0(B)$ and $HH_0(\Lambda)$ are isomorphic; this can of course be verified by a direct computation.

4. Han's Conjecture for Null-Square Projective Algebras

Our first aim is to prove that if the algebras A and B are finite dimensional and basic, and if the invariants under the action of the cyclic groups C_m on the spaces considered in Theorem 3.15 are zero, then the spaces themselves are zero.

Let A and B be finite-dimensional and basic algebras. Let E and F be complete sets of primitive orthogonal idempotents of A and B, respectively. If k is perfect, then

$$\mathrm{rad}(B \otimes A^{\mathrm{op}}) = B \otimes \mathrm{rad}\, A^{\mathrm{op}} + \mathrm{rad}\, B \otimes A^{\mathrm{op}}$$

and $\{g \otimes e\}_{(g,e) \in F \times E}$ is a complete set of primitive orthogonal idempotents of $B \otimes A^{\mathrm{op}}$. Consequently, $\{Bg \otimes eA\}_{(g,e) \in F \times E}$ is a complete set of representatives, without repetitions, of the isomorphism classes of projective $B - A$-bimodules. Let

$$(4.1) \qquad\qquad _BM_A = \bigoplus_{(g,e) \in F \times E} {_g}m_e(Bg \otimes eA)$$

be a projective finitely generated $B - A$-bimodule, where by the Krull-Schmidt theorem, the integers $_gm_e$ are uniquely determined by M. Similarly, let

$$(4.2) \qquad\qquad _AN_B = \bigoplus_{(f,h) \in E \times F} {_f}n_h(Af \otimes hB)$$

be a finitely generated projective $A - B$-bimodule.

The next definition is pictured in Figure 4.1 on page 659.

Definition 4.1. In the situation considered above, the (N, M)-*quiver* is defined as follows: its vertices are $E \cup F$, where we agree to distribute E in a first horizontal floor and F in a ground floor.

There are two sort of arrows:

- Horizontal, distributed into the following:
  - first floor ones, which provide the Peirce E-quiver of A (see Definition 2.13);
  - ground floor ones, namely, the Peirce F-quiver of B.
- Vertical, distributed into the following:
  - down ones, where there are $_g m_e$ arrows from e to g in one-to-one correspondence with the direct summands $Bg \otimes eA$ of M;
  - up ones, defined according to N in a way analogous to that of M.

We agree to write the sequence of arrows of a path from right to left, as for composition of morphisms. Recall that the *length* of a path is the length of the corresponding sequence, and that a *cycle* is a path which starts and ends at the same vertex. Next, we define some particular kinds of paths in the (N, M)-quiver.

Definition 4.2. Let γ be a path of the (N, M)-quiver. We say the following:

- γ is *balanced* if it does not contain two consecutive horizontal arrows. In case γ starts and ends at the same floor, its *revolution number* is half of the number of the vertical arrows of the sequence of γ.
- γ is *E-balanced* if it is balanced and it starts and ends at the first floor, that is, at E-vertices. The set of E-balanced paths with revolution number m is denoted by P_m^E.
- γ is an *E-vertical balanced cycle* if it is an E-balanced cycle whose first arrow is down vertical. The set of E-vertical balanced cycles with revolution number m is denoted by CV_m^E.

Theorem 4.3. *Let A and B be basic finite-dimensional algebras over a perfect field $\Bbbk$, and let M and N be projective bimodules as above. Let C_m be the cyclic group of order m with generator t acting on $H_0(A, (N \otimes_B M)^{\otimes_A m})$ by cyclic permutation as given in Definition 3.10.*

If $H_0(A, (N \otimes_B M)^{\otimes_A m})^{C_m} = 0$, then $H_0(A, (N \otimes_B M)^{\otimes_A m}) = 0$.

Proof. We assert that $H_0(A, (N \otimes_B M)^{\otimes_A m})$ is a direct sum of vector spaces indexed by CV_m^E. In order to provide an outline of the evidence, let us consider the particular

$$N = (Af \otimes hB) \oplus (Af' \otimes h'B) \quad \text{and} \quad M = (Bg \otimes eA) \oplus (Bg' \otimes e'A).$$

Notice that the (M, N)-quiver has two down arrows and two up arrows. Then,

$$(4.3) \quad N \otimes_B M = (Af \otimes hBg \otimes eA) \oplus (Af \otimes hBg' \otimes e'A)$$
$$\otimes (Af' \otimes h'Bg \otimes eA) \oplus (Af' \otimes h'Bg' \otimes e'A)$$

and

$$H_0(A, (N \otimes_B M)) = (eAf \otimes hBg) \oplus (e'Af \otimes hBg')$$
$$\otimes (eAf' \otimes h'Bg) \oplus (e'Af' \otimes h'Bg').$$

If the first summand is non-zero, then $eAf \neq 0$ and $hBg \neq 0$, and by definition there are corresponding arrows in the E and Peirce F-quivers, respectively, from f to e and from g to h. We associate with this non-zero summand the following E-vertical balanced cycle with revolution number 1:

> – the first vertical down arrow from e to g, corresponding to the projective direct summand $Bg \otimes eA$ of M;
> – the subsequent horizontal arrow at the ground floor from g to h, due to $hBg \neq 0$;
> – the vertical up arrow from h to f, corresponding to the projective bimodule $Af \otimes hB$;
> – the horizontal arrow at the first floor from f to e, due to $eAf \neq 0$.

