

Explicit Solutions to Fractional Stefan-like problems for Caputo and Riemann–Liouville Derivatives

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Abstract: Two fractional two-phase Stefan-like problems are considered by using Riemann–Liouville and Caputo derivatives of order $\alpha \in (0, 1)$ verifying that they coincide with the same classical Stefan problem at the limit case when $\alpha = 1$. For both problems, explicit solutions in terms of the Wright functions are presented. Even though the similarity of the two solutions, a proof that they are different is also given. The convergence when $\alpha \nearrow 1$ of the one and the other solutions to the same classical solution is given. Numerical examples for the dimensionless version of the problem are also presented and analyzed.

Keywords: Stefan-like problem; Caputo derivative; Riemann–Liouville derivative; Wright functions.

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1 Introduction

This paper deals with Stefan–like problems governed by fractional diffusion equations (FDE). A classical Stefan problem is a problem where a phase-change occurs, usually linked to melting (change from solid to liquid) or freezing (change from liquid to solid). In these problems the diffusion, considered as a heat flow, is expressed in terms of instantaneous local flow of temperature modeled by the Fourier Law. Therefore, the governing equations related to each phase are the well-known heat equations. There is also a latent heat-type condition at the interface connecting the velocity of the free boundary and the heat flux of the temperatures in both phases known as “Stefan condition”. A vast literature on Stefan problems is given in [1, 4, 5, 24, 25].

For example, the following is the mathematical formulation for a classical one-dimensional two-phase Stefan problem: *Find the triple $\{u_1, u_2, s\}$ such that they have sufficiently regularity and they verify that:*

$$\begin{aligned}
 (i) \quad & \frac{\partial}{\partial t} u_2(x, t) = \lambda_2^2 \frac{\partial^2}{\partial x^2} u_2(x, t), & 0 < x < s(t), 0 < t < T, \\
 (ii) \quad & \frac{\partial}{\partial t} u_1(x, t) = \lambda_1^2 \frac{\partial^2}{\partial x^2} u_1(x, t), & x > s(t), 0 < t < T, \\
 (iii) \quad & u_1(x, 0) = U_i, & 0 \leq x, \\
 (iv) \quad & u_2(0, t) = U_0, & 0 < t \leq T, \\
 (v) \quad & u_1(s(t), t) = u_2(s(t), t) = U_m, & 0 < t \leq T, \\
 (vi) \quad & \rho l \frac{d}{dt} s(t) = k_1 \frac{\partial}{\partial x} u_1(s(t), t) - k_2 \frac{\partial}{\partial x} u_2(s(t), t), & 0 < t \leq T, \\
 (vii) \quad & s(0) = 0,
 \end{aligned} \tag{1}$$

where $U_i < U_m < U_0$, $\lambda_j^2 = \frac{k_j}{\rho c_j}$, $j = 1$ (solid), $j = 2$ (liquid) and we have assumed that the thermo-physical properties are constant as well as the free boundary can be represented by an increasing

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function of time.

Problem (1) is clearly governed by the heat equations $(1 - i)$ and $(1 - ii)$, and has a phase-change condition (namely the Stefan condition) given by equation $(1 - vi)$.

When the governing equations $(1 - i)$ and $(1 - ii)$, or the Stefan condition $(1 - vi)$ are replaced by other equations involving fractional derivatives in problems like (1), we will refer to them as fractional Stefan-like problems.

For example, the heat equation can be replaced by a fractional diffusion equation (FDE), which is closely linked to the study of anomalous diffusion. A detailed explanation about the relation between anomalous diffusion and random walk processes can be founded at the work done by Metzler and Klafter [12]. As we know, the diffusion equation is connected to the Brownian motion, where the mean square displacement (msd) of particles is proportional to time. However, in Random Walks the msd is proportional to a power of time. It is also interesting the approach given in [2, 8, 22] where it is suggested that anomalous diffusion could be caused by heterogeneities in the domain.

For the relation between fractional diffusion equations and their applications, we refer the reader to [11, 14, 16] and references therein where applications to the theory of linear viscoelasticity or thermoelasticity, among other, are presented.

In this paper, two approaches leading to subdiffusion are considered. The first one linked to the mathematical interest as generalized operators which interpolates classical derivatives (see [6]), and the second one related to Fourier's generalization laws (see [15]). These two approaches derived in two different formulations for the FDE. In order to present them, let a function $u = u(x, t)$ be defined for given one-dimensional variables x and time t . A first formulation for the FDE given in terms of fractional integrals (see [7]) is given by:

$${}_0I_t^\alpha u_{xx}(x, t) = u(x, t) - u(x, 0) \quad (2)$$

where, ${}_0I_t^\alpha$ is the fractional integral of Riemann–Liouville of order α in the t -variable defined as

$${}_0I_t^\alpha u(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(x, \tau) d\tau$$

for every u such that $u(x, \cdot) \in L^1(0, T)$ for every $x > 0$. Equation (2) is derived also in [12], when a fractal time random walk is considered. As it can be seen, no partial derivative in time is part of equation (2), but differentiating respect on time to both members we get a second formulation for a FDE

$${}^{RL}D_t^{1-\alpha} u_{xx}(x, t) = u_t(x, t), \quad (3)$$

where ${}^{RL}D_t^{1-\alpha}$ is the fractional derivative of Riemann–Liouville in the t -variable defined for every $\alpha \in (0, 1)$ as

$${}^{RL}D_t^{1-\alpha} u(x, t) = \frac{\partial}{\partial t} {}_0I_t^\alpha u(x, t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t - \tau)^{\alpha-1} u(x, \tau) d\tau$$

for every $u \in AC_t[0, T] = \{u \mid u(x, \cdot) \text{ is absolutely continuous on } [0, T] \text{ for every } x \in \mathbb{R}^+\}$. Nevertheless, when discussing about FDE associated to fractional time derivatives, the reader may retract on the FDE for the Caputo derivative, that is

$${}_0^C D_t^\alpha u(x, t) = u_{xx}(x, t). \quad (4)$$

Here, the partial time derivative has been replaced by a fractional derivative in the sense of Caputo respect on time. The Caputo derivative ${}_0^C D_t^\alpha$ is defined for every $\alpha \in (0, 1)$ as

$${}_0^C D_t^\alpha u(x, t) = [{}_0I_t^{1-\alpha} (u_t)](x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - \tau)^{-\alpha} u_t(x, \tau) d\tau$$

for every $u \in AC_t[0, T]$.

As we said before, in this paper, problems like (1) governed by equations like (3) or (4) will be studied. The literature on fractional phase-change problems is rather scant. In [3] a fractional two-phase moving-boundary problem is approximated by a scale Brownian motion model for sub-diffusion. In [26] sharp and diffuse interface models of fractional Stefan problems are discussed. In [17] a formulation of a one-phase fractional phase-change problem is given arising a time dependence on the initial extreme of the fractional derivative. When the starting time considered in the fractional derivative of the governing equation is equal to 0, the mathematical point of view becomes interesting because they admit self-similar solutions in terms of the Wright functions (see [9, 10, 13, 18, 19]). It is worth noting that this kind of problems are not deduced as in [17, 27].

This paper is a continuation of a previous work [20], related to fractional one-phase change problems. In Section 2 some basic definitions and properties on fractional calculus are given. In Section 3, two fractional two-phase Stefan-like problems are considered, admitting both exact self-similar solutions. While the two governing equations are equivalent under certain assumptions for boundary-value-problems, when different “fractional Stefan conditions” are considered, the solutions obtained seem to be different. The uniqueness of the self-similar solution for one of the problems is obtained while it is an open problem for the other (see [19]). Finally, numerical examples and graphics of the solutions are presented by considering a dimensionless model in Section 4.

2 Basic definitions and properties

Proposition 1. [6] *The following properties involving the fractional integrals and derivatives hold:*

1. *The fractional derivative of Riemann–Liouville is a left inverse operator of the fractional integral of Riemann–Liouville of the same order $\alpha \in \mathbb{R}^+$. If $f \in AC[a, b]$, then*

$${}^RL D^\alpha {}_a I^\alpha f(t) = f(t) \quad \text{for every } t \in (a, b)$$

2. *The fractional integral of Riemann–Liouville is not, in general, a left inverse operator of the fractional derivative of Riemann–Liouville.*

In particular, if $0 < \alpha < 1$, then ${}_a I^\alpha ({}^RL D^\alpha f)(t) = f(t) - \frac{{}_a I^{1-\alpha} f(a^+)}{\Gamma(\alpha)(t-a)^{1-\alpha}}$.

3. *If there exist some $\phi \in L^1(a, b)$ such that $f = {}_a I^\alpha \phi$, then*

$${}_a I^\alpha {}^RL D^\alpha f(t) = f(t) \quad \text{for every } t \in (a, b).$$

4. *If $f \in AC[a, b]$, then*

$${}^RL D^\alpha f(t) = \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + {}_a^C D^\alpha f(t).$$

The fractional integral and derivatives of power functions can be easily calculated (see e.g. [14]). In fact, for every $t \geq a$ we have that

$${}_a I^\alpha ((t-a)^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \quad \text{for every } \beta > -1, \quad (5)$$

and that

$${}_a^RL D^\alpha ((t-a)^\beta) = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha} & \text{if } \beta \neq \alpha - 1, \\ 0 & \text{if } \beta = \alpha - 1. \end{cases} \quad (6)$$

In particular, if $\beta > 0$, ${}_a^RL D^\alpha ((t-a)^\beta) = {}_a^C D^\alpha ((t-a)^\beta)$ due to Proposition 1 item 4 and the Caputo derivative of $(t-a)^\beta$ is not defined for $-1 < \beta < 0$.

