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A necessary condition ensuring the strong hyperbolicity of first-order systems

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Abstract. We study strong hyperbolicity of first-order partial differential equations for systems with differential constraints. In these cases, the number of equations is larger than the unknown fields, therefore, the standard Kreiss necessary and sufficient conditions of strong hyperbolicity do not directly apply. To deal with this problem, one introduces a new tensor, called a reduction, which selects a subset of equations with the aim of using them as evolution equations for the unknown. If that tensor leads to a strongly hyperbolic system we call it a hyperbolizer. There might exist many of them or none. A question arises on whether a given system admits any hyperbolization at all. To sort-out this issue, we look for a condition on the system, such that, if it is satisfied, there is no hyperbolic reduction. To that purpose we look at the singular value decomposition of the whole system and study certain one parameter families (ε) of perturbations of the principal symbol. We look for the perturbed singular values around the vanishing ones and show that if they behave as $O(\varepsilon^l)$, with $l \geq 2$, then there does not exist any hyperbolizer. In addition, we further notice that the validity or failure of this condition can be established in a simple and invariant way. Finally, we apply the theory to examples in physics, such as Force-Free Electrodynamics in Euler potentials form and charged fluids with finite conductivity. We find that they do not admit any hyperbolization.

Keywords: Strong hyperbolicity; evolution equation; constraint equation; singular value decomposition; force-free electrodynamics; charged fluids.

Mathematics Subject Classification 2010: 35L03

1. Introduction

One of the most important characteristic of partial derivative equations (PDE) describing physical processes is that they should have a well-posed formulation. This property is fundamental since it guarantees the predictability of the theory. This condition asserts that the solutions exist, are unique and are continuous with respect to the initial values [13, 19]. We shall consider quasilinear first-order system of equations, since most equations in physics can be put in that form. Beside evolution equations, these usually contain constraint equations, i.e. the system has typically more equations than variables. We shall request that these systems are well-posed and stable with respect to any lower-order perturbation.

Within this class of well-posed PDEs, we find the so-called strongly hyperbolic systems which are the ones we are going to study in this paper. In particular, we will answer to the question as to when a general first-order system fails to be strongly hyperbolic? This question was studied and answered for the case when there is no constraints [13, 19] and the corresponding condition was found to be necessary and sufficient. Here, we provide a necessary condition for the general case. In addition, we shall apply our machinery to some physical examples, which are currently under study, and we shall show that they do not have a well-posed initial value formulation, resulting they being only weakly hyperbolic.

It is important to note that while most of the physical systems are symmetric hyperbolic, e.g. in general relativity [6, 7, 9], Maxwell electrodynamics [10, 25], nonlinear Maxwell versions [1], etc., they constitute a subclass wihin the class of strongly hyperbolic systems, therefore if our condition is not fulfilled then the system will not admit a symmetric hyperbolic formulation.

Following [10], we consider a first-order system of partial differential equations on a fiber bundle b (real or complex) with base manifold M (real) of dimension n

$$\mathfrak{N}^{Aa}_{\alpha}(x,\phi)\nabla_a\phi^{\alpha} = J^A(x,\phi). \tag{1.1}$$

Here, M is the space-time and x are points of it. We call X_x the fiber of b at point x and its dimension u. A cross-section ϕ is a map from open sets of M to b, i.e. $\phi : U \to b$, they are the unknown fields. Here $\mathfrak{N}^{Aa}_{\alpha}$ and J^A are giving fields on b, called the principal symbol and the current of the theory, respectively. That is, they do not depend on the derivative of ϕ, but can depend on ϕ and x. The multi-tensorial index A belongs to a new vector space E_x that indicates the space of equations. We call the dimension of this space e, and from now on we shall assume it is equal or greater than the dimension of X_x i.e. $e = \dim$ "A" $\geq \dim$ "α" $= u$.

In many examples of physical interest, system (1.1) can be splitted into evolution and constraint equations. The first ones define an initial value problem, namely, they are a set of equations, such that, given data $\phi^{\alpha}_0 = \phi^{\alpha}|_S$ over a specific hypersurface S of dimension $n-1$, they determine a unique solution in a neighborhood of S. The second ones restrict the initial data and have to be fulfilled during evolution. For a detailed discussion see Reula's work [27].

The choice of a coherent set of evolution equations is made in terms of a new map, $h^{\alpha}{}_{A}(x,\phi) : E_x \to X_x$ called a *reduction*. It takes a linear combination of the whole set of Eq. (1.1) and reduces them to a set of dimension u, which will be used for the initial value problem,

$$h^{\alpha}{}_{A}\mathfrak{N}^{Aa}_{\gamma}\nabla_a\phi^{\gamma} = h^{\alpha}{}_{A}J^A(x,\phi). \tag{1.2}$$

We would like the above system to be well posed and stable under arbitrary lower order terms (see [10, 13–15, 19, 20, 26, 27, 29]), for that, we shall need for the tensors $\mathfrak{N}^{Aa}_{\gamma}$ and $h^{\alpha}{}_{A}\mathfrak{N}^{Aa}_{\gamma}$ to satisfy certain properties which we display in the following definitions.

Given $\omega_a \in T_xM^*$ consider the set of complex planes $S^{\mathbb{C}}_{\omega_a} = \{n(\lambda)_a := -\lambda\omega_a + \beta_a$ for each fixed $\beta_a \in T_xM^*$ not proportional to ω_a and $\lambda \in \mathbb{C}\}$. This set turns into a set of real lines when λ run over $\mathbb{R}$ and we call it S_{ω_a}.[a] So, following the covariant formulation of Reula [27] and Geroch [10], we need to study the kernel of $\mathfrak{N}^{Aa}_{\gamma} n(\lambda)_a$ with $n(\lambda)_a \in S^{\mathbb{C}}_{\omega_a}$.

Definition 1.1. System (1.1) is *hyperbolic* at the point (x,ϕ), if there exists $\omega_a \in T_xM^*$ such that for each plane $n(\lambda)_a$ in $S^{\mathbb{C}}_{\omega_a}$, the principal symbol $\mathfrak{N}^{Aa}_{\gamma} n(\lambda)_a$ can only have a non-trivial kernel when λ is real.

An important concept for hyperbolic systems are their *characteristic structure*, it is the set of all covectors $n_a \in T_xM^*$ such that $\mathfrak{N}^{Aa}_{\gamma} n_a$ has non-trivial kernel. In addition, we call characteristic covectors to these n_a. The hyperbolicity condition is not sufficient for well-posedness, and we now strengthen it.

Definition 1.2.[b] System (1.1) is *strongly hyperbolic* at (x,ϕ) (some background solution) if there exist a covector ω_a and a reduction $h^{\alpha}{}_{A}(x,\phi)$, such that:

(i) $A^{\alpha a}_{\gamma}\omega_a := h^{\alpha}{}_{A}\mathfrak{N}^{Aa}_{\gamma}\omega_a$ is invertible, and
(ii) For each $n(\lambda)_a \in S_{\omega_a}$,

$$\dim\left(\operatorname{span}\left\{\bigcup_{\lambda\in\mathbb{R}} \operatorname{Ker}\{A^{\alpha a}_{\gamma} n(\lambda)_a\}\right\}\right) = u. \tag{1.3}$$

In order to guarantee the well-posedness for Eq. (1.2), it is necessary to impose some other smoothness conditions in x, ϕ, β_a associated with the reduction[c] (see [29]). However, smoothness property shall not play any role in what follows, but it is an issue that should be addressed at some point of the development of the theory. We are only considering the algebraic part (Eq. (1.3)) of the usual definitions of strong hyperbolicity. Since, in general, it is the part that fail in physical systems and it is enough to obtain the necessary condition for systems with constrains in Theorem 2.2.

When this definition holds we refer to reduction $h^{\gamma}{}_{A}$ as a *hyperbolizer*. In general the hyperbolizer can also depend on β_a, so Eq. (1.2) becomes a pseudo-differential expression. Notice that when the system is strongly hyperbolic, it is hyperbolic too.

Note that from i, $h^{\alpha}{}_{A}$ is surjective and there exists ω_a such that $\mathfrak{N}^{Aa}_{\gamma}\omega_a$ has no kernel. And because $A^{\alpha a}_{\gamma}\omega_a$ is invertible, the set $\{\lambda_i\}$, such that $A^{\alpha a}_{\gamma} n(\lambda_i)_a = A^{\alpha a}_{\gamma}\beta_a - \lambda_i A^{\alpha a}_{\gamma}\omega_a$ has kernel, are the eigenvalues of $(A^{\alpha a}_{\beta}\omega_a)^{-1}A^{\beta a}_{\gamma}\beta_a$ and they

[a]Notice that these planes or lines do not cross the origin for any λ.
[b]That the well-posedness property follows from studying the hyperbolicity can be seen by considering a high frequency limit perturbation of a background solution of (1.2) as $\tilde{\phi}^{\alpha} = \phi^{\alpha} + \varepsilon\delta\phi^{\alpha} e^{i\frac{f(x)}{\varepsilon}}$ with ε approaching zero, and resulting in a equation for $\delta\phi^{\alpha}$ and $n_a := \nabla_a f$ (see [8]).
[c]Unlike the usual terminology, we exclude the smoothness condition from Definition 1.2.

are functions of ω_a and β_a. Condition (ii) request that these must be real and $(A^{\alpha a}_{\beta}\omega_a)^{-1}A^{\beta a}_{\gamma}\beta_a$ diagonalizable for any β_a. These eigenvalues are given by the roots of the polynomial equation $\det(h^{\alpha}{}_{A}\mathfrak{N}^{Aa}_{\beta}n(\lambda)_a) = 0$ and the solution $n_a(\lambda)$ are called *characteristic structure of the evolution equations.*

Therefore, an important question is: What are necessary and sufficient conditions for the principal symbol $\mathfrak{N}^{Aa}_{\gamma}n_a : X_x \to E_x$ to admit a hyperbolizer? We find a partial answer, namely an algebraic necessary condition (and sufficient condition for the case without constraints), which is of practical importance for ruling out theories as unphysical, when they do not satisfy it.

To find this condition we shall use the Singular Value Decomposition (SVD) (we give a covariant formalism of SVD in Appendix A) of $\mathfrak{N}^{Aa}_{\gamma}n_a$ in the neighborhood of a characteristic covector, and conclude that the way in which the singular values approach to this covector gives information about the size of the kernel.

2. Main Results

In this section, we introduce our main results. Consider any fixed $\theta \in [0, 2\pi]$ and a line $n(\lambda)_a \in S_{\omega_a}$ for some ω_a. So we define the extended two-parameter line $n_{\varepsilon,\theta}(\lambda)_a = -\varepsilon e^{i\theta}\omega_a + n(\lambda)_a$ with ε real and $0 \leq |\varepsilon| \ll 1$. Then the perturbed principal symbol results in

$$\mathfrak{N}^{Aa}_{\beta}n_{\varepsilon,\theta}(\lambda)_a = (-\lambda\mathfrak{N}^{Aa}_{\beta}\omega_a + \mathfrak{N}^{Aa}_{\beta}\beta_a) - \varepsilon(e^{i\theta}\mathfrak{N}^{Aa}_{\beta}\omega_a). \tag{2.1}$$

Moro *et al.* [21] and Soderstrom [30] proved that the singular values of this perturbed operator admit a Taylor expansion at least up to second order in $|\varepsilon|$, also they showed explicit formulas to calculate them (see Theorem B.2 in Appendix B). We use their results to prove a necessary condition for strong hyperbolicity.

Consider first the case that no constraints are present. This is dim "A" $=$ dim "α", and all equations should be considered as evolution equations. We call it, the "Square" case, since the principal symbol $\mathfrak{N}^{Aa}_{\gamma}n_a$, maps between spaces of equal dimensions, and hence it is a square matrix.

In this case, any invertible reduction tensor $h^{\alpha}{}_{A}$ that we use, would keep the same kernels. Thus strong hyperbolicity is a sole property of the principal symbol.

For this type of systems the Kreiss's Matrix Theorem (in Theorem 3.5 we show the so-called resolving condition) lists several necessary and sufficient conditions for well posedness of constant coefficient systems [13, 18, 19]. In Sec. 3.1, we shall prove the theorem below, which incorporates to the Kreiss's Matrix Theorem a further necessary condition. This becomes in a sufficient conditions if the eigen-projectors of $(A^{\alpha a}_{\beta}\omega_a)^{-1}A^{\beta a}_{\gamma}\beta_a$ are uniformly bounded for all β_a, with $|\beta| = 1$.

