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Subdirectly Irreducible IKt-Algebras

Abstract. The IKt-algebras that we investigate in this paper were introduced in the paper *An algebraic axiomatization of the Ewald's intuitionistic tense logic* by the first and third author. Now we characterize by topological methods the subdirectly irreducible IKt-algebras and particularly the simple IKt-algebras. Finally, we consider the particular cases of finite IKt-algebras and complete IKt-algebras.

Keywords: Heyting algebras, Tense operators, IKt-algebras.

1. Introduction and Preliminaries

In this paper, we take for granted the concepts and results on distributive lattices, Heyting algebras, category theory, universal algebra and Priestley duality. To obtain more information about these topics, we direct the reader to the bibliography indicated in [1,6,8,18,30,33–35]. However, in order to simplify reading, in this section we shall summarize the fundamental concepts we use.

If X is a poset (i.e. partially ordered set) and $Y \subseteq X$, then we shall denote by $\downarrow Y$ ($\uparrow Y$) the set of all $x \in X$ such that $x \leq y$ ($y \leq x$) for some $y \in Y$. If $x \in X$ we shall denote by $\downarrow x$ ($\uparrow x$) instead of $\downarrow \{x\}$ ($\uparrow \{x\}$).

Let X, Y be sets. Given a relation $R \subseteq X \times Y$, for each $Z \subseteq X$, $R(Z)$ will denote the image of Z by R. If $Z = \{x\}$, we will write $R(x)$ instead of $R(\{x\})$. Moreover, for each $V \subseteq Y$, $R^{-1}(V)$ will denote the inverse image of V by R, i.e. $R^{-1}(V) = \{x \in X : R(x) \cap V \neq \emptyset\}$. If $V = \{y\}$, we will write $R^{-1}(y)$ instead of $R^{-1}(\{y\})$. Besides, if $R, T \subseteq X \times X$ then the relation $R \circ T$ is defined by setting $(x, y) \in R \circ T$ if and only if there is $z \in X$ such that $(x, z) \in R$ and $(z, y) \in T$. If Y is a subset of X, then Y^c will denote the set-theoretic complement of Y, i.e. $Y^c = X \backslash Y$.

Let us recall that an algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra if the following conditions hold:

Presented by **Jacek Malinowski**; *Received* June 22, 2016

Studia Logica
DOI: 10.1007/s11225-017-9707-2

Published online: 04 February 2017

(1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a lattice with 0, 1,

(2) $x \wedge (x \to y) = x \wedge y$,

(3) $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$,

(4) $(x \wedge y) \to x = 1$.

In [18] a duality theory for Heyting algebras was developed. In order to determine this duality, the category **HS** of Heyting space and their corresponding morphisms (or p-functions) was introduced, which we will describe below. Specifically, a Heyting space is a Priestley space $(X, \leq, \tau)$ such that if U is clopen in $'X'$, then $\downarrow U$ is clopen. Alternatively, $(X, \leq, \tau)$ is a Heyting space if for every open set U, then downset $\downarrow U$ is open. From now on, whenever it is deemed convenient, Heyting space will be denoted simply by X.

On the other hand, a p-function from a Heyting space X_1 into another one X_2 is an order preserving continuous function $f : X_1 \longrightarrow X_2$, which verifies the following condition:

(pf) for all $x \in X_1$, $z \in X_2$ such that $f(x) \leq z$, there is $y \in X_1$ such that $x \leq y$ and $f(y) = z$.

Besides, if A is a Heyting algebra and $\mathfrak{X}(A)$ is the set of all prime filters of A, then the Priestley space associated with A, $(\mathfrak{X}(A), \subseteq, \tau)$ (see [33–35]), is a Heyting space. If X is a Heyting space and $D(X)$ is the set of all increasing and clopen subsets of X, then $(D(X), \cup, \cap, \to, \emptyset, X)$ is a Heyting algebra, where for all U, $V \in D(X)$,

$$U \to V = \{x \in X : \uparrow x \cap U \subseteq V\}. \tag{I}$$

Furthermore, $(\mathcal{A}, \to) \cong (D(\mathfrak{X}(A)), \to)$ for all Heyting algebra $\mathcal{A}$ and $X \cong \mathfrak{X}(D(X))$ for all Heyting space X, via natural isomorphisms denoted by σ_A and ε_X respectively, where $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$ is defined by

$$\sigma_A(a) = \{S \in \mathfrak{X}(A) :\, a \in S\}, \tag{II}$$

and $\varepsilon_X : X \longrightarrow \mathfrak{X}(D(X))$ is defined by

$$\varepsilon_X(x) = \{U \in D(X) :\, x \in U\}. \tag{III}$$

The correspondences between the morphisms of both categories are defined in the usual way. Then, it is concluded that the category **HS** of Heyting spaces and p-functions is dually equivalent to the category **HA** of Heyting algebras and their corresponding homomorphisms. The above duality was taken into account to characterize the congruence lattice on a Heyting algebra as it is indicated in the following theorem:

THEOREM 1.1. *Let $\mathcal{A}$ be a Heyting algebra, $\mathfrak{X}(A)$ be the Heyting space associated with $\mathcal{A}$, $\mathcal{C}_I(\mathfrak{X}(A))$ be the lattice of increasing and closed subsets of $\mathfrak{X}(A)$ and $Con_H(\mathcal{A})$ be the lattice of Heyting congruence on $\mathcal{A}$. If $Y \in \mathcal{C}_I(\mathfrak{X}(A))$ and*

$$\Theta_I(Y) = \{(a,b) \in A \times A : \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\}, \tag{IV}$$

then $\Theta_I(Y) \in Con_H(\mathcal{A})$. Conversely, if $\theta \in Con_H(\mathcal{A})$, $h : A \longrightarrow A/\theta$ is the natural epimorphism and

$$Y = \{h^{-1}(S) : S \in \mathfrak{X}(\mathcal{A}/\theta)\}, \tag{V}$$

then $Y \in \mathcal{C}_I(\mathfrak{X}(A))$ and $\theta = \Theta_I(Y)$. Therefore, the lattice $\mathcal{C}_I(\mathfrak{X}(A))$ is isomorphic to the dual lattice $Con_H(\mathcal{A})$, and the isomorphism is the function Θ_I defined by the prescription IV.

Let us recall that under the Priestley duality, the lattice of all filters of a bounded distributive lattice is dually isomorphic to the lattice of all increasing closed subsets of the dual space. Under that isomorphism, any filter T of a bounded distributive lattice A corresponds to the increasing closed

$$Y_T = \{S \in \mathfrak{X}(A) : T \subseteq S\} = \bigcap \{\sigma_A(a) : a \in T\} \tag{VI}$$

and any increasing closed subset Y of $\mathfrak{X}(A)$ corresponds to the filter

$$T_Y = \{a \in A : Y \subseteq \sigma_A(a)\}. \tag{VII}$$

A direct consequence of these last results is the well-known fact that there exists a lattice isomorphism between the lattice of all filters of a Heyting algebra $\mathcal{A}$ and the lattice of all congruences on $\mathcal{A}$. Under that isomorphism, any congruence θ on $\mathcal{A}$ corresponds to the filter $S_\theta = \{a \in A : (a,1) \in \theta\}$ and any filter S of $\mathcal{A}$ corresponds to the congruence θ_S defined by $(a,b) \in \theta_S$ iff $(a \to b) \wedge (b \to a) \in S$. Therefore there exists a lattice isomorphism between the lattice of all congruences determined by filters of a Heyting algebra $\mathcal{A}$ and the lattice of all congruences on $\mathcal{A}$.

It should be noted that the following characterization of subdirectly irreducible Heyting algebras have been found useful for characterizing the finite and complete subdirectly irreducible IKt-algebras.

THEOREM 1.2. *A Heyting algebra $\mathcal{A}$ is subdirectly irreducible if and only if there is $u \in A\backslash\{1\}$, such that $a \leq u$, for all $a \in A\backslash\{1\}$.*

On the other hand, it is known that propositional logics, both classic or nonclassic, do not incorporate the dimension of time. To obtain a tense logic, we enrich the given propositional logic by new unary operators (called tense operators) which are usually denoted by G, H, F and P. Study of tense operators has originated in 1980's [5,13]. Recall that for a classical propositional calculus represented by means of a Boolean algebra $\mathcal{B} = \langle B, \vee, \wedge, \neg, 0, 1\rangle$, tense operators were axiomatized [16] by the following axioms:

$$G(1) = 1, H(1) = 1,$$
$$G(x \wedge y) = G(x) \wedge G(y), H(x \wedge y) = H(x) \wedge H(y),$$
$$x \leq GP(x), x \leq HF(x),$$

where $P(x) = \neg H(\neg x)$ and $F(x) = \neg G(\neg x)$.

In the last few years tense operators have been considered by different authors for various classes of algebras. Some contributions in this area have been the papers by Diaconescu and Georgescu [16], Botur et al. [3], Chiriţă [14,15], Chajda [7], Chajda and Kolařik [9], Figallo and Pelaitay [20,23, 25,26], Chajda and Paseka [12], Botur and Paseka [4], Paseka [32], Menni and Smith [31], Dzik et al. [17]. In particular, intuitionistic tense logic IKt was introduced by Ewald [19] by extending the language of intuitionistic propositional logic with the unary operators P (it was the case), F (it will be the case), H (it has always been the case) and G (it will always be the case). The Hilbert-style axiomatization of IKt can be found in [19] [p. 171]. It is well-known that the axiomatization of Ewald is not minimal because several axioms can be deduced from the other axioms. Besides, in contrast to classical tense logic, F and P cannot be defined in terms of G and H (see [17,23]). In [26], Figallo and Pelaitay introduced the variety **IKt** of IKt-algebras and proved that the IKt system has IKt-algebras as algebraic counterpart. These algebras were defined as described below.

Definition 1.3. Let $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$ be a Heyting algebra, let G, H, F and P be unary operations on A satisfying:

(t1) $G(1) = 1$ and $H(1) = 1$,

(t2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,

(t3) $x \leq GP(x)$ and $x \leq HF(x)$,

(t4) $F(0) = 0$ and $P(0) = 0$,

(t5) $F(x \vee y) = F(x) \vee F(y)$ and $P(x \vee y) = P(x) \vee P(y)$,

(t6) $PG(x) \leq x$ and $FH(x) \leq x$,

(t7) $F(x \rightarrow y) \leq G(x) \rightarrow F(y)$ and $P(x \rightarrow y) \leq H(x) \rightarrow P(y)$.

Then the algebra $(\mathcal{A}, G, H, F, P)$ will be called an IKt-algebra and G, H, F and P will be called tense operators.

Independently, in [17] (see also [31]), two algebraic models of the IKt system were obtained in terms of Heyting algebras expanded with two Galois connections verifying the Dunn's axioms, in one case, and the Fisher-Servi's axioms, in another case. It is not difficult to check that all three algebraic models of the IKt system are equivalent. In [17], a relational representation theorem for IKt-algebras was obtained. In this paper, taking into account the results obtained in [17], we give a topological duality for IKt-algebras. By means of this duality, we characterize the IKt-congruences lattice which allowed us to determine the simple and subdirectly irreducible IKt-algebras.

The paper is organized as follows: In Section 1, we briefly summarize the main definitions and results needed throughout this article. In Section 2, we describe a topological duality for IKt-algebras, extending the one obtained in [17] for Heyting algebras. For this purpose we introduce the category **IKtS** whose objects are IKt-spaces and whose morphisms are increasing continuous functions verifying certain additional conditions and we prove that this category is equivalent to the dual of the category **IKtA**, whose objects are IKt-algebras and whose morphisms are IKt-homomorphisms. In Section 3, which is the core of this paper, the results of Section 2 are applied. Firstly, we characterize congruences on IKt-algebras by means of the mentioned duality and certain closed and increasing subsets of the space associated with them. This enables us to obtain a new characterization of congruences on IKt-algebras. This result allows us to describe the subdirectly irreducible IKt-algebras and in particular the simple IKt-algebras. Finally in this section, we describe the simple and subdirectly irreducible IKt-algebras in the cases of finite algebras and complete algebras.