Proposition 2. [21] *The following limits hold:*

1. *If we set ${}_a I^0 = Id$ for the identity operator, then for every $f \in L^1(a, b)$,*

$$\lim_{\alpha \searrow 0} {}_a I^\alpha f(t) = {}_a I^0 f(t) = f(t), \quad a.e.$$

2. *For every $f \in AC[a, b]$, we have*

$$\lim_{\alpha \nearrow 1} {}_a^C D^\alpha f(t) = f'(t) \quad \text{and} \quad \lim_{\alpha \searrow 1} {}_a^C D^\alpha f(t) = f'(t) - f'(a^+) \quad \text{for all } t \in (a, b).$$

3. *For every $f \in AC[a, b]$,*

$$\lim_{\alpha \nearrow 1} {}_a^{RL} D^\alpha f(t) = f'(t) \quad \text{and} \quad \lim_{\alpha \searrow 1} {}_a^{RL} D^\alpha f(t) = f'(t) \quad a.e.$$

Definition 1. *For every $x \in \mathbb{R}$, the Wright function is defined as*

$$W(x; \rho; \beta) = \sum_{k=0}^{\infty} \frac{x^k}{k! \Gamma(\rho k + \beta)}, \quad \rho > -1 \text{ and } \beta \in \mathbb{R}. \quad (7)$$

An important particular case of the Wright function is the Mainardi function defined by

$$M_\rho(x) = W(-x, -\rho, 1 - \rho) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n! \Gamma(-\rho n + 1 - \rho)}, \quad 0 < \rho < 1.$$

Proposition 3. [16, 29] *Let $\alpha > 0$, $\rho \in (0, 1)$ and $\beta \in \mathbb{R}$. Then the next assertions follows:*

1. *For every $x \in \mathbb{R}$ we have*

$$\frac{\partial}{\partial x} W(x, \rho, \beta) = W(x, \rho, \rho + \beta).$$

2. *For every $x > 0$ and $c > 0$,*

$${}_0 I^\alpha [x^{\beta-1} W(-cx^{-\rho}, -\rho, \beta)] = x^{\beta+\alpha-1} W(-cx^{-\rho}, -\rho, \beta + \alpha). \quad (8)$$

Proposition 4. [20, 29] *For every $\beta \geq 0$, $\rho \in (0, 1)$:*

1. *The Wright function $W(-\cdot, -\rho, \beta)$ is positive and strictly decreasing in \mathbb{R}^+ .*

2. *For every $x \geq 0$ the following equality holds*

$$\rho x W(-x, -\rho, \beta - \rho) = W(-x, -\rho, \beta - 1) + (1 - \beta) W(-x, -\rho, \beta).$$

3. *If, in addition $0 < \rho \leq \mu < \delta$, then for every $x > 0$ the following inequality holds*

$$\Gamma(\delta) W(-x, -\rho, \delta) < \Gamma(\mu) W(-x, -\rho, \mu). \quad (9)$$

Proposition 5. [30] *For every $\beta \geq 0$ and $\rho \in (0, 1)$ the following limit holds*

$$\lim_{x \rightarrow \infty} W(-x, -\rho, \beta) = 0.$$

Proposition 6. [18, 20] *Let $x \in \mathbb{R}_0^+$ be. Then the following limits hold:*

$$\lim_{\alpha \nearrow 1} M_{\alpha/2}(2x) = \lim_{\alpha \nearrow 1} W\left(-2x, -\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right) = M_{1/2}(2x) = \frac{e^{-x^2}}{\sqrt{\pi}}, \quad (10)$$

$$\lim_{\alpha \nearrow 1} W\left(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right) = \frac{e^{-x^2}}{\sqrt{\pi}}, \quad (11)$$

$$\lim_{\alpha \nearrow 1} \left[1 - W \left(-2x, -\frac{\alpha}{2}, 1 \right) \right] = \operatorname{erf}(x), \quad (12)$$

and

$$\lim_{\alpha \nearrow 1} \left[W \left(-2x, -\frac{\alpha}{2}, 1 \right) \right] = \operatorname{erfc}(x), \quad (13)$$

where $\operatorname{erf}(\cdot)$ is the error function defined by $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$ and $\operatorname{erfc}(\cdot)$ is the complementary error function defined by $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$. Moreover, the convergence is uniform over compact sets.

Proposition 7. *The fractional initial-boundary-value problems (14) and (15) for the quarter plane are equivalent if there exists $\beta > 0$ and $\delta > 0$ such that $\beta < \alpha < 1$ and $u_{xx}(x, \cdot)$ is an $O(t^{-\beta})$ in $(0, \delta)$:*

$$\begin{aligned} (i) \quad & {}_0^C D_t^\alpha u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t), \quad 0 < x, 0 < t, \\ (ii) \quad & u(x, 0) = u_0(x), \quad 0 \leq x, \\ (iii) \quad & u(0, t) = g(t), \quad 0 < t, \end{aligned} \quad (14)$$

$$\begin{aligned} (i) \quad & \frac{\partial}{\partial t} u(x, t) = {}_0^{RL} D_t^{1-\alpha} \left(\frac{\partial^2}{\partial x^2} u(x, t) \right), \quad 0 < x, 0 < t, \\ (ii) \quad & u(x, 0) = u_0(x), \quad 0 \leq x, \\ (iii) \quad & u(0, t) = g(t), \quad 0 < t, \end{aligned} \quad (15)$$

Proof. Let $u = u(x, t)$ be a function satisfying equation (14 - i). Applying ${}_0^{RL} D_t^{1-\alpha}$ to both sides and using Proposition 1 item 1 we get (15 - i).

Let now, for the inverse suppose that u satisfies equation (15 - i). Applying ${}_0 I_t^{1-\alpha}$ to both sides and using Proposition 1 item 2 yields that

$${}_0^C D_t^\alpha u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) - \frac{\lim_{t \searrow 0} {}_0 I_t^\alpha \left(\frac{\partial^2}{\partial x^2} u(x, t) \right)}{\Gamma(1-\alpha)t^\alpha}, \quad 0 < x, \quad 0 < t. \quad (16)$$

Now, for every x fixed we have that $u_{xx}(x, \cdot)$ is an $O(t^{-\beta})$ in $(0, \delta)$, then for $t > 0$ small it holds that

$$-C\tau^{-\beta} \leq u_{xx}(x, \tau) \leq C\tau^{-\beta}, \quad 0 < \tau \leq t < \delta. \quad (17)$$

Multiplying by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$ in (17), integrating between 0 and t and applying formula (5) yields that

$$-C \frac{\Gamma(1-\beta)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \leq {}_0 I_t^\alpha u_{xx}(x, t) \leq C \frac{\Gamma(1-\beta)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}, \quad t < \delta. \quad (18)$$

Taking the limit when t tends to zero in (18) and being $\beta < \alpha$ we conclude that equation (14 - i) holds as we wanted to see. \square

Remark 1. *Equations (14-i) and (15-i) has been treated as equivalent in literature, as it can be seeing at [11, 12, 15], but the condition*

$$\lim_{t \searrow 0} {}_0 I_t^\alpha \left(\frac{\partial^2}{\partial x^2} u(x, t) \right) = 0 \quad (19)$$

must be considered and should not be forget it.

Remark 2. *It is easy to check that the following functions verifies equation (14-i) and (15-i) (we have taken $\lambda = 1$ without loss of generality)*

$$w_1(x, t) = x^2 + \frac{2}{\Gamma(\alpha+1)} t^\alpha. \quad (20)$$

$$w_2(x, t) = E_\alpha(t^\alpha) \exp\{-x\} \quad (21)$$

and

$$w_3(x, t) = W \left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1 \right). \quad (22)$$

The condition (19) trivially holds for function w_1 and w_2 and it is no difficult to check it for w_3 (by differentiating first and using Proposition 3 then).

3 The Fractional Stefan-like Problems

In this section, two fractional Stefan-like problems admitting both explicit self-similar solutions will be treated. Before that, some clarification about the used terminology is presented.

We refer to fractional Stefan problems when the governed equations in such problem are derived from physical assumptions, like considering memory fluxes.

For example, suppose that a process of melting of a semi-infinite slab ($0 \leq x < \infty$) of some material is taking place, and the flux involved is a flux with memory. The melt temperature is U_m , and a constant temperature $U_0 > U_m$ is imposed on the fixed face $x = 0$. Let $u_1 = u_1(x, t)$ and $u_2 = u_2(x, t)$ be the temperatures at the solid and liquid phases respectively. Let $J_1 = J_1(x, t)$ and $J_2 = J_2(x, t)$ be the respective functions for the fluxes at position x and time t and let $x = s(t)$ be the function representing the (unknown) position of the free boundary at time t . Suppose further that:

- (i) All the thermophysical parameters are constants.
- (ii) The function s is an increasing function and consequently, an invertible function.
- (iii) J_1 and J_2 are fluxes modeling the material with memory which verifies that “the weighted sum of the fluxes back in time at the current time, is proportional to the gradient of temperature”, that is, the following equations hold

$$\nu_\alpha {}_0 I_t^{1-\alpha} J_1(x, t) = -k_1 \frac{\partial u_1}{\partial x}(x, t) \quad (23)$$

and

$$\nu_\alpha {}_{h(x)} I_t^{1-\alpha} J_2(x, t) = -k_2 \frac{\partial u_2}{\partial x}(x, t) \quad (24)$$

where the initial time in the fractional integral (24) is given by function h which gives us the time when the phase change occurs. That is,

$$t = h(x) = s^{-1}(x) \quad (\text{i.e. } x = s(t))$$

The number ν_α is a parameter with physical dimension (see(70)) such that

$$\lim_{\alpha \nearrow 1} \nu_\alpha = 1, \quad (25)$$

which has been added in order to preserve the consistency with respect to the units of measure in equations (23) and (24). Also, the parameter

$$\mu_\alpha = \frac{1}{\nu_\alpha} \quad (26)$$

will be used in the following equations. More details about these parameters are given in Section 4.