The decomposition of $H_0(A, (N \otimes_B M)^{\otimes_A 2})$ contains the direct summand

$$(eAf' \otimes h'Bg' \otimes e'Af \otimes hBg).$$

This corresponds to the E-vertical balanced cycle γ with revolution number 2, described by the following sequence of vertices (from right to left) and drawn below:
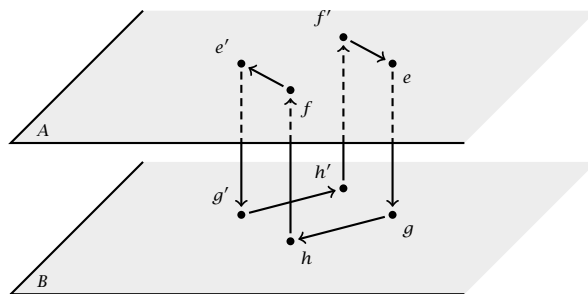
$$e, f', h', g', e', f, h, g, e.$$

FIGURE 4.1. (N, M)-quiver

The direct summands of $H_0(A, (N \otimes_B M)^{\otimes_A 2})$ originate from the indecomposable direct summands of M and N. The vertical arrows of the E-vertical balanced cycle keep track of them. For instance, the vertical arrow from e' to g' corresponds to the projective direct summand $Bg' \otimes e'A$ of M.

Note that the E-vertical balanced cycle drawn above is not the square of a vertical balanced cycle of revolution number 1. On the other hand, E-vertical balanced cycles which are powers of shorter ones do exist.

Let $y \in \mathrm{CV}_m^E$. We consider the non-zero vector subspaces of A and B corresponding to the horizontal arrows of y, which belong to the respective E and Peirce F-quivers. Let V_y be their tensor product, obtained by following the order of the arrows of y.

Conversely, as sketched above, a non-zero vector space direct summand of $H_0(A, (N \otimes_B M)^{\otimes_A m})$ determines an E-vertical balanced cycle of revolution number m. Then,

$$H_0(A, (N \otimes_B M)^{\otimes_A m}) = \bigoplus_{y \in \mathrm{CV}_m^E} V_y.$$

We describe now the transported action of C_m on CV_m^E. Let y be an E-vertical balanced cycle at a vertex e, of revolution number m. Let y' be the E-balanced path obtained from y by removing at its beginning the balanced oriented path α defined as follows: α is the first arrow of y followed by the next ones until reaching the source of the second vertical down arrow of y. Note that α begins at e, it has revolution number 1, and in general α is not a cycle. The target of y' is still e, and we have $t \cdot y = \alpha y'$.

We suppose now $H_0(A, (N \otimes_B M)^{\otimes_A m}) \neq 0$, that is, $\mathrm{CV}_m^E \neq \varnothing$. Let $y \in \mathrm{CV}_m^E$, and let $\underline{y}$ be the E-vertical balanced cycle of smallest length such that $y = (\underline{y})^\ell$; in particular, $\underline{y}$ is not a power of a shorter E-vertical balanced cycle.

The stabilizer subgroup of y in C_m is generated by $t^{m/\ell}$. That is, $t^{m/\ell} \cdot y = y$ and $\{t^i y\}_{i=0,\dots,m/\ell-1}$ are distinct. Let $k[\mathrm{CV}_m^E]$ be the vector space with basis CV_m^E. The trace element $\hat{y} = y + t \cdot y + t^2 \cdot y + \dots + t^{m/\ell-1} \cdot y \in k[\mathrm{CV}_m^E]$ is a sum of different basis elements, and so $\hat{y} \neq 0$. Moreover, $t \cdot \hat{y} = \hat{y}$. We will infer from $\underline{y}$ a non-zero element of $H_0(A, (N \otimes_B M)^{\otimes_A m})$.