2. Topological Duality for IKt-Algebras

In this section, we will develop a topogical duality for IKt-algebras taking into account the results established by Dzik et al. [17]. In order to determine this duality, we introduce a topological category whose objects and their corresponding morphisms will be describe below.

Definition 2.1. An IKt-space is a system $(X, \leq, R)$ where $(X, \leq)$ is a Heyting space, R is a binary relation on X and R^{-1} is the converse of R such that the following conditions are satisfied:

(tS1) for each $x \in X$, $R(x)$ and $R^{-1}(x)$ are closed subsets of X,

(tS2) for each $x \in X$, $R(x) = \downarrow R(x) \cap R(\uparrow x)$, (tS3) for each $x \in X$, $R(\uparrow x)$ and $R^{-1}(\uparrow x)$ are increasing and closed subsets of X,

(tS4) for each $U \in D(X)$, $G_R(U), H_{R^{-1}}(U), F_R(U), P_{R^{-1}}(U) \in D(X)$, where G_R, $H_{R^{-1}}, F_R$ and $P_{R^{-1}}$ are operators on $\mathcal{P}(X)$ defined for all $Y \subseteq X$ as follows:
$G_R(U) = \{x \in X \mid \ R(\uparrow x) \subseteq U\}$,
$H_{R^{-1}}(U) = \{x \in X \mid \ R^{-1}(\uparrow x) \subseteq U\}$,
$F_R(U) = \{x \in X \mid R(x) \cap U \neq \emptyset\}$,
$P_{R^{-1}}(U) = \{x \in X \mid \ R^{-1}(x) \cap U \neq \emptyset\}$.

DEFINITION 2.2. An IKt-function from an IKt-space $(X_1, \leq_1, R_1)$ into another one, $(X_2, \leq_2, R_2)$, is a p-function $f : X_1 \longrightarrow X_2$, which satisfies the following conditions:

(tf1) $f(R_1(x)) \subseteq R_2(f(x))$,

(tf2) $R_2(\uparrow f(x)) = f(R_1(\uparrow x))$,

(tf3) ${R_2}^{-1}(\uparrow f(x)) = f({R_1}^{-1}(\uparrow x))$.

The category that has IKt-spaces as objects and IKt-functions as morphisms will be denoted by **IKtS**. By **IKtA**, we denote the category of IKt-algebras and IKt-homomorphisms. Our next task will be given to determine that the category **IKtS** is naturally equivalent to the dual category of **IKtA**.

Now we shall show a characterization of IKt-functions which will be useful later.

LEMMA 2.1. *Let $(X_1, \leq_1, R_1)$ and $(X_2, \leq_2, R_2)$ be two IKt-spaces. Then, the following conditions are equivalents:*

(i) *$f : X_1 \longrightarrow X_2$ is an IKt-function,*

(ii) *$f : X_1 \longrightarrow X_2$ is a p-function such that:*

(tf1′) *$(x, y) \in R_1$ implies $(f(x), f(y)) \in R_2$ for any $x, y \in X_1$,*
(tf2′) *$f^{-1}(G_{R_2}(U)) = G_{R_1}(f^{-1}(U))$ for any $U \in D(X_2)$,*
(tf3′) *$f^{-1}(H_{R_2^{-1}}(U)) = H_{R_1^{-1}}(f^{-1}(U))$ for any $U \in D(X_2)$.*

PROOF. (i) $\Rightarrow$ (ii):
(tf1′): Assume that $(x, y) \in R_1$. So, $f(y) \in f(R_1(x))$. Then, by (tf1), we have that $f(y) \in R_2(f(x))$, i.e., $(f(x), f(y)) \in R_2$.
(tf2′): Let $x \in f^{-1}(G_{R_2}(U))$. Hence, $R_2(\uparrow f(x)) \subseteq U$. By (tf2), we have that $f(R_1(\uparrow x)) \subseteq U$. Since $R_1(\uparrow x) \subseteq f^{-1}(f(R_1(\uparrow x)))$ we obtain $R_1(\uparrow$

$x) \subseteq f^{-1}(U)$. Thus, $x \in G_{R_1}(f^{-1}(U))$. On the other hand, suppose that $x \in G_{R_1}(f^{-1}(U))$. Hence, $R_1(\uparrow x) \subseteq f^{-1}(U)$. Since $f(f^{-1}(U)) \subseteq U$ we obtain $f(R_1(\uparrow x)) \subseteq U$. By (tf2), we have that $R_2(\uparrow f(x)) \subseteq U$. Thus, $x \in f^{-1}(G_{R_2}(U))$. (tf3′) can be proved in a similar way.

(ii) $\Rightarrow$ (i):

(tf1): Let $z \in f(R_1(x))$. Then, there exists $y \in R_1(x)$ such that $f(y) = z$. Since $(x, y) \in R_1$, by (tf1), we have that $(f(x), z) \in R_2$. Thus, $z \in R_2(f(x))$.

(tf2): Let $y \in R_2(\uparrow f(x))$. Suppose that for any $z \in R_1(\uparrow x)$ it is verified that $f(z) \not\leq_2 y$. Then for any $z \in R_1(\uparrow x)$ there exists $U_z \in D(X_2)$ such that $f(z) \in U_z$ and $y \notin U_z$. As X_1 is an IKt-space we infer that $R_1(\uparrow x)$ is compact in X_1. Since f is a continuous function we have that $f(R_1(\uparrow x))$ is compact in X_2. Consequently, there exists $U \in D(X_2)$ such that $f(R_1(\uparrow x)) \subseteq U$ and $y \notin U$. Hence, we obtained that $f(x) \notin G_{R_2}(U)$ and therefore $x \notin f^{-1}(G_{R_2}(U))$. By (tf2′), we have that $x \notin G_{R_1}(f^{-1}(U))$. From this statement there exists $z \in R_1(\uparrow x)$ such that $z \notin f^{-1}(U)$. Thus, $f(R_1(\uparrow x)) \not\subseteq U$, which is a contradiction. The inclusion $f(R_1(\uparrow x)) \subseteq R_2(\uparrow f(x))$ follows by (tf1′) and the fact that f is an increasing function. (tf3) can be proved in a similar way. ■

Next, we will describe some properties of IKt-spaces which will be quite useful for determining the duality for IKt-algebras that we are interested. For this purpose, recall the notion of IKt-frame, which was introduced in [26] (see also [17]) as we indicate below:

Definition 2.3. A structure $\langle X, \leq, R\rangle$ is an IKt-frame if $\langle X, \leq\rangle$ is a quasi-ordered set, R is a binary relation on X, and R^{-1} is the converse of R such that:

(K1) $R(\downarrow x) \subseteq\downarrow R(x)$ for any $x \in X$,

(K2) $R^{-1}(\downarrow x) \subseteq\downarrow R^{-1}(x)$ for any $x \in X$.

Remark 2.1. In every IKt-frame $\langle X, \leq, R\rangle$ the following conditions are satisfied:

(K3) $\uparrow R^{-1}(x) \subseteq R^{-1}(\uparrow x)$ for any $x \in X$,

(K4) $\uparrow R(x) \subseteq R(\uparrow x)$ for any $x \in X$.

Proposition 2.1. *If $(X, \leq, R)$ is an IKt-space, then $\langle X, \leq, R\rangle$ is an IKt-frame.*

Proof. We will only prove (K1). Let $x, z \in X$ such that $z \in R(\downarrow x)$. Suppose that $z \notin\downarrow R(x)$. Then, for all $y \in R(x)$, $z \not\leq y$. As X is a Priestley space, for each $y \in R(x)$ there exists $U_y \in D(X)$ such that $y \notin U_y$ and $z \in U_y$. Then $R(x) \subseteq \bigcup_{y \in R(x)}(X \backslash U_y)$. Since $R(x)$ is compact, from the preceding assertion we infer that there is $U \in D(X)$ such that $z \in U$ and $R(x) \cap U = \emptyset$. But as $z \in R(\downarrow x)$, then there exists $w \in X$ such that $w \leq x$ and $z \in R(w)$. It follows that $z \in R(w) \cap U$, i.e., $w \in F_R(U)$. Since $F_R(U)$ is an increasing set we have that $x \in F_R(U)$, which is a contradiction. Similarly, we can prove that $(X, \leq, R)$ satisfies (K2). ■

Next we will define a contravariant functor from **IKtS** onto **IKtA**.

LEMMA 2.2. *Let $(X, \leq, R)$ be an IKt-space. Then,*

$$\Psi(X) = (D(X), G_R, H_{R^{-1}}, F_R, P_{R^{-1}})$$

is an IKt-algebra.

PROOF. By virtue of the results established in [18], $D(X)$ is a Heyting algebra. Besides, from (tS4), $D(X)$ is closed under the operations G_R, $H_{R^{-1}}$, F_R and $P_{R^{-1}}$. From the definition of the operations G_R, $H_{R^{-1}}$, F_R and $P_{R^{-1}}$ we infer that (t1), (t2), (t4) and (t5) hold. Then, we only have to prove that $D(X)$ satisfies the following remaining axioms:
(t3): It is verified that $U \subseteq G_R(P_{R^{-1}}(U))$. Indeed, let $x, y \in X$ such that $x \in U$ and $y \in R(\uparrow x)$. Then, there exists $z \in \uparrow x$ such that $(z, y) \in R$. Since $U \in D(X)$ we have that $z \in U$ and since $z \in R^{-1}(y)$, we obtain that $R^{-1}(y) \cap U \neq \emptyset$, that is, $y \in P_{R^{-1}}(U)$. In a similar way, we can prove that $U \subseteq H_{R^{-1}}(F_R(U))$. Therefore, (t3) holds.
(t6): We have that $P_{R^{-1}}(G_R(U)) \subseteq U$. Indeed, let $x \in P_{R^{-1}}(G_R(U))$. Then, $R^{-1}(x) \cap G_R(U) \neq \emptyset$. Hence, there exists $z \in R^{-1}(x)$ such that $R(\uparrow z) \subseteq U$. Since $\leq$ is reflexive we have that $(z, x) \in (\leq \circ R)$, i.e., $x \in R(\uparrow z)$. Therefore, $x \in U$. In a similar way, we can prove that $F_R(H_{R^{-1}}(U)) \subseteq U$. Then, (t6) holds.
(t7): Suppose that $x \in F_R(U \to V)$, $x \leq z$ and $z \in G_R(U)$. Let us to prove that $z \in F_R(V)$, i.e. , $R(z) \cap V \neq \emptyset$. Since $x \in F_R(U \to V)$ then there exists $y \in R(x) \cap (U \to V)$, $z \geq x$, $(x, y) \in R$, and so from (K3) we have that $(z, v) \in R$ and $v \geq y$ for some $v \in X$. Since $v \in R(z) \subseteq R(\uparrow z) \subseteq U$, $y \leq v$ and $y \in U \to V$ then $v \in V$. Therefore, $R(z) \cap V \neq \emptyset$. In a similar way we can prove $P_{R^{-1}}(U \to V) \subseteq H_{R^{-1}}(U) \to P_{R^{-1}}(V)$. Finally, (t7) holds. ■

LEMMA 2.3. *Let $f : (X_1, \leq_1, R_1) \longrightarrow (X_2, \leq_2, R_2)$ be a morphism of IKt-spaces. Then, $\Psi(f) : D(X_2) \longrightarrow D(X_1)$, defined by $\Psi(f)(U) = f^{-1}(U)$ for all $U \in D(X_2)$, is an IKt-homomorphism.*

PROOF. It follows from the results established in [18] and Lemma 2.1. ■

The previous two lemmas show that Ψ is a contravariant functor from **IKtS** to **IKtA**. To achieve our goal we need to define a contravariant functor from **IKtA** to **IKtS**.

