

THE STRUCTURE OF GROUP PRESERVING OPERATORS

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ABSTRACT. In this paper, we prove the existence of a particular diagonalization for normal bounded operators defined on subspaces of $L^2(\mathfrak{S})$ where \mathfrak{S} is a second countable LCA group. The subspaces where the operators act are invariant under the action of a group Γ which is a semi-direct product of a uniform lattice of \mathfrak{S} with a discrete group of automorphisms. This class includes the crystal groups which are important in applications as models for images. The operators are assumed to be Γ preserving. i.e. they commute with the action of Γ . In particular we obtain a spectral decomposition for these operators.

This generalizes recent results on shift-preserving operators acting on lattice invariant subspaces where \mathfrak{S} is the Euclidean space.

1. INTRODUCTION

Let Γ be a discrete group, not necessarily commutative, acting on $L^2(\mathfrak{S})$, where \mathfrak{S} is a second countable LCA group.

In this article, we study the structure of bounded operators acting on subspaces of $L^2(\mathfrak{S})$ that are invariant under the action of the group Γ (which we call Γ -invariant subspaces). Our operators are required to be Γ -preserving. This means that they commute with the action of Γ .

A recent paper [1], studied the case where \mathfrak{S} is the d -dimensional additive group \mathbb{R}^d and the group Γ is the lattice $\Delta = \mathbb{Z}^d$ acting by translations on $L^2(\mathbb{R}^d)$. The authors considered Δ -preserving operators acting on finitely generated Δ -invariant spaces.

They introduced the notion of Δ -eigenvalue and Δ -diagonalization (see definition in Section 3) and proved that if $L : V \rightarrow V$ is a bounded normal Δ -preserving operator defined on a finitely generated Δ -invariant space V , then L is Δ -diagonalizable, that is, there exist $r \in \mathbb{N}$ and Δ -invariant subspaces V_1, \dots, V_r such that:

$$V = V_1 \dot{\oplus} \dots \dot{\oplus} V_r,$$

where the sum is orthogonal and the subspaces are invariant under L . The action of L on each $f \in V_j$ is given by $Lf = \Lambda_{a_j} f$ with $\Lambda_{a_j} = \sum_{k \in \Delta} a_j(k) T_k$, and T_k denotes the translation by $k \in \Delta$. The operator Λ_{a_j} is called a Δ -eigenvalue, and is defined by some sequence $a_j \in \ell_2(\Delta)$, $j = 1, \dots, r$. As a consequence we have the following formula for L :

$$L = \sum_{j=1}^r \Lambda_{a_j} P_{V_j},$$

where P_{V_j} denotes the orthogonal projection onto V_j .

This type of decomposition of a Δ -preserving operator describes in a simple and compressed way the action of L and is reminiscent of the spectral theorem for normal matrices. The finiteness of the decomposition is a consequence of the fact that the invariant space on which L acts is finitely generated.

2010 *Mathematics Subject Classification.* 41A65, 47A15, 43A70, 20H15.

Key words and phrases. Invariant subspaces; Parseval frames, normal operators, diagonalization, range operators, direct integrals.

In the first part of this paper, we extend this result to the group setting (Δ is a uniform lattice of a second countable group \mathfrak{S}) and we remove the requirement that the invariant space is finitely generated. We are able to prove the Δ -diagonalization for normal bounded Δ -preserving operators, mentioned before, satisfying some additional properties.

The main difficulty in pursuing this, is a question about measurability that can not be solved using the arguments of the finitely generated case. We need to resort to the theory of set-valued maps and results on measurable selections, in particular, Castaign's Selection Theorem (see subsection 2.3).

The key tool in the analysis is the characterization of invariant spaces by Helson [20] through *measurable range functions* and the decomposition of Δ -preserving operators by *measurable range operators* (or direct integrals of operators), see [12].

In the second part of the paper, we extend this decomposition to Γ -invariant spaces. More exactly, we consider a non-commutative group Γ that is a semidirect product $\Gamma = \Delta \rtimes G$. Here, Δ is a discrete cocompact subgroup of \mathfrak{S} and G is a discrete and countable group that acts on \mathfrak{S} by continuous automorphisms preserving Δ . (See Section 4 for details). These groups are important in applications as models for images since they include, as a particular case, rigid movements. See [7] and [22] for applications to image processing.

The structure of Γ -invariant spaces has been studied in great detail in a recent paper [8]. In this article, we consider Γ -preserving operators in this general setting. In order to obtain a diagonalization for these operators we need to define what we mean by Γ -eigenvalues, defined previously for the case of uniform lattices.

Because Γ -invariant subspaces are a particular subclass of Δ -invariant subspaces with extra restrictions, Γ -eigenvalues should be Δ -eigenvalues with some special property, due to the action of the group G (see Proposition 4.13). Finally, using this, we are able to obtain the desired diagonalization that we call Γ -diagonalization,

The paper is organized as follows: Section 2 sets the notation that we will use throughout the paper and contains all the definitions and properties needed for the diagonalization results. We start in Subsection 2.1 describing the structure of Δ -invariant subspaces and its characterizations through measurable range functions. We consider Δ -preserving operators and its associated range operators in Subsection 2.2. In Subsection 2.3, the definition and basic properties of measurable set-valued maps and a result on measurable selections, are described. Then, we show results on the relationships between the spectrum of a Δ -preserving operator and the spectrum of its range operator in Subsection 2.4. Sections 3 and 4 contain the main results of this paper. In Section 3 we prove the Δ -diagonalization in the setting of groups for Δ -invariant spaces not necessarily finitely generated and finally in Section 4 we treat the Γ -diagonalization case.

2. PRELIMINARIES

Throughout this work, \mathfrak{S} will be a second countable LCA group and Δ will be a uniform lattice of \mathfrak{S} (that is, a discrete subgroup such that \mathfrak{S}/Δ is compact). We will denote by $\widehat{\mathfrak{S}}$ the Pontryagin dual of \mathfrak{S} and we will write the characters of \mathfrak{S} indistinctly by

$$\langle \xi, x \rangle = e^{2\pi i \xi \cdot x}, \quad \xi \in \widehat{\mathfrak{S}}, x \in \mathfrak{S}.$$

Moreover, the annihilator of Δ will be denoted by

$$\Delta^\perp = \{\ell \in \widehat{\mathfrak{S}} : \langle \ell, k \rangle = 1 \forall k \in \Delta\}.$$

The Haar measure on \mathfrak{S} of a measurable set $E \subset \mathfrak{S}$ will be denoted by $|E|$. Furthermore, $\Omega \subset \widehat{\mathfrak{S}}$ will always denote a Borel section of $\widehat{\mathfrak{S}}/\Delta^\perp$.

We will use the following notation for the Fourier transform of $f \in L^1(\mathfrak{G})$:

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathfrak{G}} e^{-2\pi i \xi \cdot x} f(x) dx = \int_{\mathfrak{G}} \overline{\langle \xi, x \rangle} f(x) dx,$$

which extends by density to an isometric isomorphism in $L^2(\mathfrak{G})$.

We denote by $T : \Delta \rightarrow \mathcal{U}(L^2(\mathfrak{G}))$ the unitary representation defined by

$$T_k f(x) = f(x - k), \quad f \in L^2(\mathfrak{G}), k \in \Delta.$$

Note that for all $f \in L^2(\mathfrak{G})$ and all $k \in \Delta$, the following relation holds:

$$\widehat{T_k f}(\xi) = e^{-2\pi i \xi \cdot k} \widehat{f}(\xi), \quad \xi \in \widehat{\mathfrak{G}}. \quad (2.1)$$

If \mathcal{H} is a Hilbert space, we denote by $\mathcal{B}(\mathcal{H})$ the linear and bounded operators from \mathcal{H} into \mathcal{H} . Given an operator $A \in \mathcal{B}(\mathcal{H})$ we denote by $\sigma(A)$ its operator spectrum. The point spectrum of A , that is the set of its eigenvalues, is denoted by $\sigma_p(A)$.

A normal operator $A \in \mathcal{B}(\mathcal{H})$ is called diagonalizable if \mathcal{H} admits an orthonormal basis consisting of eigenvectors of A . We will use the symbol \oplus to denote an orthogonal sum.

2.1. Δ -invariant spaces. We begin by introducing some important notions on Δ -invariant spaces, also known as shift-invariant spaces by translations of Δ .

Definition 2.1. *A closed subspace $V \subset L^2(\mathfrak{G})$ is Δ -invariant if $T_k V \subset V$ for all $k \in \Delta$.*

Given a countable set of functions $\Phi \subset L^2(\mathfrak{G})$, we will denote

$$S(\Phi) := \overline{\text{span}}\{T_k \varphi : k \in \Delta, \varphi \in \Phi\}.$$

Since $L^2(\mathfrak{G})$ is separable, if V is a Δ -invariant subspace of $L^2(\mathfrak{G})$, there exists a countable set $\Phi \subset L^2(\mathfrak{G})$ such that $V = S(\Phi)$. In this case, we say that Φ is a set of generators of V . Moreover, if V admits a finite set of generators, we say that V is finitely generated, and if $V = S(\varphi)$ for $\varphi \in L^2(\mathfrak{G})$ we say that V is principal.

Definition 2.2. *For $f \in L^2(\mathfrak{G})$ and $\omega \in \widehat{\mathfrak{G}}$ define formally the fiberization map \mathcal{T} as*

$$\mathcal{T}[f](\omega) = \{\widehat{f}(\omega + \ell)\}_{\ell \in \Delta^\perp}. \quad (2.2)$$

The fiberization map \mathcal{T} is an isometric isomorphism between the Hilbert spaces $L^2(\mathfrak{G})$ and $L^2(\Omega, \ell_2(\Delta^\perp))$, see [15, Proposition 3.3]. Observe that by (2.1), for every $f \in L^2(\mathfrak{G})$ and $k \in \Delta$, we have the following intertwining property:

$$\mathcal{T}[T_k f](\omega) = e^{-2\pi i \omega \cdot k} \mathcal{T}[f](\omega), \quad \omega \in \Omega. \quad (2.3)$$

The following map, first introduced by Helson [20], is fundamental in the theory of shift-invariant spaces.

Definition 2.3. *A range function is a map*

$$\mathcal{J} : \Omega \rightarrow \{\text{closed subspaces of } \ell_2(\Delta^\perp)\}.$$

We say that a range function \mathcal{J} is *measurable* if $\omega \mapsto \langle \mathcal{P}_{\mathcal{J}(\omega)} a, b \rangle_{\ell_2(\Delta^\perp)}$ is measurable for all $a, b \in \ell_2(\Delta^\perp)$, where $\mathcal{P}_{\mathcal{J}(\omega)} \in \mathcal{B}(\ell_2(\Delta^\perp))$ is the orthogonal projection onto $\mathcal{J}(\omega)$.

The next theorem is due to Helson [20] and Bownik [11] in the Euclidean setting. We state its generalization to the setting of LCA groups as it appears in [15].

