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SOME SCALE-TIME LOCALIZATION PROPERTIES OF THE CONTINUOUS WAVELET TRANSFORM.

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Abstract: Time-Frequency localization or concentration principles are fundamental concepts of signal processing and related fields. We shall prove some simultaneous localization or concentration inequalities for the Continuous Wavelet Transform. We will also show that simultaneous localization in the scale-time(space) is impossible, in the sense that the scale sections of the support of the wavelet transform of a non null L^p-function can not have finite Lebesgue measure.

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1 INTRODUCTION

The Lebesgue L^p spaces have proved to be useful in modelling signals which are not necessarily restricted to the usual finite energy case of $L^2(\mathbb{R})$, and on the other hand, Wavelet transforms are nowadays a standard signal analysis-synthesis tool [13]. Let $f \in L^p(\mathbb{R}^n)$, $p \in (1, +\infty)$, then its continuous wavelet transform (CWT) is defined as (here we use the L^1 normalization) [10, 14]:

$$\mathcal{W}f(a,b) = \frac{1}{a^n} \int_{\mathbb{R}^n} \psi\left(\frac{x-b}{a}\right) f(x)dx\,, \qquad (a,b) \in \mathbb{R}_{>0} \times \mathbb{R}^n,\ \mathbb{R}_{>0} = (0,+\infty)\,, \tag{1}$$

for an admissible wavelet ψ [13]. The variable a represents, in some sense, the scales of the signal f "*acting*" in an interval of time centered in the location parameter b. In view, that this integral transform gives an alternative to the ordinary windowed Fourier transform time-frequency decomposition of f, it is of interest to describe its simultaneous time-frequency or time-scale localization properties. We shall see that if we want to study the time-scale localization in terms of the Lebesgue measure of the support of $\mathcal{W}f$ we have a similar restriction to that given by Benedicks for the Fourier transform, which says:

Theorem 1 *[2] Let $f \in L^1(\mathbb{R}^n)$, such that both supports of f and $\hat{f}$, have finite Lebesgue measure, then $f = 0$ a.e.*

Similar localization principles for wavelets are given in e.g. [9][16]. On the other hand, localization properties can also be stated as uncertainty inequalities like Heisenberg's classical principle for the Fourier transform [9]:

Theorem 2 *Given $f \in L^2(\mathbb{R})$, then : $\frac{\|f\|_{L^2}^2}{4\pi} \leq \|xf(x)\|_{L^2} \left\|\lambda \hat{f}(\lambda)\right\|_{L^2}$. The inequality becomes an equality if and only if $f(x) = Ce^{-kx^2}$, with $C \in \mathbb{C}$ and $k > 0$.*

This type of inequalities are studied for example in [16], for the Cohen class transforms in [11], for the Linear Canonical Transform in [15, 19, 4] and in [8] for the Gabor-STFT transforms. Other generalizations can be found in [12]. We shall first study some localization properties of the CWT of an L^p function-signal in terms of some norm inequalities relating its CWT and its Fourier transform. Afterwords, in a similar way to [2], we shall prove that the -scale- sections of the support of the wavelet transform of a non null L^p-function cannot have finite Lebesgue measure.

2 Preliminaries.

2.1 Fourier Transform.

If $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of functions [7], for $f \in \mathcal{S}(\mathbb{R}^n)$ we shall define its Fourier transform as : $\mathcal{F}(f)(\lambda) = \hat{f}(\lambda) = \int\limits_{\mathbb{R}^n} f(x)e^{-i\,2\pi\,x.\lambda}dx$. This linear operator extends to the Lebesgue spaces $L^p(\mathbb{R})$ and to the dual of $\mathcal{S}(\mathbb{R}^n)$: $\mathcal{S}'(\mathbb{R}^n)$. A well known property is the Plancherel's identity, of which an immediate generalization is the *Hausdorff-Young Inequality*: if $p \in (1,2]$, then $\left\|\hat{f}\right\|_{L^q} \leq \|f\|_{L^p}$, with $\frac{1}{p}+\frac{1}{q}=1$. A sharper inequality is the *Babenko-Beckner inequality* [1]: if $p \in (1,2]$, then

$$\left\|\hat{f}\right\|_{L^q} \leq B(p,q)\,\|f\|_{L^p}\ , \tag{2}$$

with $B(p,q) = \left(\frac{p^{1/p}}{q^{1/q}}\right)^{\frac{n}{2}}$, $\frac{1}{p}+\frac{1}{q}=1$.