LEMMA 2.4. ([17]) *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra and let R^A be the relation defined on $\mathfrak{X}(A)$ by the prescription:*

$$(S,T) \in R^A \iff G^{-1}(S) \subseteq T \subseteq F^{-1}(S). \tag{VIII}$$

Then, for all $S, T \in \mathfrak{X}(A)$,

$$(S,T) \in R^A \iff H^{-1}(T) \subseteq S \subseteq P^{-1}(T). \tag{IX}$$

Lemma 2.4 means that we have two ways to define the relation R^A, either by using G and F, or by using H and P.

LEMMA 2.5. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra and $\mathfrak{X}(A)$ be the associated Priestley space of A. Then,*

(i) *for all $S \in \mathfrak{X}(A)$, $R^A(S)$ is closed in $\mathfrak{X}(A)$,*

(ii) *R^A is closed in $\mathfrak{X}(A) \times \mathfrak{X}(A)$,*

(iii) *for all $S \in \mathfrak{X}(A)$, ${R^A}^{-1}(S)$ is closed in $\mathfrak{X}(A)$,*

(iv) *${R^A}^{-1}$ is closed in $\mathfrak{X}(A) \times \mathfrak{X}(A)$.*

PROOF. We will only prove (i) and (ii). Similarly we can prove (iii) and (iv).
(i): Suppose that $T \notin R^A(S)$. Then, by definition of the relation R^A, there exists $x \in G^{-1}(S)$ such that $x \notin T$, or there exists $y \in T$ such that $y \notin F^{-1}(S)$. In the first case, $T \notin \sigma_A(x)$ and $G^{-1}(S) \in \sigma_A(x)$. Then taking into account that $\sigma_A(x)$ is an increasing set we infer that $R^A(S) \subseteq \sigma_A(x)$. From this assertion we deduce that $T \in \sigma_A(x)^c$ and $\sigma_A(x)^c \subseteq R^A(S)^c$. In the second case, $T \in \sigma_A(y)$ and $F^{-1}(S) \in \sigma_A(y)^c$. Since $\sigma_A(y)^c$ is a decreasing set we infer that $R^A(S) \subseteq \sigma_A(y)^c$. So, $T \in \sigma_A(y)$ and $\sigma_A(y) \subseteq R^A(S)^c$ and the proof is complete.
(ii): Let S, $T \in \mathfrak{X}(A)$ such that $(S,T) \notin R^A$. So, $T \notin R^A(S)$. Then, by (i) we have that:
(a) there exists $x \in A$ such that $S \in \sigma_A(G(x))$, $T \in \sigma_A(x)^c$ and $\sigma_A(x)^c \cap R^A(S) = \emptyset$ or
(b) there exists $y \in X$ such that $T \in \sigma_A(y)$, $S \in \sigma_A(F(y))^c$ and $\sigma_A(y) \cap R^A(S) = \emptyset$.

In the case (a), we have that $(S,T) \in \sigma_A(G(x)) \times \sigma_A(x)^c$ and $R^A \cap (\sigma_A(G(x)) \times \sigma_A(x)^c) = \emptyset$. In the case (b), we have that $(S,T) \in \sigma_A(F(y))^c \times \sigma_A(y)$ and $R^A \cap (\sigma_A(F(y))^c \times \sigma_A(y)) = \emptyset$.

Therefore, from the last assertions in both cases we can conclude that R^A is closed in $\mathfrak{X}(A) \times \mathfrak{X}(A)$. ■

LEMMA 2.6. *Let* $(\mathcal{A}, G, H, F, P)$ *be an* IKt*-algebra and* $\mathfrak{X}(A)$ *be the associated Priestley space of* A*. Then,*

(i) *for all closed subset* M *of* $\mathfrak{X}(A)$, $R^A(M)$ *is closed in* $\mathfrak{X}(A)$.

(ii) *for all closed subset* M *of* $\mathfrak{X}(A)$, $R^{A^{-1}}(M)$ *is closed in* $\mathfrak{X}(A)$.

PROOF. (i): Let $M \subseteq \mathfrak{X}(A)$ closed. Then, $R^A(M) = p_2(R^A \cap (M \times \mathfrak{X}(A)))$, where $p_2 : \mathfrak{X}(A) \times \mathfrak{X}(A) \to \mathfrak{X}(A)$ is the function defined by $p_2(x, y) = y$ for all $(x, y) \in \mathfrak{X}(A) \times \mathfrak{X}(A)$. From Lemma 2.5 we have that $R^A \cap (M \times \mathfrak{X}(A))$ is a closed subset of $\mathfrak{X}(A) \times \mathfrak{X}(A)$. Since $\mathfrak{X}(A)$ is a compact and Hausdorff space, we have that p_2 is a closed function, from which it follows that $p_2(R^A \cap (M \times \mathfrak{X}(A)))$ is closed in $\mathfrak{X}(A)$, i.e., $R^A(M)$ is closed in $\mathfrak{X}(A)$. (ii) can be proved in a similar way to (i). ■

COROLLARY 2.4. *Let* $(\mathcal{A}, G, H, F, P)$ *be an* IKt*-algebra and* $\mathfrak{X}(A)$ *be the associated Priestley space of* A*. Then, for all* $S \in \mathfrak{X}(A)$, $R^A(\uparrow S)$ *and* $R^{A^{-1}}(\uparrow S)$ *are closed subsets of* $\mathfrak{X}(A)$.

PROOF. Since $\uparrow S$ is a closed subset of $\mathfrak{X}(A)$, by Lemma 2.6, we have that $R^A(\uparrow S)$ and $R^{A^{-1}}(\uparrow S)$ are closed in $\mathfrak{X}(A)$. ■

The following lemma, whose proof can be found in [17, Lemma 4.5], will be essential for the proof of Lemma 2.10.

LEMMA 2.7. *Let* $(\mathcal{A}, G, H, F, P)$ *be an* IKt*-algebra and* $(\mathfrak{X}(A), \subseteq, R^A)$ *be the* IKt*-space associated with* $(\mathcal{A}, G, H, F, P)$*. Then, for all* $S, T \in \mathfrak{X}(A)$*, the following properties are verified:*

(a) $(S, T) \in (\subseteq \circ R^A) \iff G^{-1}(S) \subseteq T$,

(b) $(S, T) \in (R^A \circ \supseteq) \iff T \subseteq F^{-1}(S)$,

(c) $(S, T) \in (R^A \circ \supseteq) \iff H^{-1}(S) \subseteq T$,

(d) $(S, T) \in (\subseteq \circ R^A) \iff T \subseteq P^{-1}(S)$.

LEMMA 2.8. *Let* $(\mathcal{A}, G, H, F, P)$ *be an* IKt*-algebra and let* $S \in \mathfrak{X}(A)$ *and* $a \in A$*. Then,*

(i) $G(a) \notin S$ *iff there exists* $T \in \mathfrak{X}(A)$ *such that* $(S, T) \in (\subseteq \circ R^A)$ *and* $a \notin T$,

(ii) $H(a) \notin S$ *iff there exists* $T \in \mathfrak{X}(A)$ *such that* $(S, T) \in (\subseteq \circ R^{A^{-1}})$ *and* $a \notin T$,

(iii) $F(a) \in S$ *iff there exists* $T \in \mathfrak{X}(A)$ *such that* $(S,T) \in (R^A \circ \supseteq)$ *and* $a \in T$,

(iv) $P(a) \in S$ *iff there exists* $T \in \mathfrak{X}(A)$ *such that* $(S,T) \in (R^{A^{-1}} \circ \supseteq)$ *and* $a \in T$.

Proof. (i):
($\Rightarrow$): Suppose that $G(a) \notin S$. Then, $(a] \cap G^{-1}(S) = \emptyset$. So, by the Birkhoff–Stone theorem, there exists $T \in \mathfrak{X}(A)$ such that $G^{-1}(S) \subseteq T$ and $a \notin T$. Hence, by Lemma 2.7, $(S,T) \in (\subseteq \circ R^A)$.
($\Leftarrow$): Suppose that $(S,T) \in (\subseteq \circ R^A)$ and $a \notin T$, for some $S \in \mathfrak{X}(A)$. By Lemma 2.7, we have that $G^{-1}(S) \subseteq T$. It follows that $G(a) \notin S$. In a similar way, we can prove (ii).
(iii):
($\Rightarrow$): Suppose that $F(a) \in S$. Then, $[a) \cap F^{-1}(S)^c = \emptyset$. So, by the Birkhoff–Stone theorem, there exists $T \in \mathfrak{X}(A)$ such that $T \subseteq F^{-1}(S)$ and $a \in T$. Hence, by Lemma 2.7, $(S,T) \in (R^A \circ \supseteq)$.
($\Leftarrow$): Suppose that $(S,T) \in (R^A \circ \supseteq)$ and $a \in T$, for some $S \in \mathfrak{X}(A)$. From Lemma 2.7, we have that $T \subseteq F^{-1}(S)$. We conclude that $F(a) \in S$. In a similar way we can prove (iv). ∎

Lemma 2.9. *Let* $(\mathcal{A}, G, H, F, P)$ *be an* IKt*-algebra. Then,* $\Phi(A) = (\mathfrak{X}(A), \subseteq, R^A)$ *is an* IKt*-space and* $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$ *is an* IKt*-isomorphism.*

Proof. From the duality for Heyting algebras, we have that $(\mathfrak{X}(A), \subseteq)$ is a Heyting space. First, we will prove that the following assertions hold for all $a \in A$:

$$(*) \quad G_{R^A}(\sigma_A(a)) = \sigma_A(G(a));\ F_{R^A}(\sigma_A(a)) = \sigma_A(F(a));$$
$$H_{R^{A^{-1}}}(\sigma_A(a)) = \sigma_A(H(a));\ P_{R^{A^{-1}}}(\sigma_A(a)) = \sigma_A(P(a)).$$

$G_{R^A}(\sigma(a)) = \sigma_A(G(a))$: Let us take a prime filter S such that $G(a) \notin S$. By Lemma 2.8, there exists $T \in \mathfrak{X}(A)$ such that $(S,T) \in (\subseteq \circ R^A)$ and $a \notin T$. Then, $R^A(\uparrow S) \not\subseteq \sigma_A(a)$. So, $S \notin G_{R^A}(\sigma_A(a))$ and, therefore, $G_{R^A}(\sigma(a)) \subseteq \sigma_A(G(a))$. Moreover, it is immediate that $\sigma_A(G(a)) \subseteq G_{R_A}(\sigma_A(a))$.
$F_{R^A}(\sigma_A(a)) = \sigma_A(F(a))$: Suppose that $S \in F_{R^A}(\sigma_A(a))$. From this there exists $T \in \mathfrak{X}(A)$ such that $(S,T) \in R^A$ and $a \in T$. Since $G^{-1}(S) \subseteq T \subseteq F^{-1}(S)$, we have that $F(a) \in S$, that is, $S \in \sigma_A(F(a))$ and therefore $F_{R^A}(\sigma_A(a)) \subseteq \sigma_A(F(a))$. On the other hand, suppose that $S \in \sigma_A(F(a))$, that is, $F(a) \in S$. Then, by Lemma 2.8, there exists $T \in \mathfrak{X}(A)$ such that $(S,T) \in (R^A \circ \supseteq)$ and $a \in T$. Hence, there exists $Z \in \mathfrak{X}(A)$ such that $(S,Z) \in R^A$ and $T \subseteq Z$. From which it follows that $Z \in R^A(S) \cap \sigma_A(a)$. Therefore, $S \in F_{R^A}(\sigma_A(a))$, from which we conclude $\sigma_A(F(a)) \subseteq$

$F_{R^A}(\sigma_A(a))$. In a similar way we can prove that $H_{R^{A^{-1}}}(\sigma_A(a)) = \sigma_A(H(a))$ and $P_{R^{A^{-1}}}(\sigma_A(a)) = \sigma_A(P(a))$. Next, we will show that $(\mathfrak{X}(A), \subseteq, R^A)$ satisfies the axioms (tS1), (tS2), (tS3) and (tS4).