Theorem 2.4 ([11, 15]). *Let V be a closed subspace of $L^2(\mathfrak{G})$ and \mathcal{T} the map of Definition 2.2. The subspace V is Δ -invariant if and only if there exists a unique measurable range function \mathcal{J}_V such that*

$$V = \{f \in L^2(\mathfrak{G}) : \mathcal{T}[f](\omega) \in \mathcal{J}_V(\omega), \text{ a.e. } \omega \in \Omega\}.$$

Moreover, if $V = S(\Phi)$ for some countable set Φ of $L^2(\mathfrak{S})$, the measurable range function associated to $S(\Phi)$ satisfies

$$\mathcal{J}_V(\omega) = \overline{\text{span}}\{\mathcal{T}[\varphi](\omega) : \varphi \in \Phi\}, \quad \text{a.e. } \omega \in \Omega.$$

Remark 2.5. Uniqueness of the measurable range function is understood in the following sense: two range functions \mathcal{J}_1 and \mathcal{J}_2 are equal if $\mathcal{J}_1(\omega) = \mathcal{J}_2(\omega)$ a.e. $\omega \in \Omega$.

From now on, given a Δ -invariant space V , we will simply write its associated range function as \mathcal{J} when it is clear from the context that we are referring to \mathcal{J}_V .

Given a range function \mathcal{J} , the space

$$\mathcal{M}_{\mathcal{J}} = \{F \in L^2(\Omega, \ell_2(\Delta^\perp)) : F(\omega) \in \mathcal{J}(\omega), \text{ for a.e. } \omega \in \Omega\}, \quad (2.4)$$

is a closed *multiplicative-invariant* subspace of $L^2(\Omega, \ell_2(\Delta^\perp))$, i.e. for every $F \in \mathcal{M}_{\mathcal{J}}$ we have that $\psi F \in \mathcal{M}_{\mathcal{J}}$ for all $\psi \in L^\infty(\Omega)$. By Theorem 2.4, if V is a Δ -invariant space with range function \mathcal{J} , we have that $\mathcal{T}[V] = \mathcal{M}_{\mathcal{J}}$.

The following identity is due to Helson [20],

$$(P_{\mathcal{M}_{\mathcal{J}}}F)(\omega) = P_{\mathcal{J}(\omega)}(F(\omega)), \quad \forall F \in L^2(\Omega, \ell_2(\Delta^\perp)), \text{ a.e. } \omega \in \Omega, \quad (2.5)$$

and as a consequence, the next proposition holds.

Proposition 2.6. Let $V \subset L^2(\mathfrak{S})$ be a Δ -invariant space with range function \mathcal{J}_V . Then V^\perp is also a Δ -invariant space with range function

$$\mathcal{J}_{V^\perp}(\omega) = (\mathcal{J}_V(\omega))^\perp, \quad \text{a.e. } \omega \in \Omega.$$

The result below gives a characterization of frames of translations of a Δ -invariant space V in terms of its fibers, see [11, 15].

Theorem 2.7 ([11, 15]). Let $\Phi \subset L^2(\mathfrak{S})$ be a countable set. Then the following conditions are equivalent:

- (1) The system $\{T_k\varphi : k \in \Delta, \varphi \in \Phi\}$ is a frame of V with bounds $A, B > 0$.
- (2) The system $\{\mathcal{T}[\varphi](\omega) : \varphi \in \Phi\} \subset \ell_2(\Delta^\perp)$ is a frame of $\mathcal{J}(\omega)$ with uniform bounds $A, B > 0$ for a.e. $\omega \in \Omega$.

Definition 2.8. The spectrum of a Δ -invariant space V with range function \mathcal{J} is defined by

$$\Sigma(V) = \{\omega \in \Omega : \dim \mathcal{J}(\omega) > 0\}.$$

The result we state next gives a decomposition of a Δ -invariant space into an orthogonal sum of principal Δ -invariant spaces satisfying some additional properties. In [11] the theorem was proved in the Euclidean case but the proof can be extended in a straightforward manner to our setting.

Theorem 2.9 ([11, Theorem 3.3]). Let V be a Δ -invariant space of $L^2(\mathfrak{S})$. Then V can be decomposed as an orthogonal sum

$$V = \bigoplus_{i \in \mathbb{N}} S(\varphi_i), \quad (2.6)$$

where φ_i is a Parseval frame generator of $S(\varphi_i)$, and $\Sigma(S(\varphi_{i+1})) \subset \Sigma(S(\varphi_i))$ for all $i \in \mathbb{N}$.

As a consequence, one obtains the following lemma which for the case of $\dim \mathcal{J}(\omega) < \infty$ for a.e. $\omega \in \Omega$ has been proved in [1], and extends with little effort to the general case.

Lemma 2.10. Let V be a Δ -invariant space with range function \mathcal{J} . Then, there exist functions $\{\varphi_i\}_{i \in \mathbb{N}}$ of $L^2(\mathfrak{S})$ and a family of disjoint measurable sets $\{A_n\}_{n \in \mathbb{N}_0}$ and A_∞ , such that $\Omega = (\bigcup_{n \in \mathbb{N}_0} A_n) \cup A_\infty$ and the following statements hold:

- (1) $\{T_k\varphi_i : i \in \mathbb{N}, k \in \Delta\}$ is a Parseval frame of V ,
- (2) for every $n \in \mathbb{N}$ and for every $i > n$, $\mathcal{T}\varphi_i(\omega) = 0$ a.e. $\omega \in A_n$,
- (3) for every $n \in \mathbb{N}$, $\{\mathcal{T}\varphi_1(\omega), \dots, \mathcal{T}\varphi_n(\omega)\}$ is an orthonormal basis of $\mathcal{J}(\omega)$ for a.e. $\omega \in A_n$, and $\{\mathcal{T}\varphi_i(\omega)\}_{i \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{J}(\omega)$ for a.e. $\omega \in A_\infty$,
- (4) for every $n \in \mathbb{N}_0$, $\dim \mathcal{J}(\omega) = n$ for a.e. $\omega \in A_n$ and $\dim \mathcal{J}(\omega) = \infty$ for a.e. $\omega \in A_\infty$.

2.2. Δ -preserving operators. The natural operators acting on Δ -invariant spaces are the Δ -preserving operators. These are the ones that commute with translations by elements of Δ , also known in the literature as shift-preserving operators.

Definition 2.11. Let $V, V' \subset L^2(\mathfrak{G})$ be two Δ -invariant spaces and let $L : V \rightarrow V'$ be a bounded operator. We say that L is Δ -preserving if $LT_k = T_kL$ for all $k \in \Delta$.

The structure of Δ -preserving operators can be understood through the concept of its range operator, which was first introduced in the Euclidean context in [11].

Definition 2.12. Given measurable range functions

$$\mathcal{J}, \mathcal{J}' : \Omega \rightarrow \{\text{closed subspaces of } \ell_2(\Delta^\perp)\},$$

a range operator $O : \mathcal{J} \rightarrow \mathcal{J}'$ is a choice of linear operators $O(\omega) : \mathcal{J}(\omega) \rightarrow \mathcal{J}'(\omega)$, $\omega \in \Omega$.

A range operator O is said to be bounded if $\text{ess sup}_{\omega \in \Omega} \|O(\omega)\|_{op} < \infty$, and is measurable if $\omega \mapsto \langle O(\omega)P_{\mathcal{J}(\omega)}a, b \rangle_{\ell_2(\Delta^\perp)}$ is measurable for all $a, b \in \ell_2(\Delta^\perp)$.

There exists a correspondence between bounded Δ -preserving operators and bounded measurable range operators. In what follows we describe how this correspondence can be deduced.

Definition 2.13. For $\psi \in L^\infty(\Omega)$, denote as $M_\psi : L^2(\Omega, \ell_2(\Delta^\perp)) \rightarrow L^2(\Omega, \ell_2(\Delta^\perp))$ the multiplication operator

$$M_\psi F(\omega) = \psi(\omega)F(\omega), \quad F \in L^2(\Omega, \ell_2(\Delta^\perp)), \omega \in \Omega,$$

which is well defined and bounded.

Let $D = \{e^{-2\pi i\omega \cdot k}\}_{k \in \Delta} \subseteq L^\infty(\Omega)$, then D is a determining set for $L^1(\Omega)$ (see [12, Definition 3.3]). If $L : V \rightarrow V'$ is a bounded Δ -preserving operator, then the operator

$$\tilde{L} = \mathcal{T}L\mathcal{T}^{-1} : \mathcal{M}_{\mathcal{J}_V} \rightarrow \mathcal{M}_{\mathcal{J}_{V'}} \quad (2.7)$$

is bounded and, by (2.3), satisfies that $\tilde{L}M_\psi = M_\psi\tilde{L}$ for every $\psi \in D$. By [12, Theorem 3.7], there exists a bounded measurable range operator $O : \mathcal{J}_V \rightarrow \mathcal{J}_{V'}$ such that $\tilde{L}F(\omega) = O(\omega)F(\omega)$, for every $F \in \mathcal{M}_{\mathcal{J}_V}$, $\omega \in \Omega$. That is,

$$\mathcal{T}[Lf](\omega) = O(\omega)\mathcal{T}[f](\omega), \quad f \in V, \omega \in \Omega. \quad (2.8)$$

Moreover, the correspondence between L and O is one-to-one if we identify range operators that agree a.e. $\omega \in \Omega$.

A different way to understand range operators is through the direct integral theory. In fact, it can be proved (see [12]) that

$$\mathcal{M}_{\mathcal{J}_V} = \int_{\Omega}^{\oplus} \mathcal{J}_V(\omega) d\omega$$

and that the operator \tilde{L} defined in (2.7) is a decomposable operator such that

$$\tilde{L} = \int_{\Omega}^{\oplus} O(\omega) d\omega.$$

In the following theorem we enumerate some results that relate the properties of L with the pointwise properties of its range operator O (see [12] for proofs).

Theorem 2.14 ([12]). *Let $V, V' \subset L^2(\mathfrak{S})$ be two Δ -invariant spaces. Let $L : V \rightarrow V'$ be a Δ -preserving operator with corresponding range operator $O : \mathcal{J}_V \rightarrow \mathcal{J}_{V'}$. Then the following are true:*

- (1) $\|L\|_{op} = \text{ess sup}_{\omega \in \Omega} \|O(\omega)\|_{op}$.
- (2) *The adjoint $L^* : V' \rightarrow V$ is also Δ -preserving with corresponding range operator $O^* : \mathcal{J}_{V'} \rightarrow \mathcal{J}_V$ given by $O^*(\omega) = (O(\omega))^*$ for a.e. $\omega \in \Omega$.*
- (3) *L is normal (self-adjoint) if and only if $O(\omega)$ is normal (self-adjoint) for a.e. $\omega \in \Omega$.*
- (4) *L is injective if and only if $O(\omega)$ is injective for a.e. $\omega \in \Omega$.*
- (5) *L is a (partial) isometry if and only if $O(\omega)$ is a (partial) isometry for a.e. $\omega \in \Omega$.*
- (6) *The space $V'' = \overline{L(V)} \subseteq L^2(\mathfrak{S})$ is Δ -invariant and its range function is given by*

$$\mathcal{J}_{V''}(\omega) = \overline{O(\omega)\mathcal{J}_V(\omega)},$$

for a.e. $\omega \in \Omega$.

