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SYMMETRIC STRUCTURE FOR CLOSURE ALGEBRAS

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ABSTRACT. The aim of this paper is to investigate the variety of symmetric closure algebras, that is, closure algebras endowed with a De Morgan operator. Some general properties are derived. Particularly, the lattice of subvarieties of the subvariety of monadic and linear symmetric algebras is described and an equational basis for each subvariety is given.

1. INTRODUCTION

This paper deals with the variety of symmetric closure algebras ($\mathscr{SC}$), that is, closure algebras with a symmetric operator (a De Morgan negation), introduced in [9]. An important characteristic of these algebras is that the set of open elements forms a symmetric Heyting algebra [21] (a Heyting algebra with a de Morgan negation [24]). Section 2 provides all necessary background on closure algebras and symmetric Heyting algebras. We describe the subdirectly irreducible algebras of the variety of symmetric closure algebras in Section 3. The rest of the paper is devoted to the study of the subvariety $\mathscr{SM}$ of monadic symmetric algebras and the variety $\mathscr{SC}_L$ of linear closure algebras. We determine its lattice of subvarieties and we find equational bases for each subvariety of $\mathscr{SC}_L$.

Throughout this paper, $\mathscr{B}$ and $\mathscr{H}$ will denote the equational classes of all Boolean algebras and all Heyting algebras, respectively. If $\mathscr{K}$ is a class of similar algebras we will use the following notation: $\mathbf{Si}(\mathscr{K})$ (resp. $\mathbf{Si_{fin}}(\mathscr{K})$, $\mathbf{Simp_{fin}}(\mathscr{K})$) for the class of subdirectly irreducible (resp. finite subdirectly irreducible, finite simple) algebras in $\mathscr{K}$. $\mathbf{H}(\mathscr{K})$ for the class of algebras that are homomorphic images of algebras in $\mathscr{K}$; and $\mathbf{S}(\mathscr{K})$ for the class of algebras that are subalgebras of algebras in $\mathscr{K}$. The lattice of congruences of an algebra $A \in \mathscr{K}$ is denoted by $Con_{\mathscr{K}}(A)$ or $Con(A)$.

In general, for a variety $\mathscr{K}$ and $A, B \in \mathscr{K}$, $A \lhd_{\mathscr{K}} B$ means that A is a $\mathscr{K}$-subalgebra of B. The subalgebra generated by a part X of $A \in \mathscr{K}$ is denoted by $[X]_{\mathscr{K}}$. If $\mathscr{K}$ is a subclass in a variety $\mathscr{V}$, we will denote $V(\mathscr{K})$ the subvariety generated by $\mathscr{K}$, and $\Lambda(\mathscr{V})$ (or simply Λ) the lattice of subvarieties of $\mathscr{V}$.

2. PRELIMINARIES

In a paper of paramount importance titled "The algebra of topology" J. C. C. McKinsey and A. Tarski [17] started the investigation of a class of algebraic structures which they named *closure algebras*. A closure algebra is an algebra $(A; \vee, \wedge, -, \nabla, 0, 1)$ such that $(A; \vee, \wedge, -, 0, 1)$ is a Boolean algebra and ∇ is a closure operator, that is, ∇ is a unary

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operator on A that satisfies the "Kuratowski axioms", for all $x,\ y \in A$: $\nabla(0) = 0,\ x \leq \nabla(x),\ \nabla(\nabla(x)) = \nabla(x),\ \nabla(x \vee y) = \nabla(x) \vee \nabla(y)$.

The simplest example of closure algebras is the variety $\mathscr{M}$ of monadic Boolean algebras.

Closure algebras have been extensively studied by several authors. Particularly, W. Blok in an exhaustive and very deep work, developed in [6] the general properties of the lattice of subvarieties of the variety of closure algebras.

An important feature in the structure of a closure algebra is the set of open elements. In a continuation of their work on closure algebras, McKinsey and Tarski showed in [18] that the set of open elements of a closure algebra is a Heyting algebra. Conversely, any Heyting algebra can be embedded as the lattice of open elements of a closure algebra.

With the operators ∇ and $-$ we can define a new unary operator Q (interior operator) by means of $Q(x) = -\nabla(-x)$, for all $x \in A$. This operator satisfies the conditions: $Q(1) = 1,\ x \geq Q(x),\ Q(Q(x)) = Q(x),\ Q(x \wedge y) = Q(x) \wedge Q(y)$. Closure algebras can be defined by means of these equations and in that case, by defining $\nabla(x) = -Q(-x)$ we obtain the closure operator.

The equational class of closure algebras will be denoted by $\mathscr{C}$. These algebras were named interior algebras by W. Blok in [6]. If $A \in \mathscr{C}$ then $Q(A)$ is a $(0,1)$-sublattice of A, and it is a Heyting algebra if we define $a \to b = Q(-a \vee b)$, for every $a, b \in A$. If $b \in Q(A)$, b is said to be *open*.

It is known that if $A \in \mathscr{H}$, then the lattice $Con(A)$ of congruences of A is isomorphic to the lattice $\mathbf{F}(A)$ of all filters of A. If $F \in \mathbf{F}(A)$ then the congruence θ associated to F is defined by $(a,b) \in \theta \Leftrightarrow a \wedge u = b \wedge u$ for some $u \in F$.

If $A \in \mathscr{C}$ and F is a filter in A, F is said to be an *open filter* if $Q(x) \in F$ whenever $x \in F$. It is known ([18]) that $Con_{\mathscr{C}}(A)$ is isomorphic to the lattice $\mathbf{F}_Q(A)$ of all open filters of A, and it is not difficult to see that $\mathbf{F}_Q(A)$ and $\mathbf{F}(Q(A))$ are isomorphic. So we have:

Theorem 2.1. ([5]) *Let $A \in \mathscr{C}$. Then $Con_{\mathscr{C}}(A)$ and $Con_{\mathscr{H}}(Q(A))$ are isomorphic.*

Recall that a Heyting algebra A is subdirectly irreducible if and only if $A = A_1 \oplus 1$, with $A_1 \in \mathscr{H}$ and $A_1 \oplus 1$ is the lattice obtained by adjoining a new 1 to A_1.