(tS1): By Lemma 2.5, $R^A(S)$ and $R^{A^{-1}}(S)$ are closed subsets of $\mathfrak{X}(A)$ for all $S \in \mathfrak{X}(A)$.

(tS2): For any $S \in \mathfrak{X}(A)$, $R^A(S) =\downarrow R^A(S) \cap R^A(\uparrow S)$. Indeed, since $R^A(S) \subseteq\downarrow R^A(S)$ and $R^A(S) \subseteq R^A(\uparrow S)$ we have that $R^A(S) \subseteq\downarrow R^A(S) \cap R^A(\uparrow S)$. On the other hand, let $T \in R^A(\uparrow S) \cap \downarrow R^A(S)$. Then, there exists $S_1, S_2 \in \mathfrak{X}(A)$ such that $S \subseteq S_1$, $S_1 R^A T$, $S R^A S_2$ and $S_2 \supseteq T$. Hence, $G^{-1}(S) \subseteq G^{-1}(S_1) \subseteq T$ and $T \subseteq S_2 \subseteq F^{-1}(S)$. Therefore, $T \in R^A(S)$.

(tS3): By Corollary 2.4 it is verified that for each $S \in \mathfrak{X}(A)$, $R^A(\uparrow S)$ and $R^{A^{-1}}(\uparrow S)$ are closed subsets of $\mathfrak{X}(A)$. Then, we only need to prove that $R^A(\uparrow S)$ and $R^{A^{-1}}(\uparrow S)$ are increasing subsets of $\mathfrak{X}(A)$ for all $S \in \mathfrak{X}(A)$.

Let $T \in R^A(\uparrow S)$ and $W \in \mathfrak{X}(A)$ such that $T \subseteq W$. Then, there exists $V \in \mathfrak{X}(A)$ such that $S \subseteq V$ and $T \in R^A(V)$. From this we obtain that $W \in\uparrow R^A(V)$. By (K4) we have that $W \in R^A(\uparrow V)$. On the other hand, we infer that $\uparrow V \subseteq\uparrow S$ and in consequence $R^A(\uparrow V) \subseteq R^A(\uparrow S)$, from which we conclude that $W \in R^A(\uparrow S)$. Therefore, $R^A(\uparrow S)$ is an increasing subset of $\mathfrak{X}(A)$.

(tS4): For any $U \in D(\mathfrak{X}(A))$ there exists $a \in A$ such that $U = \sigma_A(a)$. Then, the equalities $(*)$ allow us to affirm that $G_{R^A}(U), F_{R^A}(U), H_{R^{A^{-1}}}(U), P_{R^{A^{-1}}}(U) \in D(\mathfrak{X}(A))$. So, $(\mathfrak{X}(A), \subseteq, R^A)$ is an IKt-space. By Lemma 2.2 we have that $D(\mathfrak{X}(A))$ is an IKt-algebra. By virtue of the results established in [18, Proposition 5.8] and the assertions $(*)$ we conclude that σ_A is an IKt-isomorphism. ∎

Lemma 2.10. *Let $(\mathcal{A}_1, G_1, H_1, F_1, P_1)$ and $(\mathcal{A}_2, G_2, H_2, F_2, P_2)$ be IKt-algebras and $h : A_1 \longrightarrow A_2$ be an IKt-homomorphism. Then, the application $\Phi(h) : \mathfrak{X}(A_2) \longrightarrow \mathfrak{X}(A_1)$, defined by $\Phi(h)(S) = h^{-1}(S)$ for all $S \in \mathfrak{X}(A_2)$, is an IKt-function.*

Proof. From the duality for Heyting algebras, it holds that the application $\Phi(h) : \mathfrak{X}(A_2) \to \mathfrak{X}(A_1)$ is a p-function. We will only prove (tf1) and (tf2); (tf3) can be proved in a similar way to (tf2).

(tf1): Let $S, T \in \mathfrak{X}(A_2)$. Let us prove that if $(S,T) \in R^{A_2}$, then $G^{-1}(h^{-1}(S)) \subseteq h^{-1}(T) \subseteq F^{-1}(h^{-1}(S))$. Suppose that $a \in G^{-1}(h^{-1}(S))$. Then we have that $h(G(a)) = G(h(a)) \in S$, from which it follows that $h(a) \in S$, i.e., $a \in h^{-1}(S)$. In a similar way we can prove $h^{-1}(T) \subseteq F^{-1}(h^{-1}(S))$.

(tf2): Let $S \in \mathfrak{X}(A_1)$ and $T \in \mathfrak{X}(A_2)$ such that $(h^{-1}(S), T) \in (\subseteq \circ R^{A_2})$. By Lemma 2.7, $G^{-1}(h^{-1}(S)) \subseteq T$. Suppose that $h(T^c) \cap [G^{-1}(S)) \neq \emptyset$. Then, there exists $a, b \in A$ such that $a \notin T$, $G(b) \in S$ and $b \leq h(a)$. From this last assertion we obtain that $G(b) \leq Gh(a) = h(G(a))$. Since, $G(b) \in S$ we have that $h(G(a)) \in S$. So, $a \in G^{-1}(H^{-1}(S)) \subseteq T$, which is a contradiction. Therefore, $h(T^c) \cap [G^{-1}(S)) = \emptyset$. Then, by the Birkhoff–Stone Theorem, there is a $Z \in \mathfrak{X}(A_1)$ such that $[G^{-1}(S)) \subseteq Z$ and $Z \cap h(T^c) = \emptyset$. From which we conclude $G^{-1}(S) \subseteq Z$ and $h^{-1}(Z) \subseteq T$. Then, $(S, Z) \in (\subseteq \circ R^{A_1})$ and $h^{-1}(Z) \subseteq T$, which completes the proof. ■

Lemmas 2.9 and 2.10 show that Φ is a contravariant functor from **IKtA** to **IKtS**.

The following characterization of isomorphisms in the category **IKtS** will be used to determine the duality we were looking for.

PROPOSITION 2.2. *Let $(X_1, \leq_1, R_1)$ and $(X_2, \leq_2, R_2)$ be two IKt-spaces. Then, for every function $f : X_1 \longrightarrow X_2$ the following conditions are equivalents:*

(i) *f is an isomorphism in the category* **IKtS**,

(ii) *f is a bijective p-function such that for all $x, y \in X_1$:*

(itf) $(x, y) \in R_1 \iff (f(x), f(y)) \in R_2$.

PROOF. It is routine. ■

The application $\varepsilon_X : X \longrightarrow \mathfrak{X}(D(X))$ defined by the prescription III leads to another characterization of IKt-space, which also allow us to assert that this application is an isomorphism in the category **IKtS**, as we will describe below:

LEMMA 2.11. *Let $(X, \leq, R)$ be an IKt-space, $\varepsilon_X : X \longrightarrow \mathfrak{X}(D(X))$ defined the prescription III and let $R^{D(X)}$ be the relation defined on $\mathfrak{X}(D(X))$ by means of the operators G_R and F_R as follows:*

$$(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \iff G_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_X(y) \subseteq F_R^{-1}(\varepsilon_X(x)). \quad \text{(X)}$$

Then, the following property holds:

(tS5) $(x, y) \in R$ *implies* $(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)}$.

PROOF. Let us consider $x, y \in X$ such that $(x, y) \in R$ and prove that $G_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_X(y) \subseteq F_R^{-1}(\varepsilon_X(x))$. Let $U \in D(X)$ such that $G_R(U) \in \varepsilon_X(x)$. Then, $R(\uparrow x) \subseteq U$. Since $R(x) \subseteq R(\uparrow x)$ we have that $R(x) \subseteq U$. This last assertion allows us to infer that $U \in \varepsilon_X(y)$. Therefore, $G_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_X(y)$. Conversely, let us consider $V \in \varepsilon_X(y)$. Then, $y \in R(x) \cap V$. So $x \in F_R(V)$, i.e., $F_R(V) \in \varepsilon_X(x)$. Therefore, we have that $\varepsilon_X(y) \subseteq F_R^{-1}(\varepsilon_X(x))$. ■

Proposition 2.3. *Let* $(X, \leq, R)$ *be an* IKt*-space and let* $R^{D(X)}$ *be the relation defined on* $\mathfrak{X}(D(X))$ *by the prescription* X. *Then, the condition* (tS2) *can be replaced by the following one:*

(tS6) $(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \Longleftrightarrow (x, y) \in R$.

Proof. (tS2) $\Rightarrow$ (tS6): Let $x, y \in X$ such that $y \notin R(x)$. From (tS2) we have that $y \notin R(\uparrow x)$ or $y \notin \downarrow R(x)$. In the first case, by (tS3), we obtain that $R(\uparrow x) \subseteq U$ and $y \notin U$ for some $U \in D(X)$. Then, $x \in G_R(U)$ and it follows that $U \in G_R^{-1}(\varepsilon_X(x))$ and $U \notin \varepsilon_X(y)$. Therefore, $(\varepsilon_X(x), \varepsilon_X(y)) \notin R^{D(X)}$. In the other case, we have that $y \in U$ and $R(x) \cap U = \emptyset$ for some $U \in D(X)$. So, $x \notin F_R(U)$. This last assertion allows us to infer that $U \notin F_R^{-1}(\varepsilon_X(x))$ and $U \in \varepsilon_X(y)$. Therefore, $(\varepsilon_X(x), \varepsilon_X(y)) \notin R^{D(X)}$.
(tS6) $\Rightarrow$ (tS2): We have to prove $\downarrow R(x) \cap R(\uparrow x) \subseteq R(x)$; the other inclusion always holds. Suppose that $y \in \downarrow R(x) \cap R(\uparrow x)$. Then there exists $z_1, z_2 \in X$ such that $x \leq z_1$, $z_1 \, R \, y$, $x \, R \, z_2$ and $z_2 \geq y$. Hence, by the property (tS5) in Lemma 2.11, $\varepsilon_X(x) \subseteq \varepsilon_X(z_1)$, $\varepsilon_X(z_1) \; R^{D(X)} \; \varepsilon_X(y)$, $\varepsilon_X(x) \; R^{D(X)} \; \varepsilon_X(z_2)$ and $\varepsilon_X(z_2) \supseteq \varepsilon_X(y)$. So $G_R^{-1}(\varepsilon_X(x)) \subseteq G_R^{-1}(\varepsilon_X(z_1)) \subseteq \varepsilon_X(y)$ and $\varepsilon_X(y) \subseteq \varepsilon_X(z_2) \subseteq F_R^{-1}(\varepsilon_X(x))$. Hence, $(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)}$, and so by assumption it follows that $(x, y) \in R$. ∎

Corollary 2.5. *Let* $(X, \leq, R)$ *be an* IKt*-space. Then, the application* $\varepsilon_X : X \longrightarrow \mathfrak{X}(D(X))$ *is an isomorphism in the category* **IKtS**.

Proof. It follows from the results established in [18], Lemma 2.11, Propositions 2.2 and 2.3. ∎

Then, from the above results and using the usual procedures we can prove that the functors $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are naturally equivalent to the identity functors on **IKtS** and **IKtA**, respectively, being the families $\{\sigma_A(a) : a \in A\}$ and $\{\varepsilon_X(x) : x \in X\}$ the natural equivalence in each case, from which we conclude

Theorem 2.6. *The category* **IKtS** *is naturally equivalent to the dual of the category* **IKtA**.