- (7) *The space $\ker(L)$ is Δ -invariant and its range function is given by $K(\omega) = \ker(O(\omega))$ for a.e. $\omega \in \Omega$.*

2.3. Measurable set-valued maps. Given $L : V \rightarrow V$ a bounded Δ -preserving operator, there is a relation between the spectrum of L and the pointwise spectrum of its range operator, as we will discuss in the next subsection. For that, we need to introduce the definition of measurable set-valued maps. We refer the reader to [6] for a detailed exposition on the set-valued maps' theory.

Definition 2.15. *Let (X, \mathcal{M}) be a measurable space and Y a topological space. A set-valued map from X to Y is a map $F : X \rightsquigarrow Y$ whose values are sets in Y . That is, $F(x) \subseteq Y$ for every $x \in X$. If $F(x)$ is closed (compact) for every $x \in X$, then F is said to be a set-valued map to closed (compact) values.*

A set-valued map is said to be measurable if for every open set $O \subset Y$, the set

$$F^{-1}(O) := \{x \in X : F(x) \cap O \neq \emptyset\} \in \mathcal{M}.$$

For example, in [12] it was proved that a measurable range function \mathcal{J} is a measurable set-valued map $\Omega \rightsquigarrow \ell_2(\Delta^\perp)$ to non-empty closed values.

One very important result that we will need later is the existence of a dense set of measurable selections for measurable set-valued maps, which is known as Castaign's Selection Theorem (see [6] for a proof).

Definition 2.16. *Let (X, \mathcal{M}) be a measurable space and Y a topological space. Given $F : X \rightsquigarrow Y$ a measurable set-valued map, we say that a measurable function $f : X \rightarrow Y$ is a measurable selection of F if $f(x) \in F(x)$ for every $x \in X$.*

Theorem 2.17 (Castaign's Selection Theorem). *Let (X, \mathcal{M}) be a measurable space, Y a complete separable metric space and $F : X \rightsquigarrow Y$ a measurable set-valued map to non-empty closed values, then there exists a sequence of measurable selections $f_j : X \rightarrow Y$, $j \in \mathbb{N}$ such that for every $x \in X$.*

$$F(x) = \overline{\{f_j(x) : j \in \mathbb{N}\}}. \quad (2.9)$$

2.4. The spectrum of Δ -preserving operators. We start this subsection with Theorem 2.18 and Theorem 2.19 whose proofs appeared in [12] and also in [16, 24] in the context of decomposable operators on direct integral Hilbert spaces.

The first theorem establishes that the spectra of the fibers of a Δ -preserving operator L define a measurable set-valued map and describe the relationship between those spectra and the spectrum of L .

Theorem 2.18 ([12]). *Let $L : V \rightarrow V$ be a Δ -preserving operator with range operator $O : \mathcal{J} \rightarrow \mathcal{J}$. Then $F : \Omega \rightsquigarrow \mathbb{C}$ defined by $F(\omega) = \sigma(O(\omega))$, $\omega \in \Omega$ is a measurable set-valued map to non-empty compact values and $F(\omega) \subseteq \sigma(L)$ for a.e. $\omega \in \Omega$.*

Moreover, when L is normal, $\sigma(L)$ coincides with the smallest closed subset of \mathbb{C} that contains $F(\omega)$ for a.e. $\omega \in \Omega$.

Suppose now that $L : V \rightarrow V$ is a normal, bounded and Δ -preserving operator. Then, there exists a spectral measure E of L and we have that

$$L = \int_{\sigma(L)} \lambda dE(\lambda).$$

Since the range operator O satisfies that $O(\omega)$ is normal for a.e. $\omega \in \Omega$, then there exists a spectral measure E_ω of $O(\omega)$ for a.e. $\omega \in \Omega$ and

$$O(\omega) = \int_{\sigma(O(\omega))} \lambda dE_\omega(\lambda).$$

In this direction, the following result was obtained.

Theorem 2.19 ([12]). *Let $L : V \rightarrow V$ a normal, bounded and Δ -preserving operator with range operator O . Let E be the spectral measure of L and E_ω the spectral measure of $O(\omega)$ for a.e. $\omega \in \Omega$. Then, for any Borel set $B \subset \mathbb{C}$, $E(B)$ is a Δ -preserving operator and its range operator is given by $E_\omega(B)$.*

In the next section, we will discuss to what extent the diagonalization properties of the range operator can provide the Δ -preserving operator with a special structure. For this purpose, we are interested in finding measurable selections of the eigenvalues of the range operator.

Assume first that L is acting on a Δ -invariant space V such that $\dim \mathcal{J}(\omega) < \infty$ for a.e. $\omega \in \Omega$. Then, we have that $O(\omega) : \mathcal{J}(\omega) \rightarrow \mathcal{J}(\omega)$ is an operator acting on a finite-dimensional space for a.e. $\omega \in \Omega$. In [1], a construction of a measurable selection of the eigenvalues of O was obtained in the following sense.

Theorem 2.20 ([1]). *Let $O : \mathcal{J} \rightarrow \mathcal{J}$ be a bounded measurable range operator on a range function satisfying $\dim \mathcal{J}(\omega) < \infty$ for a.e. $\omega \in \Omega$. Then, there exist functions $\lambda_j \in L^\infty(\Omega)$, $j \in \mathbb{N}$, such that*

- (1) $\lambda_j(\omega) \neq \lambda_{j'}(\omega)$ for $j \neq j'$ and for a.e. $\omega \in \Omega$,
- (2) $\sigma(O(\omega)) = \{\lambda_1(\omega), \dots, \lambda_i(\omega)\}$ for a.e. $\omega \in A_{n,i}$ and for every $i \leq n$, $i, n \in \mathbb{N}$,

where $A_{n,i}$ are the measurable sets:

$$A_{n,i} := \{\omega \in A_n : \#\sigma(O(\omega)) = i\}, \quad (2.10)$$

and $\{A_n\}_{n \in \mathbb{N}}$ are the sets defined in Lemma 2.10.

Remark 2.21. *If $r = \text{ess sup}_{\omega \in \Omega} \#\sigma(O(\omega)) < \infty$, then $|A_{n,i}| = 0$ for every $i > r$. Thus if we discard the functions λ_j such that $\lambda_j(\omega)$ is not an eigenvalue of $O(\omega)$ for a.e. $\omega \in \Omega$, the number of measurable functions constructed, after discarding, will be r in total.*

We remark that $\dim \mathcal{J}(\omega) < \infty$ for a.e. $\omega \in \Omega$ does not imply that V is finitely generated, as we show in the example below.

Example 2.22. *Let $\mathfrak{S} = \mathbb{R}$, $\Delta = \mathbb{Z}$ and $\Omega = [0, 1)$. For every $n \in \mathbb{N}$, define the set $E_n := [0, \frac{1}{n})$ and $\psi_n \in L^2(\mathbb{R})$ by the map \mathcal{T} as $\mathcal{T}[\psi_n] := e_n \chi_{E_n}$, where e_n is the n th canonical sequence of $\ell_2(\mathbb{Z})$. Then $V := S(\{\psi_n : n \in \mathbb{N}\})$ is a \mathbb{Z} -invariant space which is not finitely generated and its range function \mathcal{J} satisfy that $\dim \mathcal{J}(\omega) < \infty$ for a.e. $\omega \in [0, 1)$. Indeed, let $A_n := E_n \setminus E_{n+1}$ for $n \in \mathbb{N}$, then*

$[0, 1) = \bigcup_{n \in \mathbb{N}} A_n \cup \{0\}$. For every $n \in \mathbb{N}$ and for a.e. $\omega \in A_n$, the dimension of $\mathcal{J}(\omega)$ is n since $\{\mathcal{T}[\psi_1](\omega), \dots, \mathcal{T}[\psi_n](\omega)\}$ is an orthonormal basis of $\mathcal{J}(\omega)$.

To remove the condition $\dim \mathcal{J}(\omega) < \infty$ for a.e. $\omega \in \Omega$, Theorem 2.20 is no longer useful since its proof strongly relies on the fact that the dimension of $\mathcal{J}(\omega)$ is finite for a.e. $\omega \in \Omega$. In the following section we will see that, under certain conditions, Castaign's Selection Theorem (Theorem 2.17) will be helpful in this endeavor.

3. Δ -DIAGONALIZATION

We are interested in studying the structure of bounded, normal and Δ -preserving operators whose fibers are diagonalizable operators almost everywhere. The question that arises is the following. Suppose that $L : V \rightarrow V$ is a bounded, normal and Δ -preserving operator with range operator $O : \mathcal{J} \rightarrow \mathcal{J}$. If $O(\omega)$ is diagonalizable for a.e. $\omega \in \Omega$, does this induce any simpler kind of decomposition for L ?

This question has been studied in [1] in the Euclidean setting with $\Delta = \mathbb{Z}^d$, where a positive answer was obtained for the case of normal Δ -preserving operators acting on finitely generated Δ -invariant spaces. For this, the authors introduced three new concepts which they called s -eigenvalue, s -eigenspace and s -diagonalization.

3.1. Background. In this subsection we will review part of the work done in [1]. Along the way, we will translate the statements to the general group setting and we will show the results that can be effortlessly extended to Δ -invariant spaces that are not finitely generated.

Given a sequence $a = \{a(s)\}_{s \in \Delta} \in \ell_1(\Delta)$, we denote its Fourier transform as

$$\widehat{a}(\omega) = \sum_{s \in \Delta} a(s) e^{-2\pi i \omega \cdot s}, \quad \omega \in \Omega.$$

This extends to $\ell_2(\Delta)$ and it holds that $a \in \ell_2(\Delta)$ if and only if $\widehat{a} \in L^2(\Omega)$. Moreover, if $\widehat{a} \in L^\infty(\Omega)$, we will say that a is of bounded spectrum.

Definition 3.1. Given $a \in \ell_2(\Delta)$ of bounded spectrum, let $M_{\widehat{a}} : L^2(\Omega, \ell_2(\Delta^\perp)) \rightarrow L^2(\Omega, \ell_2(\Delta^\perp))$ be the multiplication operator by \widehat{a} , as in Definition 2.13. We denote by $\Lambda_a : L^2(\mathfrak{S}) \rightarrow L^2(\mathfrak{S})$ the operator defined by

$$\Lambda_a := \mathcal{T}^{-1} M_{\widehat{a}} \mathcal{T},$$

which is clearly well-defined and bounded.

Let us denote by \mathfrak{B} the following set:

$$\mathfrak{B} := \{\varphi \in L^2(\mathfrak{S}) : \{T_k \varphi\}_{k \in \Delta} \text{ is a Bessel sequence}\}.$$

Recall that if $a \in \ell_2(\Delta)$ and $f \in \mathfrak{B}$, then the series

$$\sum_{s \in \Delta} a(s) T_s f \tag{3.1}$$

converges in $L^2(\mathfrak{S})$.

Proposition 3.2. Let $a = \{a(s)\}_{s \in \Delta} \in \ell_2(\Delta)$ be of bounded spectrum. If $f \in \mathfrak{B}$, then

$$\Lambda_a f = \sum_{s \in \Delta} a(s) T_s f,$$

with convergence in $L^2(\mathfrak{S})$.