Thus from Theorem 2.1 and this remark it follows immediately the following corollary:

Corollary 2.2. *Let $A \in \mathscr{C}$. Then A is subdirectly irreducible if and only if $Q(A)$ is subdirectly irreducible as Heyting algebra. Moreover, A is subdirectly irreducible if and only if $Q(A) = A_1 \oplus 1$, for some $A_1 \in \mathscr{H}$.*

A *symmetric Heyting algebra* is a system $(A; \vee, \wedge, \Rightarrow, \sim, 0, 1)$ such that $(A; \vee, \wedge, \Rightarrow, 0, 1)$ is a Heyting algebra and $(A; \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra ([21], [12], [24], [4]). The variety of symmetric Heyting algebras will be denoted by $\mathscr{S}\mathscr{H}$.

Observe that if $A, B \in \mathscr{S}\mathscr{H}$ and $h : A \to B$ is a homomorphism then $h^{-1}(1) = Ker(h)$ is a filter with the property that if $a \Rightarrow b \in Ker(h)$, then $\sim b \Rightarrow \sim a \in Ker(h)$. Any filter F of an algebra $A \in \mathscr{S}\mathscr{H}$ satisfying the condition that if $a \Rightarrow b \in F$ then $\sim b \Rightarrow \sim a \in F$, is called a *kernel* of A. A. Monteiro [21] proved that if, for a given kernel F in a symmetric Heyting algebra A we define $x\theta_F y$ if and only if $x \Rightarrow y \in F$ and $y \Rightarrow x \in F$, then θ_F is a congruence.

Moreover, every congruence on A is determined by a kernel, and the mapping $F \mapsto \theta_F$ is a lattice isomorphism between the lattice of kernels of A and the lattice $Con_{\mathscr{SH}}(A)$.

A slightly more amenable to work with, the following theorem gives an equivalent condition for a filter to be a kernel. As usual, $\neg a = a \Rightarrow 0$.

Theorem 2.3. *([21]) A filter F of a symmetric Heyting algebra A is a kernel if and only if $\neg\sim a \in F$ whenever $a \in F$.*

In the case of a principal filter $F_a = \{x \in A : a \leq x\}$, this condition is equivalent to the condition $a \wedge \sim a = 0$.

Consider the following terms: $(\neg\sim)^0 x = x$ and $(\neg\sim)^{n+1} x = \neg\sim(\neg\sim)^n x$. Observe that $\neg\sim(a \wedge b) = \neg\sim a \wedge \neg\sim b$ and $\neg\sim\neg\sim a \leq a$, for all $a, b \in L \in \mathscr{SH}$. Thus, for $n \in \omega$, $(\neg\sim)^n (a \wedge \neg\sim a) \geq (\neg\sim)^{n+1}(a \wedge \neg\sim a)$.

We say that $L \in \mathscr{SH}$ is *of finite range* if for all $a \in L$ there exists $n \in \omega$ such that $(\neg\sim)^n (a \wedge \neg\sim(a)) = (\neg\sim)^{n+1}(a \wedge \neg\sim a)$.

Let $Cen(A) = \{x \in A : \neg\sim x = \sim\neg x\}$ be the center of A, that is, the sublattice of complemented elements of A.

Theorem 2.4. ([24]) *For $A \in \mathscr{SH}$ of finite range and $|A| \geq 1$, the following are equivalent:*

1. *A is simple.*
2. *A is subdirectly irreducible.*
3. *A is directly indecomposable.*
4. *$Cen(A) = \{0, 1\}$ or $Cen(A) = \{0, a = \sim a, \neg a, 1\}$.*

If A satisfies the Kleene condition ($\sim x \wedge \sim y \leq \sim x \vee \sim y$) then in Theorem 2.4, condition [4] is just $Cen(A) = \{0, 1\}$.

To close this section we recall the following important results.

Theorem 2.5. ([21]) *Any finite symmetric Heyting algebra is a direct product of (finite) simple symmetric Heyting algebras.*

Corollary 2.6. *Every finite subdirectly irreducible symmetric Heyting algebra is simple.*

The variety $\mathscr{SH}$ is generated by its finite members ([21]). Thus

Corollary 2.7. $\mathscr{SH} = V(\mathbf{Si}_{fin}(\mathscr{SH})) = V(\mathbf{Simp}_{fin}(\mathscr{SH}))$.

3. Symmetry on closure algebras

In this section we study the variety of *symmetric closure algebras* introduced in [9]. This variety consists of closure algebras with a De Morgan negation "$\sim$" such that $Q(\sim Q(x)) = \sim Q(x)$, and in particular, the set of open elements form a symmetric Heyting algebra.

First we need some properties of the variety of symmetric Boolean algebras $\mathscr{SB}$ ([4], [21]). A symmetric Boolean algebra is an algebra $(A; \sim)$ such that A is a Boolean algebra and $\sim$ is a De Morgan negation.

Observe that these algebras are symmetric Heyting algebras. In addition, $\sim -x \vee \sim x = \sim(-x \wedge x) = \sim 0 = 1$, and $\sim -x \wedge \sim x = \sim(-x \vee x) = \sim 1 = 0$. So $\sim -x = -\sim x$. Then $\sim$ is a dual isomorphism of Boolean algebras.

A *cyclic Boolean algebra of order two* is an algebra $(A;T)$ such that A is a Boolean algebra and T is a unary operation which is an automorphism such that $T^2 = I_d$ (see [22]). If $(A;\sim) \in \mathscr{SB}$ and we put $T(x) = \sim x \Rightarrow 0 = - \sim x \vee 0 = - \sim x = \sim -x$, then T is an automorphism such that $T^2 = I_d$, that is, $(A;T)$ a cyclic Boolean algebra of order two. Conversely, if $(A;T)$ is a cyclic Boolean algebra or order two and we put $\sim x = -T(x)$, then $(A;\sim)$ is a symmetric Boolean algebra. We say that this two varieties are equivalent in the sense of R. Lewin ([14]).

Congruences in the variety $\mathscr{SB}$ (or in the variety of cyclic Boolean algebras of order two) are given by *symmetric filters*, that is, by those filters F satisfying the condition $T(x) \in F$ (or equivalently $- \sim x \in F$) whenever $x \in F$.

There are two subdirectly irreducible algebras in $\mathscr{SB}$, which in addition, are simple, namely: the algebra $\mathbf{2} = \{0,1\}$ and the algebra $\mathbf{2}\times\mathbf{2}$, where $\sim(0,1) = (0,1)$ and $\sim(1,0) = (1,0)$, (or $T(0,1) = (1,0)$).

Now we are in a position to define symmetric closure algebras.

Definition 3.1. *We say that an algebra* $(A;\vee,\wedge,-,Q,\sim,0,1)$ *is a symmetric closure algebra if the following conditions are satisfied:*
1. $(A;\vee,\wedge,-,Q,0,1) \in \mathscr{C}$.
2. $(A;\vee,\wedge,-,\sim,0,1) \in \mathscr{SB}$.
3. $Q(\sim Q(x)) = \sim Q(x)$.