Making an analogous reasoning for the two-phase free-boundary problem, than the one made in [17] for the one-phase free-boundary problem, the mathematical model for the problem described above is given by

$$\begin{aligned}
(i) \quad & \frac{\partial}{\partial t} u_2(x, t) = \lambda_2^2 \mu_{\alpha_2} \frac{\partial}{\partial x} \left(\frac{RL}{h(x)} D_t^{1-\alpha} \left(\frac{\partial}{\partial x} u_2(x, t) \right) \right), & 0 < x < s(t), 0 < t < T, \\
(ii) \quad & \frac{\partial}{\partial t} u_1(x, t) = \lambda_1^2 \mu_{\alpha_1} \frac{\partial}{\partial x} \left({}^RL D_t^{1-\alpha} \left(\frac{\partial}{\partial x} u_1(x, t) \right) \right), & x > s(t), 0 < t < T, \\
(iii) \quad & u_1(x, 0) = U_i, & 0 \leq x, \\
(iv) \quad & u_2(0, t) = U_0, & 0 < t \leq T, \\
(v) \quad & u_1(s(t), t) = u_2(s(t), t) = U_m, & 0 < t \leq T, \\
(vi) \quad & \rho l \frac{d}{dt} s(t) = k_1 \mu_{\alpha_1} {}^RL D_t^{1-\alpha} \frac{\partial}{\partial x} u_1(x, t) \Big|_{(s(t)^+, t)} \\
& \quad \quad \quad - k_2 \mu_{\alpha_2} \frac{RL}{h(x)} D_t^{1-\alpha} \frac{\partial}{\partial x} u_2(x, t) \Big|_{(s(t)^-, t)}, & 0 < t \leq T.
\end{aligned}$$

(27)

where $U_i < U_m < U_0$ and $\mu_\alpha = \frac{1}{\nu_\alpha}$, (note that the parameter μ_α can be the same in equations (27)–i and (27)–ii, and without loss of generality we will take from now on that $\mu_{\alpha_2} = \mu_{\alpha_1}$). Note that self-similar solutions to problem (27) had not been yet founded, due to the difficulty imposed by the variable button limit in the fractional derivative for the liquid phase. As it was said at the beginning of this section, this paper deals with Stefan-like problems admitting explicit self-similar solutions. These problems come from the assumption of consider the button limit $t_0 = 0$ in the fractional time derivatives in the Caputo or Riemann–Liouville sense.

The Stefan-Like Problem for the Caputo derivative. The next problem was treated in [19] and can be obtained by replacing all the times derivatives in (1) by fractional derivatives in the Caputo sense of order $\alpha \in (0, 1)$, i.e.

$$\begin{aligned}
(i) \quad & {}^C D_t^\alpha u_2(x, t) = \lambda_{\alpha_2}^2 \frac{\partial^2}{\partial x^2} u_2(x, t), & 0 < x < s(t), 0 < t < T, \\
(ii) \quad & {}^C D_t^\alpha u_1(x, t) = \lambda_{\alpha_1}^2 \frac{\partial^2}{\partial x^2} u_1(x, t), & x > s(t), 0 < t < T, \\
(iii) \quad & u_1(x, 0) = U_i, & 0 \leq x, \\
(iv) \quad & u_2(0, t) = U_0, & 0 < t \leq T, \\
(v) \quad & u_1(s(t), t) = u_2(s(t), t) = U_m, & 0 < t \leq T, \\
(vi) \quad & \rho l {}^C D_t^\alpha s(t) = k_{\alpha_1} \frac{\partial}{\partial x} u_1(s(t)^+, t) - k_{\alpha_2} \frac{\partial}{\partial x} u_2(s(t)^-, t), & 0 < t \leq T, \\
(vii) \quad & s(0) = 0.
\end{aligned}$$

(28)

where $U_i < U_m < U_0$, λ_{α_i} are positive parameters named as “subdiffusion coefficients” given by $\lambda_{\alpha_i} = \lambda_i \sqrt{\mu_\alpha}$ for $i = 1, 2$, and k_{α_i} are positive parameters named as “subdiffusion thermal conductivities” given by $k_{\alpha_i} = k_i \mu_\alpha$, $i = 1, 2$.

Definition 2. *The triple $\{u_1, u_2, s\}$ is a solution to problem (28) if the following conditions are satisfied*

1. u_1 is continuous in the region $\mathcal{R}_T = \{(x, t) : 0 \leq x \leq s(t), 0 < t \leq T\}$ and at the point $(0, 0)$, u_1 verifies that

$$0 \leq \liminf_{(x,t) \rightarrow (0,0)} u_1(x, t) \leq \limsup_{(x,t) \rightarrow (0,0)} u_1(x, t) < +\infty.$$

2. u_2 is continuous in the region $\{(x, t) : x > s(t), 0 < t \leq T\}$ and at the point $(0, 0)$, u_2 verifies that

$$0 \leq \liminf_{(x,t) \rightarrow (0,0)} u_2(x, t) \leq \limsup_{(x,t) \rightarrow (0,0)} u_2(x, t) < +\infty.$$

3. $u_1 \in C((0, \infty) \times (0, T)) \cap C_x^2((0, \infty) \times (0, T))$, such that $u_1 \in AC_t[0, T]$

4. $u_2 \in C((0, \infty) \times (0, T)) \cap C_x^2((0, \infty) \times (0, T))$, such that $u_2 \in AC_t[0, T]$.

5. $s \in AC[0, T]$.

6. u_1, u_2 and s satisfy (28).

Theorem 1. [19] A self-similar solution to problem (28) is given by

$$\begin{cases} u_2(x, t) = U_0 - \frac{U_0 - U_m}{1 - W\left(-2\xi_\alpha \lambda, -\frac{\alpha}{2}, 1\right)} \left[1 - W\left(-\frac{x}{\lambda_{\alpha_2} t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right] \\ u_1(x, t) = U_i + \frac{U_m - U_i}{W\left(-2\xi_\alpha, -\frac{\alpha}{2}, 1\right)} W\left(-\frac{x}{\lambda_{\alpha_1} t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) \\ s(t) = 2\xi_\alpha \lambda_{\alpha_1} t^{\alpha/2} \end{cases} \quad (29)$$

where ξ_α is a solution to the equation

$$\frac{k_{\alpha_2}(U_0 - U_m)\Gamma(1 - \frac{\alpha}{2})}{\lambda_{\alpha_2}} F_2(2\lambda x) - \frac{k_{\alpha_1}(U_m - U_i)\Gamma(1 - \frac{\alpha}{2})}{\lambda_{\alpha_1}} F_1(2x) = \Gamma\left(1 + \frac{\alpha}{2}\right) \lambda_{\alpha_1} \rho l 2x, \quad x > 0 \quad (30)$$

where $\lambda = \frac{\lambda_{\alpha_1}}{\lambda_{\alpha_2}} = \frac{\lambda_1 \sqrt{\mu_\alpha}}{\lambda_2 \sqrt{\mu_\alpha}} = \frac{\alpha_1}{\alpha_2} > 0$, and $F_1 : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ and $F_2 : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ are the functions defined by

$$F_1(x) = \frac{M_{\alpha/2}(x)}{W\left(-x, -\frac{\alpha}{2}, 1\right)} \quad \text{and} \quad F_2(x) = \frac{M_{\alpha/2}(x)}{1 - W\left(-x, -\frac{\alpha}{2}, 1\right)}. \quad (31)$$

Note 1. The uniqueness of solution to equation (30) is still an open problem. However, the uniqueness of similarity solution will be achieved next for the Riemann–Liouville Stefan–like problem.

The Stefan-Like Problem for the Riemann–Liouville derivative. Consider now the following problem:

$$\begin{aligned} (i) \quad & \frac{\partial}{\partial t} w_2(x, t) = \lambda_{\alpha_2}^2 \frac{\partial}{\partial x} \left({}_0^{RL} D_t^{1-\alpha} \left(\frac{\partial}{\partial x} w_2(x, t) \right) \right), & 0 < x < r(t), \quad 0 < t < T, \\ (ii) \quad & \frac{\partial}{\partial t} w_1(x, t) = \lambda_{\alpha_1}^2 \frac{\partial}{\partial x} \left({}_0^{RL} D_t^{1-\alpha} \left(\frac{\partial}{\partial x} w_1(x, t) \right) \right), & x > r(t), \quad 0 < t < T, \\ (iii) \quad & w_1(x, 0) = U_i, & 0 \leq x, \\ (iv) \quad & w_2(0, t) = U_0, & 0 < t \leq T, \\ (v) \quad & w_1(r(t), t) = w_2(r(t), t) = U_m, & 0 < t \leq T, \\ (vi) \quad & \rho l \frac{d}{dt} r(t) = k_{\alpha_1} {}_0^{RL} D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x, t) \Big|_{(r(t)^+, t)} \\ & \quad - k_{\alpha_2} {}_0^{RL} D_t^{1-\alpha} \frac{\partial}{\partial x} w_2(x, t) \Big|_{(r(t)^-, t)}, & 0 < t \leq T, \\ (vii) \quad & r(0) = 0. \end{aligned} \quad (32)$$

where, as before, $U_i < U_m < U_0$, $\lambda_{\alpha_i} = \lambda_i \sqrt{\mu_\alpha}$ for $i = 1, 2$, and $k_{\alpha_i} = k_i \mu_\alpha$, $i = 1, 2$.