Theorem 2.1. *System* (1.1) *with* dim "A" $=$ dim "α" *is strongly hyperbolic if and only if the following conditions are valid*:

(1) *There exists* ω_a *such that the system is hyperbolic and* $\mathfrak{N}^{Aa}_{\gamma}\omega_a$ *has no kernel.*

(2) *For each line* $n(\lambda)_a$ *in* S_{ω_a} *consider any extended one* $n_{\varepsilon,\theta}(\lambda)_a$ *then the principal symbol* $\mathfrak{N}^{Aa}_{\beta} n_{\varepsilon,\theta}(\lambda)_a$ *has only singular values of orders* $O(|\varepsilon|^0)$ *and* $O(|\varepsilon|^1)$.

In general we consider systems that fulfill 1, and we refer to 2 as "the condition for strong hyperbolicity".

We shall also give a couple of examples on how to apply these results: A simple matrix case of 2×2, in Sec. 4.1, and a physical example, charged fluids with finite conductivity in Sec. 4.3. We shall show that conductivity case is only weakly hyperbolic.

Consider now dim "A" $>$ dim "α". In this case, we want to find a suitable subset of evolution equations. In general if we consider $n(\lambda)_b \in S_{\omega_a}$ and count the dimension of the kernel of $\mathfrak{N}^{Bb}_{\gamma} n(\lambda)_b$ (the physical propagation directions), over $\lambda \in \mathbb{R}$, we find that this number is less than u. As a consequence we need to introduce a hyperbolizer in order to increase the kernel and fulfill condition (1.3).

We call it the "rectangular case" and we find only a necessary condition for strong hyperbolicity.

Theorem 2.2. *When* dim "A" $>$ dim "α" *in system* 1.1, *conditions in Theorem* 2.1 *are still necessary.*

As we said before, this condition has practical importance since it can be checked with a simple calculation (see Theorem 4.1), thus discarding as unphysical those systems that do not satisfy it. We prove this theorem in Sec. 3.2, and present its application to a physically motivated example, namely Force Free electrodynamics in Euler potentials description, in Sec. 4.2. We shall show that this system does not admit a hyperbolizer (it is weakly hyperbolic) for any choice of reduction, and we emphasize how simple it is to show that one of their singular values is order $O(|\varepsilon|^l)$ with $l \geq 2$, using Theorem 4.1 in Sec. 4.

3. Singular Value Decomposition, Perturbation Theory and Diagonalization of Linear Operator

In this section, we shall use the SVD to find conditions for Jordan diagonalization. Those will be used to prove Theorems 2.1 and 2.2, obtaining conditions for strong hyperbolicity. In Appendix A we describe the SVD theory in detail. We included it because our approach to the topic is a bit different than the standard one, as presented in the literature.

In order to prove our main results we shall study the principal symbol $\mathfrak{N}^{Aa}_{\gamma}(x,\phi) n(z)_a = -z\mathfrak{N}^{Ba}_{\alpha}\omega_a + \mathfrak{N}^{Ba}_{\alpha}\beta_a$ with $n(z)_a \in S^{\mathbb{C}}_{\omega_a}$ [d] for some $\omega_a \in T^*_x M$, and perturbations as in Eq. (2.1). We shall assume that there exists ω_a such that $\mathfrak{N}^{Bb}_{\alpha}\omega_b$ has no kernel and show a necessary (and sufficient in the square case) condition for the existence of a reduction $h^{\alpha}{}_{A}$ such that $(A^{\alpha a}_{\mu}\omega_a)^{-1} A^{\mu a}_{\gamma}\beta_a$ is diagonalizable, even

[d] Notice that we changed λ for z to remember that z belong to $\mathbb{C}$.

with complex eigenvalues; recall that $A^{\alpha a}_{\gamma} = h^{\alpha}{}_{A} \mathfrak{N}^{Aa}_{\gamma}$. In addition, if we also request that the system is hyperbolic with this ω_a, we would have completed the above theorems.

In what follows, we present the notation that we will use through the paper. We shall name square and rectangular operators to those that map spaces of equal or different dimensions respectively. We call $K^{A}_{\alpha}(x, \phi, \beta_a) := \mathfrak{N}^{Ba}_{\alpha} \beta_a$ and $B^{A}_{\alpha}(x, \phi, \omega_a) := \mathfrak{N}^{Bb}_{\alpha} \omega_b$. Notice that these operators change with x, ϕ, β_a and x, ϕ, ω_a, respectively. However, the condition we are looking for are algebraic, so they hold at each particular point, which we shall assert from now on. In addition, we define

$$T^{A}_{\alpha}(z) := K^{A}_{\alpha} - zB^{A}_{\alpha} = (\mathfrak{N}^{Bb}_{\beta} \beta_a) - z(\mathfrak{N}^{Bb}_{\beta} \omega_b) : X \to E. \tag{3.1}$$

Note that we have suppressed subindex x in vectorial spaces X_x and E_x.

These operators $K^{A}_{\alpha}, B^{A}_{\alpha} : X \to E$ take elements ϕ^{α} in the vector space X, with $\dim(X) = u$, and give elements l^A in the vector space E, with $\dim(E) = e$. Because we are interested in systems with constraints, we shall consider operators with $\dim(E) \geq \dim(X)$. From now on Greek indices go to $1, \ldots, u$ and capital Latin to $1, \ldots, e$. We call X' and E' to the dual spaces of X and E and ϕ_{α} and l_A to their elements, respectively. We call right kernel of T^{A}_{α} to the vectors ϕ^{α} such that $T^{A}_{\alpha}\phi^{\alpha} = 0$; and we call left kernel to the covector l_A such that $l_A T^{A}_{\alpha} = 0$. We refer to T^{A}_{1}, $T^{A}_{2}, \ldots,$ T^{A}_{u} as the columns of T^{A}_{α} and $T^{1}_{\alpha}, T^{2}_{\alpha}, \ldots, T^{3}_{\alpha}$ as the rows of T^{A}_{α}. We call $\sigma_i[T^{A}_{\alpha}(z)]$ to the singular values of $T^{A}_{\alpha}(z)$ for any choice of the Hermitian forms G_{1AB} and $G_{2\alpha\beta}$. Finally we use a bar $\bar{T}^{A}_{\alpha}$ to denote the complex conjugate of T^{A}_{α}.

The key idea of this section is to perturb the operator (3.1) with another appropriate operator (as in Eq. (2.1)), linear in a real, small, parameter factor ε, and study how the singular values change. For that, we follow [16, 21, 30, 32]. In particular Söderström [30] and Moro *et al.* [21] show that the singular values have Taylor expansion in $|\varepsilon|$, at least up to order two and this will be crucial for the following results. They also give closed form expressions for the first-order term, using left and right eigenvectors.

Roughly speaking our first two results are for square operators. We shall show that an operator is Jordan diagonalizable if and only if, each of their perturbed singular values are order $O(|\varepsilon|^0)$ or $O(|\varepsilon|^1)$. In addition, we shall extend this result and show that: a perturb singular value is order $O(|\varepsilon|^l)$ if and only if the operator has an l-Jordan block,[e] associated to some eigenvalue, in the Jordan decomposition.

These results lead us to obtain a necessary condition for strong hyperbolicity on rectangular operators. This is a necessary condition for the existence of a reduction from rectangular to square operators, such that, the reduced one is diagonalizable.

[e]We called l-Jordan block to the matrix $J_l(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \in \mathbb{C}^{l \times l}$ with eigenvalue λ.

The conclusion is analogous to the square case, if any singular value of the perturbed operator is order $O(|\varepsilon|^l)$ with $l \geq 2$, then the system cannot be reduced to a diagonalizable operator i.e. strong hyperbolicity only admits orders $O(|\varepsilon|^0)$ and $O(|\varepsilon|^1)$. Moreover, if the singular values are order $O(|\varepsilon|^l)$ then any reduction leads to operators with l-Jordan blocks or larger.

3.1. *Square operators*

We consider first the space of equations such that $\dim(E) = \dim(X)$. For simplicity we shall identify E with X, but in general there is no natural identification between them. We also consider a square operator $T^{\alpha}{}_{\beta}(z) = K^{\alpha}{}_{\beta} - zB^{\alpha}{}_{\beta} : X \to X$ with $z \in \mathbb{C}$ and $B^{\alpha}{}_{\beta}$ invertible (without right kernel). We call λ_i $i = 1, \ldots, k$ the different eigenvalues of $(B^{-1})^{\gamma}{}_{\alpha} K^{\alpha}{}_{\beta}$; q_i, r_i their respective geometric and algebraic multiplicities, and $D_\lambda := \{\lambda_i \text{ with } i = 1, \ldots, k\}$.

In the following lemma, we shall use the SVD of $T^{\alpha}{}_{\beta}(z)$ and show for which z the operator $T^{\alpha}{}_{\beta}(z)$ has vanishing singular values and how many there are.

Lemma 3.1. (1) *$T^{\alpha}{}_{\beta}(\lambda_i)$ has exactly q_i null singular values. The rest of $u - q_i$ singular values of $T^{\alpha}{}_{\beta}(\lambda_i)$ are positive.*

(2) *$\sigma_i[T^{\alpha}{}_{\beta}(z)] > 0$ for all singular values of $T^{\alpha}{}_{\beta}(z)$ if and only if $z \notin D_\lambda$.*

(3) *Consider any given subset $L \subset \mathbb{C}$, then*

$$\sigma_i[T^{\alpha}{}_{\beta}(z)] > 0 \quad \forall\, z \in L \quad \text{and} \quad \forall\, i = 1, \ldots, u$$

if and only if $D_\lambda \cap L = \phi$.

Proof. (1) Notice that

$$T^{\alpha}{}_{\beta}(z) = B^{\alpha}{}_{\eta}(B^{-1})^{\eta}{}_{\gamma} T^{\gamma}{}_{\beta}(z) = B^{\alpha}{}_{\eta}((B^{-1})^{\eta}{}_{\gamma} K^{\gamma}{}_{\beta} - z\delta^{\eta}{}_{\beta}).$$

It is clear from this expression that $right\_\ker(T(z)) = right\_\ker(B^{-1}K - z\delta)$. Therefore $T^{\alpha}{}_{\beta}(z)$ has kernel only when z is equal to one eigenvalue of $B^{-1} \circ K$.

On the other hand, the singular value decomposition of T is

$$T^{\alpha}{}_{\beta}(z) = U^{\alpha}{}_{i'}(z) \Sigma^{i'}{}_{j'}(z) (V^{-1})^{j'}{}_{\beta}(z).$$

Now U, Σ, V^{-1} are operators that depend on z, and from the orthogonality conditions (A.1), (A.2) in Appendix A, $U(z)$ and $V^{-1}(z)$ are always invertible $\forall\, z \in \mathbb{C}$. Thus $\Sigma(z)$ is diagonal and controls the kernel of T (this argument is valid for the rectangular case too). Consider now the case $z = \lambda_i$ we know that $\dim(right\_\ker T(\lambda_i)) = q_i$ but from Corollary A.2 in Appendix A, it is the number of vanishing singular values.

(2) and (3) are particular cases of (1). □

The operator $(B^{-1})^{\gamma}{}_{\alpha} K^{\alpha}{}_{\beta}$ is Jordan diagonalizable when $q_i = r_i$ $\forall\, i$ and from the previous Lemma, this is only possible if the dimension of the right kernel[f] of

[f] Or left kernel, since for square operators the dimension of right and left kernels are equal.

$T^{\alpha}_{\ \beta}(\lambda_i)\,\forall\, i$ is maximum. We shall see under which conditions this becomes true. But first we need a previous lemma.

Point (1) in the following Lemma is valid for rectangular operators too. We shall use it also in Sec. 4 to give a condition for hyperbolicity in the general case.