Proof. Given that (3.1) is convergent in $L^2(\mathfrak{S})$, for a.e. $\omega \in \Omega$,

$$\begin{aligned} \mathcal{T} \left(\sum_{s \in \Delta} a(s) T_s f \right) (\omega) &= \left\{ \mathcal{F} \left(\sum_{s \in \Delta} a(s) T_s f \right) (\omega + \ell) \right\}_{\ell \in \Delta^\perp} \\ &= \left\{ \sum_{s \in \Delta} a(s) e^{-2\pi i \omega \cdot s} \widehat{f}(\omega + \ell) \right\}_{\ell \in \Delta^\perp} \\ &= \widehat{a}(\omega) \{ \widehat{f}(\omega + \ell) \}_{\ell \in \Delta^\perp} \\ &= M_{\widehat{a}} \mathcal{T} f(\omega). \end{aligned}$$

Thus, $\Lambda_a f = \sum_{s \in \Delta} a(s) T_s f$. \square

On the other hand, \mathfrak{B} is a dense set of $L^2(\mathfrak{S})$ since the functions of compact support in $L^2(\mathfrak{S})$ belong to \mathfrak{B} (see [17, Proposition 9.3.4], where a proof is given in the Euclidean case and can be immediately extended to our group context). Then, if $a \in \ell_2(\Delta)$ is of bounded spectrum, it is possible to give an alternative definition for Λ_a as the continuous extension of the bounded operator $\tilde{\Lambda}_a : \mathfrak{B} \rightarrow L^2(\mathfrak{S})$, defined by

$$\tilde{\Lambda}_a f := \sum_{s \in \Delta} a(s) T_s f. \quad (3.2)$$

For this reason, sometimes we will write $\Lambda_a f$ as the series (3.2), even for functions which are not in \mathfrak{B} , meaning the extension of $\tilde{\Lambda}_a$ to $L^2(\mathfrak{S})$.

Notice that in the particular case when $a \in \ell_1(\Delta)$ (and thus of bounded spectrum), an easy computation shows that $\Lambda_a = \sum_{s \in \Delta} a(s) T_s$ where the convergence is in the strong operator topology of $\mathcal{B}(L^2(\mathfrak{S}))$.

Observe that if V is Δ -invariant, then $\Lambda_a(V) \subseteq V$ whenever it is bounded, and in this case $\Lambda_a : V \rightarrow V$ is a Δ -preserving operator with corresponding range operator $O_a(\omega) = \widehat{a}(\omega) \mathcal{I}_\omega$, a.e. $\omega \in \Omega$, where \mathcal{I}_ω denotes the identity operator on $\mathcal{J}(\omega)$ for a.e. $\omega \in \Omega$.

The following corresponds to the definition of s -eigenvalue and s -eigenvector in [1].

Definition 3.3. *Let $V \subset L^2(\mathfrak{S})$ be a Δ -invariant space and $L : V \rightarrow V$ a bounded Δ -preserving operator. Given $a \in \ell_2(\Delta)$ a sequence of bounded spectrum, we say that Λ_a is a Δ -eigenvalue of L if*

$$V_a := \ker(L - \Lambda_a) \neq \{0\}.$$

We call V_a the Δ -eigenspace associated to Λ_a .

These Δ -eigenspaces V_a are Δ -invariant spaces and satisfy that $LV_a \subseteq V_a$. The proposition below was proved in [1] showing that the Δ -eigenvalues of L are intrinsically related to the eigenvalues of the range operator of L .

Proposition 3.4. *Let $V \subset L^2(\mathfrak{S})$ be a Δ -invariant space with range function \mathcal{J} , $L : V \rightarrow V$ a bounded Δ -preserving operator with range operator O and $a \in \ell_2(\Delta)$ a sequence of bounded spectrum. Then, the following statements hold:*

- (1) *If Λ_a is a Δ -eigenvalue of L , then $\widehat{a}(\omega)$ is an eigenvalue of $O(\omega)$ for a.e. $\omega \in \Sigma(V_a)$.*
- (2) *The mapping $\omega \mapsto \ker(O(\omega) - \widehat{a}(\omega) \mathcal{I}_\omega)$, $\omega \in \Omega$ is the measurable range function of V_a , which we will denote by \mathcal{J}_a .*

Remark 3.5. *In fact, the converse for statement (1) in Proposition 3.4 is true in the following sense: if $\widehat{a}(\omega)$ is an eigenvalue of $O(\omega)$ for a.e. ω in a set of positive measure, then Λ_a is a Δ -eigenvalue of L .*

The following is an extension of the definition of s -diagonalization given in [1], which was originally stated for finitely generated Δ -invariant spaces. Here, we extend the definition to any Δ -invariant space.

Definition 3.6. *Let $V \subset L^2(\mathfrak{S})$ be a Δ -invariant space and $L : V \rightarrow V$ a bounded, Δ -preserving operator. We say that L is Δ -diagonalizable if there exists a set of sequences of bounded spectrum $\{a_j\}_{j \in I} \subseteq \ell_2(\Delta)$, where I is at most countable, such that Λ_{a_j} is a Δ -eigenvalue of L for every $j \in I$ and V can be decomposed into the orthogonal sum*

$$V = \bigoplus_{j \in I} V_{a_j}. \quad (3.3)$$

Given such a decomposition, we will say that $\{a_j\}_{j \in I} \subseteq \ell_2(\Delta)$ is a Δ -diagonalization of L .

If an operator L is Δ -diagonalizable, a decomposition as in (3.3) exists but is not unique. Observe that if $\{a_j\}_{j \in I}$ is a Δ -diagonalization of L , then

$$L = \sum_{j \in I} \Lambda_{a_j} P_{V_{a_j}}, \quad (3.4)$$

where $P_{V_{a_j}}$ is the orthogonal projection of V onto V_{a_j} and, if I is an infinite set, the convergence is in the strong operator topology sense.

The next theorem enumerates some results regarding Δ -diagonalizable operators. Statements (1) and (3) are extended versions of [1, Theorem 6.4] and [1, Theorem 6.16] respectively.

Theorem 3.7. *Let V be a Δ -invariant space with range function \mathcal{J} and $L : V \rightarrow V$ a bounded Δ -preserving operator with range operator O . Then, the following statements hold:*

- (1) *If L is Δ -diagonalizable, $O(\omega)$ is diagonalizable for a.e. $\omega \in \Omega$. Moreover, if $\{a_j\}_{j \in I}$ is a Δ -diagonalization of L , then $\sigma_p(O(\omega)) \subset \{\widehat{a}_j(\omega) : j \in I\}$ for a.e. $\omega \in \Omega$.*
- (2) *If L is Δ -diagonalizable, L is normal.*
- (3) *If $\dim \mathcal{J}(\omega) < \infty$ for a.e. $\omega \in \Omega$ and L is normal, then L is Δ -diagonalizable.*

Proof. The statement in (1) follows straightforwardly from the Definition 3.6 and Proposition 3.4.

In order to see (2), observe that if L is Δ -diagonalizable, then $O(\omega)$ is normal for a.e. $\omega \in \Omega$ since it is diagonalizable for a.e. $\omega \in \Omega$. Thus, by Theorem 2.14, L is normal.

Finally, we prove (3). If L is normal then $O(\omega)$ is normal for a.e. $\omega \in \Omega$ due to Theorem 2.14. Thus, we have that $O(\omega)$ is a normal operator acting on a finite-dimensional space $\mathcal{J}(\omega)$, and hence diagonalizable for a.e. $\omega \in \Omega$.

By Theorem 2.20, there exist functions $\lambda_j \in L^\infty(\Omega)$, $j \in \mathbb{N}$, such that for a.e. $\omega \in \Omega$ we have the following orthogonal decomposition

$$\mathcal{J}(\omega) = \bigoplus_{j \in \mathbb{N}} \ker(O(\omega) - \lambda_j(\omega)\mathcal{I}_\omega). \quad (3.5)$$

We discard the functions such that $\ker(O(\omega) - \lambda_j(\omega)\mathcal{I}_\omega) = \{0\}$ for a.e. $\omega \in \Omega$. Since for every j , $\lambda_j \in L^\infty(\Omega)$, there exists a sequence of bounded spectrum $a_j \in \ell_2(\Delta)$ such that $\widehat{a}_j = \lambda_j$. Then, Λ_{a_j} is a Δ -eigenvalue of L for every j and by (3.5) we have the orthogonal decomposition

$$V = \bigoplus_j V_{a_j}.$$

□

Notice that because of Remark 2.21, if V is finitely generated and L is Δ -diagonalizable, then there exists a Δ -diagonalization where the sum in (3.4) is finite.

In general it is uncertain if the diagonalization of $O(\omega)$ for a.e. $\omega \in \Omega$ implies that L is Δ -diagonalizable. This is due to the fact that it depends on the possibility to obtain a measurable selection of the eigenvalues of the range operator.

3.2. Δ -diagonalization on general Δ -invariant spaces. In this subsection we establish conditions on the range operator of a Δ -preserving operator acting on a general Δ -invariant space in order to admit a Δ -diagonalization. For this, we will make use of the following lemma.

Lemma 3.8. *Let (X, \mathcal{M}) be a measurable space, and let $F : X \rightsquigarrow \mathbb{C}$ be a measurable set-valued map to non-empty closed values such that $F(x) \subseteq K$ for every $x \in X$ and $K \subset \mathbb{C}$ a compact set. Then, there exists a sequence of measurable bounded functions $g_j : X \rightarrow \mathbb{C}$, $j \in \mathbb{N}$ such that for every $j \neq j'$, $g_j(x) \neq g_{j'}(x)$ for every $x \in X$ and*

$$F(x) \subset \overline{\{g_j(x) : j \in \mathbb{N}\}}, \quad x \in X. \quad (3.6)$$

Proof. By Theorem 2.17 we have a sequence of measurable functions $f_j : X \rightarrow Y$, $j \in \mathbb{N}$ such that $F(x) = \overline{\{f_j(x) : j \in \mathbb{N}\}}$ for every $x \in X$. We construct the functions g_j , $j \in \mathbb{N}$ inductively. Choose $z_0 \notin K$ such that $z_0 + j \notin K$ for every $j \in \mathbb{N}$.

Let $g_1 := f_1$. Now, consider the set $E_2 := \{x \in X : f_2(x) = g_1(x)\}$. Since both f_2 and g_1 are measurable functions, we have that E_2 is measurable. Now, define $g_2 : X \rightarrow \mathbb{C}$ as follows,

$$g_2(x) := \begin{cases} f_2(x) & x \notin E_2 \\ z_0 + 2 & \text{c.c.} \end{cases} \quad (3.7)$$

It is clear that g_2 is a measurable bounded function and $g_1(x) \neq g_2(x)$ for every $x \in X$.

Now, let $E_3 := \{x \in X : f_3(x) = g_2(x)\} \cup \{x \in X : f_3(x) = g_1(x)\}$. Again, since f_3, g_2 and g_1 are measurable functions, E_3 is a measurable set. Hence, we define $g_3 : X \rightarrow \mathbb{C}$ as

$$g_3(x) := \begin{cases} f_3(x) & x \notin E_3 \\ z_0 + 3 & \text{c.c.} \end{cases} \quad (3.8)$$

Again, g_3 is a measurable bounded function and $g_3(x) \neq g_2(x) \neq g_1(x)$ for every $x \in X$.