Remark 3. The expression $k_{\alpha_1} {}_0^{RL} D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x, t) \Big|_{(r(t)^+, t)}$ is equivalent to

$$\lim_{x \rightarrow r(t)^+} k_{\alpha_1} {}_0^{RL} D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x, t), \quad (33)$$

which should not coincide with

$$k_{\alpha_1} {}_0^{RL} D_t^{1-\alpha} \left(\lim_{x \rightarrow r(t)^+} \frac{\partial}{\partial x} w_1(x, t) \right). \quad (34)$$

Definition 3. The triple $\{w_1, w_2, r\}$ is a solution of problem (32) if the following conditions are satisfied

1. w_1 is continuous in the region $\mathcal{R}_T = \{(x, t) : 0 \leq x \leq s(t), 0 < t \leq T\}$ and at the point $(0, 0)$, w_1 verifies that

$$0 \leq \liminf_{(x,t) \rightarrow (0,0)} w_1(x, t) \leq \limsup_{(x,t) \rightarrow (0,0)} w_1(x, t) < +\infty.$$

2. w_2 is continuous in the region $\{(x, t) : x > r(t), 0 < t \leq T\}$ and at the point $(0, 0)$, w_2 verifies that

$$0 \leq \liminf_{(x,t) \rightarrow (0,0)} w_2(x, t) \leq \limsup_{(x,t) \rightarrow (0,0)} w_2(x, t) < +\infty.$$

3. $w_1 \in C((0, \infty) \times (0, T)) \cap C_x^2((0, \infty) \times (0, T))$, such that $w_{1x} \in AC_t(0, T)$.
4. $w_2 \in C((0, \infty) \times (0, T)) \cap C_x^2((0, \infty) \times (0, T))$, such that $w_{2x} \in AC_t[0, T]$.
5. $r \in C^1(0, T)$.
6. There exist ${}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} w_2(x, t)|_{(s(t)^+, t)}$ and ${}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x, t)|_{(r(t)^-, t)}$ for all $t \in (0, T]$.
7. w_1, w_2 and s satisfy (32).

Theorem 2. An explicit solution for the two-phase fractional Stefan-like problem (32) is given by

$$\begin{cases} w_2(x, t) = U_0 - \frac{U_0 - U_m}{1 - W(-2\eta_\alpha \lambda, -\frac{\alpha}{2}, 1)} \left[1 - W\left(-\frac{x}{\lambda_{\alpha_2} t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) \right] \\ w_1(x, t) = U_i + \frac{U_m - U_i}{W(-2\eta_\alpha, -\frac{\alpha}{2}, 1)} W\left(-\frac{x}{\lambda_{\alpha_1} t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) \\ r(t) = 2\eta_\alpha \lambda_{\alpha_1} t^{\alpha/2} \end{cases} \quad (35)$$

where η_α is the unique positive solution to the equation

$$\frac{k_{\alpha_2}(U_0 - U_m)}{\lambda_{\alpha_1} \lambda_{\alpha_2}} G_2(2\lambda x) - \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} G_1(2x) = \left(\rho l + \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} \right) 2x, \quad (36)$$

where $\lambda = \frac{\lambda_{\alpha_1} \sqrt{\mu_\alpha}}{\lambda_{\alpha_2} \sqrt{\mu_\alpha}} = \frac{\lambda_1}{\lambda_2} > 0$, $U_i < U_m < U_0$ and $G_1 : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ and $G_2 : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ are the functions defined by

$$G_1(x) = \frac{W(-x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2})}{W(-x, -\frac{\alpha}{2}, 1)} \quad \text{and} \quad G_2(x) = \frac{2/\alpha W(-x, -\frac{\alpha}{2}, \frac{\alpha}{2})}{1 - W(-x, -\frac{\alpha}{2}, 1)}. \quad (37)$$

Proof. Let the functions

$$\begin{aligned} w_i : \mathbb{R}_0^+ \times (0, T) &\rightarrow \mathbb{R} \\ (x, t) &\rightarrow w_i(x, t) = A_i + B_i \left[1 - W\left(-\frac{x}{\lambda_{\alpha_i} t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right) \right] \end{aligned} \quad (38)$$

be the proposed solutions for $i = 1, 2$. Rewriting expression (8) for the variable t and taking $c = \frac{x}{\lambda_{\alpha_i}}$ gives

$${}_0 I_t^\alpha t^{\beta-1} W\left(-\frac{x}{\lambda_{\alpha_i}} t^{-\rho}, -\rho, \beta\right) = t^{\beta+\alpha-1} W\left(-\frac{x}{\lambda_{\alpha_i}} t^{-\rho}, -\rho, \beta + \alpha\right). \quad (39)$$

Then, by using (39) for $\beta = 1 - \frac{\alpha}{2}$ and Proposition 3 it is easy to check that w_i verifies equations (32 - i) and (32 - ii) respectively for $i = 1, 2$.

From condition (32 - v) we deduce that $r(t)$ must be proportional to $t^{\alpha/2}$. Therefore we set

$$r(t) = 2\eta_\alpha \lambda_{\alpha_1} t^{\alpha/2}, \quad t \geq 0 \quad (40)$$

where η_α is a constant to be determined and λ_{α_1} was added for simplicity in the next calculations. Now, from conditions (32 - iii), (32 - iv) and (32 - v) it holds that

$$\begin{aligned} A_1 &= U_i + \frac{U_m - U_i}{W(-2\eta_\alpha, -\frac{\alpha}{2}, 1)}, & B_1 &= -\frac{U_m - U_i}{W(-2\eta_\alpha, -\frac{\alpha}{2}, 1)} \\ A_2 &= U_0, & B_2 &= -\frac{U_0 - U_m}{1 - W(-2\eta_\alpha \lambda, -\frac{\alpha}{2}, 1)} \end{aligned}$$

As before, by considering (39) for $\beta = 1 - \frac{\alpha}{2}$ and Proposition 3, it holds that

$$\begin{aligned} &{}_0^{RL}D_t^{1-\alpha} w_{i_x}(x, t) = \\ &\frac{B_i \alpha / 2}{\lambda_{\alpha_1} \lambda_{\alpha_i} t^{1-\alpha/2}} W\left(-\frac{x}{\lambda_{\alpha_i} t^{\alpha/2}}, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right) + \frac{B_i \alpha / 2}{\lambda_{\alpha_1} \lambda_{\alpha_i}} \frac{x}{t} W\left(-\frac{x}{\lambda_{\alpha_i} t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right), \quad i = 1, 2. \end{aligned} \quad (41)$$

Then replacing (41) and (40) in equation (32 – *vii*), and evaluating the limits following (33) it yields that η_α must verify the next equality

$$\begin{aligned} \rho l 2\eta_\alpha \lambda_{\alpha_1} = & -\frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} \frac{W(-2\eta_\alpha, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2})}{W(-2\eta_\alpha, -\frac{\alpha}{2}, 1)} - \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} 2\eta_\alpha - \\ & + \frac{k_{\alpha_2}(U_0 - U_m)}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{W(-2\lambda\eta_\alpha, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2})}{1 - W(-\lambda 2\eta_\alpha, -\frac{\alpha}{2}, 1)} + \frac{k_{\alpha_2}(U_0 - U_m)}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{2\lambda\eta_\alpha W(-\lambda 2\eta_\alpha, -\frac{\alpha}{2}, 1)}{1 - W(-\lambda_\alpha 2\eta_\alpha, -\frac{\alpha}{2}, 1)}. \end{aligned} \quad (42)$$

which leads to conclude that $\{w_1, w_2, r\}$ is a solution to (32) if and only if η_α is a solution to the equation

$$\begin{aligned} & \frac{k_{\alpha_2}(U_0 - U_m)}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{W(-\lambda 2x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}) + 2\lambda x W(-\lambda 2x, -\frac{\alpha}{2}, 1)}{1 - W(-\lambda 2x, -\frac{\alpha}{2}, 1)} - \\ & - k_{\alpha_1} \frac{U_m - U_i}{\lambda_{\alpha_1}^2} \frac{W(-2x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2})}{W(-2x, -\frac{\alpha}{2}, 1)} = \left(\rho l + \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} \right) 2x, \quad x > 0. \end{aligned} \quad (43)$$

which, by using Proposition 4 – 2 leads to equation (36).