Finally, in our case we can prove the following generalization of theorem 2:

Lemma 1 *Let $f \in L^q(\mathbb{R})$ and $p \in [2,\infty)$, then if $\frac{1}{p}+\frac{1}{q}=1$:*

$$\frac{1}{2\pi p B(p,q)}\left\|\hat{f}\right\|_{L^p}^p \leq \left(\int_{\mathbb{R}}|\lambda|^q|\hat{f}(\lambda)|^p d\lambda\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}}|x|^q|f(x)|^q dx\right)^{\frac{1}{q}},$$

where $B(p,q) = \left(\frac{p^{1/p}}{q^{1/q}}\right)^{\frac{1}{2}}$ (the constant of the Babenko-Beckner inequality). If $p \neq 2$, equality is attained if and only if $f=0$ a.e.

2.2 Wavelet Transform.

We recall that, for suitable wavelet function ψ, eq. 1 defines a bounded linear operator and indeed the following characterization of the L^p spaces in terms of the continuous Wavelet transform holds [10, 14, 17]:

Theorem 3 *Let $f \in L^p(\mathbb{R})$, $p \in (1,+\infty)$, then there exists positive constants $c_\psi(p), C_\psi(p)$, only depending on ψ and p, such that:*

$$A_\psi(p)\,\|f\|_{L^p} \leq \left(\int\limits_{\mathbb{R}_{>0}}\left(\int\limits_{\mathbb{R}}|\mathcal{W}f(a,b)|^2\frac{da}{a}\right)^{\frac{p}{2}}db\right)^{\frac{1}{p}} \leq B_\psi(p)\,\|f\|_{L^p}\ .$$

Some properties of the continuous wavelet transform in L^p spaces are also studied in e.g. [14] for more references. We will assume that the wavelet function $\psi \in L^1(\mathbb{R}^n)\bigcap L^2(\mathbb{R}^n)$ verifies the admissibility condition on ψ [13]: $C_\psi = \int\limits_{[0,\infty)}\frac{|\hat{\psi}(\lambda)|^2}{|\lambda|}d\lambda < \infty$.

3 Uncertainty Principles for the Wavelet Transform.

We shall analyse the concentration of $\mathcal{W}f$. We will compare the localization of f (resp. $\hat{f}$) with the localization of its wavelet transform proving the following Heisenberg type uncertainty principles for the L^p Wavelet transform (in the variable b):

Theorem 4 *Let $f \in L^p(\mathbb{R})$, $p \in [2,+\infty)$ and $\hat{f} \in L^q(\mathbb{R})$ with $\frac{1}{p}+\frac{1}{q}=1$, then:*

$$\frac{A_\psi^p(p)\,\|f\|_{L^p}^p}{(B(p,q)\,C_\psi 2\pi p)^{\frac{1}{q}}} \leq \left\|\lambda\hat{f}\right\|_{L^q}\left(\int\limits_{\mathbb{R}_{>0}}\left(\int\limits_{\mathbb{R}}|b|^q|\mathcal{W}f(a,b)|^p db\right)^{\frac{2}{p}}\frac{da}{a}\right)^{\frac{p-1}{2}}.$$

REMARK.

The constant $A_\psi(p)$ is the same of theorem 3. This result, if $p = 2$, reduces to one obtained in [18].

Proof. (sketch) Checking that $\psi \in L^q(\mathbb{R})$ with $q \in (1, 2]$, and as f is a real function $\overset{\vee}{f} = \overline{\hat{f}} \in L^q(\mathbb{R})$, thus by means of duality argument and since $f = \overset{\vee}{\hat{f}}$, eq. 1 can be written as:

$$\mathcal{W}f(a,b) = \frac{1}{a}\int\limits_{\mathbb{R}} \psi\left(\frac{x-b}{a}\right) f(x)dx = \frac{1}{a}\int\limits_{\mathbb{R}} \psi\left(\frac{x-b}{a}\right) \overset{\vee}{\hat{f}}(x)dx = \int\limits_{\mathbb{R}} \hat{\psi}\,(a\lambda)\,\overline{\hat{f}}(\lambda)e^{-i2\pi\lambda b}d\lambda\,. \qquad (3)$$

thus, if $\frac{1}{p} + \frac{1}{q} = 1$, by lemma 1 :

$$\int\limits_{\mathbb{R}} |\mathcal{W}f(a,b)|^p db \leq 2\pi\, p\, B(p,q)\left(\int\limits_{\mathbb{R}} |b|^q |\mathcal{W}f(a,b)|^p db\right)^{\frac{1}{q}} \left(\int\limits_{\mathbb{R}} |\lambda|^q |\hat{\psi}\,(a\lambda)\,\overline{\hat{f}}(\lambda)|^q d\lambda\right)^{\frac{1}{q}},$$

therefore:

$$I_1 = \int\limits_{\mathbb{R}_{>0}} \left(\int\limits_{\mathbb{R}} |\mathcal{W}f(a,b)|^p db\right)^{\frac{2}{p}} \frac{da}{a}$$