Let $\mathscr{SC}$ denote the variety of symmetric closure algebras. Observe that if $A \in \mathscr{SC}$, then $Q(A) \in \mathscr{SH}$.

Lemma 3.2. *For* $A, A_1 \in \mathscr{SC}$ *and* $h : A \to A_1$ *an* $\mathscr{SC}$*-homomorphism, it holds that (i)* $h(Q(A)) \subseteq Q(A_1)$*, (ii)* $h_{\upharpoonright Q(A)} : Q(A) \to Q(A_1)$ *is an* $\mathscr{SH}$*-homomorphism, and (iii) if* h *is onto, then* $h_{\upharpoonright Q(A)}$ *is onto.*

Our objective now is to characterize the congruences and the subdirectly irreducible algebras in $\mathscr{SC}$.

A filter F is called an *open kernel* if F is open, that is, $Q(x) \in F$ whenever $x \in F$, and $\sim b \to \sim a \in F$ whenever $a \to b \in F$.

Observe that an open filter F is an open kernel if and only if F is symmetric. Indeed, if F is an open kernel and $x \in F$, then $1 \to x \in F$ and thus $\sim x \to 0 \in F$. So $Q(- \sim x) \in F$ and then $- \sim x \in F$. Consequently F is symmetric. Conversely, let F be an open filter such that if $x \in F$, $- \sim x \in F$, and suppose that $x \to y \in F$. Then $Q(-x \vee y) \in F$, and thus $-x \vee y \in F$. This implies that $\sim -(-x \vee y) = \sim x \vee \sim -y \in F$. Hence $Q(\sim x \vee \sim -y) \in F$, and consequently, $\sim y \to \sim x \in F$. Therefore F is a kernel.

Let $N_{\mathscr{SC}}(A)$ denote the family of all open kernels of A. Then we have the following theorem ([9, 10]).

Theorem 3.3. *For any* $A \in \mathscr{SC}$*,* $Con_{\mathscr{SC}}(A) \cong N_{\mathscr{SC}}$*.*

It is easy to see that if F is an open kernel of an algebra $A \in \mathscr{SC}$, then $Q(F)$ is a kernel of $Q(A)$ considered in $\mathscr{SH}$. Conversely, if F is an kernel of $Q(A)$, then the filter generated in A by F, $[F)_A$, is an open kernel of A and $Q([F)_A) = F$. Then we have the following corollary.

Corollary 3.4. *Let $A \in \mathscr{SC}$. Then $Con_{\mathscr{SC}}(A) \cong N_{\mathscr{SC}}(A) \cong N_{\mathscr{SH}}(Q(A)) \cong Con_{\mathscr{SH}}(Q(A))$.*

Corollary 3.5. *Let $A \in \mathscr{SC}$. Then A is subdirectly irreducible if and only if $Q(A)$ is subdirectly irreducible in $\mathscr{SH}$.*

Then by Theorem 2.4 we have the following theorem.

Theorem 3.6. *Let $|A| \geq 1$, with $A \in \mathscr{SC}$ and $Q(A)$ of finite range. Then the following conditions are equivalent:*

1. *A is simple.*
2. *A is subdirectly irreducible.*
3. *A is directly indecomposable.*
4. *$Cen(Q(A)) = \{0, 1\}$ or $Cen(Q(A)) = \{0, a =\sim a, \neg a, 1\}$.*

Corollary 3.7. *Let $A \in \mathscr{SC}$, A finite. Then A is subdirectly irreducible if and only if A is simple.*

As a consequence of the above results, we have the following.

Theorem 3.8. *Let $A \in \mathscr{SC}$, A finite. Then A is a direct product of finite simple algebras.*

4. SUBVARIETIES AND FINITE GENERATION OF $\mathscr{SC}$

In this section we will prove that the variety $\mathscr{SC}$ is generated by its finite members. We will follow a path similar to that of McKinsey and Tarski in [18] for closure algebras. We will also give some general results about the relationship between the lattice of subvarieties of symmetric closure algebras and the lattice of subvarieties of symmetric Heyting algebras.

Lemma 4.1. *Let $(A; \wedge, \vee, -, Q, \sim, 0, 1)$ be a symmetric closure algebra and let $a_1, \ldots, a_r \in A$. Then there exists $A_1 \subseteq A$ and an interior operator Q_1 on A_1 such that the following conditions hold:*

(i) *$(A_1; \wedge, \vee, -, Q_1, \sim, 0, 1)$ is a symmetric closure algebra.*
(ii) *$a_i \in A_1$ for $i = 1, \ldots, r$.*
(iii) *A_1 contains at most $2^{2^{2r}}$ elements.*
(iv) *If $x \in A_1$ and $Q(x) \in A_1$ then $Q_1(x) = Q(x)$.*

Proof. Let A_1 be the symmetric Boolean subalgebra generated in A by $a_1, \ldots, a_r$. Then $|A_1| \leq 2^{2^{2r}}$, and (*ii*) and (*iii*) hold. Let $K = \{x \in A_1 : Q(x) \in A_1\}$ and $Q(K) = \{Q(x) : x \in K\}$. Observe that if $x \in Q(K)$ then $\sim x \in Q(K)$. Indeed, since A_1 is a symmetric Boolean algebra, $\sim x \in A_1$, whenever $x \in Q(K)$. But A is a symmetric closure algebra, so $Q(x) = \sim x \in Q(K)$. Let L be the De Morgan algebra generated by $Q(K)$ in A_1. Then L defines an interior operator Q_1 on A_1. Let us see that if $x \in A_1$ and $Q(x) \in A_1$ then $Q(x) = Q_1(x)$. First we prove that $L = [Q(K)]_{\boldsymbol{D}_{01}}$. Indeed, if $x \in L$ then there exist subsets $\{H_x\} \subseteq Q(K)$ and $\{S_x\} \subseteq \sim Q(K)$ such that

$$x = \bigwedge_{H_x} \bigvee_{y_i \in H_x} y_i \vee \bigwedge_{S_x} \bigvee_{y_j \in S_x} y_j.$$

But we know that $\sim (Q(K)) \subseteq Q(K)$. So we can state that there exist $\{R_x\} \subseteq Q(K)$ such that

$$x = \bigwedge_{R_x} \bigvee_{y_i \in R_x} y_i$$

and then $L = [Q(K)]_{\mathbf{D}_{01}}$. Observe that if $x \in L$, $Q_1(x) = x = Q(x)$. If $x \in A_1$

$$Q_1(x) = \bigvee \{a \in L : \ a \leq x\}.$$

Since $Q(a) = a \leq Q(x)$, it follows that $Q_1(x) \leq Q(x)$. In addition, $x, Q(x) \in A_1$, and thus $Q(x) \in Q(K)$. So $Q(x) \in \{a \in L : a \leq x\}$, which implies that $Q(x) \leq Q_1(x)$. □

Definition 4.2. *A finite sequence of terms in the language of $\mathscr{SC}$, $f_1, \ldots, f_r$, (all with the same number of variables) is called a chain for the term f if:*
(i) $f_r = f$
(ii) $f_h(x_1, \ldots, x_n) = x_i$, or $f_h = f_i \wedge f_j$, $(h > i, j)$, or $f_h = f_i \vee f_j$, or $f_h = \sim f_i$, or $f_h = -f_i$, or $f_h = Q(f_i)$, or $f_h = 1$.