Let $\underline{u}$ be a non-zero element of $V_{\underline{y}}$, and let $u = \underline{u}^{\otimes \ell} \in V_y$. This way, $u \neq 0$ and $t^{m/\ell} \cdot u = u$. Moreover, $t^i \cdot u \in V_{t^i y}$. Observe that the vector spaces $V_{t^i \cdot y}$ are distinct for $i = 0, \dots, m/\ell - 1$ since the corresponding E-vertical balanced paths are different. Consequently, $\hat{u} = u + t \cdot u + \dots + t^{\ell-1} \cdot u \neq 0$. Moreover, $t \cdot \hat{u} = \hat{u}$; then, $H_0(A, (N \otimes_B M)^{\otimes_A m})^{C_m} \neq 0$. $\square$

Theorem 4.4. *Let* $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix} \right)$ *be a null-square projective algebra where A and B are basic finite-dimensional algebras over a perfect field k, and let M and N be finitely generated projective bimodules, given as in* (4.1) *and* (4.2). *If $HH_n(\Lambda) = 0$ for n large enough, then $H_0(A, (N \otimes_B M)^{\otimes_A n} = 0$ for all $n > 0$, and $(N \otimes_B M)^{\otimes_A n} = 0$ for n large enough.*

Proof. The hypothesis that Hochschild homology of Λ vanishes in sufficiently large degrees implies by Corollary 3.16 that $H_0(A, (N \otimes_B M)^{\otimes_A m})^{C_m} = 0$ for m

large enough. By Theorem 4.3, we infer $H_0(A, (N \otimes_B M)^{\otimes_A m}) = 0$ for the same values of m, so $\mathrm{CV}_m^E = \varnothing$ for m large enough.

However, if $\mathrm{CV}_{n_0}^E \neq \varnothing$ for some n_0, then $\mathrm{CV}_{rn_0}^E \neq \varnothing$ for all $r > 0$. Hence, $\mathrm{CV}_n^E = \varnothing$ for all $n > 0$. As a consequence, $H_0(A, (N \otimes_B M)^{\otimes_A n}) = 0$ for all $n > 0$.

We assert that in the same way as in the proof of Theorem 4.3, $(N \otimes_B M)^{\otimes_A m}$ is a direct sum of non-zero vector spaces which are in one-to-one correspondence with P_m^E (see Definition 4.2). For instance, in the decomposition (4.3), the first summand $Af \otimes hBg \otimes eA$ corresponds to the E-balanced paths which contain the vertical down arrow from e to g and the vertical up arrow from h to f. More precisely, there is a subsequent decomposition

$$Af \otimes hBg \otimes eA = \bigoplus_{y,x \in E} yAf \otimes hBg \otimes eAx,$$

and for each non-zero summand $yAf \otimes hBg \otimes eAx$, the E-balanced path is determined by the sequence of vertices y, f, h, g, e, x.

In particular, $(N \otimes_B M)^{\otimes_A m} = 0$ if and only if $\mathrm{P}_m^E = \varnothing$.

We have shown before that the (N, M)-quiver has no E-balanced vertical cycles. Since the (N, M)-quiver is finite, the E-balanced paths have a maximal length. Then, $\mathrm{P}_n^E = \varnothing$ for n large enough, and $(N \otimes_B M)^{\otimes_A n} = 0$ for the same set of values of n. □

The long exact sequence of Theorem 3.15 provides then the following result.

Corollary 4.5. *Let* $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ *be a null-square projective algebra where A and B are basic finite-dimensional algebras over a perfect field k, where M and N are finitely generated projective bimodules. If* $HH_n(\Lambda) = 0$ *for n large enough, then for all n,*

$$HH_n(\Lambda) = HH_n(A) \oplus HH_n(B).$$

Our next aim is to provide a tool for bounding above the global dimension of a null-square projective algebra. For this purpose, we first briefly recall the *mapping cone* construction. Let

$$(C_\bullet, c) = \{C_n \xrightarrow{c_n} C_{n-1}\}_{n \in \mathbb{Z}} \quad \text{and} \quad (D_\bullet, d) = \{D_n \xrightarrow{d_n} D_{n-1}\}_{n \in \mathbb{Z}}$$

be complexes with differentials c and d. Let $f : C_\bullet \to D_\bullet$ be a map of complexes. Let $C_\bullet[1]$ be the complex defined by $C_n[1] = C_{n-1}$. There exists a complex $(\mathrm{co}(f)_\bullet, e)$ called the mapping cone of f, and a short exact sequence of complexes

$$0 \to C_\bullet[1] \to \mathrm{co}(f)_\bullet \to D_\bullet \to 0$$

such that the connecting homomorphism in the long exact sequence of cohomology is the morphism induced by f. In particular, f induces isomorphisms (i.e., f

is a *quasi-isomorphism*) if and only if the mapping cone complex is acyclic. Actually, $\mathrm{co}(f)_n = C_n \oplus D_{n-1}$ with differential $e = \begin{pmatrix} c & f \\ 0 & -d \end{pmatrix}$; note that the change of sign for d guarantees $e^2 = 0$, since $fc = df$.