Lemma 3.2. (1) *Given* $P : X \to E$ *a linear rectangular operator with* $\dim(E) \geq \dim(X)$*. Then*

$$\sqrt{\det(P^* \circ P)} = \prod_{i=1}^{u} \sigma_i[P]. \tag{3.2}$$

(2) *Consider the square operator* $T^{\alpha}_{\ \beta}(z) = K^{\alpha}_{\ \beta} - zB^{\alpha}_{\ \beta} : X \to X$*. Then*

$$\sqrt{\det(T^* \circ T)} = \prod_{i=1}^{u} \sigma_i[B^{\alpha}_{\ \beta}]|\lambda_1 - z|^{r_1} \cdots |\lambda_k - z|^{r_k} = \prod_{i=1}^{u} \sigma_i[T^{\alpha}_{\ \beta}(z)]. \tag{3.3}$$

Proof. (1) Consider the SVD of $P^{A}_{\ \alpha} = (U_P)^{A}_{\ i}(\Sigma_P)^{i}_{\ i'}(V_P^{-1})^{i'}_{\ \alpha}$. Here $(U_P)^{A}_{\ i} \in \mathbb{C}^{e\times e}$, $(\Sigma_P)^{i}_{\ i'} \in \mathbb{R}^{e\times u}$ and $(V_P^{-1})^{i'}_{\ \alpha} \in \mathbb{C}^{u\times u}$ (see Theorem A.1). Then

$$\begin{aligned}(P^* \circ P)^{\beta}_{\ \alpha} &= G_2^{\beta\alpha_2}(\bar{V}_P^{-1})^{i'_2}_{\ \alpha_2}(\Sigma_P)^{j_2}_{\ i'_2}(\bar{U}_P)^{A_2}_{\ j_2}G_{1A_2A_1}(U_P)^{A_1}_{\ j_1}(\Sigma_P)^{j_1}_{\ i'_1}(V_P^{-1})^{i'_1}_{\ \alpha}\\ &= V^{\beta}_{\ i'_1}\delta_2^{i'_1 i'_2}(\Sigma_P)^{j_2}_{\ i'_2}\delta_{1j_2j_1}(\Sigma_P)^{j_1}_{\ i'_1}(V_P^{-1})^{i'_1}_{\ \alpha},\end{aligned}$$

where we have used the orthogonality conditions $(\bar{U}_P)^{A_2}_{\ i_2}G_{1A_2A_1}(U_P)^{A_1}_{\ i_1} = \delta_{i_2i_1}$ and $G_2^{\beta\alpha_2}(\bar{V}_P^{-1})^{i'_2}_{\ \alpha_2} = V^{\beta}_{\ j'_2}\delta_2^{j'_2 i'_2}$.

Taking determinant and square root

$$\begin{aligned}\sqrt{\det((P^* \circ P)^{\beta}_{\ \alpha})} &= \sqrt{\det(V^{\beta}_{\ i'_2}\delta_2^{i'_2 j'_2}(\Sigma_P)^{i_2}_{\ j'_2}\delta_{1i_2i_1}(\Sigma_P)^{i_1}_{\ j'_1}(V_P^{-1})^{j'_1}_{\ \alpha})}\\ &= \sqrt{\det(\delta_2^{i'_2 j'_2}(\Sigma_P)^{i_2}_{\ j_{2'}}\delta_{i_2i_1}(\Sigma_P)^{i_1}_{\ j'_1})}\\ &= \prod_{i=1}^{u}\sigma_i[P^{A}_{\ \alpha}].\end{aligned}$$

(2) Similarly, taking the determinant of $T^* \circ T$ we get,

$$\begin{aligned}\sqrt{\det(T^* \circ T)} &= \sqrt{\det(G_2^{\eta\beta}\bar{T}^{\alpha}_{\ \beta}(z)G_{1\alpha\gamma}T^{\gamma}_{\ \beta}(z))}\\ &= \sqrt{\det(G_2^{\alpha\rho}((\bar{B}^{-1})^{\mu}_{\ \gamma}\bar{K}^{\gamma}_{\ \rho} - \bar{z}\delta^{\mu}_{\ \rho})\bar{B}^{\upsilon}_{\ \mu}G_{1\upsilon\gamma}B^{\gamma}_{\ \eta}((B^{-1})^{\eta}_{\ \gamma}K^{\gamma}_{\ \beta} - z\delta^{\eta}_{\ \beta}))}\\ &= \sqrt{\det(G_2^{\alpha\mu}\bar{B}^{\upsilon}_{\ \mu}G_{1\upsilon\gamma}B^{\gamma}_{\ \eta})}\\ &\quad\times\sqrt{\det((\bar{B}^{-1})^{\mu}_{\ \gamma}\bar{K}^{\gamma}_{\ \alpha} - \bar{z}\ \delta^{\mu}_{\ \alpha})\det((B^{-1})^{\eta}_{\ \gamma}K^{\gamma}_{\ \beta} - z\delta^{\eta}_{\ \beta})}\\ &= \sqrt{\det(G_2^{\alpha\mu}\bar{B}^{\upsilon}_{\ \mu}G_{1\upsilon\gamma}B^{\gamma}_{\ \eta})}|\lambda_1 - z|^{r_1} \cdots |\lambda_k - z|^{r_k}\\ &= \sigma_1[B]\cdots\sigma_u[B]|\lambda_1 - z|^{r_1} \cdots |\lambda_k - z|^{r_k}.\end{aligned}$$

In the fourth line we have used

$$\det((B^{-1})^{\eta}{}_{\gamma}K^{\gamma}{}_{\beta} - z\delta^{\eta}{}_{\beta}) = (\lambda_1 - z)^{r_1} \cdots (\lambda_k - z)^{r_k}$$

and on the last line we have used the first point of the lemma for B. Therefore, using it again for T, we conclude

$$\left(\prod_{i=1}^{u} \sigma_i[B]\right) |\lambda_1 - z|^{r_1} \cdots |\lambda_k - z|^{r_k} = \sqrt{\det(T^* \circ T)} = \sigma_1[T] \cdots \sigma_u[T]. \qquad \square$$

Notice that if in Eq. (3.3) we set $z = \lambda_1 + \varepsilon$ (with ε real and small), then the product of the singular values are order $O(|\varepsilon|^{r_1})$. Since these singular values have Taylor expansions in $|\varepsilon|$, if all singular values are $O(|\varepsilon|^l)$ with $l < 2$, then we need r_1 of them to vanish (that is $O(|\varepsilon|^1)$). Therefore by the previous lemma $q_1 = r_1$. If this happens for all λ_i then $q_i = r_i \ \forall\, i$ and the operator $(B^{-1})^{\gamma}{}_{\alpha}K^{\alpha}{}_{\beta}$ is Jordan diagonalizable.

A formalization of this idea is given in the next theorem. Notice that the orders of the singular values are invariant under different choices of Hermitian forms, although the singular values are not. We show this in Appendix B.

Theorem 3.3. *The following conditions are equivalent*:

(1) $(B^{-1})^{\gamma}{}_{\alpha}K^{\alpha}{}_{\beta}$ *is Jordan diagonalizable.*
(2) $T^{\alpha}{}_{\beta}(\lambda_i) = K^{\alpha}{}_{\beta} - \lambda_i B^{\alpha}{}_{\beta}$ *has* r_i *vanishing singular values for each* λ_i.
(3) *For at least one fixed* $\theta \in [0, 2\pi]$ *and* $0 \le |\varepsilon| \ll 1$ *with* ε *real, the singular values of the perturbed operators* $T^{\alpha}{}_{\beta}(\lambda_i + \varepsilon e^{i\theta}) = T^{\alpha}{}_{\beta}(\lambda_i) - \varepsilon e^{i\theta}B^{\alpha}{}_{\beta}$ *are either of two forms*

$$\sigma_j[T^{\alpha}{}_{\beta}(\lambda_i + \varepsilon e^{i\theta})] = \sigma_j[T^{\alpha}{}_{\beta}(\lambda_i)] + \xi_j\varepsilon + O(\varepsilon^2) \ \textit{with} \ \sigma_j[T^{\alpha}{}_{\beta}(\lambda_i)] \neq 0 \quad \textit{or}$$

$$\sigma_j[T^{\alpha}{}_{\beta}(\lambda_i + \varepsilon e^{i\theta})] = \xi_j|\varepsilon| + O(|\varepsilon|^2) \quad \textit{with} \ \xi_j \neq 0 \tag{3.4}$$

[g]*for all* $\lambda_i \in D_\lambda$ *i.e. none of them is* $\sigma[T^{\alpha}{}_{\beta}(\lambda_i + \varepsilon e^{i\theta})] = O(|\varepsilon|^l)$ *with* $l \ge 2$.

Proof. (1) $\Leftrightarrow$ (2) Since the geometric and algebraic multiplicities are equal for all eigenvalues, i.e. $q_i = r_i \ \forall\, i = 1, \ldots, k$.

(3) $\Leftrightarrow$ (1) Using Lemma 3.2 we have

$$\left(\prod_{i=1}^{u} \sigma_i[B]\right) |\lambda_1 - z|^{r_1} \cdots |\lambda_k - z|^{r_k} = \sigma_1[K - zB] \cdots \sigma_u[K - zB]. \tag{3.5}$$

Set $z = \lambda_i + \varepsilon e^{i\alpha}$ with ε less than any distance between the eigenvalues to λ_i

$$\varepsilon < \min\{|\lambda_i - \lambda_j| \text{ with } j = 1, \ldots, u \text{ and } i \neq j\}. \tag{3.6}$$

By Lemma 3.1, we know that q_i singular values have to vanish for $z = \lambda_i$. Suppose they are the first q_i, we call them $(\sigma_{\lambda_i})_j[K - zB]$, with $j = 1, \ldots, q_i$ then

[g]The orders in ε are independent of θ.

we rewrite Eq. (3.5)

$$\varepsilon^{r_i} = (\sigma_{\lambda_i})_1[(K-\lambda_i B)+\varepsilon(-e^{i\theta}B)] \cdots (\sigma_{\lambda_i})_{q_i}[(K-\lambda_i B)+\varepsilon(-e^{i\theta}B)]p(z)|_{z=\lambda_i+\varepsilon e^{i\theta}}, \tag{3.7}$$

where

$$p(z)|_{z=\lambda_i+\varepsilon e^{i\alpha}} = \frac{\sigma_{q_i+1}(K-zB)\cdots\sigma_u(K-zB)|_{z=\lambda_i+\varepsilon e^{i\theta}}}{(\prod_{i=1}^{u}\sigma_i[B])|\lambda_1-\lambda_i-\varepsilon e^{i\alpha}|^{r_1}\cdots|\lambda_{i-1}-\lambda_i-\varepsilon e^{i\alpha}|^{r_{i-1}} \, |\lambda_{i+1}-\lambda_i-\varepsilon e^{i\alpha}|^{r_{i+1}}\cdots|\lambda_k-\lambda_i-\varepsilon e^{i\alpha}|^{r_k}}.$$

Note that $p(z)|_{z=\lambda_i+\varepsilon e^{i\alpha}}$ does not vanish for Lemma 3.1 and does not blow up for an ε small enough because of Eq. (3.6).

We know for (2) in Theorem B.2, in Appendix B, that for $|\varepsilon| \ll 1$ the $\sigma's$ can be expanded as

$$\begin{aligned}(\sigma_{\lambda_i})_j[K-(\lambda_i+\varepsilon e^{i\theta})B] &= (\sigma_{\lambda_i})_j[(K-\lambda_i B)+\varepsilon(-e^{i\theta}B)] \\ &= |\varepsilon|\xi_j + O(\varepsilon^2).\end{aligned}$$

For some ξ_j as in Eq. (B.4).

If we replace the last expression in (3.7) we obtain

$$|\varepsilon|^{r_i-q_i} = \xi_1\cdots\xi_{q_i}p(z)|_{z=\lambda_i+\varepsilon e^{i\alpha}} + O(\varepsilon). \tag{3.8}$$

Therefore:

♦ (3) $\Rightarrow$ (1) By hypothesis $\xi_j \neq 0$ for $j=1,\ldots,q_i$; then Eq. (3.8) can only be valid if $q_i = r_i\ \forall\, i=1,\ldots,k$ (taking small enough ε). Therefore $(B^{-1})^{\gamma}{}_{\alpha}K^{\alpha}{}_{\beta}$ is diagonalizable.