3. Simple and Subdirectly Irreducible IKt-Algebras

In this section, our first objective is the characterization of the congruence lattice on an IKt-algebra by means of certain closed and increasing subsets of its associated IKt-space, which allows us to describe the congruences on IKt-algebras. Later, this result will be taken into account to obtain simple and subdirectly irreducible IKt-algebras. With this purpose, we shall start by introducing the following notion.

DEFINITION 3.1. Let $(X, \leq, R)$ be an IKt-space. A subset Y of X is an IKt-subset if it satisfies the following conditions for all $y, z \in X$:

(IKt1) if $y \in Y$ and $z \in R(\uparrow y)$, then, there is $w \in Y$ such that $w \in R(\uparrow y)$ and $w \leq z$,

(IKt2) if $y \in Y$ and $z \in R^{-1}(\uparrow y)$, then, there is $v \in Y$ such that $v \in R^{-1}(\uparrow y)$ and $v \leq z$.

The notion of an increasing IKt-subset of an IKt-space has several equivalent formulations, which will be useful later:

PROPOSITION 3.1. *Let $(X, \leq, R)$ be an IKt-space. If Y is an increasing subset of X, then, the following conditions are equivalent:*

(i) *Y is an IKt-subset,*

(ii) *for all $y \in Y$, the following conditions are satisfied:*

(IKt3) $R(\uparrow y) \subseteq Y$,
(IKt4) $R^{-1}(\uparrow y) \subseteq Y$,

(iii) *$Y = G_R(Y) \cap Y \cap H_{R^{-1}}(Y)$, where $G_R(Y) = \{x \in X : R(\uparrow x) \subseteq Y\}$ and $H_{R^{-1}}(U) = \{x \in X : R^{-1}(\uparrow x) \subseteq Y\}$.*

PROOF. (i) $\Rightarrow$ (ii): Let $y \in Y$ and $z \in R(\uparrow y)$, then by (IKt1) there is $w \in Y$ such that $w \in R(\uparrow y)$ and $w \leq z$. Since Y is increasing, it follows that $z \in Y$ and therefore $R(\uparrow y) \subseteq Y$. The proof that $R^{-1}(\uparrow y) \subseteq Y$ is similar

(ii) $\Rightarrow$ (i): It is immediate

(ii) $\Leftrightarrow$ (iii): It is immediate ■

The closed and increasing IKt-subsets of the IKt-space associated with an IKt-algebra perform a fundamental roll in the characterization of the IKt-congruences on these algebras as we shall show next.

THEOREM 3.2. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra, and $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. Then, the lattice $\mathcal{C}_{IT}(\mathfrak{X}(A))$ of closed and increasing IKt-subsets of $\mathfrak{X}(A)$ is isomorphic to the dual lattice $Con_{IKt}(\mathcal{A})$ of IKt-congruences on $\mathcal{A}$, and the isomorphism is the function Θ_{IT} defined by the same prescription as in Theorem 1.1.*

PROOF. By the results established in Theorem 1.1, for the completion of the proof it remains to show that Θ_{IT} is an application from $\mathcal{C}_{IT}(\mathfrak{X}(A))$ onto

$Con_{IKt}(\mathcal{A})$. Let $Y \in \mathcal{C}_{IT}(\mathfrak{X}(A))$, then, from Theorem 1.1 it follows that $\Theta_{IT}(Y)$ is a Heyting congruence. In what follows we only prove that this congruence preserves the operations G and F. Let a, $b \in A$ such that $(a, b) \in \Theta_{IT}(Y)$, then (1) $\sigma_A(a) \cap Y = \sigma_A(b) \cap Y$. Suppose that $S \in \sigma_A(G(a)) \cap Y$. Since σ_A is an IKt-isomorphism, we get that $S \in G_{R^A_G}(\sigma_A(a)) \cap Y$, from which it follows that $R^A(\uparrow S) \subseteq \sigma_A(a)$. Also, taking into account that $S \in Y$, the fact that Y is an increasing IKt-subset and Proposition 3.1, we obtain that $R^A(\uparrow S) \subseteq Y$. Therefore, $R^A(\uparrow S) \subseteq \sigma_A(a) \cap Y$. This last assertion and (1) allow us to infer that $R^A(\uparrow S) \subseteq \sigma_A(b)$ and consequently, $S \in G_{R^A}(\sigma_A(b))$. Since $G_{R^A}(\sigma_A(b)) = \sigma_A(G(b))$, we conclude that $S \in \sigma_A(G(b)) \cap Y$, and so $\sigma_A(G(a)) \cap Y \subseteq \sigma_A(G(b)) \cap Y$. The proof of the other inclusion is similar, which implies that $(G(a), G(b)) \in \Theta_{IT}(Y)$. Now suppose that $Q \in \sigma_A(F(a)) \cap Y$. Since the application σ_A is an IKt-isomorphism, then $Q \in F_{R^A}(\sigma_A(a)) \cap Y$, and so $R^A(Q) \cap \sigma_A(a) \neq \emptyset$, from which it follows that there is $T \in \sigma_A(a) \cap R^A(Q)$. Then, taking into account that Y is an increasing IKt-subset, $Q \in Y$, $T \in R^A(Q)$ and Proposition 3.1, we obtain that $T \in Y$. Consequently, $T \in \sigma_A(a) \cap Y$, and from (1) we have that $T \in \sigma_A(b) \cap Y$. Therefore, $T \in R^A(Q) \cap \sigma_A(b)$, which implies that $Q \in F_{R^A}(\sigma_A(b)) \cap Y$. Since $F_{R^A}(\sigma_A(b)) = \sigma_A(F(b))$, we conclude that $\sigma_A(F(a)) \cap Y \subseteq \sigma_A(F(b)) \cap Y$. The other inclusion is proved in a similar way, which implies that $(F(a), F(b)) \in \Theta_{IT}(Y)$. Analogously, $\Theta_{IT}(Y)$ preserves H and P and so $\Theta_{IT}(Y) \in Con_{IKt}(\mathcal{A})$. Conversely, let $\varphi \in Con_{IKt}(\mathcal{A})$ and $h : A \longrightarrow A/\varphi$ be the natural epimorphism. Since φ is a Heyting congruence on $\mathcal{A}$, then by Theorem 1.1, $Y = \{h^{-1}(M) : M \in \mathfrak{X}(\mathcal{A}/\varphi)\}$ is a closed and increasing subset of $\mathfrak{X}(A)$ and $\varphi = \Theta_I(Y)$. Then, it only remains to prove that Y is IKt-subset of $\mathfrak{X}(A)$. More precisely, by Proposition 3.1, we only have to prove that Y satisfies the following conditions:

(IKt3) For all $S \in Y$, $R^A(\uparrow S) \subseteq Y$: Indeed, let $S, T \in \mathfrak{X}(A)$ such that $S \in Y$ and $T \in R^A(\uparrow S)$. Also let $\Phi(h) : \mathfrak{X}(\mathcal{A}/\varphi) \longrightarrow \mathfrak{X}(A)$ be the application defined by $\Phi(h)(Q) = h^{-1}(Q)$ for all $Q \in \mathfrak{X}(\mathcal{A}/\varphi)$. Hence $Y = \Phi(h)(\mathfrak{X}(\mathcal{A}/\varphi))$ and $T \in R^A(\uparrow \Phi(h)(Q))$ for some $Q \in \mathfrak{X}(\mathcal{A}/\varphi)$. Besides, by Lemma 2.10, $\Phi(h)$ is an IKt-function. From the preceding assertions and the property (If2) of the IKt-functions, we infer that $T \in \Phi(h)(\uparrow R^{\mathcal{A}/\varphi}(Q))$. Consequently, since $\Phi(h)(\uparrow R^{\mathcal{A}/\varphi}(Q)) \subseteq Y$, we conclude that $T \in Y$ and therefore $R^A(\uparrow S) \subseteq Y$.

(IKt4) For all $S \in Y$, $R^{A^{-1}}(S) \subseteq Y$: it can be proved in a similar way to one used in the proof of (IKt3). Finally, we conclude that $Y = \{h^{-1}(M) : M \in \mathfrak{X}(A)/\varphi)\}$ is a closed and increasing IKt-subset of $\mathfrak{X}(A)$ and $\varphi = \Theta_{IT}(Y)$.

■

Next, we shall use the results already obtained in order to determine the simple and subdirectly irreducible IKt-algebras

Corollary 3.3. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra, and $(\mathfrak{X}(A), \subseteq, R^A)$ be the IKt-space associated with $\mathcal{A}$. Then, the following conditions are equivalent:*

(i) *$(\mathcal{A}, G, H, F, P)$ is a simple IKt-algebra,*

(ii) *$\mathcal{C}_{IT}(\mathfrak{X}(A)) = \{\emptyset, \mathfrak{X}(A)\}$.*

Proof. It is a direct consequence of Theorem 3.2. ■

Corollary 3.4. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra, and $(\mathfrak{X}(A), R^A)$ be the IKt-space associated with $\mathcal{A}$. Then, the following conditions are equivalent:*

(i) *$(\mathcal{A}, G, H, F, P)$ is a subdirectly irreducible IKt-algebra,*

(ii) *there is $Y \in \mathcal{C}_{IT}(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$ such that $Z \subseteq Y$ for all $Z \in \mathcal{C}_{IT}(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$.*

Proof. It is a direct consequence of Theorem 3.2. ■

The characterization of increasing IKt-subsets given in Proposition 3.1 suggests us to introduce the following definition:

Definition 3.5. Let $(X, \leq, R)$ be an IKt-space and let the function $d_X : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, defined for all $Z \in \mathcal{P}(X)$, by:

$$d_X(Z) = G_R(Z) \cap Z \cap H_{R^{-1}}(Z), \tag{XI}$$

For each $n \in \omega$, let $d_X^n : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, defined for all $Z \in \mathcal{P}(X)$, by:

$$d_X^0(Z) = Z,\ d_X^{n+1}(Z) = d_X(d_X^n(Z)). \tag{XII}$$

By using the above aplications d_X, d_X^n, $n \in \omega$, we obtain another equivalent formulation of the notion of increasing IKt-subset of an IKt-space.

Lemma 3.1. *Let $(X, \leq, R)$ be an IKt-space. If Y is an increasing IKt-subset of X, then the following conditions are equivalent:*

(i) *Y is an IKt-subset,*

(ii) *$Y = d_X^n(Y)$ for all $n \in \omega$,*

(iii) *$Y = \bigcap_{n \in \omega} d_X^n(Y)$.*

Proof. It is an immediate consequence of Definition 3.5 and Proposition 3.1. ■

Taking into account that the restrictions to $D(X)$ of the functions defined in XI and XII are functions from $D(X)$ into $D(X)$, we obtain the following lemma:

LEMMA 3.2. *Let* $(X, \leq, R)$ *be an* IKt*-space and* $(D(X), G_R, H_{R^{-1}}, F_R, P_{R^{-1}})$ *be the* IKt*-algebra associated with* X*. Then for all* $n \in \omega$ *and for all* $U, V \in D(X)$*, the following conditions are satisfied:*

(d1) $d_X^n(X) = X$ *and* $d_X^n(\emptyset) = \emptyset$,

(d2) $d_X^{n+1}(U) \subseteq d_X^n(U)$,

(d3) $d_X^n(U \cap V) = d_X^n(U) \cap d_X^n(V)$,

(d4) $U \subseteq V$ *implies* $d_X^n(U) \subseteq d_X^n(V)$,

(d5) $d_X^n(U) \subseteq U$,

(d6) $d_X^{n+1}(U) \subseteq G_R(d_X^n(U))$ *and* $d_X^{n+1}(U) \subseteq H_{R^{-1}}(d_X^n(U))$,

(d7) $\bigcap_{n\in\omega} d_X^n(U) \in \mathcal{C}_{IT}(X)$ *and therefore* $d(\bigcap_{n\in\omega} d_X^n(U)) = \bigcap_{n\in\omega} d_X^n(U)$.