We simplify the tensor product notation as follows: let U be a C-B-bimodule, and let V be a B-A-bimodule; we will write UV instead of $U \otimes_B V$ and VU instead of $V \otimes_A U$.

Theorem 4.6. *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be a null-square projective algebra where A and B are k-algebras, and M and N are $B - A$ and $A - B$-projective bimodules, respectively. Let X be a left A-module and $P_\bullet \to X$ be a projective resolution.*

Associated with $P_\bullet \to X$ there is a Λ-projective resolution $Q_\bullet \to (X \rightleftharpoons 0)$ such that if $P_\bullet \to X$ is finite and if $(N \otimes_B M)^{\otimes_A n} = 0$ for n large enough, then $Q_\bullet \to (X \rightleftharpoons 0)$ is finite.

Proof. We define the modules of $Q_\bullet$ as follows:

$$Q_0 = \left(P_0 \underset{0}{\overset{1}{\rightleftharpoons}} MP_0\right)$$

$$Q_1 = \left(P_1 \underset{0}{\overset{1}{\rightleftharpoons}} MP_1\right) \oplus \left(NMP_0 \underset{1}{\overset{0}{\rightleftharpoons}} MP_0\right)$$

$$Q_2 = \left(P_2 \underset{0}{\overset{1}{\rightleftharpoons}} MP_2\right) \oplus \left(NMP_1 \underset{1}{\overset{0}{\rightleftharpoons}} MP_1\right) \oplus \left(NMP_0 \underset{0}{\overset{1}{\rightleftharpoons}} M(NM)P_0\right)$$

$$Q_3 = \left(P_3 \underset{0}{\overset{1}{\rightleftharpoons}} MP_3\right) \oplus \left(NMP_2 \underset{1}{\overset{0}{\rightleftharpoons}} MP_2\right) \oplus \left(NMP_1 \underset{0}{\overset{1}{\rightleftharpoons}} M(NM)P_1\right)$$
$$\oplus \left((NM)^2P_0 \underset{1}{\overset{0}{\rightleftharpoons}} M(NM)P_0\right)$$

$$Q_4 = \left(P_4 \underset{0}{\overset{1}{\rightleftharpoons}} MP_4\right) \oplus \left(NMP_3 \underset{1}{\overset{0}{\rightleftharpoons}} MP_3\right) \oplus \left(NMP_2 \underset{0}{\overset{1}{\rightleftharpoons}} M(NM)P_2\right)$$
$$\oplus \left((NM)^2P_1 \underset{1}{\overset{0}{\rightleftharpoons}} M(NM)P_1\right) \oplus \left((NM)^2P_0 \underset{0}{\overset{1}{\rightleftharpoons}} M(NM)^2P_0\right)$$

$$\vdots$$

$$Q_{2m} = \left(P_{2m} \underset{0}{\overset{1}{\rightleftharpoons}} MP_{2m}\right) \oplus \left(NMP_{2m-1} \underset{1}{\overset{0}{\rightleftharpoons}} MP_{2m-1}\right)$$
$$\oplus \cdots \oplus \left((NM)^m P_0 \underset{0}{\overset{1}{\rightleftharpoons}} M(NM)^m P_0\right)$$

$$Q_{2m+1} = \left(P_{2m+1} \underset{0}{\overset{1}{\rightleftharpoons}} MP_{2m+1}\right) \oplus \left(NMP_{2m} \underset{1}{\overset{0}{\rightleftharpoons}} MP_{2m}\right)$$
$$\oplus \cdots \oplus \left((NM)^{m+1}P_0 \underset{1}{\overset{0}{\rightleftharpoons}} M(NM)^m P_0\right)$$

$$\vdots$$

We observe that the Q_i are projective Λ-modules. Indeed, first we note that the free rank-one bimodule $B \otimes A$ is projective as a left (or right) module, so any

projective bimodule (e.g., M) is projective as a left (or right) module. Consequently, for any left A-module X, the left B-module $M \otimes_A X$ is projective. Finally, Lemma 2.8 shows that each direct summand of Q_i is a projective Λ-module.

The differentials are defined in Figure 4.2.

It is immediate to check that the differentials are morphisms of Λ-modules, that is, the corresponding squares commute (see Definition 2.6 and Proposition 2.7).

The column with X in the bottom is the projective resolution $P_\bullet \to X$. We observe that the two columns on its right give the mapping cone of the identity of the complex $(MP_\bullet, -1 \otimes p)$. Since the identity is an isomorphism, the mapping cone is exact. Similarly, the next two columns on the right provide the mapping cone of the identity for the complex $(M(NM)P_\bullet, -1 \otimes p)$, and so forth.