The number r is the length of the chain. The smallest length of the chain is called the order of f. In the next theorem we use the fact that any equation in $\mathscr{SC}$, $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$, can be written in the form $t(x_1, \ldots, x_n) = 1$.

Theorem 4.3. *If $f(x_1, \ldots, x_n) = 1$ is an identity for every finite algebra in $\mathscr{SC}$, then it is an identity for every algebra in $\mathscr{SC}$.*

Proof. Let $A \in \mathscr{SC}$ infinite such that $f(x_1, \ldots, x_n) = 1$ is not an identity for A, that is, there exist $a_1, a_2, \ldots, a_n \in A$ such that $f(a_1, \ldots, a_n) \neq 1$. Let $f_1, \ldots, f_r$ be a chain for f, of length r, where r is the order of f. Let us put $f_1(a_1, \ldots, a_n) = b_1$, $f_2(a_1, \ldots, a_n) = b_2$, $\ldots$, $f_r(a_1, \ldots, a_n) = f(a_1, \ldots, a_n) = b_r \neq 1$. By the previous lemma, there exists a finite closure algebra $A_1 \subseteq A$ such that $a_1, \ldots, a_n, b_1, \ldots, b_r \in A_1$. Then $f_1(a_1, \ldots, a_n) = b_1 \in A_1$, $f_2(a_1, \ldots, a_n) = b_2 \in A_1$, $\ldots$, $f_r(a_1, \ldots, a_n) = f(a_1, \ldots, a_n) = b_r \in A_1$ and $b_r \neq 1$. So $f(x_1, \ldots, x_n) = 1$ is not an identity for the finite algebra A_1, a contradiction. □

Corollary 4.4. *The variety $\mathscr{SC}$ is generated by its finite members.*

Corollary 4.5.

$$\mathscr{SC} = V\left(\mathbf{Si_{fin}}(\mathscr{SC})\right) = V\left(\mathbf{Simp_{fin}}(\mathscr{SC})\right).$$

Next, as in the case of closure algebras and Heyting algebras (see Blok [6]), we will find a strong relationship between the lattice of subvarieties of $\mathscr{SH}$ and the lattice of $\mathscr{SC}$. If $\mathscr{K} \in \Lambda(\mathscr{SC})$ then we denote

$$Q(\mathscr{K}) = \{Q(A) : A \in \mathscr{K}\}$$

and for $\mathscr{K} \in \Lambda(\mathscr{SH})$, we put

$$\mathscr{K}_{\mathscr{SC}} = \{A \in \mathscr{SC} : Q(A) \in \mathscr{K}\}.$$

It is long but computational to check that

Theorem 4.6.

- **(i)** *If $\mathscr{K}$ is an equational class of symmetric closure algebras, then $Q(\mathscr{K})$ is an equational class of symmetric Heyting algebras.*
- **(ii)** *If $\mathscr{K}$ is an equational class of symmetric Heyting algebras, then $\mathscr{K}_{\mathscr{SC}}$ is an equational class of symmetric closure algebras.*

Theorem 4.7.

- **(i)** *If $\mathscr{K} \subseteq \mathscr{SC}$, then $Q(V(\mathscr{K})) = V(Q(\mathscr{K}))$.*
- **(ii)** *if $\mathscr{K} \subseteq \mathscr{SH}$ is such that $\mathbf{Si}(V(\mathscr{K})) \subseteq \mathscr{K}$, then $V(\mathscr{K})_{\mathscr{SC}} = V(\mathscr{K}_{\mathscr{SC}})$.*

(In this case, $\mathscr{K}$ is not necessarily a variety).

Then we define the functions

$$\mathscr{O}_1 : \Lambda(\mathscr{S}\mathscr{C}) \longrightarrow \Lambda(\mathscr{S}\mathscr{H}), \quad \mathscr{O}_1(\mathscr{K}) = Q(\mathscr{K}).$$

and

$$\mathscr{O}_2 : \Lambda(\mathscr{S}\mathscr{H}) \longrightarrow \Lambda(\mathscr{S}\mathscr{C}), \quad \mathscr{O}_2(\mathscr{K}) = \mathscr{K}_{\mathscr{S}\mathscr{C}}.$$

The functions $\mathscr{O}_1$ and $\mathscr{O}_2$ are well defined and they commute with the operators **P**, **H** and **S**. By Jónsson's Theorems [13] we have that the distributive lattices $\Lambda(\mathscr{S}\mathscr{C})$ and $\Lambda(\mathscr{S}\mathscr{H})$ are complete. Now we will see some important properties of these functions.

Theorem 4.8. *$\mathscr{O}_1$ is a complete surjective lattice homomorphism. $\mathscr{O}_2$ is a lattice embedding. Moreover, $\mathscr{O}_1 \circ \mathscr{O}_2 = Id$ and consequently $\Lambda(\mathscr{S}\mathscr{H})$ is a retract of $\Lambda(\mathscr{S}\mathscr{C})$.*