The next step is to prove that Eq. (36) has unique solution. For that purpose we define function G in \mathbb{R}^+ as

$$G(x) = \frac{k_{\alpha_2}(U_0 - U_m)}{\lambda_{\alpha_1} \lambda_{\alpha_2}} G_2(2\lambda x) - \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} G_1(2x) - \left(\rho l + \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} \right) 2x.$$

Note that G is continuous function such that

$$G(0^+) = +\infty. \quad (44)$$

From Proposition 4 – 3 for every $x > 0$ we have that

$$0 < \frac{W(-2x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2})}{W(-2x, -\frac{\alpha}{2}, 1)} < \frac{1}{\Gamma(\frac{\alpha}{2} + 1)}, \quad (45)$$

then G_1 is bounded. Also, from (45) it holds that

$$\begin{aligned} & -\frac{k_{\alpha_1}(U_i - U_m)}{\lambda_{\alpha_1}^2} \frac{1}{\Gamma(\frac{\alpha}{2} + 1)} + \frac{k_{\alpha_2}(U_0 - U_m)}{\lambda_{\alpha_1} \lambda_{\alpha_2}} G_2(2\lambda x) - \left(\rho l + \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} \right) 2x < \\ & G(x) < \frac{k_{\alpha_2}(U_0 - U_m)}{\lambda_{\alpha_1} \lambda_{\alpha_2}} G_2(2\lambda x) - \left(\rho l + \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} \right) 2x, \end{aligned} \quad (46)$$

and taking the limit when $x \rightarrow \infty$ in (46) and using Proposition 5 we obtain that

$$G(+\infty) = -\infty. \quad (47)$$

Finally, consider the function $K: \mathbb{R}^+ \rightarrow \mathbb{R}$ defined as

$$K(x) = -\frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2} [G_1(2x) + 2x] - \rho l 2x. \quad (48)$$

Applying Proposition 3 item 1 and being $\frac{(U_m - U_i)}{\lambda_{\alpha_1}^2} > 0$ it results that K is a strictly decreasing function. By the other side, from Proposition 4 item 1 we have that G_2 is a strictly decreasing function. Then it can be concluded that G is a strictly decreasing function. Therefore Eq. (36) has a unique positive solution. \square

Remark 4. The limits described in Remark 3 are different if we compute them for the functions w_1 and r . In fact, by using the computation made in the previous theorem, we get

$$\lim_{x \rightarrow r(t)^+} {}^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x, t) = \frac{B_1}{\lambda_{\alpha_1}} \left[\frac{\alpha}{2} t^{\alpha/2-1} W \left(-2\eta_{\alpha}, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2} \right) + \frac{\alpha}{2} 2\eta_{\alpha} t^{\alpha/2-1} W \left(-2\eta_{\alpha}, -\frac{\alpha}{2}, 1 \right) \right]. \quad (49)$$

and from Proposition 4 – 2, we have:

$$W \left(-2\eta_{\alpha}, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2} \right) + 2\eta_{\alpha} W \left(-2\eta_{\alpha}, -\frac{\alpha}{2}, 1 \right) = \frac{2}{\alpha} W \left(-2\eta_{\alpha}, -\frac{\alpha}{2}, \frac{\alpha}{2} \right). \quad (50)$$

Then

$$\lim_{x \rightarrow r(t)^+} {}^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} w_1(x, t) = \frac{B_1}{\lambda_{\alpha_1}} t^{\alpha/2-1} W \left(-2\eta_{\alpha}, -\frac{\alpha}{2}, \frac{\alpha}{2} \right) \quad (51)$$

whereas

$${}^{RL}D_t^{1-\alpha} \left(\lim_{x \rightarrow r(t)^+} \frac{\partial}{\partial x} w_1(x, t) \right) = \frac{B_1}{\lambda_{\alpha_1}} t^{\alpha/2-1} \frac{\Gamma \left(1 - \frac{\alpha}{2} \right)}{\Gamma \left(\frac{\alpha}{2} \right)} M_{\alpha/2}(2\eta_{\alpha}). \quad (52)$$

And we know that (51) and (52) are different due to Proposition 4 – 3.

Theorem 3. If $\lambda = 1$, the explicit solutions (35) to problem (32), and (29) to problem (28) are different.

Proof. Take $U_i = -1$, $U_m = 0$ and $U_0 = 1$. Let $\{u_1, u_2, s\}$ be the solution to problem (28). Then $s(t) = 2\lambda_{\alpha_1} \xi_{\alpha} t$ where ξ_{α} is a positive solution to equation

$$\frac{k_{\alpha_2} \Gamma \left(1 - \frac{\alpha}{2} \right)}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{M_{\alpha/2}(2\lambda x)}{1 - W \left(-\lambda 2x, -\frac{\alpha}{2}, 1 \right)} - \frac{k_{\alpha_1} \Gamma \left(1 - \frac{\alpha}{2} \right)}{\lambda_{\alpha_1}^2} \frac{M_{\alpha/2}(2x)}{W \left(-2x, -\frac{\alpha}{2}, 1 \right)} = \Gamma \left(1 + \frac{\alpha}{2} \right) \rho l 2x. \quad (53)$$

By the other side, let $\{w_1, w_2, r\}$ be the solution to problem (32). Then η_{α} is the positive solution to equation

$$\frac{k_{\alpha_2} 2/\alpha}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{W \left(-2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2} \right)}{1 - W \left(-\lambda 2x, -\frac{\alpha}{2}, 1 \right)} - \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2} \frac{W \left(-2x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2} \right)}{W \left(-2x, -\frac{\alpha}{2}, 1 \right)} = \left(\rho l + \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2} \right) 2x, \quad (54)$$

or equivalently,

$$\frac{k_{\alpha_2}}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{\Gamma \left(1 + \frac{\alpha}{2} \right) 2/\alpha W \left(-2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2} \right)}{1 - W \left(-2\lambda x, -\frac{\alpha}{2}, 1 \right)} - \frac{k_{\alpha_1} \Gamma \left(1 + \frac{\alpha}{2} \right)}{\lambda_{\alpha_1}^2} \frac{W \left(-2x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2} \right) + 2x W \left(-2x, -\frac{\alpha}{2}, 1 \right)}{W \left(-2x, -\frac{\alpha}{2}, 1 \right)} = \Gamma \left(1 + \frac{\alpha}{2} \right) \rho l 2x. \quad (55)$$

From Proposition 4 – 2, for every $x > 0$ we have that

$$W \left(-2x, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2} \right) + 2x W \left(-2x, -\frac{\alpha}{2}, 1 \right) = \frac{2}{\alpha} W \left(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2} \right). \quad (56)$$

Then using the fact that the Gamma function verifies that $\frac{\Gamma \left(1 + \frac{\alpha}{2} \right)}{\frac{\alpha}{2}} = \Gamma \left(\frac{\alpha}{2} \right)$ and replacing (56) in (55) we deduce that η_{α} is the unique positive solution to the equation

$$\frac{k_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}} \frac{\Gamma(\frac{\alpha}{2})W(-2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2})}{1 - W(-2\lambda x, -\frac{\alpha}{2}, 1)} - \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2} \frac{\Gamma(\frac{\alpha}{2})W(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2})}{W(-2x, -\frac{\alpha}{2}, 1)} = \Gamma(1 + \frac{\alpha}{2})\rho l 2x, x > 0. \quad (57)$$

If we suppose then that $\xi_\alpha = \eta_\alpha$, it result that there exist $\xi_\alpha > 0$ such that

$$\begin{aligned} & \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2} \frac{\Gamma(\frac{\alpha}{2})W(-2\xi_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2})}{W(-\xi_\alpha, -\frac{\alpha}{2}, 1)} - \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2} \frac{\Gamma(1 - \frac{\alpha}{2})M_{\alpha/2}(2\xi_\alpha)}{W(-\xi_\alpha, -\frac{\alpha}{2}, 1)} = \\ & = \frac{k_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}} \frac{\Gamma(\frac{\alpha}{2})W(-\lambda 2\xi_\alpha, -\frac{\alpha}{2}, \frac{\alpha}{2})}{1 - W(-\lambda\xi_\alpha, -\frac{\alpha}{2}, 1)} - c_2 \frac{\Gamma(1 - \frac{\alpha}{2})M_{\alpha/2}(\lambda 2\xi_\alpha)}{1 - W(-\lambda\xi_\alpha, -\frac{\alpha}{2}, 1)}. \end{aligned} \quad (58)$$

By using the hypothesis that $\lambda = 1$, we conclude that

$$\frac{\frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2}}{W(-\xi_\alpha, -\frac{\alpha}{2}, 1)} = \frac{\frac{k_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}}}{1 - W(-\lambda\xi_\alpha, -\frac{\alpha}{2}, 1)}, \quad (59)$$

which leads to

$$W(-\xi_\alpha, -\frac{\alpha}{2}, 1) = \frac{1}{1 + \frac{k_{\alpha_2}\lambda_{\alpha_2}}{k_{\alpha_1}\lambda_{\alpha_1}}}. \quad (60)$$

Replacing (60) in equation (53) yields that

$$\rho l \lambda_{\alpha_1} 2\xi_\alpha = 0$$

which leads to $\xi_\alpha = 0$, contradicting the fact that $\xi_\alpha > 0$. \square

Note 2. It is worth noting that an analogous proof for Theorem 3 but considering $\lambda \neq 1$ does not holds. In fact, if we define the function $h_\alpha: \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$h_\alpha(x) = \Gamma\left(\frac{\alpha}{2}\right)W\left(-x, -\frac{\alpha}{2}, \frac{\alpha}{2}\right) - \Gamma\left(1 - \frac{\alpha}{2}\right)M_{\alpha/2}(x)$$