$$\leq \int\limits_{\mathbb{R}_{>0}} \left(2\pi\, pB(p,q)\left(\int\limits_{\mathbb{R}} |b|^q |\mathcal{W}f(a,b)|^p db\right)^{\frac{1}{q}}\right)^{\frac{2}{p}} \left(\left(\int\limits_{\mathbb{R}} |\lambda|^q |\hat{\psi}\,(a\lambda)\,\overline{\hat{f}}(\lambda)|^q d\lambda\right)^{\frac{1}{q}}\right)^{\frac{2}{p}} \frac{da}{a},$$

then by Hölder's inequality:

$$\frac{I_1}{(2\pi\, p\, B(p,q))^{\frac{2}{p}}} \leq \left(\int\limits_{\mathbb{R}_{>0}} \left(\int\limits_{\mathbb{R}} |b|^q |\mathcal{W}f(a,b)|^p db\right)^{\frac{2}{p}} \frac{da}{a}\right)^{\frac{1}{q}} \left(\int\limits_{\mathbb{R}_{>0}} \left(\int\limits_{\mathbb{R}} |\lambda|^q |\hat{\psi}\,(a\lambda)\,\overline{\hat{f}}(\lambda)|^q d\lambda\right)^{\frac{2}{q}} \frac{da}{a}\right)^{\frac{1}{p}}$$

$$= I_2 I_3\,,$$

But, since $q \leq 2$ then $\frac{2}{q} \geq 1$ and therefore by Minkowski's integral inequality we get that:

$$I_3 \leq \left(\int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}_{>0}} \left(|\lambda|^q |\hat{\psi}\,(a\lambda)\,\overline{\hat{f}}(\lambda)|^q\right)^{\frac{2}{q}} \frac{da}{a}\right)^{\frac{q}{2}} d\lambda\right)^{\frac{2}{qp}} = \left(\int\limits_{\mathbb{R}} |\lambda|^q |\hat{f}(\lambda)|^q \left(\int\limits_{\mathbb{R}_{>0}} |\hat{\psi}\,(a\lambda)\,|^2 \frac{da}{a}\right)^{\frac{q}{2}} d\lambda\right)^{\frac{2}{qp}}$$

$$\leq C_\psi^{\frac{1}{p}} \left(\int\limits_{\mathbb{R}} |\lambda|^q |\hat{f}(\lambda)|^q d\lambda\right)^{\frac{2}{qp}}.$$

Finally, as $p \geq 2$, again by Minkowski's inequality:

$$I_1 \geq \left(\int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}_{>0}} |\mathcal{W}f(a,b)|^2 \frac{da}{a}\right)^{\frac{p}{2}} db\right)^{\frac{2}{p}}$$

Combining these bounds on I_1, I_3, and theorem 3 we get the desired result.
□

Denote the Lebesgue measure of a measurable subset A: $|A|$, and a for a measurable function f define $\mathbf{C}(f) := \{f \neq 0\}$. We are dealing with $f \in L^p(\mathbb{R}^n)$ which consists of equivalence classes of functions, however as we shall actually only consider the measure of these sets any contradiction is avoided. In practice it is important to compare the time concentration versus the bandwidth of a signal. This can be done by comparison of the size of the supports of f and $\hat{f}$ or other time-frequency representation. We shall prove that the scale sections of the support of the wavelet transform of a non null L^p-function cannot have finite Lebesgue measure. For this result we will assume that the measure of the support of $\hat{\psi}$ is finite (but not necessarily band limited). In [18] a similar result is proved for $p = 2$, however that proof, which does not need $\hat{\psi}$ to be supported on a set of finite measure, relies heavily on Hilbert space methods, so it cannot be directly modified for the case $p \neq 2$.

Theorem 5 *Let $f \in L^p(\mathbb{R}^n)$, $p \in (1, 2]$, and ψ an admissible wavelet such that $|\mathbf{C}(\hat{\psi})| < \infty$. If for almost all $a \in \mathbb{R}_{>0}$: $|S_a|_{\mathbb{R}^n} = |\{b \in \mathbb{R}^n \ : \ |\mathcal{W}f(a, b)| > 0\}|_{\mathbb{R}^n} < \infty$, then $f = 0$ a.e.*

Finally, from this result one obtains immediately:

Corollary 1 *Let $f \in L^p(\mathbb{R}^n)$, $p \in (1, 2]$, and ψ an admissible wavelet such that $|\mathbf{C}(\hat{\psi})| < \infty$. If $|\{(a, b) \in \mathbb{R}_{>0} \times \mathbb{R}^n \ : \ |\mathcal{W}f(a, b)| > 0\}|_{\mathbb{R}\times\mathbb{R}^n} < \infty$, then $f = 0$ a.e.*

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