♦ (1) $\Rightarrow$ (3) If $(B^{-1})^{\gamma}{}_{\alpha}K^{\alpha}{}_{\beta}$ is diagonalizable then $r_i = q_i$ and taking $\varepsilon \to 0$ we obtain $1 = \xi_1\cdots\xi_{q_i}p(z)|_{z=\lambda_1}$. Which implies $\xi_j \neq 0$ for $j=1,\ldots,q_i$. Because $r_i = q_i$ for all i, we conclude the proof. □

An interpretation of condition (3) in the above theorem is the following, for any non-Jordan diagonalizable square operator, you can always find a right eigenvector, such that, the contraction of it with all left eigenvectors vanishes. This is clearly impossible if the operator is diagonalizable. We show this in the next example, consider the matrix

$$K^{\alpha}{}_{\beta} = P^{\alpha}{}_{i'}\begin{pmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_1 & 1 & 0\\ 0 & 0 & \lambda_1 & 1\\ 0 & 0 & 0 & \lambda_1\end{pmatrix}^{i'}_{j'}(P^{-1})^{j'}{}_{\beta}$$

and $B^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta}$ the identity matrix. We call $v_{1,2}$ to the right eigenvectors and $u_{1,2}$ to the left eigenvectors

$$(v_1)^{\alpha} = P^{\alpha}{}_{i'} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^{i'}, \quad (v_2)^{\alpha} = P^{\alpha}{}_{i'} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}^{i'},$$

$$(u_1)_{\alpha} = (1\ 0\ 0\ 0)_{j'} (P^{-1})^{j'}{}_{\alpha}, \quad (u_2)_{\alpha} = (0\ 0\ 0\ 1)_{j'} (P^{-1})^{j'}{}_{\alpha}$$

then

$$(u_{1,2})_{\alpha} (v_2)^{\alpha} = 0. \tag{3.9}$$

Theorem B.2, in Appendix B, tells us how to calculate the coefficients of first-order perturbation of the singular values of $T^{\alpha}{}_{\beta}(\lambda_1 + \varepsilon e^{i\theta}) = (K^{\alpha}{}_{\beta} - \lambda_1 \delta^{\alpha}{}_{\beta}) - \varepsilon e^{i\theta} \delta^{\alpha}{}_{\beta}$. They are given by $\xi_i = \sigma_i [L^j{}_k]$ with

$$L^j{}_k := \begin{pmatrix} (u_1)_{\alpha} \\ (u_2)_{\alpha} \end{pmatrix} \delta^{\alpha}{}_{\beta} ((v_1)^{\beta}, (v_2)^{\beta}).$$

But because of Eq. (3.9), $L^j{}_k$ has kernel $\binom{0}{1}$ and so $\xi_2 = 0$. Thus $T^{\alpha}{}_{\beta}(\lambda_1 + \varepsilon e^{i\theta})$ has a singular value of order $O(\varepsilon^2)$ and it is not diagonalizable.

Let us go back to the general case. When the eigenvalues of $(B^{-1})^{\gamma}{}_{\alpha} K^{\alpha}{}_{\beta}$ are real, then the below corollary follows. This is equivalent to Theorem 2.1.

Corollary 3.4. *The next conditions are equivalent*

(1) $(B^{-1})^{\gamma}{}_{\alpha} K^{\alpha}{}_{\beta}$ *is Jordan diagonalizable with real eigenvalues.*
(2) *All singular values satisfy*

$$\sigma_j [T^{\alpha}{}_{\beta}(x + iy)] > 0 \quad \text{with } x, y \in \mathbb{R} \quad \text{and} \quad y \neq 0. \tag{3.10}$$

For at least one fixed $\theta \in [0, 2\pi]$ and $0 \leq |\varepsilon| \ll 1$ with ε real,

$$\sigma_j [T^{\alpha}{}_{\beta}(x + i\varepsilon e^{i\theta})] = \sigma_j [T^{\alpha}{}_{\beta}(x)] + \xi_j \varepsilon + O(\varepsilon^2) \quad \text{with } \sigma_j [T^{\alpha}{}_{\beta}(x)] \neq 0 \quad \text{or}$$

$$\sigma_j [T^{\alpha}{}_{\beta}(x + i\varepsilon e^{i\theta})] = \xi_j |\varepsilon| + O(\varepsilon^2) \quad \text{with } \xi_j \neq 0 \tag{3.11}$$

for any $x \in \mathbb{R}$ i.e. none of them is $\sigma[T^{\alpha}{}_{\beta}(x + i\varepsilon e^{i\theta})] = O(|\varepsilon|^l)$ with $l \geq 2$.

Proof. (1) $\Rightarrow$ (2). It follows directly from Theorem 3.3.

(2) $\Rightarrow$ (1). Because $\sigma_i [T^{\alpha}{}_{\beta}(z)] > 0\ \forall i$ and $\forall z \in S = \{z \in \mathbb{C}/\mathrm{Im}(z) \neq 0\}$, then from Lemma 3.1, $S \cap D_{\lambda} = \phi$. Therefore the eigenvalues are real. The second part also follows from Theorem 3.3. □

An alternative proof to Theorem 3.3 can be obtained directly showing that condition (2) in Corollary 3.4, with $\theta = \frac{\pi}{2}$, is equivalent to one of the conditions in Kreiss's Matrix Theorem. Indeed, an alternative formulation is given by

Theorem 3.5 (Part of Kreiss matrix Theorem). *The square operator $K : V \to V$ is Jordan uniformly*[h] *diagonalizable with real eigenvalues if and only if for any $x, y \in \mathbb{R}$ with $y \neq 0$ there exists a constant $C > 0$ such that*

$$\|(K^{\alpha}{}_{\beta} - (x+iy)\delta^{\alpha}{}_{\beta})^{-1}\|_2 \leq \frac{C}{|y|}. \tag{3.12}$$

We recall that we are considering this theorem point to point, then the uniformly part is trivial. It means that considering $K^{\alpha}{}_{\gamma} := (A^{\alpha a}_{\mu}\omega_a)^{-1} A^{\mu a}_{\gamma}\beta_a$, for each β_a it is diagonalizable with real eigenvalues if and only if there exists $C(\beta) > 0$ such that, for each β_a equation (3.12) holds.[i] In this sense, we now show that (3.12) implies (3.13) and this is equivalent to (3.11). From [33]

$$\|T\|_2 = \max\{\sigma_i[T]\},$$

where $\max\{\sigma_i[T]\}$ is the maximum of all singular values of T.

In addition

$$\|T^{-1}\|_2 = \max\{\sigma_i[T^{-1}]\} = \max\left\{\frac{1}{\sigma_i[T]}\right\} = \frac{1}{\min\{\sigma_i[T]\}},$$

where we have used that the singular values of T are the inverse[j] of the singular values of T^{-1}.

Now, from inequality (3.12)

$$\frac{1}{\min\{\sigma_i[K^{\alpha}{}_{\beta} - (x+iy)\delta^{\alpha}{}_{\beta}]\}} = \|(K^{\alpha}{}_{\beta} - (x+iy)\delta^{\alpha}{}_{\beta})^{-1}\|_2 \leq \frac{C}{|y|}$$

let $\tilde{C} := \frac{1}{C}$, then the Kreiss's Matrix Theorem asserts that K is Jordan diagonalizable if and only if we can find $\tilde{C}$ such that

$$\tilde{C}|y| \leq \min\{\sigma_i[K^{\alpha}{}_{\beta} - (x+iy)\delta^{\alpha}{}_{\beta}]\}. \tag{3.13}$$

This equation is equivalent to condition (3.11). Since $\tilde{C}|y| \leq |y|^l$ with $l \geq 2$ in $0 \leq |y| << 1$ implies that $\tilde{C} = 0$, therefore $\min\{\sigma_i[K^{\alpha}{}_{\beta} - (x+iy)\delta^{\alpha}{}_{\beta}]\}$ must be order $O(|y|^0)$ or $O(|y|^1)$. In addition, notice that $K^{\alpha}{}_{\beta} - (x+iy)\delta^{\alpha}{}_{\beta}$, in (3.12), is invertible when $x+iy$ is not an eigenvalue of $K^{\alpha}{}_{\beta}$. Since condition (3.12) applies for all $x+iy$ with $y \neq 0$ then the eigenvalues of $K^{\alpha}{}_{\beta}$ are real, as Eq. (3.10) implies.

As a side remark, we prove the following theorem.

Theorem 3.6. $T^{\alpha}{}_{\beta}(\lambda_i + \varepsilon e^{i\theta}) : X \to X$ *has a singular value of order $O(\varepsilon^l)$ if and only if $(B^{-1})^{\gamma}{}_{\alpha}K^{\alpha}{}_{\beta}$ has a l-Jordan block, with eigenvalue λi in its Jordan decomposition.*

[h] When $K^{\alpha}{}_{\gamma} := (A^{\alpha a}_{\mu}\omega_a)^{-1} A^{\mu a}_{\gamma}\beta_a$ depends on β, the matrix $S(\beta)$, which diagonalizes it, depends on β too. We said that $K(\beta)$ is Jordan uniformly diagonalizable if it is diagonalizable and $|S(\beta)||S^{-1}(\beta)| \leq C$.

[i] It is not sufficient for well posedness. In the case of constant coefficient systems, in order to obtain well posedness a uniform lower bound $C(\beta) > \hat{C} > 0$ with $\hat{C}$ constant and for all β_a not proportional to n_a with $|\beta| = 1$ is necessary.

[j] Notice that if the SVD of T is $T = U\Sigma V^{-1}$ then $T^{-1} = V\Sigma^{-1}U^{-1}$ because V and U^{-1} are orthogonal and Σ^{-1} is diagonal, that is the SVD of T^{-1}. Therefore $\sigma_i[T^{-1}] = \frac{1}{\sigma_i[T]}$.

Proof. We consider a basis in which $(B^{-1})^{\gamma}{}_{\alpha}K^{\alpha}{}_{\beta}$ stays in its Jordan form. In this basis we choose the following two Hermitian forms $G_{1AB} = \text{diag}(1,\ldots,1)$ and $G_{2\alpha\beta} = \text{diag}(1,\ldots,1)$. Then, the calculation of $\sigma_i[T^{\alpha}{}_{\beta}(\lambda)]$ $i = 1,\ldots,u$ decouples in Jordan blocks. Therefore we only need to study the singular values of an l-Jordan block $J_l(\lambda)$. It is easy to see from Eq. (3.3), with $z = \lambda + e^{i\theta}\varepsilon$, that $J_l(\lambda)$ has a unique singular value of order $O(|\varepsilon|^l)$ and the others are order $O(|\varepsilon|^0)$. This concludes the proof. □

3.2. *Rectangular operators*

In this subsection, we consider the case $\dim E \geq \dim X$. Theorem 3.8 provides a necessary condition for reducing a rectangular operator to a square one, in such a way that the resulting operator is Jordan diagonalizable. The proof of condition (2) in Theorem 2.2 is a corollary of this theorem. A reduction is given explicitly by another linear operator $h^{\alpha}{}_{A} : E \to X$ (see Sec. 1). It selects some evolution equations from the space of equations in a physical theory. The theorem asserts under which conditions it will be impossible to find a hyperbolizer, namely, a reduction satisfying condition (1.3) for strong hyperbolicity.

The proof of this theorem is based in the following Lemma (for a proof see [33]).

Lemma 3.7. *Consider the linear operators* $T^{A}{}_{\alpha} : X \to E$, $h^{a}{}_{A} : E \to X$ *and* $H^{\alpha}{}_{\beta} := h^{\alpha}{}_{A}T^{A}{}_{\beta} : X \to X$ *then*

$$0 \leq \sigma_i[H^{\alpha}{}_{\beta}] \leq \sigma_i[T^{A}{}_{\beta}] \max\{\sigma_j[h^{\alpha}{}_{A}]\}. \tag{3.14}$$

[k]*Where the singular values have been ordered from larger to smaller for each operator.*

Consider for contradiction, that there exists a hyperbolizer. Namely a surjective reduction $h^{\alpha}{}_{A} : E \to X$, that does not depend on z, of the operator $T^{A}{}_{\alpha}(z) = K^{A}{}_{\alpha} - zB^{A}{}_{\alpha} : X \to E$, in which $B^{A}{}_{\alpha}$ has no right kernel,[l] and such that $h^{\alpha}{}_{A}B^{A}{}_{\beta}$ is invertible.

Then the next theorem follows.

Theorem 3.8. *Suppose that for at least one singular value of* $T^{A}{}_{\alpha}(\lambda + \varepsilon e^{i\theta})$, *with* $\lambda \in \mathbb{C}$, *satisfies*

$$\sigma[T^{A}{}_{\alpha}(\lambda + \varepsilon e^{i\theta})] = O(\varepsilon^l) \quad \textit{with } l \geq 2$$

[k]The singular values are $\sigma_i[H^{\alpha}{}_{\beta}] = \sqrt{\lambda_i[G_2^{-1} \circ \bar{H}' \circ G_2 \circ H]}$, $\sigma_i[T^{A}{}_{\beta}] = \sqrt{\lambda_i[G_2^{-1} \circ \bar{T}' \circ G_1 \circ T]}$ and $\sigma_i[h^{a}{}_{A}] = \sqrt{\lambda_i[G_1^{-1} \circ \bar{h}' \circ G_2 \circ h]}$. Where $\lambda_i[K]$ mean eigenvalues of K.