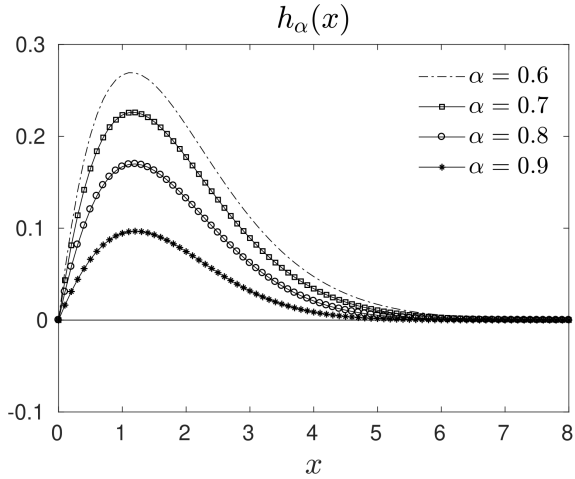


Figure 1: The function $h_\alpha(x) = \Gamma(\frac{\alpha}{2})W(-x, -\frac{\alpha}{2}, \frac{\alpha}{2}) - \Gamma(1 - \frac{\alpha}{2})M_{\alpha/2}(x)$ for different values of α ,

then equality (58) can be expressed as

$$\frac{k_{\alpha_2}}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{h_{\alpha}(\lambda 2\xi_{\alpha})}{1 - W(-\lambda 2\xi_{\alpha}, -\frac{\alpha}{2}, 1)} = \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2} \frac{h_{\alpha}(2\xi_{\alpha})}{W(-2\xi_{\alpha}, -\frac{\alpha}{2}, 1)}. \quad (61)$$

If $\lambda \neq 1$, it is not possible to cancel the expressions $h_{\alpha}(\lambda 2\xi_{\alpha})$ and $h_{\alpha}(2\xi_{\alpha})$ in equation (61). Moreover the graphics in Figure 2 lead us to suppose that there exists a positive solution to equation

$$\frac{k_{\alpha_2}}{\lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{h_{\alpha}(\lambda x)}{1 - W(-\lambda x, -\frac{\alpha}{2}, 1)} = \frac{k_{\alpha_1}}{\lambda_{\alpha_1}^2} \frac{h_{\alpha}(x)}{W(-x, -\frac{\alpha}{2}, 1)}, \quad x > 0, \quad (62)$$

then, it is not possible to get a contradiction like (60).

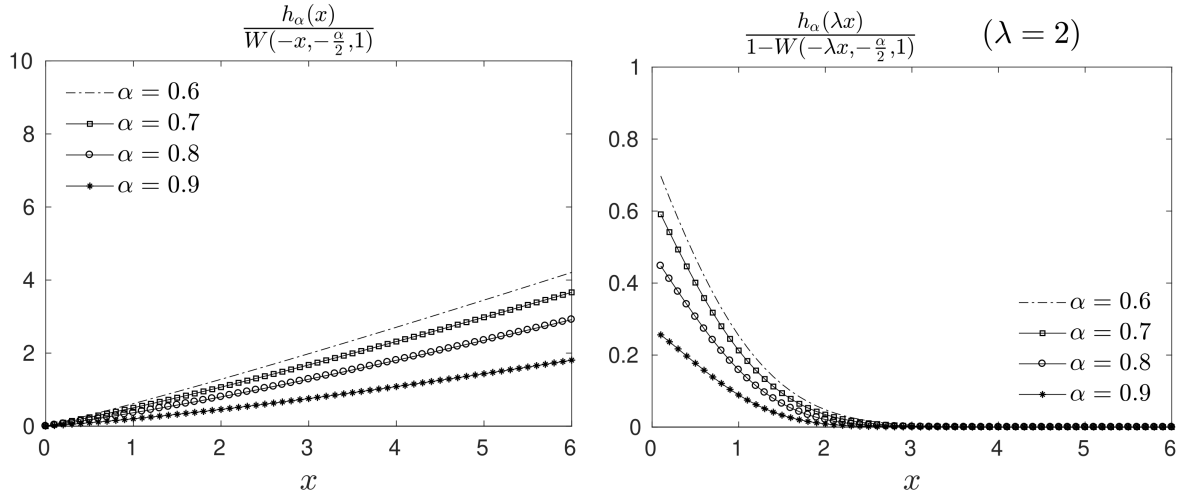


Figure 2: The left and right quotients of equation (62) for different values of α

However, if we take different values of λ (which are different to 1) and the parameters ξ_{α} and η_{α} are estimated numerically for different values of α , we show that they are different and converging both to the same value when $\alpha \nearrow 1$. Numerical examples will be given in the next section.

Theorem 4. The explicit solution (35) to problem (32) converges, when $\alpha \nearrow 1$, to the unique solution to the classical Stefan problem given by

$$\begin{aligned} (i) \quad & \frac{\partial}{\partial t} u_2(x, t) = \lambda_2^2 \frac{\partial^2}{\partial x^2} u_2(x, t), & 0 < x < s(t), 0 < t < T, \\ (ii) \quad & \frac{\partial}{\partial t} u_1(x, t) = \lambda_1^2 \frac{\partial^2}{\partial x^2} u_1(x, t), & x > s(t), 0 < t < T, \\ (iii) \quad & u_1(x, 0) = U_i, & 0 \leq x, \\ (iv) \quad & u_2(0, t) = U_0, & 0 < t \leq T, \\ (v) \quad & u_1(s(t), t) = u_2(s(t), t) = U_m, & 0 < t \leq T, \\ (vi) \quad & \frac{d}{dt} s(t) = k_1 \frac{\partial}{\partial x} u_1(s(t), t) - k_2 \frac{\partial}{\partial x} u_2(s(t), t), & 0 < t \leq T, \\ (vii) \quad & s(0) = 0 \end{aligned} \quad (63)$$

Proof. The unique solution to problem (63) is the Neumann solution given in [28],

$$\begin{cases} z_2(x, t) = U_0 - (U_0 - U_m) \frac{\operatorname{erf}\left(\frac{x}{2\lambda_2\sqrt{t}}\right)}{\operatorname{erf}(\nu_1\lambda)} \\ z_1(x, t) = U_i + (U_m - U_i) \frac{\operatorname{erfc}\left(\frac{x}{2\lambda_1\sqrt{t}}\right)}{\operatorname{erfc}(\nu_1)} \\ w(t) = 2\eta_1\lambda_1\sqrt{t} \end{cases} \quad (64)$$

where η_1 is the unique solution to the equation

$$\frac{k_2(U_0 - U_m)}{\lambda_1 \lambda_2} \frac{\exp\{-\lambda^2 x^2\}}{\sqrt{\pi} \operatorname{erf}(\lambda x)} - \frac{k_1(U_m - U_i)}{\lambda_1^2} \frac{\exp\{-x^2\}}{\sqrt{\pi} \operatorname{erfc}(x)} = \rho l x, \quad x > 0. \quad (65)$$

Reasoning like in the previous theorem we can state that the solution to problem (32) is given by (35) where η_α is the unique positive solution to the equation

$$\frac{k_{\alpha_2}(U_0 - U_m)}{\lambda_{\alpha_1} \lambda_{\alpha_2} \alpha} \frac{W(-2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2})}{1 - W(-2\lambda x, -\frac{\alpha}{2}, 1)} - \frac{k_{\alpha_1}(U_m - U_i)}{\lambda_{\alpha_1}^2 \alpha} \frac{W(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2})}{W(-2x, -\frac{\alpha}{2}, 1)} = \rho l x, \quad x > 0. \quad (66)$$

Clearly, if we take $\alpha = 1$ in equation (66) we recover equation (65). So, let the sequence $\{\eta_\alpha\}_\alpha$ be, where η_α is the unique positive solution to equation (66) for each $0 < \alpha < 1$. Defining the functions

$$f_\alpha(x) = \frac{k_{\alpha_2}(U_0 - U_m)}{\rho l \lambda_{\alpha_1} \lambda_{\alpha_2} \alpha} \frac{W(-2\lambda x, -\frac{\alpha}{2}, \frac{\alpha}{2})}{1 - W(-2\lambda x, -\frac{\alpha}{2}, 1)} - \frac{k_{\alpha_1}(U_m - U_i)}{\rho l \lambda_{\alpha_1}^2 \alpha} \frac{W(-2x, -\frac{\alpha}{2}, \frac{\alpha}{2})}{W(-2x, -\frac{\alpha}{2}, 1)}$$

for every $x \in \mathbb{R}^+$ and $0 < \alpha \leq 1$, it holds that $f_\alpha(\eta_\alpha) = \eta_\alpha$ for every $\alpha \in (0, 1]$.

From [23] we know that f_1 is a strictly decreasing function in \mathbb{R}^+ . Taking a close interval $[a, b] \subset \mathbb{R}^+$ such that $\eta_1 \in [a, b]$, using the uniform convergence over compact sets of all the positive functions given in Proposition 6 and proceeding like in [20, Theorem 2] we can state that

$$\lim_{\alpha \nearrow 1} \eta_\alpha = \eta_1. \quad (67)$$

Finally, by taking the limit when $\alpha \nearrow 1$ in solution (35) by applying Proposition 6, the thesis holds. \square

Remark 5. By using the same technique described before, we can improve the result given in [19, Theorem 3.3] by considering the functions g_α defined in \mathbb{R}^+ by

$$g_\alpha(x) = \frac{k_{\alpha_2}(U_0 - U_m)}{\rho l \lambda_{\alpha_1} \lambda_{\alpha_2}} \frac{\Gamma(1 - \alpha/2)}{\Gamma(1 + \alpha/2)} \frac{M_{\alpha/2}(-2\lambda x)}{1 - W(-2\lambda x, -\frac{\alpha}{2}, 1)} - \frac{k_{\alpha_1}(U_m - U_i)}{\rho l \lambda_{\alpha_1}^2 \alpha} \frac{\Gamma(1 - \alpha/2)}{\Gamma(1 + \alpha/2)} \frac{M_{\alpha/2}(-2x)}{W(-2x, -\frac{\alpha}{2}, 1)}$$

and a sequence $\{\xi_\alpha\}_\alpha$ were ξ_α is a solution to equation $g_\alpha(x) = x$, $x > 0$.