Proceeding this way, in countable steps we obtain a sequence of measurable bounded functions $g_j : X \rightarrow \mathbb{C}$, $j \in \mathbb{N}$ such that $g_j(x) \neq g_{j'}(x)$ for $j \neq j'$ and for every $x \in X$. Moreover, it is clear that, by construction, $\{f_j(x) : j \in \mathbb{N}\} \subset \{g_j(x) : j \in \mathbb{N}\}$ for every $x \in X$ and so $F(x) \subset \overline{\{g_j(x) : j \in \mathbb{N}\}}$ for every $x \in X$. □

Now, we are ready to state and prove the following theorem.

Theorem 3.9. *Let $V \subseteq L^2(\mathfrak{S})$ be a Δ -invariant space with range function \mathcal{J} . Let $L : V \rightarrow V$ be a bounded, normal and Δ -preserving operator with range operator $O : \mathcal{J} \rightarrow \mathcal{J}$. Suppose that $O(\omega)$ is diagonalizable and all its eigenvalues are isolated points of $\sigma(O(\omega))$ for a.e. $\omega \in \Omega$. Then, L is Δ -diagonalizable.*

Proof. By Theorem 2.18 and Lemma 3.8, there exists a sequence of measurable and bounded functions $g_j : \Omega \rightarrow \mathbb{C}$, $j \in \mathbb{N}$ such that $g_j(\omega) \neq g_{j'}(\omega)$ for $j \neq j'$ and

$\sigma(O(\omega)) \subseteq \overline{\{g_j(\omega) : j \in \mathbb{N}\}}$ for a.e. $\omega \in \Omega$. Furthermore, since all the eigenvalues of $O(\omega)$ are isolated points of $\sigma(O(\omega))$ for a.e. $\omega \in \Omega$, then

$$\sigma_p(O(\omega)) \subset \{g_j(\omega) : j \in \mathbb{N}\} \quad (3.9)$$

for a.e. $\omega \in \Omega$.

Since $O(\omega)$ is diagonalizable for a.e. $\omega \in \Omega$, the following equality holds

$$\mathcal{J}(\omega) = \bigoplus_{j \in \mathbb{N}} \ker(O(\omega) - g_j(\omega)\mathcal{I}_\omega) \quad (3.10)$$

where the sum is orthogonal. Notice that $\ker(O(\omega) - g_j(\omega)\mathcal{I}_\omega)$ could be $\{0\}$ for some j and some set of positive measure. However, we discard all the functions g_j such that $\ker(O(\omega) - g_j(\omega)\mathcal{I}_\omega) = \{0\}$ almost everywhere.

Now, since g_j is measurable and bounded, there exists a sequence of bounded spectrum $a_j \in \ell_2(\Delta)$ such that $\hat{a}_j = g_j$. Then, Λ_{a_j} is a Δ -eigenvalue of L for all j and by (3.10) we get the orthogonal decomposition

$$V = \bigoplus_j V_{a_j}.$$

□

In what follows we discuss two examples of operators satisfying that $O(\omega)$ is diagonalizable and $\sigma_p(O(\omega))$ are all isolated points of $\sigma(O(\omega))$ for a.e. $\omega \in \Omega$ and hence Δ -diagonalizable. As a first example, we give the following case.

Example 3.10. *Let $L : V \rightarrow V$ be a bounded, normal, injective and Δ -preserving operator such that $O(\omega)$ is compact for a.e. $\omega \in \Omega$. For this case, by Theorem 2.14, $O(\omega)$ is normal and injective for a.e. $\omega \in \Omega$. Hence, $O(\omega)$ is diagonalizable and its eigenvalues are all isolated points of $\sigma(O(\omega))$ for a.e. $\omega \in \Omega$, and thus, by Theorem 3.9, L is Δ -diagonalizable.*

Notice that $O(\omega)$ being compact a.e. $\omega \in \Omega$ does not imply that L is compact, this can be seen as a consequence of the following two results.

Proposition 3.11. *Let $L : V \rightarrow V$ be a bounded Δ -preserving operator. Then, L does not have eigenvalues with finite multiplicity.*

Proof. Suppose there is an eigenvalue λ of finite multiplicity. Then, $E_\lambda = \ker(L - \lambda\mathcal{I}) \neq \{0\}$. Since E_λ is a Δ -invariant space and the only finite dimensional Δ -invariant space is the zero space, we obtain a contradiction. □

Corollary 3.12. *Let $L : V \rightarrow V$ be a bounded Δ -preserving operator. Then, L is compact if and only if $L = 0$.*

Proof. If L is compact, L^*L is bounded, normal, compact and Δ -preserving. Hence, L^*L is diagonalizable. Moreover, by compactness, every eigenvalue $\lambda \neq 0$ of L^*L should be of finite multiplicity. By Proposition 3.11, we deduce that $\lambda = 0$ is the only possible eigenvalue. Hence $L^*L = 0$ and so $L = 0$. □

For instance, let V be a finitely generated Δ -invariant space with range function \mathcal{J} and take any bounded Δ -preserving operator $L \neq 0$ acting on V . Then, $O(\omega) : \mathcal{J}(\omega) \rightarrow \mathcal{J}(\omega)$ is compact since $\dim \mathcal{J}(\omega) < \infty$ for a.e. $\omega \in \Omega$ but L is not.

Remark 3.13. *Let L be a bounded, normal and Δ -preserving operator such that $O(\omega)$ is compact for a.e. $\omega \in \Omega$ but not necessarily injective. Given $V' := \ker(L)^\perp$ the orthogonal complement of $\ker(L)$ in V , we have that V' is a Δ -invariant space which is also invariant by L . Thus, define $L' := L|_{V'} : V' \rightarrow V'$, then L' is*

an operator satisfying the same properties as in Example 3.10. Hence, L' is Δ -diagonalizable on V' . Given a Δ -diagonalization $\{a_j\}_{j \in I}$ of $L' : V' \rightarrow V'$, we can decompose V as follows

$$V = \ker(L) \dot{\oplus} \bigoplus_j V'_{a_j}.$$

where V'_{a_j} are Δ -eigenspaces of L' .

We now give some sufficient conditions for a Δ -preserving operator to admit compact range operator a.e. $\omega \in \Omega$.

Proposition 3.14. *Let V be a Δ -invariant space with range function \mathcal{J} and $L : V \rightarrow V$ a bounded Δ -preserving operator with range operator O .*

- (1) *If $V' = \overline{L(V)}$ is a Δ -invariant space satisfying that $\dim \mathcal{J}_{V'}(\omega) < \infty$ for a.e. $\omega \in \Omega$, then $O(\omega)$ is of finite rank a.e. $\omega \in \Omega$. We will call these operators of finite range rank.*
- (2) *If there exists a sequence $\{L_n\}_{n \in \mathbb{N}}$ such that $L_n : V \rightarrow V$ is a bounded Δ -preserving operator of finite range rank for every $n \in \mathbb{N}$, and $L_n \rightarrow L$ when $n \rightarrow \infty$ uniformly, then $O(\omega)$ is compact for a.e. $\omega \in \Omega$.*

Proof. (1) Recall that by item (6) in Theorem 2.14 the range function associated to $\overline{L(V)}$ is the one given by $\overline{O(\omega)\mathcal{J}(\omega)}$ for a.e. $\omega \in \Omega$. Thus, $O(\omega)$ is a finite rank operator for a.e. $\omega \in \Omega$.

(2) Let O_n be the range operator associated to each L_n for every $n \in \mathbb{N}$, then by (1) in Theorem 2.14 we have that $O_n(\omega) \rightarrow O(\omega)$ when $n \rightarrow \infty$ for a.e. $\omega \in \Omega$. However using (1) of this proposition we have that $O_n(\omega)$ is of finite rank for every $n \in \mathbb{N}$, thus $O(\omega)$ is compact for a.e. $\omega \in \Omega$. \square

For the second example we need to give the following definition.

Definition 3.15. *Let \mathcal{H} be a separable Hilbert space, $A : \mathcal{H} \rightarrow \mathcal{H}$ a normal bounded operator, I a finite index set and $\{f_i\}_{i \in I} \subset \mathcal{H}$. We say that $(\mathcal{H}, A, \{f_i\}_{i \in I})$ is a DS-triple if $\{A^n f_i : n \in \mathbb{N}, i \in I\}$ is a frame of \mathcal{H} . In that case, we say that A admits a DS-triple.*

The problem of finding conditions on \mathcal{H} , A and $\{f_i\}_{i \in I}$ under which $(\mathcal{H}, A, \{f_i\}_{i \in I})$ is a DS-triple has been well studied and is of special interest in the context of dynamical sampling theory, see [3, 4, 2, 5, 14]. The following result has been proved in [14].

Theorem 3.16 ([14]). *Let \mathcal{H} be an infinite-dimensional separable Hilbert space, $A : \mathcal{H} \rightarrow \mathcal{H}$ a bounded normal operator and $\{f_i\}_{i \in I} \subset \mathcal{H}$ with I a finite index set. If $(\mathcal{H}, A, \{f_i\}_{i \in I})$ is a DS-triple, then A is diagonalizable and $\sigma_p(A) \subset \mathbb{D}$, where $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Moreover, in this case, the dimension of each eigenspace is less than or equal to $\#I$ and the cluster points of $\sigma_p(A)$ are contained in $S_1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.*

Remark 3.17. *In a similar way as in Corollary 3.12, one can also prove that if a Δ -preserving operator $L : V \rightarrow V$ admits a DS-triple $(V, L, \{f_i\}_{i \in I})$ with I a finite index set, then $L = 0$. However, a Δ -preserving operator $L \neq 0$ could satisfy that $O(\omega)$ admits a DS-triple for a.e. $\omega \in \Omega$.*

Observe that a normal compact operator acting on an infinite-dimensional Hilbert space never admits a DS-triple since the only cluster point of its eigenvalues, if any, is zero. Thus, we have the following example.

Example 3.18. Let V be a Δ -invariant space with range function \mathcal{J} such that $\dim \mathcal{J}(\omega) = \infty$ for a.e. $\omega \in \Sigma(V)$. Let $L : V \rightarrow V$ be a bounded, normal Δ -preserving operator such that $O(\omega)$ admits a DS-triple for a.e. $\omega \in \Sigma(V)$. By Theorem 2.14, $O(\omega)$ is normal for a.e. $\omega \in \Omega$. Then, by Theorem 3.16, $O(\omega)$ is diagonalizable and its eigenvalues are all isolated points of $\sigma(O(\omega))$ for a.e. $\omega \in \Omega$. Then, by Theorem 3.9, L is Δ -diagonalizable.

Finally we give a sufficient condition for a Δ -preserving operator in order to guarantee that its fibers admit a DS-triple which is an immediate consequence of Theorem 2.7.