Proof. We prove the theorem for $\mathscr{O}_2$, the other case being similar. If $0_{\mathscr{S}\mathscr{C}}$ is the trivial variety of symmetric closure algebras, then $\mathscr{O}_1(0_{\mathscr{S}\mathscr{C}}) = 0_{\mathscr{S}\mathscr{H}}$, with $0_{\mathscr{S}\mathscr{H}}$ the trivial variety of symmetric Heyting algebras. In addition, it is clear that $\mathscr{O}_1(1_{\mathscr{S}\mathscr{C}}) = \mathscr{O}_1(\mathscr{S}\mathscr{C}) = \mathscr{S}\mathscr{H} = 1_{\mathscr{S}\mathscr{H}}$. If $\{\mathscr{K}_i : i \in I\} \subseteq \Lambda(\mathscr{S}\mathscr{C})$, then

$$\mathscr{O}_1\left(\bigvee_{i\in I}\mathscr{K}_i\right) = \mathscr{O}_1\left(V\left(\bigcup_{i\in I}\mathscr{K}_i\right)\right) = Q\left(V\left(\bigcup_{i\in I}\mathscr{K}_i\right)\right) = V\left(\bigcup_{i\in I}Q(\mathscr{K}_i)\right) = \bigvee_{i\in I}\mathscr{O}_1(\mathscr{K}_i).$$

On the other hand, $\mathscr{O}_1$ preserves arbitrary infima:

$$\mathscr{O}_1\left(\bigwedge_{i\in I}\mathscr{K}_i\right) = \mathscr{O}_1\left(\bigcap_{i\in I}\mathscr{K}_i\right) = Q\left(\bigcap_{i\in I}\mathscr{K}_i\right) = \bigcap_{i\in I}Q(\mathscr{K}_i) = \bigcap_{i\in I}\mathscr{O}_1(\mathscr{K}_i) = \bigwedge_{i\in I}\mathscr{O}_1(\mathscr{K}_i)$$

Finally, $\mathscr{O}_1$ is onto, as if $\mathscr{K} \in \Lambda(\mathscr{S}\mathscr{H})$ then $\mathscr{O}_1(\mathscr{K}_{\mathscr{C}}) = Q(\mathscr{K}_{\mathscr{C}}) = \mathscr{K}$. □

Observe that an identity characterizing a subvariety of symmetric Heyting algebras can be translated into an identity characterizing a subvariety of symmetric closure algebras. Indeed, if

$$p_V(a_1,\ldots,a_n) = q_V(a_1,\ldots,a_n)$$

is the characteristic identity for a subvariety $V \subseteq \mathscr{S}\mathscr{H}$ then the subvariety $V_{\mathscr{S}\mathscr{C}}$ of $\mathscr{S}\mathscr{C}$ is determined by the equation

$$p_V(Q(a_1),\ldots,Q(a_n)) = q_V(Q(a_1),\ldots,Q(a_n)).$$

In this way we can study many subvarieties of symmetric closure algebras obtained from subvarieties of symmetric Heyting algebras.

5. Symmetric monadic algebras

In this section we study the variety $\mathscr{S}\mathscr{M}$ of symmetric monadic algebras. This variety consists of those symmetric closure algebras in which Q is a quantifier, that is, $Q(x \vee Q(y)) = Q(x) \vee Q(y)$, and it will play an important role for the study of the linear case. We reproduce here some known results, the proof of which can be found in [9].

In [4], M. Abad and L. Monteiro introduced the variety of cyclic monadic algebras. A *cyclic monadic algebra* is a Boolean algebra endowed with a unary operation which is a (monadic) automorphism of period two. We prove in this section that the variety of symmetric monadic algebras and the variety of cyclic monadic algebras are equivalent in the sense of [14]. We describe the lattice of subvarieties of $\mathscr{SM}$ and we determine an equational basis for each subvariety.

As the previous section, $A \in \mathscr{SM}$ is a simple (subdirectly irreducible) algebra if and only if $Q(A)$ is a simple symmetric Boolean algebra, that is, if and only if either $Q(A) = \mathbf{2}$ or $Q(A) = \mathbf{2} \times \mathbf{2}$, where $\sim (0,1) = (0,1)$ and $\sim (1,0) = (1,0)$.

Let $A \in \mathscr{SM}$ and let $T^* = - \sim$. We know that T^* is a Boolean automorphism. Let us prove that $T^*(Q(x)) = Q(T^*(x))$. It is easy to see that $\mathscr{SM}$ is locally finite, and consequently, $\mathscr{SM}$ is generated by its finite members. Thus it is enought to prove that the equation $T^*(Q(x)) = Q(T^*(x))$ holds in all finite subdirectly irreducible (simple) algebras of $\mathscr{SM}$.

Let B be a simple algebra in $\mathscr{SM}$ such that $Q(B) = \{0,1\}$. Let $x \in B$. If $x = 1$, then $Q(x) = 1 = x$. Thus $- \sim Q(x) = - \sim x = - \sim 1 = -0 = 1 = Q(- \sim 1) = Q(- \sim x)$. If $x \neq 1$, then $Q(x) = 0$ and $- \sim x \neq 1$. Then $- \sim Q(x) = - \sim 0 = 0 = Q(- \sim x)$. The case in which $Q(B) = \{0, 1, a = \sim a, -a\}$ is similar. So we have the following theorem.

Theorem 5.1. *Let $A \in \mathscr{SM}$. Then $(A;T^*)$ is a cyclic monadic algebra of order two.*

Now, let $(A;T)$ be a cyclic monadic algebra of order two. Let $\sim^* = -T$. We know that $\sim^*$ is a De Morgan negation. In addition, if $x \in A$, as $A \in \mathscr{M}$, $Q(\sim^* Q(x)) = Q(-T(Q(x))) = Q(-Q(T(x))) = -Q(T(x)) = -T(Q(x)) = \sim^* Q(x)$. So $(A, \sim^*)$ is a symmetric monadic algebra.

Observe that $- \sim^* = - - T = T$. We have proved the following result.

Theorem 5.2. *Let $(A;T)$ a cyclic monadic algebra of order two, and let $\sim^* = -T$. Then $(A;\sim^*) \in \mathscr{SM}$.*

If $\mathscr{M}_2$ is the variety of cyclic monadic Boolean algebra of order two, then the varieties $\mathscr{SM}$ and $\mathscr{M}_2$ are equivalent in the sense of [14]. This equivalence will allow us to use both operations $\sim$ and T, related by $T = - \sim$.

We say that a simple algebra A is *of type I* if $Q(A) = \{0,1\}$, whereas A is said to be *of type II* when $Q(A) = \{0, a = \sim a, -a, 1\}$.

Let $At(A)$ denote the set of atoms of a finite algebra A, let $\mathfrak{F}(At(A)) = \{a \in At(A) : T(a) = a\}$ denote the set of atoms fixed by the action of T, and let $\mathfrak{F}'(At(A)) = \{a \in At(A) : T(a) \neq a\}$ denote the set of atoms non-fixed by T. Observe that if $a \in \mathfrak{F}'(At(A))$ and $T(a) = b$, $(a \neq b)$, then $T(b) = T(T(a)) = a$. So $T(a) \in \mathfrak{F}'(At(A))$. Thus $|\mathfrak{F}'(At(A))|$ is an even number. It is clear that $At(A) = \mathfrak{F}(At(A)) \dot{\cup} \mathfrak{F}'(At(A))$.