[l]Notice that as in the square case, a rectangular operator $T^{A}{}_{\alpha}$ has right kernel when at least one of their singular values vanishes (see proof of Lemma 3.1). But this is equivalent to the vanishing of Eq. (3.2). Therefore, $B^{A}{}_{\alpha}$ has no right kernel if and only if $\det(B^* \circ B) = \sigma_1[B^{A}{}_{\beta}] \cdots \sigma_u[B^{A}{}_{\beta}] \neq 0$.

then there exists at least one singular value of $h^{\alpha}{}_{A}T^{A}{}_{\alpha}(\lambda+\varepsilon e^{i\theta})$ *such that*

$$\sigma[h^{\alpha}{}_{A}T^{A}{}_{\beta}(\lambda+\varepsilon e^{i\theta})] = O(\varepsilon^{m}) \quad \text{with } m \geq l \geq 2.$$

Thus $((h^{\alpha}{}_{C}B^{C}{}_{\gamma})^{-1})^{\alpha}{}_{\gamma}h^{\gamma}{}_{A}K^{A}{}_{\beta}$ *is non-diagonalizable, in particular there does not exists any hyperbolizer and system* (1.1) *is not strongly hyperbolic.*

Proof. We use Lemma 3.7 for $T^{A}{}_{\beta}(\lambda+\varepsilon e^{i\theta})$ and $h^{\alpha}{}_{A}T^{A}{}_{\beta}(\lambda+\varepsilon e^{i\theta})$, and let $\frac{1}{C} := \max\{\sigma_j[h^{\alpha}{}_{A}]\}$ (It does not vanish since $h^{\alpha}{}_{A} \neq 0$ and does not depend on λ), then for Eq. (3.14)

$$0 \leq \sigma_i[h^{\alpha}{}_{A}T^{A}{}_{\beta}(\lambda+\varepsilon e^{i\theta})] \leq \sigma_i[T^{A}{}_{\beta}(\lambda+\varepsilon e^{i\theta})]\frac{1}{C}.$$

But for some i, $\sigma_i[T^{A}{}_{\beta}(\lambda+\varepsilon e^{i\theta})] = O(\varepsilon^{l})$ with $l \geq 2$. Therefore, $\sigma_i[h^{\alpha}{}_{A}T^{A}{}_{\beta}(\lambda+\varepsilon e^{i\theta})] = O(\varepsilon^{m})$ with $m \geq l \geq 2$. Since $C\varepsilon^{m} \leq \varepsilon^{l}$ for $0 \leq \varepsilon \ll 1$ is only possible if $m \geq l$.

Applying Theorem 3.3 to

$$\tilde{T}^{\alpha}{}_{\beta} = h^{\alpha}{}_{A}K^{A}{}_{\beta} - z(h^{\alpha}{}_{A}B^{A}{}_{\beta})$$

and recalling that $h^{\alpha}{}_{A}B^{A}{}_{\beta}$ is invertible by hypothesis, we conclude that $((h^{\alpha}{}_{C}B^{C}{}_{\gamma})^{-1})h^{\gamma}{}_{A}K^{A}{}_{\beta}$ is not diagonalizable. Therefore it is not a hyperbolizer and we reach a contradiction.

This result considers perturbation of the singular values around their vanishing values. As it has been shown in Appendix B, the orders of perturbations are invariant under any choice of these Hermitian forms. Thus the result does not depend on the particularities of the SVD. □

4. Applications and Examples

In this section, we shall show how to check conditions (1) and (2) in Theorems 2.1 and 2.2. They are very simple to verify in examples.

Condition (1): We shall assume that there exists ω_a such that $\mathfrak{N}^{Bb}_{\gamma}\omega_b$ has no right kernel.

As we mentioned before, $\mathfrak{N}^{Bb}_{\gamma}n(\lambda)_b$ with $n(\lambda)_b \in S^{\mathbb{C}}_{\omega_a}$, has right kernel when at least one of its singular values vanishes. This happens if and only if, given any positive definite Hermitian forms G_1 and G_2,

$$\sqrt{\det(G_2^{\alpha\gamma}\overline{\mathfrak{N}^{Aa}_{\gamma}n(\lambda)_a}G_{1AB}\mathfrak{N}^{Bb}_{\beta}n(\lambda)_b)} = 0, \tag{4.1}$$

as it has been proved in Lemma 3.2. Therefore, the system is hyperbolic if and only if all roots λ_k of this equation are real. In addition, for any line $n(\lambda)_b$ we call characteristic eigenvalues to their corresponding $\{\lambda_k\}$.

Condition (2): In general it is not an easy task to calculate the singular values and their orders in parameter ε. Fortunately Theorem 4.1 below allows for a simpler calculation, showing when the coefficient of zero and first order of the singular values vanish.

Assuming that condition (1) has been checked for ω_a. Consider the line $n(\lambda)_a = -\lambda\omega_a + \beta_a$ belonging to S_{ω_a}, with β_a not proportional to ω_a and λ real. Let $\{\lambda_k\}$ be the characteristic eigenvalues of $n(\lambda)_a$. Then, the principal symbol $\mathfrak{N}^{Aa}_{\ \gamma} n(\lambda_k)_a$ has right and left kernels. We call $W^{\gamma}_{\ i}(\lambda_k)$ and $U^j_A(\lambda_k)$ with $i = 1, \ldots, \dim(\textit{left\_}\ker(\mathfrak{N}^{Aa}_{\ \gamma} n(\lambda_k)_a))$ and $j = 1, \ldots, \dim(\textit{right\_}\ker(\mathfrak{N}^{Aa}_{\ \gamma} n(\lambda_k)_a))$ to any basis of these spaces, respectively, namely they are linearly independent sets of vectors such that $\mathfrak{N}^{Aa}_{\ \gamma} n(\lambda_k)_a W^{\gamma}_{\ i}(\lambda_k) = 0$ and $U^j_A(\lambda_k)\mathfrak{N}^{Aa}_{\ \gamma} n(\lambda_k)_a = 0$.

Now consider a perturbation of these covectors $n_{\varepsilon,\theta}(\lambda_i)_a = -\varepsilon e^{i\theta}\omega_a + n(\lambda_i)_a$ with $0 \leq |\varepsilon| \ll 1$ and ε real, and any fixed $\theta \in [0, 2\pi]$.

Theorem 4.1. *A necessary condition for system* (1.1) *to be strongly hyperbolic is*: *The following operator*

$$L^j_{\ i}(\lambda_k) := U^j_A(\lambda_k)(\mathfrak{N}^{Aa}_{\ \gamma}\omega_a)W^{\gamma}_{\ i}(\lambda_k) \tag{4.2}$$

has no right kernel.[m]

Definition of $L^j_{\ i}(\lambda_k)$ is equivalent to $\tilde{L}^j_{\ i} = \delta^{jj}_1(0, \bar{U}_1, \bar{U}_3)^C_{\ j}\delta_{1CD}B^D_{\ \alpha}(0, V_1)^{\alpha}_{\ i}$, in Eq. (B.4) in Appendix B, under a basis transformation. If $\tilde{L}^j_{\ i}$ has right kernel then it has a singular value which vanishes, and then at least one perturbed singular value $\sigma(\mathfrak{N}^{Aa}_{\ \gamma} n_{\varepsilon,\theta}(\lambda_i)_a)$ is order $O(|\varepsilon|^l)$ with $l \geq 2$.

As we have shown in Theorem 2.1 for the square case, this is also a sufficient condition.

Now, using the tools developed, we show how to apply these results in some examples.

4.1. *Matrix example* 1

Consider the matrix

$$T(x) = K - zB := \begin{pmatrix} \lambda_1 & \kappa \\ 0 & \lambda_2 \end{pmatrix} - z\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2} \tag{4.3}$$

in which $\lambda_1, \lambda_2, \kappa$ are constants. Consider the scalar products $G_{1,2} = \delta_{1,2} = \operatorname{diag}(1,1)$, then the singular values of $T(z)$ are

$$\sigma_1[T(z)] = \sqrt{\omega(z) + \sqrt{\omega^2(x) - |z-\lambda_1|^2|z-\lambda_2|^2}},$$

$$\sigma_2[T(z)] = \sqrt{\omega(z) - \sqrt{\omega^2(x) - |z-\lambda_1|^2|z-\lambda_2|^2}}$$

with

$$\omega(z) = \frac{1}{2}(|z-\lambda_1|^2 + |z-\lambda_2|^2 + |\kappa|^2)$$

[m] $L^j_{\ i}$ has no kernel if and only if given any positive define Hermitian forms G_3 and G_4 then

$$\det\left(G^{ji_1}_3 \overline{L^{j_1}_{\ i_1}} G_{4j_1j_2} L^{j_2}_{\ i}\right) \neq 0.$$

a non-negative function of z. Notice that $\sigma_1[T(z)]$ can be vanished only when $\lambda_1 = \lambda_2$ and $\kappa = 0$.

The Taylor expansion of σ_2 centered in $\lambda_{1,2}$ is[n]:

$$\sigma_2(\lambda_{1,2} + \varepsilon) \approx 0 + \frac{|\lambda_1 - \lambda_2|}{\sqrt{|\lambda_1 - \lambda_2|^2 + |\kappa|^2}} |\varepsilon| + O(\varepsilon^2).$$

As in Theorem 2.1

- K is not diagonalizable when $\lambda_1 = \lambda_2$ and $\kappa \neq 0$. In that case $\sigma_2(\lambda_{1,2} + \varepsilon) = O(|\varepsilon|^2)$.
- K is diagonalizable for any other case and the singular values remain of order $O(|\varepsilon|)$ or $O(\varepsilon^0)$.

4.2. *Force-free electrodynamics in Euler potential description*

In this subsection, we study a description of the Force-Free Electrodynamics system based on Euler's potentials [4, 34]. When it is written as a first-order system, this is a constrained system and we shall show that it is only weakly hyperbolic. It is important to mention that Reula and Rubio [28] reached the same conclusion by another method. They used the potentials as fields obtaining a second-order system in derivatives, which then led to a pseudodifferential first-order system without constraints and finally tested the failure of strong hyperbolicity using Kreiss criteria [19]. The advantage of our technique is that we use the gradients of the potentials as fields, obtaining directly a first-order system in partial derivatives but with constraints. Then, proving that condition (2) in Theorem 2.2 fails, we conclude that there does not exist any hyperbolizer.

In this system the electromagnetic tensor F_{ab} is degenerated $F_{ab}j^b = 0$ and magnetic dominated $F := F_{ab}F^{ab} > 0$. These conditions allows us to decompose $F_{ab} = l_{1[a}l_{2b]}$, (see, [12, 24]) in terms of space-like 1-forms l_{ia} with $i = 1, 2$. For more detailed works on Force-Free electrodynamics see [2, 3, 12, 17].

In addition, Carter 1979 [4] and Uchida 1997 [34] proved that there exist two Euler potentials ϕ_1 and ϕ_2 such that $l_{ia} = \nabla_a \phi_i$.

With this ansatz, the Force Free equations in the (gradient) Euler's potentials version are

$$l_{ka}\nabla_b(l_i^a l_j^b \varepsilon^{ij}) = 0$$

$$\nabla_{[a}l_{|i|b]} = 0$$

[n]For calculate the Taylor expansion we use the identity $X - \sqrt{X^2 - Y^2} = \frac{1}{2}(\sqrt{X+Y} - \sqrt{X-Y})^2$ with real X, Y and $X + Y, X - Y > 0$.

with background metric g_{ab}. Taking a linearized version at a given point and background solution, we get the following principal symbol

$$\mathfrak{N}^{Aa}_{\alpha}(x,\phi)n_a\delta\phi^\alpha = \begin{pmatrix} (l_{1a}(l_2.n)-(l_1.l_2)n_a) & ((l_1.l_1)n_a - l_{1a}(l_1.n)) \\ (l_{2a}(l_2.n)-(l_2.l_2)n_a) & ((l_1.l_2)n_a - l_{2a}(l_1.n)) \\ n^{[b}\delta^{c]}_a & 0 \\ 0 & n^{[b}\delta^{c]}_a \end{pmatrix} \begin{pmatrix} \delta l^a_1 \\ \delta l^a_2 \end{pmatrix}.$$

The solution space $\delta\phi^\alpha = \binom{\delta l^a_1}{\delta l^a_2}$ is 8-dimensional and the associated space of equations is 14-dimensional $\delta X_A = (\delta W, \delta X, \delta Y_{bc}, \delta Z_{bc})$ where $\delta Y_{bc} = \delta Y_{[bc]}$ and $\delta Z_{bc} = \delta Z_{[bc]}$.