Proposition 3.19. Let V be a Δ -invariant space with range function \mathcal{J} . Let $L : V \rightarrow V$ a bounded Δ -preserving operator with range operator O . Assume that there exist functions $\{f_i\}_{i \in I}$, with I a finite index set, such that $\{T_k L^j f_i : k \in \Delta, j \in \mathbb{N}, i \in I\}$ is a frame of V , then $(\mathcal{J}(\omega), O(\omega), \{T[f_i](\omega)\}_{i \in I})$ is a DS-triple for a.e. $\omega \in \Omega$.

4. Γ -PRESERVING OPERATORS AND Γ -DIAGONALIZATION

Throughout this section, we will consider a discrete and at most countable group G acting on \mathfrak{S} by the continuous automorphisms $x \mapsto gx \in \mathfrak{S}$ for $g \in G$ and $x \in \mathfrak{S}$. As before, Δ is a uniform lattice of \mathfrak{S} and we will assume that the action of G on \mathfrak{S} preserves Δ , that is $g\Delta = \Delta$ for all $g \in G$. This implies in particular that the action of G preserves the Haar measure of \mathfrak{S} , i.e.

$$|gE| = |E|, \quad \forall E \subset \mathfrak{S} \text{ measurable}, \quad \forall g \in G.$$

This can be proved as follows. Since G acts on \mathfrak{S} by automorphisms, then (see e.g. [21, Theorem 15.26]) there exists a homomorphism $\delta : G \rightarrow \mathbb{R}^+$ such that, for all measurable $E \subset \mathfrak{S}$ we have $|gE| = \delta(g)|E|$. Let $Q \subset \mathfrak{S}$ be a fundamental domain for \mathfrak{S}/Δ . Its measure $|Q|$ is finite and, since the action of G preserves Δ , then for all $g \in G$ the set gQ is a fundamental domain, so $|gQ| = |Q|$. Thus, $\delta = 1$.

The action of G on \mathfrak{S} induces an action of G on $\widehat{\mathfrak{S}}$ by duality:

$$\langle g^* \xi, x \rangle := \langle \xi, gx \rangle, \quad g \in G, \xi \in \widehat{\mathfrak{S}}, x \in \mathfrak{S}.$$

This dual action of G on $\widehat{\mathfrak{S}}$ satisfies $g_1^* g_2^* = (g_2 g_1)^*$ for all $g_1, g_2 \in G$, and it preserves Δ^\perp and the Haar measure of $\widehat{\mathfrak{S}}$. Moreover, it induces an action on the quotient group $\widehat{\mathfrak{S}}/\Delta^\perp$ by

$$g^*[\xi] := [g^* \xi], \quad \xi \in \widehat{\mathfrak{S}}, \quad g \in G,$$

where $[\xi]$ is the class of ξ in $\widehat{\mathfrak{S}}/\Delta^\perp$. This implies that we can define an action of G on any Borel section $\Omega \subset \widehat{\mathfrak{S}}$ of $\widehat{\mathfrak{S}}/\Delta^\perp$, as follows. Let $q_\Omega : \widehat{\mathfrak{S}} \rightarrow \Omega$ be the canonical section, that is, $q_\Omega(\xi)$ is the unique point in $[\xi] \cap \Omega$, and denote by $\nu_\Omega : \widehat{\mathfrak{S}} \rightarrow \Delta^\perp$ the map

$$\nu_\Omega(\xi) = q_\Omega(\xi) - \xi, \quad \xi \in \widehat{\mathfrak{S}}. \quad (4.1)$$

Then, the maps $\{g^\sharp : \Omega \rightarrow \Omega, g \in G\}$ given by

$$g^\sharp \omega = q_\Omega(g^* \omega) \quad (4.2)$$

define an action, satisfying $g_1^\sharp g_2^\sharp = (g_2 g_1)^\sharp$. Indeed, we have

$$\begin{aligned} g_1^\sharp g_2^\sharp \omega &= q_\Omega(g_1^* g_2^\sharp \omega) = q_\Omega(g_1^* q_\Omega(g_2^* \omega)) = q_\Omega(g_1^* (g_2^* \omega + \nu_\Omega(g_2^* \omega))) \\ &= q_\Omega(g_1^* g_2^* \omega + g_1^* \nu_\Omega(g_2^* \omega)) = q_\Omega(g_1^* g_2^* \omega) = q_\Omega((g_2 g_1)^* \omega) \end{aligned}$$

where the second to last identity is due to the fact that $g_1^* \nu_\Omega(g_2^* \omega) \in \Delta^\perp$. The action (4.2) will coincide with the dual action of G on $\widehat{\mathfrak{S}}$ only when Ω is an invariant subset of $\widehat{\mathfrak{S}}$ for the dual action.

Given that the action of G preserves Δ , we can define the semidirect product

$$\Gamma = \Delta \rtimes G = \{(k, g) : k \in \Delta, g \in G\},$$

with composition law

$$(k, g) \cdot (k', g') = (k + gk', gg').$$

The action of Γ on \mathfrak{S} reads

$$\gamma x = gx + k, \quad \gamma = (k, g) \in \Gamma, \quad x \in \mathfrak{S}.$$

A motivational example for this setting is given by the crystal (or crystallographic) groups.

Definition 4.1. *A crystal group Γ is a discrete subgroup of the isometries of \mathbb{R}^d that has a closed and bounded Borel section P , that is,*

- (1) $\bigcup_{\gamma \in \Gamma} \gamma P = \mathbb{R}^d$.
- (2) If $\gamma \neq \gamma'$, then $|\gamma P \cap \gamma' P| = 0$.

There is a subclass of the crystal groups we are interested in.

Definition 4.2. *We say that a crystal group Γ splits if it is the semidirect product $\Gamma = \Delta \rtimes G$ of a finite group G and a uniform lattice Δ of \mathbb{R}^d .*

In particular, it can be seen that any crystal group can be embedded in a crystal group that splits. We refer the reader to [10, 18, 19] for more general results on these groups.

We will now define some unitary representations which play a fundamental role in what follows.

Definition 4.3. *We denote by $R : G \rightarrow \mathcal{U}(L^2(\mathfrak{S}))$ the unitary representation defined by*

$$R_g f(x) = f(g^{-1}x), \quad f \in L^2(\mathfrak{S}), \quad g \in G.$$

Note that, since $R_g T_k = T_{gk} R_g$, the map $(k, g) \mapsto T_k R_g$ defines a unitary representation of the semidirect product group $\Gamma = \Delta \rtimes G$ on $L^2(\mathfrak{S})$. Also, for all $f \in L^2(\mathfrak{S})$ and all $g \in G$, the following relation holds:

$$\widehat{R_g f}(\xi) = \widehat{f}(g^* \xi). \quad (4.3)$$

Definition 4.4. *We denote by $t : \Delta^\perp \rightarrow \mathcal{U}(\ell_2(\Delta^\perp))$ the left regular representation of Δ^\perp , that is*

$$t_\ell a(\ell') = a(\ell' - \ell), \quad a \in \ell_2(\Delta^\perp), \quad \ell, \ell' \in \Delta^\perp.$$

and by $r : G \rightarrow \mathcal{U}(\ell_2(\Delta^\perp))$ the unitary representation defined by

$$r_g a(\ell) = a(g^* \ell), \quad g \in G, \quad a \in \ell_2(\Delta^\perp), \quad \ell \in \Delta^\perp.$$

As for the previous case, since $r_g t_\ell = t_{(g^*)^{-1} \ell} r_g$, and recalling that $g_1^* g_2^* = (g_2 g_1)^*$, the map $(\ell, g) \mapsto t_\ell r_g$ defines a unitary representation of the semidirect product group $\Delta^\perp \rtimes G$ on $\ell_2(\Delta^\perp)$.

Finally, we introduce the following representation of G on $L^2(\Omega, \ell_2(\Delta^\perp))$.

Definition 4.5. *We denote by Π the unitary representation of G on the Hilbert space $L^2(\Omega, \ell_2(\Delta^\perp))$ defined by*

$$\Pi(g) = \mathcal{T} R_g \mathcal{T}^{-1}.$$

The explicit form of the representation Π is provided by the following proposition.

Proposition 4.6. *Let Ω be a fundamental set for $\widehat{\mathfrak{S}}/\Delta^\perp$ and let $\pi : \Omega \times G \rightarrow \mathcal{U}(\ell_2(\Delta^\perp))$ be the map*

$$\pi^\omega(g)a(\ell) = r_g t_{\nu_\Omega(g^*\omega)} a(\ell) = a(g^*\ell - \nu_\Omega(g^*\omega))$$

where ν_Ω is given by (4.1). Then the representation Π reads explicitly

$$\Pi(g)F(\omega) = \pi^\omega(g)F(g^\sharp\omega), \quad F \in L^2(\Omega, \ell_2(\Delta^\perp)), \quad g \in G, \text{ a.e. } \omega \in \Omega. \quad (4.4)$$

Moreover, the map $(\omega, g) \mapsto \pi^\omega(g)$ satisfies $\pi^\omega(e) = \mathbb{I}_{\ell_2(\Delta^\perp)}$ and

$$\pi^\omega(g_1 g_2) = \pi^\omega(g_1) \pi^{g_1^\sharp\omega}(g_2). \quad (4.5)$$

In particular, if Ω is invariant under the dual action of G on $\widehat{\mathfrak{S}}$, then $\pi^\omega(g) = r_g$.

Proof. By (4.1), (4.2) and (4.3), for all $f \in L^2(\mathfrak{S})$ we have that

$$\begin{aligned} \mathcal{T}[R_g f](\omega) &= \{\widehat{f}(g^*\omega + g^*\ell)\}_{\ell \in \Delta^\perp} = \{\widehat{f}(g^\sharp\omega + g^*\ell - \nu_\Omega(g^*\omega))\}_{\ell \in \Delta^\perp} \\ &= \pi^\omega(g)\mathcal{T}[f](g^\sharp\omega) \end{aligned}$$

which proves (4.4) using that \mathcal{T} is an isomorphism. In order to prove (4.5), recall that Π is a unitary representation, because it is defined as the intertwining of a unitary representation with an isomorphism of Hilbert spaces. Using (4.4), this implies that

$$\Pi(g_1)\Pi(g_2)F(\omega) = \Pi(g_1 g_2)F(\omega) = \pi^\omega(g_1 g_2)F(g_2^\sharp g_1^\sharp\omega).$$

On the other hand, we have

$$\Pi(g_1)\Pi(g_2)F(\omega) = \pi^\omega(g_1)\Pi(g_2)F(g_1^\sharp\omega) = \pi^\omega(g_1)\pi^{g_1^\sharp\omega}(g_2)F(g_2^\sharp g_1^\sharp\omega).$$

Since both relations hold for all $F \in L^2(\Omega, \ell_2(\Delta^\perp))$, all $g \in G$ and a.e. $\omega \in \Omega$, this proves (4.5). \square

4.1. Γ -invariant spaces. Given this setting, we are now interested in the subspaces of $L^2(\mathfrak{S})$ that are invariant under the action of the unitary representation $T_k R_g$. These spaces have been first studied in great detail in [8].