The two columns on the left of $P_\bullet \to X$ correspond to the mapping cone of the identity of the complex $(NMP_\bullet, 1 \otimes p)$. The next two columns on the left are the mapping cone of the identity of $((NM)^2 P_\bullet, 1 \otimes p)$, and so forth.

Consequently, $Q_\bullet$ is a resolution of $X \rightleftharpoons 0$ by projective Λ-modules.

Let r be an integer such that $(NM)^i = 0$ for $i > r$. Moreover, let ℓ be an integer such that $P_j = 0$ for $j > \ell$. For a given m, the module Q_m is the direct sum of vector spaces of the form $(NM)^i P_j$ for $2i + j = m$ or $2i + j = m + 1$, and of vector spaces of the form $M(NM)^i P_j$ for $2i + j = m$ or $2i + j = m + 1$. Let $m > 2r + \ell$. In case $2i + j = m$ or $m + 1$, either $i > r$ or $j > \ell$ since otherwise $2i + j \le 2r + \ell < m$. Hence, $Q_m = 0$ for all $m > 2r + \ell$. $\qquad\square$

Theorem 4.7. *Let k be a perfect field, and let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix} \right)$ be a finite-dimensional null-square projective algebra where A and B are smooth.*

If $(NM)^n = 0$ for large enough n, then Λ is smooth.

Proof. The complete list of simple Λ-modules is $\{S \rightleftharpoons 0\} \cup \{0 \rightleftharpoons T\}$ where S and T are simple modules over A and B respectively (see Proposition 2.9). The previous theorem shows that $S \rightleftharpoons 0$ is of finite projective dimension. The analogous theorem holds for Λ-modules of the form $0 \rightleftharpoons Y$ where Y is a B-module.

Then, the simple modules $0 \rightleftharpoons T$ are also of finite projective dimension. $\qquad\square$

Theorem 4.8. *Let k be a perfect field. Any finite-dimensional null-square projective k-algebra built on the class $\mathcal{H}$ of basic k-algebras verifying Han's conjecture also belongs to $\mathcal{H}$.*

Proof. Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix} \right)$, where A and B are finite-dimensional basic k-algebras which belong to $\mathcal{H}$, and M and N are projective bimodules. Suppose $HH_n(\Lambda) = 0$ for n large enough. Then, by Corollary 4.5, $HH_n(A)$ and $HH_n(B)$ vanish for the same set of values of n, so A and B are smooth. Moreover, by Theorem 4.4 we have $(NM)^n = 0$ for n large enough. The previous corollary shows that Λ is smooth. $\qquad\square$

$$
\begin{array}{ccccccccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
(NM)^2P_0 & \oplus & (NM)^2P_1 & \oplus & NMP_2 & \oplus & NMP_3 & \oplus & P_4 & \to & MP_4 & \oplus & MP_3 & \oplus & M(NM)P_2 & \oplus & M(NM)P_1 & \oplus & M(NM)^2P_0 \\
\end{array}
$$

arrows labeled: 1, $-1\otimes p_1$, $1\otimes p_2$, 1, $-1\otimes p_3$, p_4, $1\otimes p_4$, 1, $-1\otimes p_3$, $1\otimes p_2$, 1, $-1\otimes p_1$

$$
(NM)^2P_0 \;\oplus\; NMP_1 \;\oplus\; NMP_2 \;\oplus\; P_3 \;\to\; MP_3 \;\oplus\; MP_2 \;\oplus\; M(NM)P_1 \;\oplus\; M(NM)P_0
$$

arrows labeled: $1\otimes p_1$, 1, $-1\otimes p_2$, p_3, $1\otimes p_3$, 1, $-1\otimes p_2$, $1\otimes p_1$, 1

$$
NMP_0 \;\oplus\; NMP_1 \;\oplus\; P_2 \;\to\; MP_2 \;\oplus\; MP_1 \;\oplus\; M(NM)P_0
$$