4 The dimensionless problems and numerical results

In the aim to give different graphics of the solutions obtained in Section 3, the problems (28) and (32) will be rewritten in their dimensionless form.

First, we give the following table exhibiting the usual heat conduction physical dimensions related to this work. Let us write \mathbf{T} for temperature, \mathbf{t} for time, \mathbf{m} for mass and \mathbf{X} for position.

u_1, u_2, w_1, w_2	temperatures	$\left[\frac{\mathbf{T}}{\mathbf{T}} \right]$	(68)
k_1, k_2	thermal conductivities	$\left[\frac{\mathbf{m} \mathbf{X}}{\mathbf{T} \mathbf{t}^3} \right]$	
ρ	mass density	$\left[\frac{\mathbf{m}}{\mathbf{X}^3} \right]$	
c_1, c_2	specific heats	$\left[\frac{\mathbf{X}^2}{\mathbf{T} \mathbf{t}^2} \right]$	
$\lambda_i^2 = \frac{k_i}{\rho c}, i = 1, 2$	diffusion coefficients	$\left[\frac{\mathbf{X}^2}{\mathbf{t}} \right]$	
l	latent heat per unit mass	$\left[\frac{\mathbf{X}^2}{\mathbf{t}^2} \right]$	

Proposition 8. For every $\alpha \in (0, 1)$ it holds that

1. ${}_0I_t^\alpha f = [f]t^\alpha$ for every $f = f(t) \in L^1(0, T)$.
2. ${}_0^{RL}D^\alpha f = \frac{[f]}{t^\alpha}$ for every $f = f(t) \in AC[0, T]$.
3. ${}_0^CD^\alpha f = \frac{[f]}{t^\alpha}$ for every $f = f(t) \in AC[0, T]$.

Recall that the parameters ν_α and μ_α given in (25) were added to preserve the consistency with respect to the units of measure in equations (23) and (24). That is, being $[J] = [ku_x] = \frac{\mathbf{m}}{\mathbf{t}^3}$ and using Proposition 8, it holds that

$$[{}_0I_t^{1-\alpha} J(x, t)] = \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{J(x, \tau)}{(t-\tau)^\alpha} d\tau \right] = \frac{\mathbf{m}}{\mathbf{t}^{2+\alpha}}. \quad (69)$$

Then, replacing (69) in (23) one gets

$$[\nu_\alpha] = \frac{[k \frac{\partial u}{\partial x}]}{[h(x) I_t^{1-\alpha} J]} = \frac{1}{\mathbf{t}^{1-\alpha}}. \quad (70)$$

Therefore,

$$[\mu_\alpha] = \mathbf{t}^{1-\alpha}. \quad (71)$$

Proposition 9. Let x_0 be a characteristic position and let U^* be a characteristic temperature. Then, if the following rescaling variable are considered

$$y = \frac{x}{x_0}, \quad \tau = \frac{\lambda_1^2}{x_0^2} t \quad \text{and} \quad \tilde{w} = \frac{w}{U^*}, \quad (72)$$

it holds that

$${}_0I_t^\alpha(w_x(x, t)) = \frac{U^* x_0}{\lambda_1^2} \left(\frac{\lambda_1^2}{x_0^2} \right)^{1-\alpha} {}_0I_\tau^\alpha(\tilde{w}_y(y, \tau)), \quad (73)$$

$${}_0I_t^\alpha(w_{xx}(x, t)) = \frac{U^*}{\lambda_1^2} \left(\frac{\lambda_1^2}{x_0^2} \right)^{1-\alpha} {}_0I_\tau^\alpha(\tilde{w}_{yy}(y, \tau)) \quad (74)$$

and

$${}_0^{RL}D_t^{1-\alpha}(w_{xx}(x, t)) = \frac{U^*}{x_0^2} \left(\frac{\lambda_1^2}{x_0^2} \right)^{1-\alpha} {}_0^{RL}D_\tau^{1-\alpha}(\tilde{w}_{yy}(y, \tau)). \quad (75)$$

Proof. We prove here equation (73). By considering the rescaling (72), we have

$$\tilde{w}(y, \tau) = \frac{w(x(y), t(\tau))}{U^*}. \quad (76)$$

Then

$$\begin{aligned} {}_0I_t^\alpha(w_x(x, t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{w_x(x, z)}{(t-z)^{1-\alpha}} dz = \frac{U^*}{\Gamma(\alpha)} \int_0^t \frac{\frac{1}{x_0} \tilde{w}_y(y, \tau(z))}{(t-z)^{1-\alpha}} dz = \\ &= \frac{U^*}{\Gamma(\alpha)} \int_0^{\frac{\lambda_1^2}{x_0^2} t} \frac{\tilde{w}_y(y, v)}{\left(\frac{x_0^2}{\lambda_1^2} \right)^{1-\alpha} \left(\frac{\lambda_1^2}{x_0^2} t - v \right)^{1-\alpha}} \frac{x_0}{\lambda_1^2} dv = \frac{U^*}{x_0} \left(\frac{x_0^2}{\lambda_1^2} \right)^\alpha {}_0I_\tau^\alpha(\tilde{w}_y(y, \tau)). \end{aligned}$$

□

Now, let us consider problems (28) and (32). By using Proposition 9 it is easy to state that the governing equation (32 – *i*) is equivalent to the following equation

$$\frac{\partial}{\partial \tau} \tilde{w}_2(y, \tau) = \lambda^2 \mu_\alpha \left(\frac{\lambda_1^2}{x_0^2} \right)^{1-\alpha} {}_0^{RL}D_\tau^{1-\alpha} \tilde{w}_{2yy}(y, \tau). \quad (77)$$

Note that $\mu_\alpha = \left(\frac{x_0^2}{\lambda_1^2} \right)^{1-\alpha}$ is an admissible parameter because $[\mu_\alpha] = \mathbf{t}^{1-\alpha}$ and that $\lim_{\alpha \nearrow 1} \mu_\alpha = 1$. Then, the parameter $\mu_\alpha \left(\frac{\lambda_1^2}{x_0^2} \right)^{1-\alpha}$ in equation (77) can be omitted.

Analogously, transforming the governing equations, the Stefan conditions and the initial and boundary data in problems (28) and (32), and by taking $U_m = 0$ and $U^* = |U_i|$, it follows that the non-dimensional associated form are given by

$$\begin{aligned} (i) \quad & {}_0^C D_\tau^\alpha \tilde{u}_2(y, \tau) = \lambda^2 \tilde{u}_{2yy}(y, \tau), & 0 < y < \tilde{s}(\tau), \quad 0 < \tau < \tilde{T}, \\ (ii) \quad & {}_0^C D_\tau^\alpha \tilde{u}_1(y, \tau) = \tilde{u}_{2yy}(y, \tau), & y > \tilde{s}(\tau), \quad 0 < \tau < \tilde{T}, \\ (iii) \quad & \tilde{u}_1(y, 0) = -1, & 0 \leq x, \\ (iv) \quad & \tilde{u}_2(0, \tau) = \frac{U_0}{|U_i|}, & 0 < \tau \leq \tilde{T}, \\ (v) \quad & \tilde{u}_1(\tilde{s}(\tau), \tau) = \tilde{u}_1(\tilde{s}(\tau), \tau) = 0, & 0 < \tau \leq \tilde{T}, \\ (vi) \quad & {}_0^C D_\tau^\alpha \tilde{s}(\tau) = \text{Ste} \left[\tilde{u}_{1y}(\tilde{s}(\tau)^+, \tau) - \frac{k_2}{k_1} \tilde{u}_{2y}(\tilde{s}(\tau)^-, \tau) \right], & 0 < \tau \leq \tilde{T}, \\ (vii) \quad & \tilde{s}(0) = 0. \end{aligned} \quad (78)$$

and

$$\begin{aligned} (i) \quad & \tilde{w}_{2\tau}(y, \tau) = \lambda^2 {}_0^{RL}D_\tau^{1-\alpha} w_{2yy}(y, \tau), & 0 < y < \tilde{r}(\tau), \quad 0 < \tilde{t} < \tilde{T}, \\ (ii) \quad & \tilde{w}_{1\tau}(y, \tau) = {}_0^{RL}D_\tau^{1-\alpha} w_{1yy}(y, \tau), & y > \tilde{r}(\tau), \quad 0 < \tau < \tilde{T}, \\ (iii) \quad & \tilde{w}_1(y, 0) = -1, & 0 \leq y, \\ (iv) \quad & \tilde{w}_2(0, t) = \frac{U_0}{|U_i|}, & 0 < \tau \leq \tilde{T}, \\ (v) \quad & \tilde{w}_1(\tilde{r}(\tau), \tau) = \tilde{w}_2(\tilde{r}(\tau), \tau) = 0, & 0 < \tau \leq \tilde{T}, \\ (vi) \quad & \frac{d}{d\tilde{t}} \tilde{r}(\tau) = \text{Ste} \left[{}_0^{RL}D_\tau^{1-\alpha} w_{1y}(y, \tau) \Big|_{(\tilde{r}(\tau)^+, \tau)} \right. \\ & \quad \left. - \frac{k_2}{k_1} {}_0^{RL}D_\tau^{1-\alpha} \tilde{w}_{2y}(y, \tau) \Big|_{(\tilde{r}(\tau)^-, \tau)} \right], & 0 < \tau \leq \tilde{T}, \\ (vii) \quad & \tilde{r}(0) = 0. \end{aligned} \quad (79)$$

where $\lambda = \frac{\lambda_2}{\lambda_1}$ and $\text{Ste} = \frac{|U_i|c_1}{l}$ is the non-dimensional Stefan number.