(1) We shall check that the system is hyperbolic: Consider ω_a timelike and normalized $\omega_a\omega^a = -1$, since l_{ia} can be chosen orthogonal (via a gauge transformation), we define an orthonormal frame $\{e_{ia}\ i = 0,1,2,3\}$ with $e_{0a} = \omega_a$ and $l_{ia} = l_i e_{ia}$ with $i = 1,2$ such that $g_{ab} = (-1,1,1,1)$. Consider now the plane $n(\lambda)_a = -n_0\omega_a + \beta_a \in S^{\mathbb{C}}_{\omega_a}$ with $n_0 \in \mathbb{C}$, $\beta_a = n_i e_{ia}$ for $i = 1,2,3$, n_i real and let $G_{1AB} = \mathrm{diag}(1,\ldots,1)$ and $G^{\alpha\beta}_2 = \mathrm{diag}(1,\ldots,1)$, then by (4.1) the characteristic equation of the principal symbol is (notice that $\mathfrak{N}^{Ab}_{\alpha}$ is real)

$$\begin{aligned} 0 &= \sqrt{\det(G^{\alpha\gamma}_2 \mathfrak{N}^{Aa}_{\gamma}\bar{n}(\lambda)_a G_{1AB}\mathfrak{N}^{Bb}_{\beta} n(\lambda)_b)} \\ &= (|n_0|^2 + n_1^2 + n_2^2 + n_3^2)^2 |(-n_0^2 + n_3^2)| |n_a g^{ab} n_b| l_1^2 l_2^2. \end{aligned} \tag{4.4}$$

It means that the characteristic structure is given in terms of two symmetric tensors, the background metric, and $g^{ab}_1 = \mathrm{diag}(-1,0,0,1)$ i.e.

$$0 = n_a g^{ab} n_b \quad \text{and} \quad 0 = n_a g^{ab}_1 n_b. \tag{4.5}$$

The first one corresponds to the electromagnetic waves and the second one to the Alfven waves. Because the characteristic eigenvalues are real, thus the system is hyperbolic.

Note that the introduction of two unnatural scalar products lead us to a preferred Euclidean metric $g^{ab}_2 n_a n_b = |n_0|^2 + n_1^2 + n_2^2 + n_3^2$.

(2) We shall check that condition (2) in Theorem 2.2 fails: For this system, it is possible to calculate the singular values. We only show the relevant one (with n_0 real)

$$\sigma[\mathfrak{N}^{Bb}_{\beta} n_b] := \frac{1}{\sqrt{2}} \left| \sqrt{N + 2(n_0^2 - n_3^2) l_2^2} - \sqrt{N - 2(n_0^2 - n_3^2) l_2^2} \right|,$$

$$N := n_1^2 + n_2^2 + (n_0^2 + n_3^2)(1 + l_2^4).$$

Notice that it vanishes when $n_0^2 - n_3^2 = 0$.

Consider now the line $n(\lambda)_a = -\lambda\omega_a + \beta_a \in S_{\omega_a}$ with λ real, $\beta_a = n_1 e_{1a}$ and the characteristic eigenvalue $\lambda = 0$, i.e. $n(\lambda)_a g_1^{ab} n(\lambda)_b|_{\lambda=0} = 0$. Perturbing this singular value in a neighborhood of this point

$$n_{\varepsilon,\theta}(\lambda = 0)_a = -\varepsilon e^{i\theta}\omega_a - \lambda\omega_a + \beta_a|_{\substack{\lambda=0\\ \theta=0}}$$

we obtain

$$\sigma_1(\varepsilon) \approx \frac{1}{\sqrt{2}}\varepsilon^2 \left(\frac{(1+3l_2^4)}{\sqrt{n_1^2+\varepsilon^2(1+3l_2^4)}} - \frac{(1-l_2^4)}{\sqrt{n_1^2+\varepsilon^2(1-l_2^4)}} \right)$$

It is order $O(|\varepsilon|^2)$ and by Theorem 2.2 there does not exist any hyperbolizer and the system is weakly hyperbolic.

In general, explicit calculations of the singular values cannot be done. Because of that, we shall show how to reach the same conclusion using Theorem 4.1.

Consider the line $n(\lambda)_a$ as before, then we get the following principal symbol

$$\begin{aligned} \mathfrak{N}^{Aa}_{\ \alpha} n(\lambda)_a &= -\lambda \mathfrak{N}^{Aa}_{\ \alpha}\omega_a + \mathfrak{N}^{Aa}_{\ \alpha}\beta_a \\ &= -\lambda \begin{pmatrix} 0 & l_1^2 e_{0a} \\ -l_2^2 e_{0a} & 0 \\ e_0^{[b}\delta_a^{c]} & 0 \\ 0 & e_0^{[b}\delta_a^{c]} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ l_2^2 n_1 e_{1a} & -l_1 l_2 n_1 e_{2a} \\ n_1 e_1^{[b}\delta_a^{c]} & 0 \\ 0 & n_1 e_1^{[b}\delta_a^{c]} \end{pmatrix}. \end{aligned}$$

To define $L^j_{\ i}$ as (4.2), we need to calculate left and right kernel basis of $\mathfrak{N}^{Aa}_{\ \alpha} n(\lambda = 0)_a$. They are

$$\begin{pmatrix} \delta W \\ \delta X \\ \delta Y_{bc} \\ \delta Z_{bc} \end{pmatrix} = \left\langle \begin{matrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ e_{0[b}e_{2a]} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ e_{0[b}e_{3a]} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ e_{2[b}e_{3a]} \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_{0[b}e_{2a]} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_{0[b}e_{3a]} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_{2[b}e_{3a]} \end{pmatrix}, \end{matrix} \right\rangle,$$

$$\begin{pmatrix} \delta l_1^a \\ \delta l_2^a \end{pmatrix} = \left\langle \begin{pmatrix} 0 \\ e_1^a \end{pmatrix} \right\rangle.$$

We conclude that

$$L^{j}{}_{i} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.6}$$

which trivially vanishes and then it has right kernel. Thus, as we discussed before, there is a singular value that goes to zero at least quadratically in the perturbation and the system cannot be strongly hyperbolic.

If we take ω_a outside the light cone, then there will be complex characteristic eigenvalues (so the system would not be hyperbolic along those lines), so those cases are trivially not strongly hyperbolic.

4.3. *Charged fluids with finite conductivity*

In this subsection, we present the charged fluid with finite conductivity in a first order in derivative formulation, in which the relevant block of the principal part has no constraints. We shall prove that the system is weakly hyperbolic while it has finite conductivity and, of course, strongly hyperbolic with vanishing conductivity. This result is in concordance with [5, chap. IX].

The system is

$$\begin{aligned} u^m \nabla_m n + n \nabla_m u^m &= 0 \\ u^a \nabla_a \rho + (\rho + p) \nabla_a u^a &= u_b J^a F^b{}_a \\ (\rho + p) u^a \nabla_a u^b + D^b p &= -h^b_c J^a F^c{}_a \\ u^m \nabla_m q + q \nabla_m u^m + \sigma F_a{}^m \nabla_m u^a &= \sigma u^a J_a \\ \nabla_a F^{ab} &= J^b \\ \nabla_a F^{*ab} &= 0 \\ J^a &= q u^a + \sigma u_b F^{ba} \end{aligned}$$

with background metric g_{ab}, $h^b{}_c := (\delta^b_c + u^b u_c)$, $u^a u_a = -1$, $D^b := h^{bc} \nabla_a$ and $p = p(n, \rho)$. Here ρ is the proper total energy density, n the proper mass density, u^a the four-velocity, q the proper charge density, p the pressure of the fluid and σ the conductivity. For examples of this type of systems see [22, 23, 36].

The variables are $(n, \rho, u^a, q, F^{ab})$. As before, taking the linearized version at a given point and background solution of these equations, the principal symbol is given by

$$(\mathfrak{N}_{\text{fluid}})^{Aa}_{\ \ \alpha}(x,\phi)n_a\delta\phi^\alpha$$
$$= \begin{pmatrix} u.n & 0 & n\ n_b & 0 \\ 0 & u.n & (p+\rho)n_b & 0 \\ p_n h^{am}n_m & p_\rho h^{am}n_m & (\rho+p)\delta^a_{\ b}(u.n) & 0 \\ 0 & 0 & (q\delta^m_b + \sigma F_b^{\ m})n_m & u.n \end{pmatrix} \begin{pmatrix} \delta n \\ \delta\rho \\ \delta u^b \\ \delta q \\ \delta F^{ab} \end{pmatrix} = 0,$$

$$(\mathfrak{N}_{\text{Electro}})^{Aa}_{\ \ \alpha}(x,\phi)n_a\delta\phi^\alpha = \begin{pmatrix} n_a \\ n_c\varepsilon^c_{\ dab} \end{pmatrix}\delta F^{ab} = 0$$

with $u_a\delta u^a = 0$. Notice that the fluid-current part decouples of the electrodynamics part. We shall only study this fluid-current part because there is where the lack of strong hyperbolicity appears. This part of the system has no constraints.

The solution space $\delta\phi^\alpha = \begin{pmatrix} \delta n \\ \delta\rho \\ \delta u^b \\ \delta q \end{pmatrix}$ is 6-dimensional and the equation space $\delta X_A = (\delta W\ \delta X\ \delta Y_a\ \delta Z)$, with $\delta Y_a u^a = 0$, is 6-dimensional too.

The characteristic structure of the fluids-current part is

$$\det(\mathfrak{N}^{Aa}_{\text{fluid}\ \alpha}n_a) = -(\rho+p)^4(n_a u^a)^4 g_1^{ab}n_a n_b = 0. \tag{4.7}$$

This means that

$$(n_a u^a) = 0 \quad \text{and} \quad g_1^{ab}n_a n_b = 0$$

with $g_1^{ab} := (\frac{n}{(\rho+p)}p_n + p_\rho)h^{ab} - u^a u^b$ (It is a Lorentzian metric if $\frac{n}{(\rho+p)}p_n + p_\rho > 0$). In addition the characteristic structure of the electrodynamics part is

$$g^{ab}n_a n_b = 0.$$

The $n_a u^a = 0$ correspond to the material waves, $g_1^{ab}n_a n_b = 0$ to the acoustic waves and $g^{ab}n_a n_b = 0$ to the electromagnetic waves.

(1) We shall check condition (1) in Theorem 2.1: Consider now the line $n(\lambda)_a = -\lambda\omega_a + \beta_a \in S_{\omega_a}$ with $\omega_a = u_a$ and β_a spacelike and such that $\beta_a u^a = 0$. We notice from (4.7) that $\mathfrak{N}^{Aa}_{\text{fluid}\ \alpha}\omega_a$ has no right kernel if $(\rho+p) \neq 0$ and the system is hyperbolic for this ω_a if $\frac{n}{(\rho+p)}p_n + p_\rho \geq 0$. It means that the velocity of the acoustic wave $v := \sqrt{\frac{n}{(\rho+p)}p_n + p_\rho}$ is real.

(2) Condition (2) in Theorem 2.1 fails: This line has the characteristic eigenvalue $\lambda = 0$, since $u^a n(\lambda)_a|_{\lambda=0} = 0$. We choose an orthonormal frame $\{e_{ia}\ i = 0,1,2,3\}$ such that $e_{0a} = u_a$, $e_{1a} = \frac{1}{\sqrt{\beta^a\beta_a}}\beta_a$ with e_{2a} and e_{3a} space-like. In this frame the background metric looks like $g_{ab} = \text{diag}(-1,1,1,1)$.