arrows labeled: 1, $-1\otimes p_1$, p_2, $1\otimes p_2$, 1, $-1\otimes p_1$

$$
NMP_0 \;\oplus\; P_1 \;\to\; MP_1 \;\oplus\; MP_0
$$

arrows labeled: p_1, $1\otimes p_1$, 1

$$
P_0 \;\to\; MP_0
$$

arrows labeled: p_0, $1\otimes p_0$
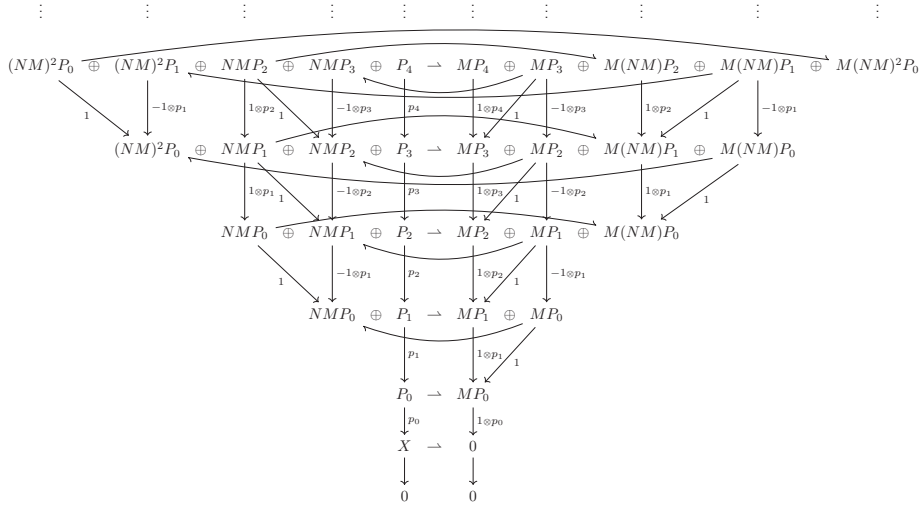
$$
X \;\to\; 0
$$

$$
0 \qquad\qquad 0
$$

FIGURE 4.2.

Remark 4.9. As much as in Remark 2.23, we observe that according to Corollary 4.5 this result agrees with the property proved by B. Keller in [14, 2.5]: namely, the Hochschild homology of a finite-dimensional smooth algebra over a perfect field is concentrated in degree zero.

5. GABRIEL QUIVER AND RELATIONS OF A NULL-SQUARE PROJECTIVE ALGEBRA

Let A be a finite-dimensional algebra such that $A/\operatorname{rad}A$ is a product of copies of k; in other words, A is basic. Equivalently, A is Morita reduced and sober; that is, the algebra of A-endomorphisms of each simple A-module is just k. Let E be a complete system of primitive and orthogonal idempotents. The set of vertices of the *Gabriel quiver* Q_A is E; the number of arrows from x to y is the dimension of the vector space $y(\operatorname{rad}A/\operatorname{rad}^2 A)x$. It is well known that Q_A is canonical, in the sense that Q_A does not depend on the choice of E.

Let Q be a quiver with finite set of vertices Q_0 and set of arrows Q_1. The vector space kQ_0 is endowed with a semisimple algebra structure where Q_0 is a complete set of primitive orthogonal idempotents. Note that kQ_0 is basic and sober. The vector space kQ_1 is a kQ_0-bimodule in the natural way. The *path algebra* kQ is by definition the tensor algebra $T_{kQ_0}(kQ_1)$; it has a canonical basis given by the oriented paths of Q. The universal property of kQ is as follows: any algebra map $\varphi : kQ \to X$ is determined by an algebra map $\varphi_0 : kQ_0 \to X$ (i.e., a set map from Q_0 to a system of X) and a kQ_0-bimodule map $\varphi_1 : kQ_1 \to X$, where the structure of X as kQ_0-bimodule is being inferred from φ_0.

A finite-dimensional algebra A as above can be *presented*; specifically, there exists a (non-canonical) algebra surjection $kQ \to A$ such that its kernel I is an admissible two-sided ideal; that is, there exists a positive integer m such that $F^m \subset I \subset F^2$, where F is the two sided ideal generated by $(Q_A)_1$. Moreover, the ideal I decomposes as $\bigoplus_{x,y\in E} yIx$ since $(Q_A)_0$ is complete. The system of generators R of I considered in a presentation is *adapted*; that is, R is graded with respect to this decomposition, and its elements are called relations. Note that any system of generators R' gives rise to a graded one, namely, $R = \bigsqcup_{x,y\in E} yR'x$, where for a set of paths Z, we denote by yZx the paths of Z starting at x and ending at y.

Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a finite-dimensional null-square projective algebra, where A and B are basic and sober, with presentations (Q_A, R_A) and (Q_B, R_B), respectively, and where the projective bimodules M and N are given as in (4.1) and (4.2).

Lemma 5.1. *The Gabriel quiver of Λ is the disjoint union of Q_A, Q_B, and new arrows as follows:*

- $_g m_e$ *arrows from $e \in E$ to $g \in F$, which we call* down *arrows,*
- $_f n_h$ *arrows from $h \in F$ to $f \in E$, which we call* up *arrows.*

Proof. The description of the Jacobson radical of Λ as given in the proof of Lemma 2.9 provides immediately the result. $\square$

Let $T_{A\times B}(M \oplus N)$ be the tensor algebra of the $A \times B$-bimodule $M \oplus N$, where, as already mentioned, the given actions are extended by zero in order to consider M and N as $A \times B$-bimodules. For instance, we infer $M \otimes_{A\times B} M = N \otimes_{A\times B} N = 0$.