In the following table there are different tests, i.e. sets of parameters for λ , $\frac{k_2}{k_1}$, $U = \frac{U_0}{|U_i|}$ and Ste . For each test in Table 1 a correpondig graphic of the comparison between the ξ_α and η_α is given in Figure 3.

	λ	$\frac{k_2}{k_1}$	$U = \frac{U_0}{ U_i }$	Ste
Test 1	0.5	0.5	1.0	0.5
Test 2	2.0	2.0	1.0	0.5
Test 3	0.5	0.5	1.0	1.2
Test 4	2.0	2.0	1.0	1.2

Table 1: Different set of tests

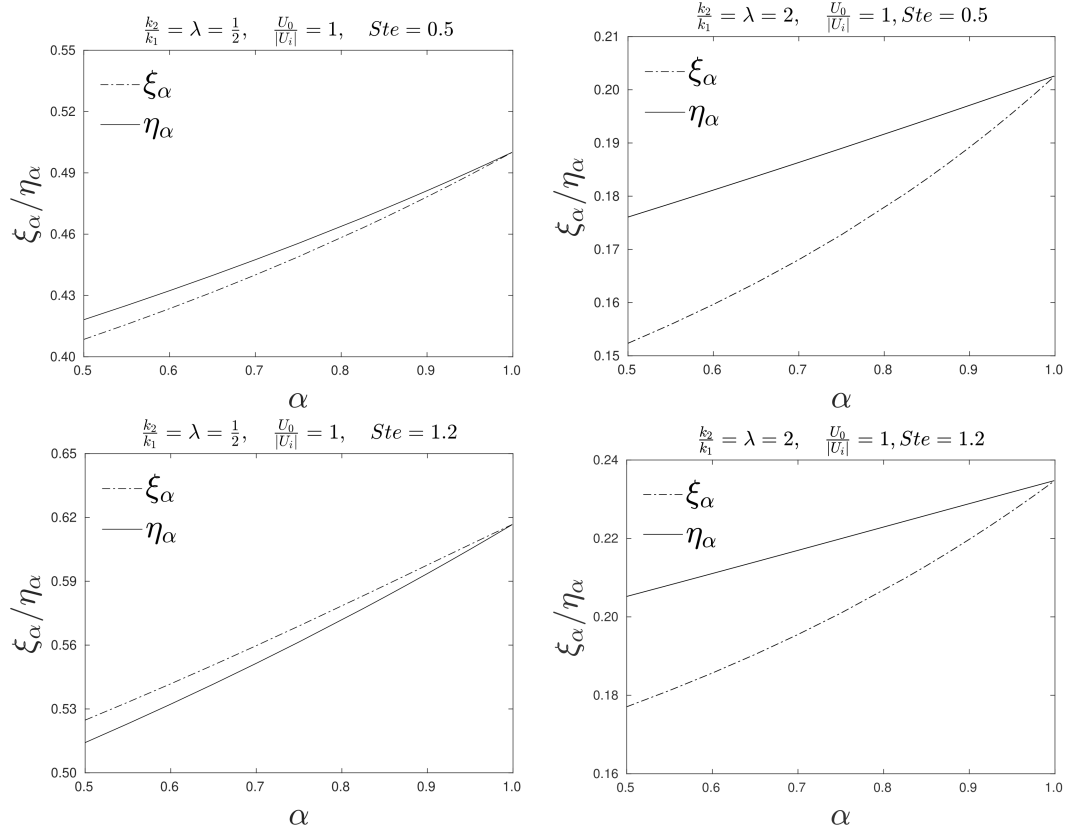


Figure 3: ξ_α vs. η_α for different values of α

At the end, we present in Figures 4 and 5 some color maps of temperature for tests 2 and 3, respectively. Three values of α are considered and as it is expected from Theorem 4, both solutions approach themselves when $\alpha \nearrow 1$.

5 Conclusion

We have presented two different fractional two-phase Stefan-like problems for the Riemann-Liouville and Caputo derivatives of order $\alpha \in (0, 1)$ with the particularity that, if the parameter $\alpha = 1$ is replaced in both problems, we recover the same classical Stefan problem. In both cases, explicit solutions in terms of self-similar variables were given. It was interesting to see that, the role of the different “fractional Stefan conditions” associated to each problem was decisive to conclude that the solutions obtained were different. Also, as it was expected, we have proved that the two different solutions converge to the same triple of limits functions when α tends to 1, where this “limit solution” is the classical solution to the classical Stefan problem mentioned before.

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Figure 4: Caputos's approach Solutions Vs. Riemann-Liouville's approach Solutions for Test 2

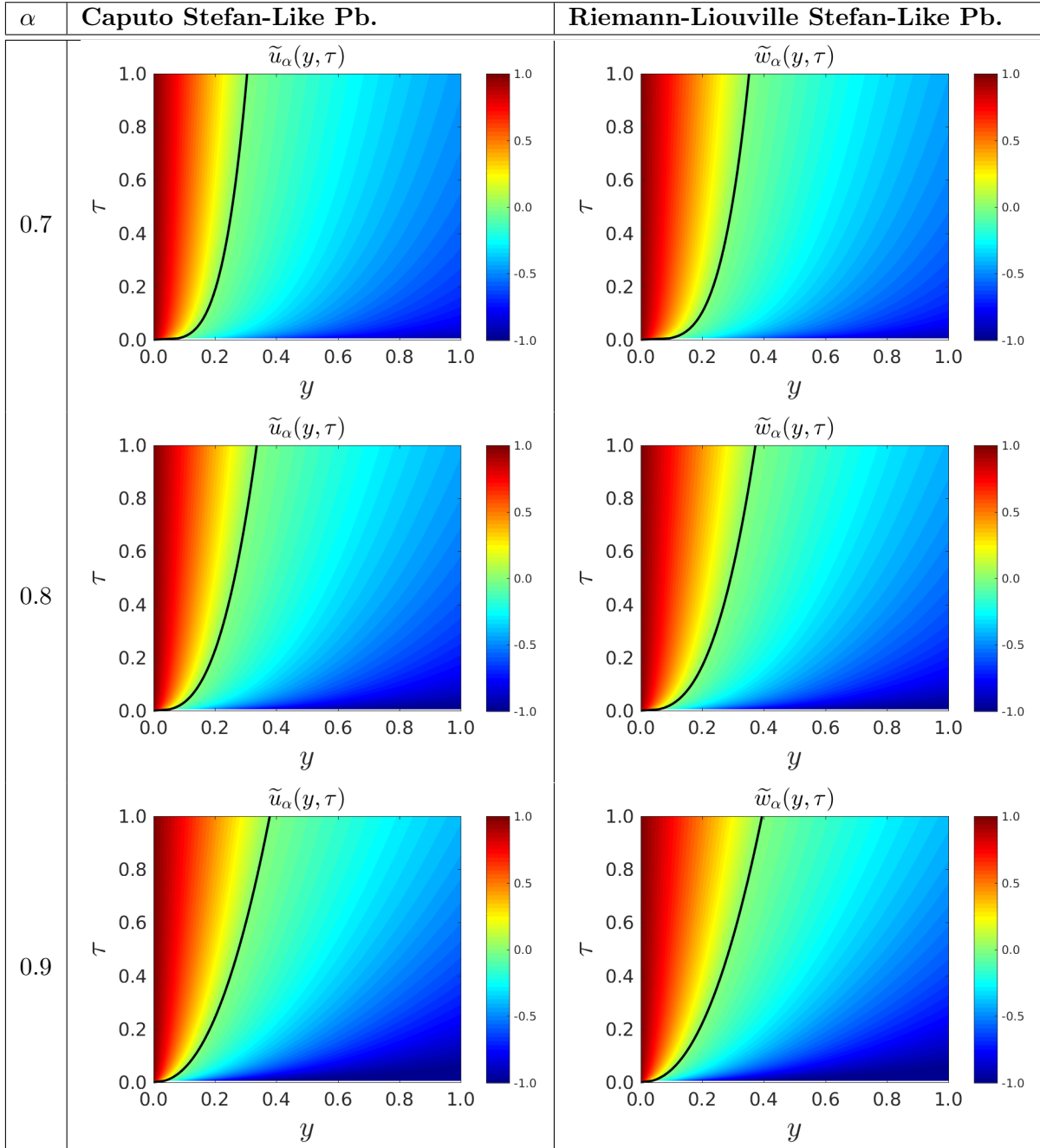


Figure 5: Caputo's approach Solutions Vs. Riemann-Liouville's approach Solutions for Test 3