PROOF. From Definition 3.5 and the fact that G_R, $H_{R^{-1}}$ and d_X^n, $n \in \omega$, are monotone operations it follows immediately that the properties (d1), (d2), (d3), (d4), (d5) and (6) hold.
(d7): Let $U \in D(X)$ and $n \in \omega$, then $d_X^n(U)$ is an increasing and closed subset of X and therefore, $\bigcap_{n\in\omega} d_X^n(U)$ is an increasing and closed subset of X. If $\bigcap_{n\in\omega} d_X^n(U) = \emptyset$, then $\bigcap_{n\in\omega} d_X^n(U) \in \mathcal{C}_{IT}(X)$. Suppose now that there is $y \in \bigcap_{n\in\omega} d_X^n(U)$, then $y \in d_X^n(U)$ for all $n \in \omega$, and so from (d5) it follows that $y \in G_R(d_X^{n-1}(U))$ and $y \in H_{R^{-1}}(d_X^{n-1}(U))$ for all $n \in \omega$. Therefore $R(\uparrow y) \subseteq d_X^{n-1}(U)$ and $R^{-1}(\uparrow y) \subseteq d_X^{n-1}(U)$ for all $n \in \omega$ and consequently $R(\uparrow y) \subseteq \bigcap_{n\in\omega} d_X^n(U)$ and $R^{-1}(\uparrow y) \subseteq \bigcap_{n\in\omega} d_X^n(U)$. From these last assertions and Proposition 3.1, we have that $\bigcap_{n\in\omega} d_X^n(U) \in \mathcal{C}_{IT}(X)$, from which we conclude, by Lemma 3.1, that $d(\bigcap_{n\in\omega} d_X^n(U)) = \bigcap_{n\in\omega} d_X^n(U)$. ■

As a consequence of Lemma 3.2 and the duality for IKt-algebras we can define $d : A \longrightarrow A$ and $d^n : A \longrightarrow A$, $n \in \omega$, by the prescriptions: $d(a) = G(a) \wedge a \wedge H(a)$, $d^0(a) = a$ and $d^{n+1}(a) = d(d^n(a))$, respectively. It should be noted that these operators were previously defined by Diaconescu and Georgescu in [16] for tense MV-algebras.

COROLLARY 3.6. ([23]) *Let* $(\mathcal{A}, G, H, F, P)$ *be an* IKt*-algebra. Then, for all* $n \in \omega$ *and for all* $a, b \in A$*, the following conditions are satisfied:*

(d1) $d^n(1) = 1$ *and* $d^n(0) = 0$,

(d2) $d^{n+1}(a) \leq d^n(a)$,

(d3) $d^n(a \wedge b) = d^n(a) \wedge d^n(b)$,

(d4) $a \leq b$ *implies* $d^n(a) \leq d^n(b)$,

(d5) $d^n(a) \leq a$,

(d6) $d^{n+1}(a) \leq G(d^n(a))$ *and* $d^{n+1}(a) \leq H(d^n(a))$.

PROOF. It is a direct consequence of Lemmas 2.9 and 3.2. ∎

LEMMA 3.3. *Let* $(\mathcal{A}, G, H, F, P)$ *be an IKt-algebra. If* $\bigwedge_{i\in I} a_i$ *exists then:*

(i) $\bigwedge_{i\in I} G(a_i)$ *exists and* $\bigwedge_{i\in I} G(a_i) = G(\bigwedge_{i\in I} a_i)$,

(ii) $\bigwedge_{i\in I} H(a_i)$ *exists and* $\bigwedge_{i\in I} H(a_i) = H(\bigwedge_{i\in I} a_i)$,

(iii) $\bigwedge_{i\in I} d(a_i)$ *exists and* $\bigwedge_{i\in I} d(a_i) = d(\bigwedge_{i\in I} a_i)$.

PROOF. (i): Assume that $a_i \in A$ for all $i \in I$ and $\bigwedge_{i\in I} a_i$ exists. Since $\bigwedge_{i\in I} a_i \leq a_i$, we have by (t2) that $G(\bigwedge_{i\in I} a_i) \leq G(a_i)$ for each $i \in I$. Thus $G(\bigwedge_{i\in I} a_i)$ is a lower bound of the set $\{G(a_i) : i \in I\}$. Assume now that b is a lower bound of the set $\{G(a_i) : i \in I\}$. By (t5) and (t6) we have that $P(b) \leq PG(a_i) \leq a_i$ for each $i \in I$. So, $P(b) \leq \bigwedge_{i\in I} a_i$. Besides, the pair (G, P) is a Galois connection, this means that $x \leq G(y) \iff P(x) \leq y$, for all $x, y \in A$. So, we can infer that $b \leq G(\bigwedge_{i\in I} a_i)$. This proves that $\bigwedge_{i\in I} G(a_i)$ exists and $\bigwedge_{i\in I} G(a_i) = G(\bigwedge_{i\in I} a_i)$.
(ii): Analogously it can be proved for the operator H.
(iii): It is a direct consequence of (i) and (ii). ∎

For invariance properties we have:

LEMMA 3.4. *Let* $(X, \leq, R)$ *be an IKt-space and* $(D(X), G_R, H_{R^{-1}}, F_R, P_{R^{-1}})$ *be the IKt-algebra associated with* X*. Then for all* $U, V \in D(X)$ *such that* $U = d_X(U)$ *and* $V = d_X(V)$*, the following properties are satisfied:*

(i) $U \cap V = d_X(U \cap V)$,

(ii) $U \cup V = d_X(U \cup V)$,

(iii) $d_X(U \to V) = U \to V$.

PROOF. (i): It follows immediately from the definition of the function d_X and the property (t2) of the IKt-algebras.
(ii): Taking into account that $U = d_X(U)$ and $V = d_X(V)$ and the fact that the operations G_R and $H_{R^{-1}}$ are monotone we infer that $U \cup V \subseteq G_R(U \cup V)$ and $U \cup V \subseteq H_{R^{-1}}(U \cup V)$, which imply that $U \cup V = d_X(U \cup V)$,
(iii): It is sufficient to prove that $U \to V \subseteq G_R(U \to V)$ and $U \to V \subseteq H_{R^{-1}}(U \to V)$. Let $x \in U \to V$, then (1) $\uparrow x \cap U \subseteq V$. Let's assume

that $y \in R(\uparrow x)$ and $z \in \uparrow y \cap U$. Since $R(\uparrow x)$ is increasing it follows that $\uparrow y \subseteq R(\uparrow x)$ and thus (2) $z \in R(\uparrow x) \cap U$. Therefore there is $t \in \uparrow x$ such that (3) $z \in R(t)$. From (2) and the hypothesis that $U \subseteq H_{R^{-1}}(U)$ we get that $z \in H_{R^{-1}}(U)$, and so by (3), $t \in U$. Therefore $t \in \uparrow x \cap U$, which implies, by (1), that $t \in V$. From this last assertion and the fact that $V \subseteq G_R(V)$, we infer that $t \in G_R(V)$ and so $R(t) \subseteq V$, from which we have, by (3), that $z \in V$. Therefore, $\uparrow y \cap U \subseteq V$, and hence $y \in U \to V$. This preceding assertion allows us to set that $R(\uparrow x) \subseteq U \to V$ and so we conclude that $U \to V \subseteq G_R(U \to V)$. It can be proved that $U \to V \subseteq H_{R^{-1}}(U \to V)$ in a similar way. Therefore, $U \to V = d_X(U \to V)$. ∎

Corollary 3.7. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then for all $a, b \in A$, such that $a = d(a)$ and $b = d(b)$, the following properties are satisfied:*

(i) $d(a \wedge b) = a \wedge b$,

(ii) $d(a \vee b) = a \vee b$,

(iii) $d(a \to b) = a \to b$.

Proof. It is a direct consequence of Lemmas 2.9 and 3.4. ∎

Proposition 3.2. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then, for all $a \in A$, the following conditions are equivalent:*

(i) $a = d(a)$,

(ii) $a = d^n(a)$ *for all* $n \in \omega$.

Proof. It follows immediately from Corollary 3.6. ∎

In what follows if $(\mathcal{A}, G, H, F, P)$ is an IKt-algebra we will denoted by $C(\mathcal{A}) = \{a \in A : d(a) = a\}$.

Lemma 3.5. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then, $\langle C(\mathcal{A}), \wedge, \vee, \to, 0, 1\rangle$ is a Heyting algebra.*

Proof. It is a direct consequence of Corollary 3.7 and the property (d1) in Lemma 3.2. ∎

Taking into account Theorem 3.2 and Priestley duality we can say that the congruences on an IKt-algebra are the lattice congruences associated with certain filters of this algebra. So our next goal is to determine the conditions that a filter of an IKt-algebra must fulfill for the associated lattice congruence to be an IKt-congruence.

Theorem 3.8. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. If S is a filter of $\mathcal{A}$, then, the following conditions are equivalent:*

(i) $\Theta(S) \in Con_{IKt}(\mathcal{A})$,

(ii) $d(a) \in S$ *for all* $a \in S$,

(iii) $d^n(a) \in S$ *for all* $a \in S$ *and for all* $n \in \omega$.

PROOF. (i) $\Rightarrow$ (ii): Let S be a filter of $\mathcal{A}$ such that $\Theta(S) \in Con_{IKt}(\mathcal{A})$. Then, from Priestley Duality and Theorem 3.2 it follows that $\Theta(S) = \Theta_{IT}(Y_S)$, where $\Theta(S)$ is the lattice congruence associated with S, $Y_S = \{x \in \mathfrak{X}(A) : S \subseteq x\} = \bigcap_{a\in S} \sigma_A(a)$ is an increasing and closed IKt-subset. From these last assertions, Lemmas 3.1 and 3.6, the fact that σ_A is an IKt-isomorphism and the application $d_{\mathfrak{X}(A)}$ is monotone, we infer that $Y_S = d_{\mathfrak{X}(A)}(Y_S) = d_{\mathfrak{X}(A)}(\bigcap_{a\in S} \sigma_A(a)) \subseteq \bigcap_{a\in S} d_{\mathfrak{X}(A)}(\sigma_A(a)) = \bigcap_{a\in S} \sigma_A(d(a)) \subseteq \bigcap_{a\in S} \sigma_A(a)$. Hence, $Y_S = \bigcap_{a\in S} \sigma_A(d(a))$, from which we conclude that $d(a) \in S$ for all $a \in S$. Indeed, suppose that $a \in S$, then $a \in x$ for all $x \in Y_S$, from which it follows that $x \in \bigcap_{a\in S} \sigma_A(d(a))$ and thus $d(a) \in x$ for all $x \in Y_S$. Therefore, $d(a) \in \bigcap_{x\in Y_S} x$, and taking into account that $S = \bigcap_{x\in Y_S} x$, we obtain that $d(a) \in S$.
(ii) $\Rightarrow$ (i): By Priestley duality and VI we have that $Y_S = \{x \in \mathfrak{X}(A) : S \subseteq x\} = \bigcap_{a\in S} \sigma_A(a)$ is an increasing and closed subset and $\Theta(S) = \Theta(Y_S)$. By Theorem 3.2, it remains to show that Y_S is an IKt-subset of $\mathfrak{X}(A)$. From the hypothesis (ii), it follows that for $a \in A$, $d(a) \in x$ for all $x \in Y_S$ and therefore, $Y_S \subseteq \bigcap_{a\in S} \sigma_A(d(a))$. Consequently, by Lemma 3.6, $Y_S = \bigcap_{a\in S} \sigma_A(d(a))$. Then, taking into account that $\sigma_A(d(a)) = d_{\mathfrak{X}(A)}(\sigma_A(a))$ and $\bigcap_{a\in S} d_{\mathfrak{X}(A)}\sigma_A(a) = d_{\mathfrak{X}(A)}(\bigcap_{a\in S} \sigma_A(a))$, we obtain than $Y_S = d_{\mathfrak{X}(A)}(Y_S)$, and so, from Lemma 3.1 and the fact that Y is increasing, we infer that Y_S is an IKt-subset of $X(\mathcal{A})$. Finally, since Y is an increasing and closed IKt-subset of $X(\mathcal{A})$ and $\Theta(S) = \Theta_{IT}(Y_S)$, we conclude, from Theorem 3.2, that $\Theta(S) \in Con_{IKt}(\mathcal{A})$.
(ii) $\Leftrightarrow$ (iii): It is trivial. ■

Theorem 3.8 leads us to introduce the following definition:

DEFINITION 3.9. Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. A filter S of $\mathcal{A}$ is an IKt-filter iff for all $a \in S$, $d(a) \in S$, or equivalently $d^n(a) \in S$ for all $n \in \omega$.