The principal symbol along this line is

$$\mathfrak{N}^{Aa}_{\alpha} n(\lambda)_a = -\lambda \mathfrak{N}^{Aa}_{\alpha}\omega_a + \mathfrak{N}^{Aa}_{\alpha}\beta_a$$

$$= -\lambda \begin{pmatrix} -1 & 0 & nu_b & 0 \\ 0 & -1 & (p+\rho)u_b & 0 \\ 0 & 0 & -(\rho+p)g^a{}_b & 0 \\ 0 & 0 & qu_b + \sigma F_b{}^m u_m & -1 \end{pmatrix}$$

$$+ \sqrt{\beta\cdot\beta} \begin{pmatrix} 0 & 0 & ne_{1b} & 0 \\ 0 & 0 & (p+\rho)e_{1b} & 0 \\ p_n e_1^a & p_\rho e_1^a & 0 & 0 \\ 0 & 0 & qe_{1b} + \sigma F_b{}^m e_{1m} & 0 \end{pmatrix}.$$

In order to find $L^j{}_i$ the basis of the left and right kernel of $\mathfrak{N}^{Aa}_{\alpha} n(\lambda = 0)_a$ are

$$\begin{pmatrix} \delta W \\ \delta X \\ \delta Y_a \\ \delta Z \end{pmatrix} = \left\langle \begin{pmatrix} 0 \\ 0 \\ e_{2a} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ e_{3a} \\ 0 \end{pmatrix}, \begin{pmatrix} -(p+\rho) \\ n \\ 0 \\ 0 \end{pmatrix} \right\rangle,$$

$$\begin{pmatrix} \delta n \\ \delta \rho \\ \delta u^b \\ \delta q \end{pmatrix} = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -p_\rho \\ p_n \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \delta u^b \\ 0 \end{pmatrix} \right\rangle$$

with $e_{1b}\delta u^b = 0$ and $\delta u^b F_b{}^m e_{1m} = 0$. Thus following Eq. (4.2)

$$L^j{}_i = \begin{pmatrix} 0 & 0 & -(\rho+p)\delta u^a e_{2a} \\ 0 & 0 & -(\rho+p)\delta u^a e_{3a} \\ 0 & -(rp_\rho + np_n) & 0 \end{pmatrix}.$$

Clearly $L^j{}_i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$. Therefore, the system is weakly hyperbolic and there is no hyperbolizer.

Notice that we chose a particular $\omega_a = u_a$. It is easy to show that condition (2) still fails for any timelike ω_a in both metrics $g^{ab}\omega_a\omega_a < 0$ and $g_1^{ab}\omega_a\omega_a < 0$. In addition, when choosing ω_a outside of these cones, complex characteristic eigenvalues appear. Thus, in both cases no hyperbolizer exists.

4.3.1. *Vanish conductivity* $\sigma = 0$

Finally, we notice that if the conductivity goes to zero $\sigma = 0$ the kernels change and the system becomes strongly hyperbolic.

Consider the above line $n(\lambda)_a$ in $\lambda = 0$, thus we shall prove that the new $L^j{}_i$ has no right kernel. The new left and right kernel basis are

$$\begin{pmatrix}\delta W\\ \delta X\\ \delta Y_a\\ \delta Z\end{pmatrix} = \left\langle \begin{pmatrix}0\\0\\e_{2a}\\0\end{pmatrix}, \begin{pmatrix}0\\0\\e_{3a}\\0\end{pmatrix}, \begin{pmatrix}-(p+\rho)\\n\\0\\0\end{pmatrix}, \begin{pmatrix}0\\q\\0\\-(p+\rho)\end{pmatrix} \right\rangle,$$

$$\begin{pmatrix}\delta n\\ \delta \rho\\ \delta u^b\\ \delta q\end{pmatrix} = \left\langle \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-p_\rho\\p_n\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\e_2^a\\0\end{pmatrix}, \begin{pmatrix}0\\0\\e_3^a\\0\end{pmatrix} \right\rangle.$$

Thus

$$L^j{}_i = \begin{pmatrix} 0 & 0 & -(\rho+p) & 0\\ 0 & 0 & 0 & -(\rho+p)\\ 0 & -((p+\rho)p_\rho + np_n) & 0 & 0\\ -(p+\rho) & -p_n q & 0 & 0 \end{pmatrix}.$$

This operator has no kernel if the determinant is different from zero

$$\det L^j{}_i = -(p+\rho)^4 \left(\frac{n}{(\rho+p)}p_n + p_\rho\right) \neq 0.$$

Therefore $p+\rho \neq 0$ and $\frac{n}{(\rho+p)}p_n + p_\rho \neq 0$. As we explained, the first condition is necessary in order for $\mathfrak{N}_{\text{fluid}}^{Aa}{}_{\alpha}\omega_a$ not to have right kernel and the second condition limits the possibility of the velocity of the acoustic waves to vanish. We conclude that the system with $\sigma = 0$ is strongly hyperbolic.

5. Conclusions

In this paper, we studied the covariant theory of strong hyperbolicity for systems with constraints and found a necessary condition for the systems to admit a hyperbolizer. If this condition, which is easy to check, is not satisfied, then there is no subset of evolution equations of the strongly hyperbolic type in the usual sense.

To find this condition we introduce the singular value decomposition of one parameter families (pencils) of principal symbols and study perturbations around the points where they have kernel. We proved that if the perturbations of a singular value, which vanishes at that point, are order 2 or larger then the system is not strongly hyperbolic, Theorem 2.2.

For systems with constraints, the rectangular case, is only a necessary condition, but in the case without constraints, namely square case, this condition becomes also a sufficient condition too, Theorem 2.1. In this case, the condition is equivalent to the ones in Kreiss's Matrix Theorem.

As an extra result, we showed that a perturbed Matrix has an l-Jordan Block if and only if it has a singular value of order $O(\varepsilon^l)$.

Although the SVD depends on the scalar products used to define the adjoint operators, we found that the asymptotic orders of the singular values are independent of them.

When the systems have constraints, their principal symbols are rectangular operators and there is not a simple way to find their characteristic structure. We proposed a way to calculate its, by connecting the kernel of an operator with the vanishing of any of their singular values (see Eq. (4.1)).

We applied these theorems to some examples of physical interest.

A simple matrix example of 2×2 in which we study its Jordan form using the perturbed SVD.

The second example is Force Free electrodynamic in the Euler potentials form, that when written in first-order form has constraints. Using our result we checked that there is no hyperbolizer, being the system only weakly hyperbolic. We did this in two alternative ways. First by computing the singular values and showing that at some point one of them is order $O(\varepsilon^2)$. Second by computing the first-order leading term using right and left kernels as in Theorem 4.1. In general, it is not an easy task to calculate the singular values, therefore, the second way simplifies the study of strong hyperbolicity.

The last example is a charged fluid with finite conductivity. For this case, it is enough to consider the fluid-current part that decouples (at the level of the principal symbol) from the electromagnetic part. We showed how the introduction of finite conductivity, hampers the possibility of a hyperbolizer.

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Appendix A. Singular Value Decomposition

In this Appendix, the SVD is defined for linear operators that map between two vector spaces of finite dimension. For introduction on topic see [11, 31, 33, 35]. One of the most significant properties of the SVD is that it allows us to characterize the image and the kernel of the operator through real quantities called singular values. They, as we showed in Sec. 3, provide data about the Jordan form of square matrices.

One problem about singular values decomposition, is that, it is necessary to introduce extra structure to the problem, namely, scalar products. When they are used in vector spaces over a manifold, they might introduce non-covariant expressions. These scalar products are two positive definite, tensorial Hermitian forms,[o] in the input and output spaces of the operator, respectively.

In this appendix, we use the notation of Sec. 3 with $K^A_{\ \alpha} : X \to E$ and assume that $e = \dim E \geq \dim X = u$.

Consider the two positive definite Hermitian forms G_{1AB} and $G_{2\alpha\beta}$ in the spaces E and X, respectively. This allows us to define *the adjoint operator*

$$K^* = G_2^{-1} \circ \bar{K}' \circ G_1 : E \to X,$$
$$(K^*)^\alpha_{\ C} = (G_2^{-1} \circ \bar{K}' \circ G_1)^\alpha_{\ C} = G_2^{\alpha\beta} \bar{K}^B_{\ \beta} G_{1BC},$$

where $G_1^{AB} G_{1BC} = \delta^A_{1C}$ and $G_2^{\alpha\gamma} G_{2\gamma\nu} = \delta^\alpha_{\ \nu}$ are the identity operators in E and X respectively, and $\bar{K}'$ is the dual complex operator of K.

With this operator, we can define

$$\begin{aligned} K^* \circ K \ &= \ G_2^{-1} \circ K' \circ G_1 \circ K : X \to X, \\ \phi^\alpha \ &\to \ G_2^{\alpha\beta} \bar{K}^B_{\ \beta} G_{1BC} K^C_{\ \gamma} \phi^\gamma, \\ K \circ K^* \ &= \ K \circ G_2^{-1} \circ K' \circ G_1 : E \to E, \\ l^A \ &\to \ K^A_{\ \alpha} G_2^{\alpha\beta} \bar{K}^B_{\ \beta} G_{1BC} l^C. \end{aligned}$$

Since $G_2 \circ K^* \circ K$ and $G_1 \circ K \circ K^*$ are semi-positive define Hermitian forms, $K^* \circ K$ and $K \circ K^*$ are diagonalizable with real and semi-positive eigenvalues. Also, the square roots of these eigenvalues are the singular values of K and K^*.

With these definitions, we assert the singular value decomposition in the form of a theorem. From now on Latin indices i, j, k go from 1 to e and primes Latin indices i', j', k' from 1 to u, unless explicitly stated. These indices indicate the different eigenvectors.

Theorem A.1. *Consider K, K^*, G_1 and G_2 as previously defined and $e \geq u$. Suppose that* $\operatorname{rank}(K) = r$ *and* $\dim(\text{ker\_right}(K)) = u - r$, *then K can be decomposed as*

$$K^A_{\ \alpha} = U^A_{\ i} \Sigma^i_{\ j'} (V^{-1})^{j'}_{\ \ \alpha},$$

where $\Sigma^i_{\ j'} = \begin{pmatrix} \Sigma_+ & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^i_{\ j'}$ *of size $e \times u$, with* $\Sigma_+ = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_1 \geq \cdots \geq \sigma_l > 0$ *$l = 1, \ldots, r$ real, and $\sigma_{r+1} = \cdots = \sigma_u = 0$. The σ_l are called singular values of K and they are the square root of the eigenvalues of $K^* \circ K$.*

[o]Consider the $\mathbb{C}$-vectorial space V of finite dimension. A Hermitian form on V is a map $G : V \times V \to \mathbb{C}$ such that $G(av, bu) = \bar{a}bG(v, u) = \bar{a}b(\bar{v}^a G_{ab} u^b)$ and $G_{ab} = \bar{G}_{ab}$ (the bar means conjugation). In addition there exists a symmetric real bilinear form g_{1ab} and an antisymmetric bilinear form g_{2ab} such that $G_{ab} = g_{1ab} + ig_{2ab}$. When the hermite form is positive definite (g_{1ab} is positive define), G defines a complex inner dot product of V.

In addition, the columns of $U^A_{\ i}$ *and* $V^\gamma_{\ i'}$ *are eigenbasis of* $K \circ K^*$ *and* $K^* \circ K$ *respectively, such that they are orthogonal*

$$\bar{U}^C_{\ i} G_{1CD} U^D_{\ j} = \delta_{1ij} := \mathrm{diag}(1, \ldots, 1, 1, \ldots, 1) \tag{A.1}$$

$$\bar{V}^\gamma_{\ i'} G_{2\gamma\eta} V^\eta_{\ j'} = \delta_{2i'j'} := \mathrm{diag}(1, \ldots, 1) \tag{A.2}$$

with $C, D, i, j = 1, \ldots, e$ *and* $\alpha, \beta\, i', j' = 1, \ldots, u$.

We are going to discriminate $V^\alpha_{\ i'} = (V_2, V_1)^\alpha_{\ i'}$ *and* $U^A_{\ i} = (U_2, U_1, U_3)^A_{\ i}$, *where* V_2 *are the first* r *columns and* V_1 *the* $u - r$ *left of* V; U_2 *are the first* r *columns,* U_1 *the following* $u - r$ *and* U_3 *the remaining* $e - u$ *of* U.

Recalling that the eigenbasis that are chosen in the Jordan decomposition are not unique we realize that they are the orthogonal factors in the SVD. For fixed $G_{1,2}$ we can select different orthogonal basis of the eigenspaces, associated to some singular value, and obtain different U, V. Nevertheless the singular values remain invariant as long as $G_{1,2}$ remain fixed.