The next two results are easy to prove, by using both that M and N are projective bimodules, and the universal properties of the algebras involved.

Lemma 5.2. *There is an algebra isomorphism*

$$\varphi : T_{A\times B}(M \oplus N) \to kQ_\Lambda/\langle R_A, R_B \rangle.$$

Lemma 5.3. *Let* $\psi : T_{A\times B}(M \oplus N) \to \Lambda$ *be the algebra map given by the inclusions of* $A \times B$ *and* $M \oplus N$; *we have*

$$\mathrm{Ker}\, \psi = \langle (M \oplus N)^{\otimes (A\times B)^2} \rangle = \langle N \otimes_B M + M \otimes_A N \rangle.$$

The set of all oriented paths of Q_A generates the vector space $kQ_A/\langle R_A \rangle$; hence, we can choose a subset P_A which is a basis of $kQ_A/\langle R_A \rangle$. Let also P_B be a basis of $kQ_B/\langle R_B \rangle$, where P_B is a subset of the oriented paths of Q_B.

Let u be a down arrow from e to g, and let v be an up arrow from h to f in Q_Λ. We define the sets $v \curlyvee u$ and $u \curlyvee v$ of oriented paths of Q_Λ as follows:

$$v \curlyvee u = v(hP_Bg)u \quad \text{and} \quad u \curlyvee v = u(eP_Af)v.$$

Let R be the disjoint union of R_A, R_B, and $v \curlyvee u$ and $u \curlyvee v$ for all pairs (u, v), where u is a down arrow and v is an up arrow.

Theorem 5.4. *Let* $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix} \right)$ *be a finite-dimensional null-square projective algebra, where A and B are basic and sober algebras with presentations (Q_A, R_A) and (Q_B, R_B) respectively, and where the projective bimodules M and N are given as in* (4.1) *and* (4.2). *The algebra Λ is presented by (Q_Λ, R).*

Proof. The key point of the proof is the following. Consider the image of $\mathrm{Ker}\, \psi$ by φ (see Lemma 5.2) in $kQ_\Lambda/\langle R_A, R_B \rangle$. Let $Bg \otimes eA$ be a direct summand of M, and $Af \otimes hB$ be a direct summand of N. They provide the direct summand $Bg \otimes eAf \otimes hB$ of $M \otimes_A N \subset (T_{A\times B}(M \oplus N))_2 \subset \mathrm{Ker}\, \psi$. In order to consider its image by φ, let u and v be the arrows in Q_Λ associated, respectively, with $Bg \otimes eA$ and $Af \otimes hB$. The image of $Bg \otimes eAf \otimes hB$ in $kQ_\Lambda/\langle R_A, R_B \rangle$ is generated by $u \curlyvee v$. $\qquad\square$

Example 5.5. Let Q_A be a *crown* quiver with three arrows a_0, a_1, and a_2; these arrows start, respectively, at e_0, e_1, and e_2, and end at, respectively, e_1, e_2, and e_0. Let $R_A = \{a_2a_0\}$. It is easy to establish that $A = kQ_A/\langle R_A \rangle$ is smooth. Let (Q_B, R_B) be a presentation of a basic and sober algebra B, and let g and h be vertices of B.

Let $M = Bh \otimes e_1A$ and $N = Ae_2 \otimes gB$, and u from e_1 to h and v from g to e_2 the corresponding arrows. Note that the (N, M)-quiver has no oriented cycles. Let Λ be the corresponding null-square projective algebra; we next describe its Gabriel quiver and a set of relations:

- $Q_\Lambda = Q_A \cup Q_B \cup \{u, v\}$.
- $R = \{a_2 a_0, R_B\} \cup \{v \gamma u\}_{\gamma \in gP_B h}$, where P_B is a basis of oriented paths of B.

Moreover, it follows from the previous results that if B is smooth, then Λ is smooth.

REFERENCES

[1] L.L. AVRAMOV and M. VIGUÉ-POIRRIER, *Hochschild homology criteria for smoothness*, Internat. Math. Res. Notices **1** (1992), 17–25. http://dx.doi.org/10.1155/S1073792892000035. MR1149001

[2] P. A. BERGH and K. ERDMANN, *Homology and cohomology of quantum complete intersections*, Algebra Number Theory **2** (2008), no. 5, 501–522. http://dx.doi.org/10.2140/ant.2008.2.501. MR2429451

[3] P. A. BERGH, Y. HAN, and D. O. MADSEN, *Hochschild homology and truncated cycles*, Proc. Amer. Math. Soc. **140** (2012), no. 4, 1133–1139. http://dx.doi.org/10.1090/S0002-9939-2011-10942-0. MR2869099