Definition 4.7. *We say that a closed subspace $V \subset L^2(\mathfrak{S})$ is Γ -invariant if $T_k R_g V \subset V$ for all $(k, g) \in \Gamma$.*

A Γ -invariant space is, in particular, Δ -invariant as it can be seen that V is Γ -invariant if

$$f \in V \Rightarrow T_k f \in V \quad \forall k \in \Delta, \text{ and } R_g f \in V \quad \forall g \in G.$$

Consequently, V is Γ -invariant if and only if V is Δ -invariant, and $\Pi(g)\mathcal{T}[V] \subset \mathcal{T}[V]$ for every $g \in G$, where Π is the representation given in Definition 4.5.

The following theorem gives a characterization of Γ -invariant closed subspaces in terms of a covariance property of the range function associated to its Δ -invariant subspace. This theorem generalizes the one in [8, Theorem 3.3] where a proof was given under the additional hypothesis that the fundamental domain Ω is invariant under the dual action of G on $\widehat{\mathfrak{S}}$. However, in that paper this additional hypothesis was omitted, as one of the reviewers of this paper pointed out.

Theorem 4.8. *A closed subspace V of $L^2(\mathfrak{S})$ is Γ -invariant if and only if it is Δ -invariant and its associated range function \mathcal{J} satisfies*

$$\mathcal{J}(\omega) = \pi^\omega(g)\mathcal{J}(g^\sharp\omega), \quad \text{a.e. } \omega \in \Omega, \quad \forall g \in G. \quad (4.6)$$

Proof. Assume that V is Δ -invariant with range function \mathcal{J} and, for $g \in G$, let us denote by $V_g = R_g(V)$. Then V_g is also Δ -invariant, and we denote by \mathcal{J}_g its range function. We claim that

$$\mathcal{J}_g(\omega) = \pi^\omega(g)\mathcal{J}(g^\sharp\omega), \text{ a.e. } \omega \in \Omega, \forall g \in G.$$

Note that, by proving this claim, we also prove the statement of the Theorem.

To prove the claim recall that, by Theorem 2.4, $F \in \mathcal{T}[V] \iff F(\omega) \in \mathcal{J}(\omega)$ for a.e. $\omega \in \Omega$ and, by (4.4), for $F \in \mathcal{T}[V]$ we have $\Pi(g)F(\omega) = \pi^\omega(g)F(g^\sharp\omega)$.

Assume first that, for a given $g \in G$, we have a function $H \in L^2(\Omega, \ell_2(\Delta^\perp))$ such that $H(\omega) \in \mathcal{J}_g(\omega)$ for a.e. $\omega \in \Omega$, that is $H \in \mathcal{T}[R_g(V)]$. We want to prove that $H(\omega) \in \pi^\omega(g)\mathcal{J}(g^\sharp\omega)$ for a.e. $\omega \in \Omega$. By Definition 4.5, we have $\Pi(g^{-1})H \in \mathcal{T}[V]$, which implies

$$\pi^\omega(g^{-1})H((g^{-1})^\sharp\omega) \in \mathcal{J}(\omega), \text{ a.e. } \omega \in \Omega.$$

Denoting by $\omega' = (g^{-1})^\sharp\omega$, this reads equivalently

$$\pi^{g^\sharp\omega'}(g^{-1})H(\omega') \in \mathcal{J}(g^\sharp\omega'), \text{ a.e. } \omega' \in \Omega.$$

Thus, by applying $\pi^{\omega'}(g)$ on both sides, and using Proposition 4.6, we obtain $H(\omega) \in \pi^\omega(g)\mathcal{J}(g^\sharp\omega)$ for a.e. $\omega \in \Omega$.

Assume now that, for a $g \in G$, we have an $H \in L^2(\Omega, \ell_2(\Delta^\perp))$ such that $H(\omega) \in \pi^\omega(g)\mathcal{J}(g^\sharp\omega)$ for a.e. $\omega \in \Omega$, and let $F = \Pi(g^{-1})H$. Then

$$H(\omega) = \pi^\omega(g)F(g^\sharp\omega).$$

Since $\pi^\omega(g)$ is a unitary operator on $\ell_2(\Delta^\perp)$, and g^\sharp is a bijection of Ω we have obtained that $F(\omega) \in \mathcal{J}(\omega)$ for a.e. $\omega \in \Omega$. That is, $F \in \mathcal{T}[V]$, or, equivalently, $H \in \Pi(g)(\mathcal{T}[V]) = \mathcal{T}[R_g V]$. Thus, $H(\omega) \in \mathcal{J}_g(\omega)$ for a.e. $\omega \in \Omega$. \square

4.2. Γ -preserving operators. In this subsection we consider operators defined on Γ -invariant spaces which commute with the unitary representation $T_k R_g$.

Definition 4.9. Let $V, V' \subset L^2(\mathfrak{S})$ be two Γ -invariant spaces. We say that a bounded operator $L : V \rightarrow V'$ is Γ -preserving if $LT_k R_g = T_k R_g L$ for every $k \in \Delta$ and $g \in G$.

Observe that, in particular, L is Γ -preserving if and only if L is Δ -preserving and $LR_g = R_g L$ for every $g \in G$. We will focus on bounded Γ -preserving operators acting on a Γ -invariant V , that is $L : V \rightarrow V$. Since L is Δ -preserving, there exists a corresponding range operator $O : \mathcal{J} \rightarrow \mathcal{J}$.

In the same spirit of Theorem 4.8, we have the following result.

Theorem 4.10. Let $V \subset L^2(\mathfrak{S})$ be a Γ -invariant space and $L : V \rightarrow V$ a bounded operator with corresponding range operator O . Then, L is Γ -preserving if and only if it is Δ -preserving and for all $g \in G$ and a.e. $\omega \in \Omega$,

$$O(g^\sharp\omega) = \pi^{g^\sharp\omega}(g^{-1})O(\omega)\pi^\omega(g). \quad (4.7)$$

Proof. Assume that L is Γ -preserving. Fix $g \in G$ and note that for every $f \in V$, and for a.e. $\omega \in \Omega$

$$\begin{aligned} O(\omega)(\Pi(g)\mathcal{T}[f](\omega)) &= O(\omega)(\mathcal{T}[R_g f](\omega)) = \mathcal{T}[LR_g f](\omega) \\ &= \mathcal{T}[R_g Lf](\omega) = \Pi(g)\mathcal{T}[Lf](\omega) \\ &= \Pi(g)O(\omega)\mathcal{T}[f](\omega). \end{aligned}$$

Hence, if $F \in \mathcal{T}[V]$, by (4.4) we have that for a.e. $\omega \in \Omega$

$$O(\omega)\pi^\omega(g)F(g^\sharp\omega) = \pi^\omega(g)O(g^\sharp\omega)F(g^\sharp\omega).$$

Since, by (4.5), $(\pi^\omega(g))^{-1} = \pi^{g^\sharp\omega}(g^{-1})$, we deduce that for a.e. $\omega \in \Omega$,

$$O(g^\sharp\omega) = \pi^{g^\sharp\omega}(g^{-1})O(\omega)\pi^\omega(g).$$

For the converse, if (4.7) holds, then for every $F \in \mathcal{T}[V]$ we have that for a.e. $\omega \in \Omega$ and for every $g \in G$,

$$O(\omega)\Pi(g)F(\omega) = \Pi(g)O(\omega)F(\omega).$$

By the computation above, this means that for every $f \in V$ and for every $g \in G$, $\mathcal{T}[LR_g f] = \mathcal{T}[R_g Lf]$. Thus, $LR_g = R_g L$ for every $g \in G$. \square

As a consequence, we see that much of the structure of O is preserved by the action of G on Ω . In particular, we have the next proposition concerning the spectra of $O(\omega)$.

Proposition 4.11. *Let $V \subset L^2(\mathfrak{S})$ be a Γ -invariant space and $L : V \rightarrow V$ a bounded Γ -preserving operator with corresponding range operator O . Then for all $g \in G$ and a.e. $\omega \in \Omega$,*

- (1) $\sigma(O(\omega)) = \sigma(O(g^\sharp\omega))$.
- (2) $\sigma_p(O(\omega)) = \sigma_p(O(g^\sharp\omega))$.

Proof. Fix $g \in G$ and $\omega \in \Omega$ where \mathcal{J} and O are defined. Assume that $\lambda \in \sigma(O(\omega))$, then $O(\omega) - \lambda\mathcal{I}_\omega$ is not invertible in $\mathcal{J}(\omega)$. Thus,

$$\begin{aligned} O(g^\sharp\omega) - \lambda\mathcal{I}_{g^\sharp\omega} &= \pi^{g^\sharp\omega}(g^{-1})O(\omega)\pi^\omega(g) - \lambda\pi^{g^\sharp\omega}(g^{-1})\pi^\omega(g) \\ &= \pi^{g^\sharp\omega}(g^{-1})(O(\omega) - \lambda\mathcal{I}_\omega)\pi^\omega(g), \end{aligned}$$

which implies that $O(g^\sharp\omega) - \lambda\mathcal{I}_{g^\sharp\omega}$ is not invertible in $\mathcal{J}(g^\sharp\omega)$, hence proving (1). Now, to prove (2), suppose $\lambda \in \mathbb{C}$ is an eigenvalue of $O(\omega)$, then there exists $v \neq 0$ and $v \in \ker(O(\omega) - \lambda\mathcal{I}_\omega)$. We will see that $\pi^{g^\sharp\omega}(g^{-1})v \in \ker(O(g^\sharp\omega) - \lambda\mathcal{I}_{g^\sharp\omega})$. Indeed, we have that

$$\begin{aligned} O(g^\sharp\omega)(\pi^{g^\sharp\omega}(g^{-1})v) &= \pi^{g^\sharp\omega}(g^{-1})O(\omega)\pi^\omega(g)(\pi^{g^\sharp\omega}(g^{-1})v) = \pi^{g^\sharp\omega}(g^{-1})O(\omega)v \\ &= \pi^{g^\sharp\omega}(g^{-1})\lambda v = \lambda(\pi^{g^\sharp\omega}(g^{-1})v). \end{aligned}$$

Since $v \neq 0$, then $\pi^{g^\sharp\omega}(g^{-1})v \neq 0$ and consequently $\ker(O(g^\sharp\omega) - \lambda\mathcal{I}_{g^\sharp\omega}) \neq \{0\}$. \square

We remark that given a measurable function $\lambda : \Omega \rightarrow \mathbb{C}$ such that $\lambda(\omega)$ is an eigenvalue of $O(\omega)$ for a.e. $\omega \in \Omega$ the proposition above does not imply that $\lambda(\omega) = \lambda(g^\sharp\omega)$ for every $g \in G$ and a.e. $\omega \in \Omega$.

Since we are interested in obtaining a diagonalization for Γ -preserving operators similar to Definition 3.6, we must find some suitable operators to play the role of Γ -eigenvalue. The natural choice would be $K_a = \sum_{(s,h) \in \Gamma} a(s,h)T_s R_h$ for some sequence $a = \{a(s,h)\}_{(s,h) \in \Gamma}$ satisfying certain conditions. However, if we require that these operators commute with $T_k R_g$ for every $k \in \Delta$ and $g \in G$, it is not difficult to see that we are left only with a multiple of the identity.