Let $\mathbf{C}_{2\times s,t}$ denote the finite simple algebras of type I with $2s$ non-fixed atoms and t fixed atoms. A simple finite algebra of type II has an even number of atoms as the number of atoms preceding a equals the number of atoms preceding $-a$. We denote $\mathbf{D}_k$ a finite simple algebra of type II with $2k$ atoms.

Since $\mathscr{SM}$ is a congruence distributive locally finite variety, we can apply the well-known results of Davey [8] for the lattice of subvarieties $\Lambda(\mathscr{SM})$. In this section we characterize the poset $J_{fin}(\Lambda(\mathscr{SM}))$ of finite join-irreducible elements of $\Lambda(\mathscr{SM})$. Recall that a variety is said to be finitely generated if it is generated by finitely many subdirectly irreducible finite algebras. We consider separately the finitely generated and non finitely generated join-irreducible varieties.

Recall that the ordering in $J_{fin}(\Lambda(\mathscr{SM}))$ is given by $V(A) \leq_\Lambda V(B) \iff A \in \mathbf{HS}(B)$, where A is a subdirectly irreducible algebra (see Jónsson [13]). Since every subdirectly irreducible algebra in $\mathscr{SM}$ is simple, then $V(A) \leq_\Lambda V(B) \iff A \in \mathbf{IS}(B)$. Thus we only have to characterize the subalgebras of a simple algebra.

Let $V_{2\times s,t} = V\left(\mathbf{C}_{2\times s,t}\right)$. The above properties give us a description of the ordering of the join-irreducible varieties generated by a simple algebra of type I.

Theorem 5.3. *(*[9, 10]*) $V_{2\times k,l} \leq_\Lambda V_{2\times s,t}$ if and only if $k \leq s$ and $k+l \leq s+t$.*

Let $\mathscr{S}$ be the variety generated by the algebras of type I. Then we can describe the poset $J_{fin}(\Lambda(\mathscr{S}))$ of finite join-irreducible elements of $\Lambda(\mathscr{S})$: Let C be a chain of type ω and $C^{[\mathbf{2}]}$ the set of increasing functions from $\mathbf{2}$ into C and consider the set $C^{[\mathbf{2}]} \setminus \{(0,0)\}$. Let $\varphi : J_{fin}(\Lambda(\mathscr{S})) \longrightarrow C^{[\mathbf{2}]} \setminus \{(0,0)\}$ be defined by $\varphi(V_{2\times s,t}) = (s, s+t)$.

Theorem 5.4. *The mapping φ is an order isomorphism.*

Observe that the subvariety $\mathscr{S}$ is determined by the equation $T(Q(x)) = Q(x)$. Indeed, this equation holds in any subdirectly algebra in $\mathscr{S}$ and does not hold in the simple algebras of type II, as $T(Q(a)) = T(a) = -a \neq a = Q(a)$ for an open element $a \notin \{0,1\}$.

Now we investigate the varieties of type II. Observe that a variety contained in $\mathscr{S}$ cannot be greater that (in $\Lambda(\mathscr{SM})$) a variety of type II. The reason is that a simple algebras of type *I* contains no subalgebras of type II.

Let us look at the ordering for join-irreducible varieties generated by algebras of type II. We denote $U_k = V(\mathbf{D}_k)$.

The following theorem completes the description of the ordering in the set of finite join-irreducibles of $\Lambda(\mathscr{SM})$.

Theorem 5.5. *(*[9, 10]*) $U_k \leq U_l$ if and only if $k \leq l$ and $V_{2\times s,t} \leq_\Lambda U_k$ if and only if $2s+t \leq k$.*

Now we will determine the infinite join-irreducible subvarieties of $\mathscr{SM}$, and we will prove that any variety is a finite join of join-irreducible varieties in $\Lambda(\mathscr{SM})$. This result will play an important role in determining equational bases for subvarieties in $\mathscr{SM}$.

Consider the following subvarieties: for $h \geq 0$,

$$S_h = V(\mathbf{C}_{2\times j,k} : k \geq 1, j \leq h) = \bigvee_{k\geq 1, j\leq h} V_{2\times j,k} = V\left(\bigcup_{k\geq 1} \mathbf{C}_{2\times h,k}\right).$$

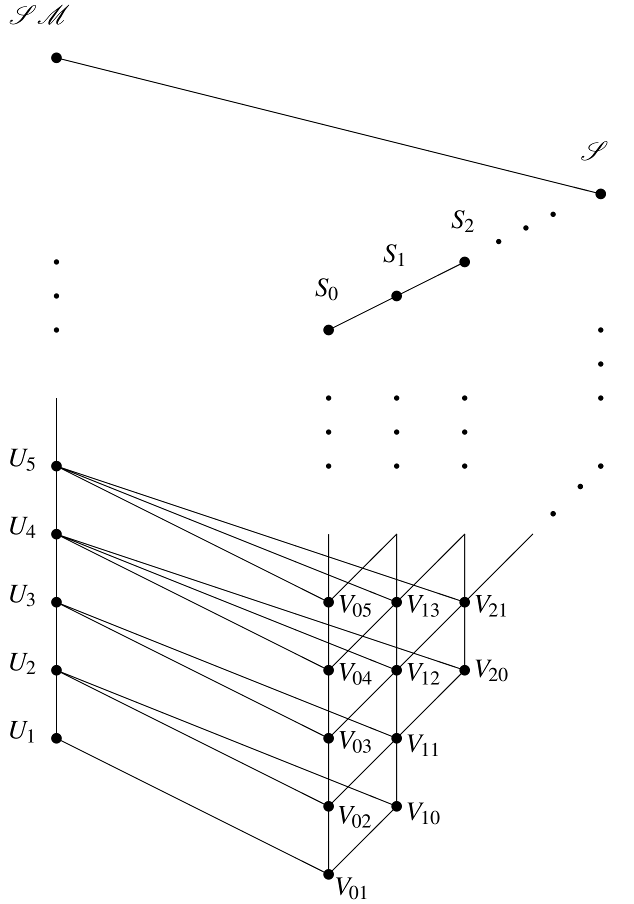
From the definition of $\mathscr{S}$, it follows that $\mathscr{S} = \bigvee_{k\geq 1, j\geq 0} V_{2\times j,k} = \bigvee_{k\geq 1} S_k$. Hence

$$S_0 \subset S_1 \subset S_2 \subset \ldots \subset S_n \subset \ldots \subset \mathscr{S}.$$

We have that $\mathscr{S}$, $\{S_i\}_{i\geq 0}$ and $\mathscr{SM}$ are the unique infinite join-irreducible varieties .