We shall denote by $\mathcal{F}_{IKt}(\mathcal{A})$ to the set of all IKt-filters of an IKt-algebra $(\mathcal{A}, G, H, F, P)$.

PROPOSITION 3.3. *Let* $(\mathcal{A}, G, H, F, P)$ *be an* IKt*-algebra. Then, the following conditions are equivalent for all* $\varphi \subseteq A \times A$*:*

(i) $\varphi \in Con_{IKt}(\mathcal{A})$,

(ii) *there is* $S \in \mathcal{F}_{IKt}(\mathcal{A})$ *such that* $\varphi = \Theta(S)$, *where* $\Theta(S)$ *is the lattice congruence associated with the filter* S.

Proof. It follows from Theorems 3.2 and 3.8 and Definition 3.9. ■

Corollary 3.10. *Let* $(\mathcal{A}, G, H, F, P)$ *be an* IKt*-algebra. Then,*

(i) $(\mathcal{A}, G, H, F, P)$ *is a simple* IKt*-algebra iff* $\mathcal{F}_{IKt}(\mathcal{A}) = \{A, \{1\}\}$.

(ii) $(\mathcal{A}, G, H, F, P)$ *is a subdirectly irreducible* IKt*-algebra iff there is* $T \in \mathcal{F}_{IKt}(\mathcal{A})$, $T \neq \{1\}$ *such that* $T \subseteq S$ *for all* $S \in \mathcal{F}_{IKt}(\mathcal{A})$, $S \neq \{1\}$.

Proof. It is a direct consequence of Proposition 3.3. ■

Finally, we shall describe the simple and subdirectly irreducible IKt-algebras.

In the proof of the following proposition we shall use the finite intersection property of compact spaces, which establishes that if X is a compact topological space, then for each family $\{M_i\}_{i\in I}$ of closed subsets in X satisfying $\bigcap_{i\in I} M_i = \emptyset$, there is a finite subfamily $\{M_{i_1}, \ldots, M_{i_n}\}$ with $\bigcap_{j=1}^{j=n} M_{i_j} = \emptyset$.

Proposition 3.4. *Let* $(\mathcal{A}, G, H, F, P)$ *be an* IKt*-algebra and the* IKt*-space associated with* $\mathcal{A}$, $(\mathfrak{X}(A), \subseteq, R^A)$. *Then, the following conditions are equivalent:*

(i) $(\mathcal{A}, G, H, F, P)$ *is a simple* IKt*-algebra,*

(ii) *for all* $U \in D(\mathfrak{X}(A))\backslash\{\mathfrak{X}(A)\}$, $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(U) = \emptyset$,

(iii) *for every* $U \in D(\mathfrak{X}(A))\backslash\{\mathfrak{X}(A)\}$ *there is* $n \in \omega$ *such that* $d^n_{\mathfrak{X}(A)}(U) = \emptyset$,

(iv) $\mathcal{F}_{IKt}(D(\mathfrak{X}(A))) = \{D(\mathfrak{X}(A)), \{\mathfrak{X}(A)\}\}$.

Proof. (i) $\Rightarrow$ (ii): Let $U \in D(\mathfrak{X}(A))\backslash\{\mathfrak{X}(A)\}$. Then, from Lemma 2.9 and (d7) in Lemma 3.2 we have that $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(U) \in \mathcal{C}_{IT}(\mathfrak{X}(A))\backslash\{\mathfrak{X}(A)\}$. From this last assertion, the hypothesis (i) and Corollary 3.3, we conclude that $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(U) = \emptyset$.
(ii) $\Rightarrow$ (iii): Let $U \in D(\mathfrak{X}(A))\backslash\{\mathfrak{X}(A)\}$. Then, from the hypothesis (ii) it is verified that (1) $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(U) = \emptyset$. Since for all $n \in \omega$, $d^n_{\mathfrak{X}(A)}(U)$ is a closed subset of $\mathfrak{X}(A)$ and $d^n_{\mathfrak{X}(A)}(U) = \bigcap_{j=1}^{n} d^j_{\mathfrak{X}(A)}(U)$, then considering (1), the fact that $\mathfrak{X}(A)$ is compact and the finite intersection property of compact spaces, we conclude that there is $n \in \omega$ such that $d^n_{\mathfrak{X}(A)}(U) = \emptyset$.
(iii) $\Rightarrow$ (iv): Assume that $S \in \mathcal{F}_{IKt}(D(\mathfrak{X}(A)))$, $S \neq \{\mathfrak{X}(A)\}$, then there is $V \in D(\mathfrak{X}(A))$, $V \neq \mathfrak{X}(A)$ such that $V \in S$. From the preceding assertion, Definition 3.9 and the hypothesis (iii), we deduce that $\emptyset \in S$, which implies that $S = D(\mathfrak{X}(A))$.

(iv) $\Rightarrow$ (i): It follows immediately from Corollary 3.10 and the fact that $(\mathcal{A}, G, H, F, P)$ is isomorphic to $(D(\mathfrak{X}(A)), G_R, H_{R^{-1}}, F_R, P_{R^{-1}})$. ■

Corollary 3.11. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then, the following conditions are equivalent:*

(i) *$(\mathcal{A}, G, H, F, P)$ is a simple IKt-algebra,*

(ii) *for every $a \in A \backslash \{1\}$ there is $n \in \omega$ such that $d^n(a) = 0$,*

(iii) *$\mathcal{F}_{IKt}(\mathcal{A}) = \{A, \{1\}\}$.*

Proof. It is a direct consequence of Proposition 3.4 and the fact that σ_A is an IKt-isomorphism (Lemma 2.9). ■

Corollary 3.12. *If $(\mathcal{A}, G, H, F, P)$ is a simple IKt-algebra, then $C(\mathcal{A}) = \{0, 1\}$ and therefore $\langle C(\mathcal{A}), \vee, \wedge, \rightarrow, 0, 1\rangle$ is a simple Heyting algebra.*

Proof. It is a direct consequence of Proposition 3.2 and Corollary 3.11. ■

Proposition 3.5. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra and $(\mathfrak{X}(A), \subseteq, R^A)$ be the IKt-space associated with $\mathcal{A}$. Then, the following conditions are equivalent:*

(i) *$(\mathcal{A}, G, H, F, P)$ is a subdirectly irreducible IKt-algebra,*

(ii) *there is $V \in D(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$, such that for all $U \in D(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$, $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(U) \subseteq V$,*

(iii) *there is $V \in D(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$, such that for each $U \in D(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$, there is $n_U \in \omega$ such that $d^{n_U}_{\mathfrak{X}(A)}(U) \subseteq V$,*

(iv) *there is $T \in \mathcal{F}_{IKt}(D(\mathfrak{X}(A))$, $T \neq \{\mathfrak{X}(A)\}$, such that $T \subseteq S$ for all $S \in \mathcal{F}_{IKt}(D(\mathfrak{X}(A)))$, $S \neq \{\mathfrak{X}(A)\}$.*

Proof. (i) $\Rightarrow$ (ii): From the hypothesis (i) and Corollary 3.4 we infer that there is $Y \in \mathcal{C}_{IT}(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$ such that for all $Z \in \mathcal{C}_{IT}(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$, (1) $Z \subseteq Y$. Therefore, there is $x \in \mathfrak{X}(A) \backslash Y$, and since Y is an increasing and closed subset of $\mathfrak{X}(A)$ and hence it is compact, then we can assert that there is $V \in D(\mathfrak{X}(A))$, such that (2) $Y \subseteq V$ and $x \notin V$ and so $V \in D(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$. On the other hand, if $U \in D(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$, then by Lemma 3.2, $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(U) \in \mathcal{C}_{IT}(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$, from which we conclude, by (1) and (2), that $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(U) \subseteq V$.
(ii) $\Rightarrow$ (iii): From the hypothesis (ii), there is $V \in D(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$, such that (1) $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(U) \subseteq V$ for all $U \in D(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$. Suppose that there is $U \in D(\mathfrak{X}(A)) \backslash \{\mathfrak{X}(A)\}$ such that for all $n \in \omega$, $d^n_{\mathfrak{X}(A)}(U) \nsubseteq V$, then for each $n \in \omega$ there is (2) $x_n \in d^n_{\mathfrak{X}(A)}(U)$ and $x_n \notin V$. Hence $(x_n)_{n\in\omega}$

is a sequence in $\mathfrak{X}(A)\backslash V$ and since $\mathfrak{X}(A)\backslash V$ is compact, then there is (3) $x \in \mathfrak{X}(A)\backslash V$ such that $(x_n)_{n\in\omega}$ accumulates at x. In addition, by (1) and (3), we have that $x \notin \bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(U)$, and therefore there is $n_0 \in \omega$ such that $x \in \mathfrak{X}(A)\backslash d^{n_0}_{\mathfrak{X}(A)}(U)$. Since x is an accumulation point of $(x_n)_{n\in\omega}$, then the preceding assertion and the fact that $\mathfrak{X}(A)\backslash d^{n_0}_{\mathfrak{X}(A)}(U)$ is an open subset of $\mathfrak{X}(A)$ allow us to infer that for all $n \in \omega$ there is $m_n \in \omega$ such that $n \leq m_n$ and $x_{m_n} \in \mathfrak{X}(A)\backslash d^{n_0}_{\mathfrak{X}(A)}(U)$. Hence $x_{m_{n_0}} \in \mathfrak{X}(A)\backslash d^{n_0}_{\mathfrak{X}(A)}(U)$ and $n_0 \leq m_{n_0}$. As a consequence of Lemma 3.2 we have that $\mathfrak{X}(A)\backslash d^{n_0}_{\mathfrak{X}(A)}(U) \subseteq \mathfrak{X}(A)\backslash d^{m_{n_0}}_{\mathfrak{X}(A)}(U)$ and so $x_{m_{n_0}} \in \mathfrak{X}(A)\backslash d^{m_{n_0}}_{\mathfrak{X}(A)}(U)$, which contradicts (2). Therefore, for each $U \in D(\mathfrak{X}(A))\backslash\{\mathfrak{X}(A)\}$ there is $n \in \omega$ such that $d^n_{\mathfrak{X}(A)}(U) \subseteq V$.
(iii) $\Rightarrow$ (iv): $V \in S$ for all $S \in \mathcal{F}_{IKt}(D(\mathfrak{X}(A)))$, $S \neq \{\mathfrak{X}(A)\}$. Indeed, let $U \in S$, then by the hypothesis (iii), there is $n_U \in \omega$ such that $d^{n_U}_{\mathfrak{X}(A)}(U) \subseteq V$. Since $d^{n_U}_{\mathfrak{X}(A)}(U) \in S$, we infer that $V \in S$, and so $V \in \bigcap_{\substack{S\in\mathcal{F}_{IKt}(D(\mathfrak{X}(A)))\\ S\neq\{\mathfrak{X}(A)\}}} S$. Therefore, considering $T = \bigcap_{\substack{S\in\mathcal{F}_{IKt}(D(\mathfrak{X}(A)))\\ S\neq\{\mathfrak{X}(A)\}}} S$, we get that $T \in \mathcal{F}_{IKt}(D(\mathfrak{X}(A)))$, $T \neq \{\mathfrak{X}(A)\}$ and $T \subseteq S$, for all $S \in \mathcal{F}_{IKt}(D(\mathfrak{X}(A)))$, $S \neq \{\mathfrak{X}(A)\}$.
(iv) $\Rightarrow$ (i): It follows immediatly from Corollary 3.10 and the fact that $(\mathcal{A}, G, H, F, P)$ is isomorphic to $(D(\mathfrak{X}(A)), G_R, H_{R^{-1}}, F_R, P_{R^{-1}})$. ■