Hence, we turn to the Δ -preserving operators Λ_a of Definition 3.1, with $a \in \ell_2(\Delta)$ of bounded spectrum. We are interested in characterizing such operators that commute with the unitary representation R_g of G on $L^2(\mathfrak{S})$.

For the next proposition, we introduce the following representation.

Definition 4.12. *We denote by $\tilde{r} : G \rightarrow \mathcal{U}(\ell_2(\Delta))$ the representation defined by*

$$(\tilde{r}_g(a))(s) = a(g^{-1}s), \quad g \in G, a \in \ell_2(\Delta), s \in \Delta.$$

Proposition 4.13. *Let $a \in \ell_2(\Delta)$ of bounded spectrum and let Λ_a an operator as in Definition 3.1. Then, the following statements are equivalent.*

- (1) $R_g \Lambda_a = \Lambda_a R_g$ for every $g \in G$.

- (2) For every $g \in G$, $\tilde{r}_g(a) = a$.
(3) For every $g \in G$, $\widehat{a}(g^\sharp\omega) = \widehat{a}(\omega)$ for a.e. $\omega \in \Omega$.

Proof. For a given $g \in G$, first compute for $\omega \in \Omega$,

$$\begin{aligned} \widehat{(\tilde{r}_g(a))}(\omega) &= \sum_{s \in \Delta} (\tilde{r}_g(a))(s) e^{-2\pi i \omega \cdot s} = \sum_{s \in \Delta} a(g^{-1}s) e^{-2\pi i \omega \cdot s} \\ &= \sum_{s \in \Delta} a(s) e^{-2\pi i \omega \cdot gs} = \sum_{s \in \Delta} a(s) e^{-2\pi i g^* \omega \cdot s} \\ &= \sum_{s \in \Delta} a(s) e^{-2\pi i g^\sharp \omega \cdot s} = \widehat{a}(g^\sharp\omega). \end{aligned} \quad (4.8)$$

Now, recalling that $\Pi(g) = \mathcal{T}R_g\mathcal{T}^{-1}$, we see that

$$R_g\Lambda_a = R_g\mathcal{T}^{-1}M_{\widehat{a}}\mathcal{T} = \mathcal{T}^{-1}\Pi(g)M_{\widehat{a}}\mathcal{T}. \quad (4.9)$$

On the other hand, observe that for $F \in L^2(\Omega, \ell_2(\Delta^\perp))$ and using (4.4)

$$\begin{aligned} \Pi(g)M_{\widehat{a}}F(\omega) &= \Pi(g)\widehat{a}(\omega)F(\omega) = \pi^\omega(g)\widehat{a}(g^\sharp\omega)F(g^\sharp\omega) \\ &= \widehat{a}(g^\sharp\omega)\pi^\omega(g)F(g^\sharp\omega) = M_{\widehat{a}(g^\sharp\cdot)}\Pi(g)F(\omega). \end{aligned}$$

From the last equality, together with (4.8) and (4.9) it follows that

$$R_g\Lambda_a = \mathcal{T}^{-1}M_{\widehat{a}(g^\sharp\cdot)}\Pi(g)\mathcal{T} = \mathcal{T}^{-1}M_{\widehat{a}(g^\sharp\cdot)}\mathcal{T}R_g = \Lambda_{\tilde{r}_g(a)}R_g.$$

Then, $R_g\Lambda_a = \Lambda_a R_g$ if and only if $\Lambda_{\tilde{r}_g(a)}R_g = \Lambda_a R_g$. Since R_g is invertible, this holds if and only if $\tilde{r}_g(a) = a$ or, equivalently, $\widehat{a}(g^\sharp\omega) = \widehat{a}(\omega)$ for a.e. $\omega \in \Omega$. \square

If $a \in \ell_2(\Delta)$ is a sequence of bounded spectrum which satisfies any of the conditions of Proposition 4.13 and $V \subset L^2(\mathfrak{S})$ is Γ -invariant, then $\Lambda_a(V) \subseteq V$ and $\Lambda_a : V \rightarrow V$ is Γ -preserving.

Remark 4.14. *If the group G is infinite, often this class of operators is very small. Indeed, for any $s_0 \in \Delta$, by the invariance (2) of Proposition 4.13, we have that*

$$\sum_{s \in \Delta} |a(s)|^2 \geq \sum_{s \in \{g^{-1}s_0 : g \in G\}} |a(s)|^2 = \#\{g^{-1}s_0 : g \in G\} \cdot |a(s_0)|^2.$$

Since the sequence a is in $\ell_2(\Delta)$, for every $s \in \Delta$ where $a(s) \neq 0$ we have

$$\#\{gs : g \in G\} < \infty. \quad (4.10)$$

For example, consider the group of translations and shears in \mathbb{R}^2 . That is, $\mathfrak{S} = \mathbb{R}^2$, $\Delta = \mathbb{Z}^2$ and $G = \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}$, which preserves the lattice \mathbb{Z}^2 . For $s = (s_1, s_2) \in \mathbb{Z}^2$ we have that $gs = (s_1 + ks_2, s_2)$. Hence, if $s_2 \neq 0$ then $\#\{gs : g \in G\} = \infty$ and so $a(s) = 0$ necessarily. Thus, the operators of this kind must be of the form

$$\Lambda_a = \sum_{s_1 \in \mathbb{Z}} a(s_1, 0)T_{(s_1, 0)}.$$

Furthermore, if G were an infinite group acting faithfully over Δ , then every operator Λ_a which commutes with R_g for every $g \in G$ must satisfy that $a(s) = 0$ for every $s \in \Delta \setminus \{0\}$.

4.3. Γ -diagonalization. Now, we aim to find conditions on Δ -preserving operators in order to obtain a Δ -diagonalization like in Definition 3.6 where each Δ -eigenvalue of the decomposition commute with the unitary representation $T_k R_g$ and each Δ -eigenspace is Γ -invariant.

Definition 4.15. Let $V \subset L^2(\mathfrak{S})$ be a Γ -invariant space and $L : V \rightarrow V$ a bounded Γ -preserving operator. Let $a \in \ell_2(\Delta)$ be of bounded spectrum, we say that $\Lambda_a : V \rightarrow V$ is a Γ -eigenvalue of L if $\tilde{r}_g(a) = a$ for every $g \in G$ and Λ_a is a Δ -eigenvalue of L , i.e.

$$V_a := \ker(L - \Lambda_a) \neq \{0\}.$$

Furthermore, we will say that L is Γ -diagonalizable if it admits a Δ -diagonalization $\{a_j\}_{j \in I}$ of L where Λ_{a_j} is a Γ -eigenvalue for every $j \in I$.

Observe that, in this case, the Δ -eigenspace V_a associated to a Γ -eigenvalue Λ_a is a Γ -invariant subspace of V . In particular, if L is Γ -diagonalizable, then

$$L = \sum_{j \in I} \Lambda_{a_j} P_{V_{a_j}}$$

where each V_{a_j} is Γ -invariant subspace and the convergence of the series is in the strong operator topology sense.

In what follows, we will assume that there exists a Borel set $\Omega_0 \subset \Omega$ which is a *transversal* for the action of G on Ω , that is, Ω_0 intersects each orbit of the action of G on Ω in exactly one point. We remark that a Borel transversal for the action of G on Ω is not necessarily a tiling of Ω , i.e. a set such that $\{g^\sharp \Omega_0\}_{g \in G}$ is an a.e. partition of Ω .

When G is finite, the existence of such set is ensured by [23, Theorem 12.16]. Moreover, the existence of a Borel transversal is equivalent to the existence of a Borel *selector* for the action of G on Ω (see [23]), that is, a Borel function $s : \Omega \rightarrow \Omega$ such that for every $\omega, \omega' \in \Omega$, we have that

$$\omega' \in O_G(\omega) \Rightarrow s(\omega) = s(\omega') \in O_G(\omega),$$

where

$$O_G(\omega) = \{\omega' \in \Omega : \omega' = g^\sharp \omega, g \in G\}.$$

Under this assumption we are able to prove our final goal. In the next theorem we show that a normal Δ -diagonalizable operator which is Γ -preserving always admits a Γ -diagonalization.

Theorem 4.16. Let $V \subset L^2(\mathfrak{S})$ be a Γ -invariant space and $L : V \rightarrow V$ a bounded normal Γ -preserving operator. Then, L is Γ -diagonalizable if and only if it is Δ -diagonalizable.

Proof. We just need to prove that if L is Δ -diagonalizable, then it admits a Δ -diagonalization conformed by Γ -eigenvalues. Assume that $\{a_j\}_{j \in I}$ is a Δ -diagonalization of L . By Theorem 3.7, $O(\omega)$ is diagonalizable and $\sigma_p(O(\omega)) \subset \{\hat{a}_j(\omega) : j \in I\}$ for a.e. $\omega \in \Omega$.

Now, let $s : \Omega \rightarrow \Omega$ be a Borel selector for the action of G on Ω . In particular, we have that $\sigma_p(O(s(\omega))) \subset \{\hat{a}_j(s(\omega)) : j \in I\}$ for a.e. $\omega \in \Omega$. Moreover, by (2) in Proposition 4.11 we see that $\sigma_p(O(\omega)) = \sigma_p(O(s(\omega)))$ for a.e. $\omega \in \Omega$. Thus, taking $\lambda_j(\omega) = \hat{a}_j \circ s(\omega)$ we get that

$$\sigma_p(O(\omega)) \subset \{\lambda_j(\omega) : j \in I\} \tag{4.11}$$

for a.e. $\omega \in \Omega$. Since s is a selector, we see that $\lambda_j(g^\sharp \omega) = \lambda_j(\omega)$ for every $j \in I$, $g \in G$ and a.e. $\omega \in \Omega$. Also, given that s is a Borel selector and $\hat{a}_j \in L^\infty(\Omega)$, we obtain that $\lambda_j \in L^\infty(\Omega)$.

Since $O(\omega)$ is diagonalizable the following orthogonal decomposition holds

$$\mathcal{J}(\omega) = \bigoplus_{j \in I} \ker(O(\omega) - \lambda_j(\omega)\mathcal{I}_\omega) \tag{4.12}$$

for a.e. $\omega \in \Omega$. We now discard those functions λ_j such that $\ker(O(\omega) - \lambda_j(\omega)\mathcal{I}_\omega) = \{0\}$ for a.e. $\omega \in \Omega$. For each remaining j , there exists a sequence $b_j \in \ell_2(\Delta)$ of

bounded spectrum such that $\widehat{b}_j = \lambda_j$. So, Λ_{b_j} is a Γ -eigenvalue of L for every j and by (4.12) we conclude that

$$V = \bigoplus_j V_{b_j}. \quad (4.13)$$

Hence, L is Γ -diagonalizable. \square

Acknowledgements: We thank the anonymous referees for the very careful reading of the manuscript and many positive suggestions which helped improve the paper.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 777822. In addition, D. Barbieri and E. Hernández were supported by Grants MTM2016-76566-P and PID2019-105599GB-I00 (Ministerio de Ciencia e Innovación, Spain). C. Cabrelli, D. Carbajal and U. Molter were supported by Grants UBACyT 20020170100430BA (University of Buenos Aires), PIP11220150100355 (CONICET) and PICT 2018-03399 (Ministry of Science and Technology from Argentina).

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