This decomposition allows us to control right and left kernels and images of any linear operator, as we show in the next corollary.

Corollary A.2. *Consider* $V = (V_2, V_1)^\alpha_{\ i'}$ *and* $U = (U_2, U_1, U_3)^A_{\ i}$ *as in the previous theorem, then the orthogonal conditions* (A.1) *and* (A.2) *are*

$$(\bar{V}_i)^\gamma_{\ s} G_{2\gamma\eta} (V_j)^\eta_{\ r} = 0 \quad \text{with } i, j = 1, 2 \text{ and } i \neq j \tag{A.3}$$

$$(\bar{U}_i)^C_{\ s} G_{1CD} (U_j)^D_r = 0 \quad \text{with } i, j = 1, 2, 3 \text{ and } i \neq j. \tag{A.4}$$

In addition

$$\dim(\text{right\_ker}(K)) = u - r$$

$$\dim(\text{left\_ker}(K)) = e - r$$

$$\dim(\text{rank\_Columns}(K)) = \dim(\text{rank\_Rows}(K)) = r.$$

And the explicit right and left kernels of K *are*

$$K^A_{\ \beta} V^\beta_{1s} = 0 \quad \text{with } s = 1, \ldots, u - r, \tag{A.5}$$

$$(\delta_1^{lm} \bar{U}^C_{1m} G_{1CB}) K^B_{\ \beta} = 0 \quad \text{with } m, l = 1, \ldots, u - r, \tag{A.6}$$

$$(\delta_1^{lm} \bar{U}^C_{3m} G_{1CB}) K^B_{\ \beta} = 0 \quad \text{with } m, l = 1, \ldots, e - u, \tag{A.7}$$

where δ_1^{lm} *is the inverse of* δ_{1lm}.

SVD is similar to Jordan decomposition for square operators. In particular, they coincide when the operators are diagonalizable with real and semipositive eigenvalues and particular $G_{1,2}$ are used.

Suppose $A^\alpha_{\ \beta} : X \to X$ can be decomposed as $A^\alpha_{\ \beta} = P^\alpha_{\ i'} \Lambda^{i'}_{\ j'} (P^{-1})^{j'}_{\ \beta}$ with $\Lambda^{i'}_{\ j'} = \mathrm{diag}(\lambda_1, \ldots, \lambda_u)$ and λ_i real and semipositive. If we choose $G_{1\alpha\beta} = G_{2\alpha\beta} =$

$\overline{(P^{-1})^{i'}{}_{\alpha}}\delta_{i'j'}(P^{-1})^{j'}{}_{\beta}$ then

$$
\begin{aligned}
(A^* \circ A)^{\alpha}{}_{\beta} &= G_2^{\alpha\gamma}\overline{A^{\eta}{}_{\gamma}}G_{1\eta\varphi}A^{\varphi}{}_{\beta} \\
&= (P^{\alpha}{}_{i'_1}\delta^{i'_1j'_1}\bar{P}^{\gamma}{}_{j'_1})(\bar{P}^{\eta}{}_{i'_2}\ \bar{\Lambda}^{i'_2}{}_{j'_2}\ (\bar{P}^{-1})^{j'_2}{}_{\gamma})((\bar{P}^{-1})^{i'_3}{}_{\eta}\delta_{i'_3j'_3}(P^{-1})^{j'_3}{}_{\varphi}) \\
&\quad \times (P^{\varphi}{}_{i'}\Lambda^{i'}{}_{j'}(P^{-1})^{j'}{}_{\beta}) \\
&= P^{\alpha}{}_{i'}(\delta^{i'j'}\overline{\Lambda^{k'}{}_{j'}}\delta_{k'i'_1}\Lambda^{i'_1}{}_{j'_2})(P^{-1})^{j'_2}{}_{\beta} \\
&= P^{\alpha}{}_{i'}(\Lambda^2)^{i'}{}_{j'}(P^{-1})^{j'}{}_{\beta}.
\end{aligned}
$$

Therefore the eigenvalues of $(A^* \circ A)$ are λ_i^2, thus the singular values of A are λ_i. In addition $U^{\alpha}{}_{i'} = V^{\alpha}{}_{i'} = P^{\alpha}{}_{i'}$ and the orthogonal condition is $\overline{P^{\gamma}{}_{i'}}G_{1,2\gamma\eta}P^{\eta}{}_{j'} = \delta_{i'j'}$.[p]

Notice that, in the deduction, we used $(\delta^{i'j'}\overline{\Lambda^{k'}{}_{j'}}\delta_{k'i'_1}\Lambda^{i'_1}{}_{j'_2}) = (\Lambda^2)^{i'}{}_{j'}$ because $A^{\alpha}{}_{\beta}$ is diagonalizable, with real eigenvalues. But if $\Lambda^{i'}{}_{j'}$ is a non-trivial Jordan form, then the singular values are the square roots of the eigenvalues of $\delta^{i'j'}\overline{\Lambda^{k'}{}_{j'}}\delta_{k'i'_1}\Lambda^{i'_1}{}_{j'_2}$, the explicit calculation becomes hard even for simple examples. In Chap. 4, we present an analysis of the 2×2 matrix case.

Appendix B. Invariant Orders of Singular Values

In this paper, we study perturbed singular values in terms of some parameter ε. We use the orders in this parameter, to decide when a system is strongly hyperbolic or not. But as we showed in Appendix A, the singular values depend on two Hermitian forms, therefore we need to show that these orders remain invariant when we select different Hermitian forms. Thus we shall prove it in Lemma B.1. Also, in Theorem B.2, we shall show explicit expressions for the first order, when particular bases are chosen.

Consider two pairs of positive definite Hermitian forms G_{1AB}, $G_{2\alpha\beta}$ and $\widehat{G}_{1CD}$, $\widehat{G}_{2\alpha\beta}$ in the spaces E and X. Since they are positive define, they are equivalent, i.e.

$$\bar{U}^{C}{}_{A}\widehat{G}_{1CD}U^{D}{}_{B} = G_{1AB}, \tag{B.1}$$

$$\bar{V}^{\gamma}{}_{\alpha}\widehat{G}_{2\gamma\eta}V^{\eta}{}_{\beta} = G_{2\alpha\beta} \tag{B.2}$$

with some $U : E \to E$ and $V : X \to X$ invertible.

Consider now the linear operator $T^{A}{}_{\alpha} : X \to E$ with $\dim(X) \le \dim(E)$. We call $\hat{\sigma}_i[T^{A}{}_{\alpha}]$ the singular values of $T^{A}{}_{\alpha}$ defined using $\widehat{G}_{1CD}$, $\widehat{G}_{2\alpha\beta}$ and $\sigma_i[T^{A}{}_{\alpha}]$ using G_{1AB}, $G_{2\alpha\beta}$.

Lemma B.1. *The operator* $T^{A}{}_{\alpha}(\varepsilon) = K^{A}{}_{\alpha} + \varepsilon e^{i\theta}B^{A}{}_{\alpha} : X \to E$, *has singular values* $\sigma_i[T^{A}{}_{\alpha}] = O(\varepsilon^{l_i})$ *with* $i = 1, \ldots, \dim(X)$ *for some* l_i *if and only if* $\hat{\sigma}_i[T^{A}{}_{\alpha}] = O(\varepsilon^{l_i})$.

[p]The standard result in textbook is when $G_{1\alpha\beta} = G_{2\alpha\beta} = \delta_{\alpha\beta} = \overline{(P^{-1})^{i'}{}_{\alpha}}\delta_{i'j'}(P^{-1})^{j'}{}_{\beta}$ it means that $(P)^{\alpha}{}_{i'}$ is orthogonal, and the matrix $A^{\alpha}{}_{\beta}$ is "symmetric".

Proof. If we call $\lambda_i[T^* \circ T]$ to the eigenvalues of $T^* \circ T$ then

$$\hat{\sigma}_i[T] = \sqrt{\lambda_i[T^* \circ T]} = \sqrt{\lambda_i[V^{-1} \circ T^* \circ T \circ V]} = \sigma_i[U^{-1} \circ T \circ V]. \qquad \text{(B.3)}$$

The last equality is easy to prove.

Considering $T^A_{\ \alpha} : X \to E$ with rank r, we recall that from definition of SVD

$$\sigma_1[T^A_{\ \alpha}] \geq \sigma_2[T^A_{\ \alpha}] \geq \cdots \geq \sigma_r[T^A_{\ \alpha}] > 0 = \sigma_{r+1}[T^A_{\ \alpha}] = \cdots = \sigma_{\dim(X)}[T^A_{\ \alpha}].$$

In [33] it is proved that

$$\sigma_e[U^{-1}]\sigma_u[V]\sigma_i[T] \leq \sigma_i[U^{-1} \circ T \circ V] \leq \sigma_i[T]\sigma_1[U^{-1}]\sigma_1[V] \quad \forall i = 1, \ldots, u$$

with $\dim X = u$, $\dim E = e$ and $\sigma_e[U^{-1}]\sigma_u[V] \neq 0$ since U and V are invertible. By this expression, we see that if $\sigma_i[T] = O(\varepsilon^{l_i})$ then $\sigma_i[U^{-1} \circ T \circ V] = O(\varepsilon^{l_i})\ \forall i$. Thus, due to Eq. (B.3) $\hat{\sigma}_i[T]$ this is exactly $\sigma_i[U^{-1} \circ T \circ V]$ and we conclude the proof. □

Theorem B.2. *Let the operator be* $T^A_{\ \alpha} = K^A_{\ \alpha} + \varepsilon e^{i\theta} B^A_{\ \alpha} : X \to E$ *where* $\varepsilon e^{i\theta} B^A_{\ \alpha}$ *represents a perturbation of* K *with any* $\theta \in [0, 2\pi]$, ε *real,* $0 \leq |\varepsilon| \ll 1$ *and* K *has rank* r. *Consider* $T^A_{\ \alpha}$ *in basis in which* $G_{1AB} = \mathrm{diag}(1, \ldots, 1)$ *and* $G_{2\alpha\beta} = \mathrm{diag}(1, \ldots, 1)$ *and the SVD of* K *is*

$$K^A_{\ \beta} = (U_2, U_1, U_3)^A_{\ i} \begin{pmatrix} \Sigma_+ & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^{i}_{\ i'} \delta_2^{i'j'} (\bar{V}_2, \bar{V}_1)^\tau_{\ j'} \delta_{2\tau\beta}$$

$$= (U_2, 0, 0)^A_{\ i} \begin{pmatrix} \Sigma_+ & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^{i}_{\ i'} \delta_2^{i'j'} (\bar{V}_2, 0)^\tau_{\ j'} \delta_{2\tau\beta}.$$

Therefore

(1) *If* $\sigma_i[K^A_{\ \beta}] > 0$ $i = 1, \ldots, r$ *are the singular values of* K, *then the singular value of* $T^A_{\ \alpha}$ *can be expanded as*

$$\sigma_i[T^A_{\ \alpha}] = \sigma_i[K^A_{\ \beta}] + |\varepsilon|\xi_i + O(\varepsilon^2) \quad \text{with } i = 1, \ldots, r$$

for some ξ_i (*see* [30] *for explicit formulas*)

(2) *When* $K^A_{\ \beta}$ *has* $\sigma_i[K^A_{\ \beta}] = 0$ *for* $i = r+1, \ldots, u$ *then the corresponding* $u - r$ *singular values of* $T^A_{\ \alpha}$ *are*

$$\sigma_i\left[K^A_{\ \alpha} + \varepsilon e^{i\theta} B^A_{\ \alpha}\right]$$
$$= 0 + |\varepsilon|\ \sigma_i\left[\delta_1^{lj} (0, \bar{U}_1, \bar{U}_3)^C_{\ j} \delta_{1CD} B^D_{\ \alpha} (0, V_1)^\alpha_{\ m}\right] + O(\varepsilon^2) \qquad \text{(B.4)}$$

[q]*with* $j = 1, \ldots, e$; $l = r+1, \ldots, e$, $m = r+1, \ldots, u$ *and* $i = r+1, \ldots, u$.

Notice that Eq. (B.4) *does not depend on* θ.

[q]These singular values are not differentiable respect to ε in $\varepsilon = 0$, due to the presence of the module $|\varepsilon|$. However, it is possible differentiate respect to the module.

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