Corollary 3.13. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then, the following conditions are equivalent:*

(i) *$(\mathcal{A}, G, H, F, P)$ is a subdirectly irreducible IKt-algebra,*
(ii) *there is $b \in A\backslash\{1\}$ such that for all $a \in A\backslash\{1\}$, there is $n \in \omega$ such that $d^n(a) \leq b$,*
(iii) *there is $T \in \mathcal{F}_{IKt}(\mathcal{A})$, $T \neq \{1\}$ such that $T \subseteq S$ for all $S \in \mathcal{F}_{IKt}(\mathcal{A})$, $S \neq \{1\}$.*

Proof. It is a direct consequence of Proposition 3.5 and the fact that σ_A is an IKt-isomorphism (Lemma 2.9). ■

Corollary 3.14. *Let $(\mathcal{A}, G, H, F, P)$ be a subdirectly irreducible IKt-algebra such that for all $a \in A\backslash\{1\}$, $d^n(a) = d^{n_a}(a)$ for all $n \in \omega$, $n_a \leq n$ for some $n_a \in \omega$. Then, $\langle C(\mathcal{A}), \vee, \wedge, \rightarrow, 0, 1\rangle$ is a subdirectly irreducible Heyting algebra.*

Proof. From Corollary 3.13 it follows that there is $b \in A\backslash\{1\}$ such that for all $a \in A\backslash\{1\}$, there is $n'_a \in \omega$ such that $d^{n'_a}(a) \leq b$. Also, from hypothesis we have that there is $n_b \in \omega$ such that $d^n(b) = d^{n_b}(b)$ for all $n \in \omega$, $n_b \leq n$ and so considering $u = d^{n_b}(b)$, then $u \in C(\mathcal{A})$, $u \neq 1$. In addition, let $c \in C(\mathcal{A})$,

$c \neq 1$, then $c = d^n(c)$ for all $n \in \omega$, and thus $c = d^{n'_c}(c) \leq b$, from which we infer that $c = d^{n_c}(c) \leq d^{n_b}(b) = u$. Consequently, by Theorem 1.2 and Lemma 3.5, it results that $\langle C(\mathcal{A}), \vee, \wedge, \to, 0, 1\rangle$ is a subdirectly irreducible Heyting algebra. ■

In the sequel we shall describe the finite simple and subdirectly irreducible IKt-algebras.

THEOREM 3.15. *Let $(\mathcal{A}, G, H, F, P)$ be a finite IKt-algebra Then, the following conditions are equivalent:*

(i) *$(\mathcal{A}, G, H, F, P)$ is a simple IKt-algebra,*

(ii) *$C(\mathcal{A}) = \{0, 1\}$,*

(iii) *$\langle C(\mathcal{A}), \vee, \wedge, \to, 0, 1\rangle$ is a simple Heyting algebra.*

PROOF. (i) $\Rightarrow$ (iii): It follows directly from Corollary 3.12.
(ii) $\Leftrightarrow$ (iii): It follows immediately from Lemma 3.5 and the fact that a Heyting algebra $\mathcal{B}$ is simple iff, $B = \{0, 1\}$.
(ii) $\Rightarrow$ (i): Since A is finite set, then for all $a \in A\backslash\{1\}$, there is $n_a \in \omega$ such that $d_n(a) = d_{n_a}(a) < a$ for all $n \in \omega$, $n_a \leq n$. Therefore, $d(d_{n_a}(a)) = d_{n_a}(a)$, which implies that $d_{n_a}(a) \in C(\mathcal{A})\backslash\{1\}$, and so by the hypothesis (ii) we infer that $d_{n_a}(a) = 0$. From the preceding assertion and Corollary 3.11 we obtain that $(\mathcal{A}, G, H, F, P)$ is a simple IKt-algebra. ■

THEOREM 3.16. *Let $(\mathcal{A}, G, H, F, P)$ be a finite IKt-algebra. Then, the following conditions are equivalent:*

(i) *$(\mathcal{A}, G, H, F, P)$ is a subdirectly irreducible IKt-algebra,*

(ii) *there is $u \in C(\mathcal{A})\backslash\{1\}$, such that $c \leq u$, for all $c \in C(\mathcal{A})\backslash\{1\}$,*

(iii) *$\langle C(\mathcal{A}), \vee, \wedge, \to, 0, 1\rangle$ is a subdirectly irreducible Heyting algebra.*

PROOF. (i) $\Rightarrow$ (iii): Since A is finite set, then for all $a \in A\backslash\{1\}$, there is $n_a \in \omega$ such that $d_n(a) = d_{n_a}(a)$ for all $n \in \omega$, $n_a \leq n$. Therefore, from this last assertion, the hypothesis (i) and Corollary 3.14, we conclude that $\langle C(\mathcal{A}), \vee, \wedge, \to, 0, 1\rangle$ is a subdirectly irreducible Heyting algebra.
(ii) $\Leftrightarrow$ (iii): It is a direct consequence of Theorem 1.2 and Lemma 3.5.

(ii) $\Rightarrow$ (i): Since A is finite set, then for all $a \in A\backslash\{1\}$, there is $n_a \in \omega$ such that $d_{n_a}(a) \in C(\mathcal{A})\backslash\{1\}$, and so by the hypothesis (ii) we infer that $d_{n_a}(a) \leq u$, $u \neq 1$. The preceding assertion and Corollary 3.13 allow us to set that $(\mathcal{A}, G, H, F, P)$ is a subdirectly irreducible IKt-algebra. ■

Now, we are interested in the characterization of the complete simple and subdirectly irreducible IKt-algebras whose filters are complete. To this end, we recall that if A is a complete lattice whose filters are complete, then for all $S \subseteq A$, $\sigma_A(\bigwedge_{a\in S} a) = \bigcap_{a\in S} \sigma_A(a)$.

PROPOSITION 3.6. *Let $(\mathcal{A}, G, H, F, P)$ be a complete IKt-algebra. Then, the following conditions are equivalent:*

(i) $a = d(a)$,

(ii) $a = d^n(a)$ *for all* $n \in \omega$,

(iii) $a = \bigwedge_{n\in\omega} d^n(a)$,

(iv) $a = \bigwedge_{n\in\omega} d^n(b)$ *for some* $b \in A$.

PROOF. It follows from Proposition 3.2, the fact that $\bigwedge_{n\in\omega} d^n(a) \in A$, and Lemma 3.3. ∎

THEOREM 3.17. *Let $(\mathcal{A}, G, H, F, P)$ be a complete IKt-algebra whose filters are complete. Then, the following conditions are equivalent:*

(i) $(\mathcal{A}, G, H, F, P)$ *is a simple IKt-algebra,*

(ii) $C(\mathcal{A}) = \{0, 1\}$,

(iii) $\langle C(\mathcal{A}), \vee, \wedge, \rightarrow, 0, 1\rangle$ *is a simple Heyting algebra.*

PROOF. (i) $\Rightarrow$ (iii): It follows directly from Corollary 3.12.
(ii) $\Leftrightarrow$ (iii): It follows immediately from Lemma 3.5 and the fact that a Heyting algebra $\mathcal{B}$ is simple iff, $B = \{0, 1\}$.
(ii) $\Rightarrow$ (i): Taking into account that $\mathcal{A}$ is complete and Proposition 3.6, we have that $C(\mathcal{A}) = \{\bigwedge_{n\in\omega} d^n(a) : a \in A\}$. Then, from the hypothesis (ii), we obtain that for all $a \in A\backslash\{1\}$, $\bigwedge_{n\in\omega} d^n(a) = 0$. In addition, from the hypothesis, it follows that $\sigma_A(\bigwedge_{n\in\omega} d^n(a)) = \bigcap_{n\in\omega} \sigma_A(d^n(a))$. Consequently $\bigcap_{n\in\omega} \sigma_A(d^n(a)) = \emptyset$ and thus $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(\sigma_A(a)) = \emptyset$ for all $a \in A$ such that $\sigma_A(a) \neq \mathfrak{X}(A)$. Taking into account that σ_A is an IKt-isomorphism, and Proposition 3.4 the proof is complete. ∎

THEOREM 3.18. *Let $(\mathcal{A}, G, H, F, P)$ be a complete IKt-algebra whose filters are complete. Then, the following conditions are equivalent:*

(i) $(\mathcal{A}, G, H, F, P)$ *is a subdirectly irreducible IKt-algebra,*

(ii) *there is* $u \in C(\mathcal{A})\backslash\{1\}$, *such that* $c \leq u$, *for all* $c \in C(\mathcal{A})\backslash\{1\}$,

(iii) $\langle C(\mathcal{A}), \vee, \wedge, \rightarrow, 0, 1\rangle$ *is a subdirectly irreducible Heyting algebra.*

PROOF. (i) $\Rightarrow$ (ii): From the hypothesis (i), Proposition 3.6 and Corollary 3.13, we can assert that there is $b \in A\backslash\{1\}$ such that (1) $c \leq b$, for all $c \in C(\mathcal{A})$, $c \neq 1$. Since A is complete, we have that $\bigwedge_{n\in\omega} d^n(b) \in C(\mathcal{A})$,

$\bigwedge_{n\in\omega} d^n(b) \neq 1$. In addition, from (1) and the fact that for all $c \in C(\mathcal{A})$, $c \neq 1$, $c = \bigwedge_{n\in\omega} d^n(c)$, we conclude that $c \leq \bigwedge_{n\in\omega} d^n(b)$. Therefore, considering $u = \bigwedge_{n\in\omega} d^n(b)$, the proof is complete.

(ii) $\Rightarrow$ (i): Taking into account that $\mathcal{A}$ is complete and Proposition 3.6, we have that $C(\mathcal{A}) = \{\bigwedge_{n\in\omega} d^n(a),\ a \in A\}$. Then, from the hypothesis (ii), we obtain that there is $b \in A$, $b \neq 1$, such that $\bigwedge_{n\in\omega} d^n(b) \neq 1$ and $\bigwedge_{n\in\omega} d^n(a) \leq \bigwedge_{n\in\omega} d^n(b)$, for all $a \in A\backslash\{1\}$ and so $\bigwedge_{n\in\omega} d^n(a) \leq b$. Therefore, considering that $\sigma_A(\bigwedge_{n\in\omega} d^n(a)) = \bigcap_{n\in\omega} \sigma_A(d^n(a))$, the fact that $\sigma_A : A \to D(\mathfrak{X}(A))$ is an IKt-isomorphism and the preceding assertion we infer that $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(\sigma_A(a)) \subseteq \sigma_A(b) \subset \mathfrak{X}(A)$, and this implies that for all $U \in D(\mathfrak{X}(A))$, $U \neq \mathfrak{X}(A)$, $\bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(U) \subseteq \sigma_A(b)$ and hence Proposition 3.5 allows us to conclude the proof.

(ii) $\Leftrightarrow$ (iii): It is a direct consequence of Theorem 1.2, Lemma 3.5 and Corollary 3.13. ∎

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