Theorem 5.6. *(*[9, 10]*) The unique infinite join-irreducible varieties in $\Lambda(\mathscr{SM})$ are $\mathscr{SM}$, $\mathscr{S}$ and $\{S_i\}_{i\geq 0}$.*

The poset $J(\Lambda(\mathscr{SM}))$ of join-irreducible elements of $\Lambda(\mathscr{SM})$ looks like the diagram in the following figure, in which we simply write V_{st} instead of $V_{2\times s,t}$.



Observe that U_t is covered only by U_{t+1}. Also, $s+t \neq 0$. So, in $C^{[\mathbf{2}]} \setminus \{(0,0)\}$, $(s,s+t+1)$ covers $(s,s+t)$. For $t \geq 1$, $(s+1,s+t)$ also covers $(s,s+t)$, and these are the only pairs that cover $(s,s+t)$. Recall that $\varphi(V_{2\times s,t}) = (s,s+t)$. Hence $V_{2\times s,t+1}$, $V_{2\times s+1,t-1}$ and U_{2s+t} are the unique elements in $J(\Lambda(\mathscr{S}\mathscr{M}))$ which cover $V_{2\times s,t}$.

The following theorem will be used in the next section.

Theorem 5.7. *(*[9, 10]*) Any variety in $\Lambda(\mathscr{S}\mathscr{M})$ is a finite join of finitely many join-irreducible varieties in $\Lambda(\mathscr{S}\mathscr{M})$.*

6. Equational Bases for $\mathscr{S}\mathscr{M}$

In this section an equation that characterizes each subvariety in $\mathscr{S}\mathscr{M}$ is given. First we characterize equationally the join-irreducible varieties, and then we determine an equation for the join of finitely many join-irreducible varieties.

We have already shown that the variety $\mathscr{S}$ is characterized, relative to $\mathscr{S}\mathscr{M}$, by the equation $T(Q(x)) = Q(x)$.

Observe that $s(x,y,1,0) = x \leftrightarrow y = Q(x \to y) \wedge Q(y \to x)$ is a switching term for the variety $\mathscr{S}$ (see [24]). Then $x \leftrightarrow y \in \{0,1\}$ for any x, y in a subdirectly irreducible algebra $A \in \mathscr{S}$. This remark will be strongly used in what follows.

Let $\Gamma_{S_n}(x_1,\ldots,x_{2n})$ denote the term

$$\left[(\bigwedge_{i=1}^{n}(T(x_{2i-1}) \leftrightarrow x_{2i})) \wedge (\bigwedge_{i\neq j,\ j,i=1}^{2n} ((x_i \wedge x_j) \leftrightarrow 0)) \wedge (\bigwedge_{i=1}^{2n+1} -(x_i \leftrightarrow 0)) \right] \rightarrow \left[T(\bigvee_{i=1}^{2n+1} x_i) \leftrightarrow (\bigvee_{i=1}^{2n+1} x_i) \right]$$

The next theorem shows a set of equations that characterize the varieties S_n within $\mathscr{SM}$.

Theorem 6.1. *(see* [9, 10]*) The variety S_n is characterized by the following equations: If $n=0$, $T(x)=x$, and if $n>0$, $T(Q(x))=Q(x)$ and $\Gamma_{S_n}(x_1,\ldots,x_{2n})=1$*

Consider now the following terms:

$$T_1^1(x_1) = x_1,$$

$$T_1^p(\vec{x}) = \left[\bigvee_{i,j=1,\ i\neq j}^{p} \nabla(x_i \wedge x_j) \wedge (\bigwedge_{k\neq i,j,\ k=1}^{p} \nabla(x_k)) \right] \vee \left[\bigvee_{i=1}^{p}(x_i \wedge (\bigwedge_{k\neq i,\ k=1}^{p} \nabla(x_k))) \right],$$

$$T_2^1(x_1) = \nabla x_1 \quad \textit{and} \quad T_2^p(\vec{x}) = \bigwedge_{i=1}^{p} \nabla(x_i),$$

where $T_i^p(\vec{x})$ stands for $T_i^p(x_1,x_2,\ldots,x_p)$.

In [23], [1] and [9] it is shown that the equation $T_1^p(\vec{x}) = T_2^p(\vec{x})$ is an equational basis for the variety generated by the simple algebra $\mathbf{B}_p \in \mathscr{M}$, for a fixed p.

Let $\Gamma_{2n,t}(x_1,\ldots,x_{2n})$ denote de term

$$\left[(\bigwedge_{i=1}^{n+t} -(x_i \leftrightarrow 0)) \wedge (\bigwedge_{i\neq j,\ j,i=1}^{n+t} ((x_i \wedge x_j) \leftrightarrow 0)) \wedge (\bigwedge_{i=1}^{n+t}(T(x_i) \leftrightarrow x_i)) \right] \rightarrow \left[(\bigvee_{i=1}^{n+t} x_i) \leftrightarrow 1 \right].$$

The following theorem gives an equational basis for the join-irreducible subvarieties of type I, $V_{2\times s,t}$.

Theorem 6.2. *Let $n>0$. The equations $T_1^{2n+t}(\vec{x}) = T_2^{2n+t}(\vec{x})$ and $\Gamma_{2n,t}(x_1,\ldots,x_{2n}) = 1$ characterize the subvariety $V_{2\times n,t}$ within S_n.*

Now we give an equational basis for the subvarieties $U_k = V(\mathbf{D}_k)$.

Lemma 6.3. *(*[9, 10]*) The subvariety $U_k = V(\mathbf{D}_k)$ is characterized by the equation $T_1^k(\vec{x}) = T_2^k(\vec{x})$.*

We have determined an equational basis for each join-irreducible subvariety. Now we are going to give an equational basis for every subvariety. From Theorem 5.7, we only have to find an equation for every finite join of join-irreducible subvarieties.

Observe that any variety $V \in J(\Lambda(\mathscr{SM}))$ is determined by a single equation, that is, there exists an equation of the form $\gamma_V(x_1,\ldots,x_r) = 1$, with r as needed, that determines V within $\mathscr{SM}$.