[4] P. A. BERGH and D. O. MADSEN, *Hochschild homology and global dimension*, Bull. Lond. Math. Soc. **41** (2009), no. 3, 473–482. http://dx.doi.org/10.1112/blms/bdp018. MR2506831

[5] ———, *Hochschild homology and split pairs*, Bull. Sci. Math. **134** (2010), no. 7, 665–676. http://dx.doi.org/10.1016/j.bulsci.2010.07.003. MR2725172

[6] ———, *Hochschild homology and trivial extension algebras*, Proc. Amer. Math. Soc. **145** (2017), no. 4, 1475–1480. http://dx.doi.org/10.1090/proc/13363. MR3601540

[7] R. O. BUCHWEITZ, *Morita contexts, idempotents, and Hochschild cohomology—with applications to invariant rings*, Commutative Algebra (Grenoble/Lyon, 2001), Contemp. Math., vol. 331, Amer. Math. Soc., Providence, RI, 2003, pp. 25–53. http://dx.doi.org/10.1090/conm/331/05901. MR2011764

[8] BUENOS AIRES CYCLIC HOMOLOGY GROUP, *Cyclic homology of algebras with one generator*, with research participantion by J. A. GUCCIONE, J. J. GUCCIONE, M. J. REDONDO, A. SOLOTAR, and O. E. VILLAMAYOR, *K*-Theory **5** (1991), no. 1, 51–69. http://dx.doi.org/10.1007/BF00538879. MR1141335

[9] H. CARTAN and S. EILENBERG, *Homological Algebra*, Princeton University Press, Princeton, NJ, 1956. MR0077480

[10] C. CIBILS, *Tensor Hochschild homology and cohomology*, Interactions Between Ring Theory and Representations of Algebras (Murcia), Lecture Notes in Pure and Appl. Math., vol. 210, Dekker, New York, 2000, pp. 35–51. MR1758400

[11] J. CUNTZ and D. QUILLEN, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), no. 2, 251–289. http://dx.doi.org/10.2307/2152819. MR1303029

[12] S. EILENBERG, A. ROSENBERG, and D. ZELINSKY, *On the dimension of modules and algebras VIII*: *Dimension of tensor products*, Nagoya Math. J. **12** (1957), 71–93. MR98774

[13] Y. HAN, *Hochschild (co)homology dimension*, J. London Math. Soc. (2) **73** (2006), no. 3, 657–668. http://dx.doi.org/10.1112/S002461070602299X. MR2241972

[14] B. KELLER, *Invariance and localization for cyclic homology of DG algebras*, J. Pure Appl. Algebra **123** (1998), no. 1–3, 223–273. http://dx.doi.org/10.1016/S0022-4049(96)00085-0. MR1492902

[15] J. L. LODAY, *Cyclic Homology*, 2nd ed., with Appendix E by M.O. RONCO and Chapter 13 by T PIRASHVILI and J.-L. LODAY, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1998. http://dx.doi.org/10.1007/978-3-662-11389-9. MR1600246

[16] S. MACLANE, *Homology*, 1st ed., Die Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 114, Springer-Verlag, Berlin-New York, 1967. MR0349792

[17] J. P. SERRE, *Local Algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000, Translated from the French by CheeWhye Chin and revised by the author. http://dx.doi.org/10.1007/978-3-662-04203-8. MR1771925

[18] A. SOLOTAR, M. SUÁREZ-ALVAREZ, and Q. VIVAS, *Hochschild homology and cohomology of generalized Weyl algebras: The quantum case*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 3, 923–956 (English, with English and French summaries). http://dx.doi.org/10.5802/aif.2780. MR3137476

[19] A. SOLOTAR and M. VIGUÉ-POIRRIER, *Two classes of algebras with infinite Hochschild homology*, Proc. Amer. Math. Soc. **138** (2010), no. 3, 861–869. http://dx.doi.org/10.1090/S0002-9939-09-10168-5. MR2566552

[20] C. A. WEIBEL, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. http://dx.doi.org/10.1017/CBO9781139644136. MR1269324

CLAUDE CIBILS:
IMAG - Institut Montpelliérain Alexander Grothendieck
Univ Montpellier, CNRS
Montpellier, France
E-MAIL: Claude.Cibils@umontpellier.fr

MARÍA JULIA REDONDO:
Instituto de Matemática (INMABB, UNS/CONICET)
Universidad Nacional del Sur
Av. Alem 1253
8000, Bahía Blanca, Argentina
E-MAIL: mredondo@criba.edu.ar

ANDREA SOLOTAR:
IMAS-CONICET y Departamento de Matemática
Facultad de Ciencias Ezactas y Naturales,
Universidad de Buenos Aires,
Ciudad Universitaria, Pabellón 1
1428, Buenos Aires, Argentina
E-MAIL: asolotar@dm.uba.ar