Let $\mathbf{V} = \bigvee_{i=1}^{n} V_i$ with $V_i \in J(\Lambda(\mathscr{S}\mathscr{M}))$, for all $i = 1, \ldots, n$. Let

$$\gamma_{\mathbf{V}}(x_1, \ldots, x_{r_{\mathbf{V}}}) = \bigvee_{i=1}^{n} \left(\gamma_{V_i}(x_1^i, \ldots, x_{r_i}^i) \leftrightarrow 1\right) \wedge \left(\sim \gamma_{V_i}(x_1^i, \ldots, x_{r_i}^i) \leftrightarrow 0\right).$$

In [9, 10] it is shown that $\gamma_{\mathbf{V}}(x_1, \ldots, x_{r_{\mathbf{V}}}) = 1$ is an equational basis for $\mathbf{V}$.

7. Linear Symmetric Closure Algebras

This part of the work is devoted to an exhaustive investigation of the variety of those closure algebras whose open elements form a linear symmetric Heyting algebra. We first present some general results obtained in [10] and then we characterize the subvarieties of linear closure algebras. In 7.2 we will carry out a deep study of finitely generated subdirectly irreducible algebras and of the ordering between the varieties generated by them. This will allow us to give in 7.3 a precise description of finitely generated subvarieties. In 7.4 we will describe the infinitely generated subvarieties of the locally finite subvarieties. Finally in 7.5 we will give an equational bases for each subvariety of locally finite subvarieties.

7.1. **Linear Symmetric Heyting Algebras.** A. Monteiro comprehensively investigated the variety of symmetric Heyting algebras and several of its subvarieties in his very important work “Sur les algèbres de Heyting symétriques” [21]. Particularly, he studied the subvariety of linear symmetric Heyting algebras, that is, symmetric Heyting algebras satisfying the identity

$$(x \Rightarrow y) \vee (y \Rightarrow x) = 1.$$

Linear symmetric Heyting algebras form an equational class $\mathscr{S}\mathscr{H}_L$. In this section we consider the lattice of subvarieties of this variety. We describe the structure of the poset of its join-irreducible elements and we find equational bases for each subvariety of $\mathscr{S}\mathscr{H}_L$. This description is based on that given in [2] and consequently some of the proofs will be ommited.

Linear symmetric Heyting algebras can be characterized by the condition that the poset of filters containing a prime filter is a chain [21].

The importance of the following examples of linear symmetric Heyting algebras will be clear later.

Let C_n, $n \geq 2$, be the Heyting algebra of all fractions $\dfrac{i}{n-1}$, $i = 0, 1, \ldots, n-1$ ([21], p. 136), with $\sim x = 1 - x$, and let D_n be the Heyting algebra $C_n \times C_n$, with $\sim (x, y) = (1 - y, 1 - x)$. C_n and D_n are linear symmetric Heyting algebras.

An I_n-algebra is a symmetric Heyting algebra satisfying the Ivo Thomas identity:

$$\gamma_n(x_0, x_1, \ldots, x_{n-1}) = \beta_{n-2} \Rightarrow (\beta_{n-3} \Rightarrow (\ \ldots \Rightarrow (\beta_0 \Rightarrow x_0)\ \ldots\ )) = 1,$$

where $\beta_i = (x_i \Rightarrow x_{i+1}) \Rightarrow x_0$ for $i = 0, 1, \ldots, n-2$ (see [21], p. 136).

The algebras C_n and D_n are examples of symmetric Heyting algebras that satisfy the identity $\gamma_n = 1$.

The following result is a characterization of the I_n-algebras.

Lemma 7.1. [21] *For a linear Heyting algebra A, the following are equivalent:*

(1) *The identity* $\gamma_n = 1$ *holds.*

(2) *The poset of proper prime filters containing a prime filter P is a chain of length at most $n-1$.*

A. Monteiro proved ([21], p. 138, Th.1.6) that the variety $\underset{\sim}{I_n}$ of I_n-algebras is generated in $\mathscr{SH}_L$ by $D_n \times D_{n-1}$.

It is clear that a finite linear symmetric algebra is an I_n-algebra for some n. Then we have the following theorem:

Theorem 7.2. *If A is a finite algebra in $\mathscr{SH}_L$, then A is subdirectly irreducible if and only if there exists n such that either A is isomorphic to D_n or A is isomorphic to C_n.*

Since the variety of linear Heyting algebras is locally finite, then it is easy to see that $\mathscr{SH}_L$ is locally finite. In addition, $\mathscr{SH}_L$ has the congruence-distributive property, being that the lattice of congruences in an algebra A is a sublattice of the lattice of congruences of the Heyting algebra A, and the latter is congruence-distributive.

We conclude this section by recalling the characterization of subalgebras of the algebras C_n and D_n.

Let $n \geq 2$. If n is even, then the subalgebras of C_n are the algebras C_{2k}, $k \leq n/2$. If n is odd, then C_k is a subalgebra of C_n for every $k \leq n$.

Let $S_Y = C_n - Y$, where $Y \subseteq C_n - \{0,1\}$. Let $S_{\mathscr{H}}$ be the set of Heyting subalgebras of C_n. Then $S_{\mathscr{H}} = \{S_Y : Y \subseteq C_n - \{0,1\}\}$. For every j, $2 \leq j \leq n$, let $Y \in S_{\mathscr{H}}$ be such that $|Y| = n-j$. Then $A = S_Y \times S_{\sim Y}$ is a subalgebra of D_n isomorphic to D_j. In addition, $D_i \subseteq D_j$ if and only if $i \leq j$. If A is a subalgebra of D_n and A is not isomorphic to D_k, for any k, then $A \simeq C_t$, for $t \leq n$. We have that $A = \{(x, \alpha(x)), x \in p_1(A)\}$, where α is an isomorphism from $p_1(A)$ onto $p_2(A)$, p_1, p_2 the projections in $D_n = C_n \times C_n$.

The order in $\mathbf{Si}(\mathscr{L})$ (and in $\mathbf{Si}_{\text{fin}}(\mathscr{L})$) is the following:

$$A \leq B \text{ if and only if } A \in \mathbf{IS}(B)$$

being that if $A \in \mathbf{Si}(\mathscr{L})$ then A is simple, that is, the unique homomorphic images are the trivial ones.

Let $\underset{\sim}{D_n}$ and $\underset{\sim}{C_n}$ denote the varieties generated by D_n and C_n, respectively, that is, $\underset{\sim}{D_n} = V(D_n)$ and $\underset{\sim}{C_n} = V(C_n)$, and for a distributive lattice R, $\mathscr{J}(R)$ denotes the ordered set of all join-irreducible elements of the distributive lattice R.

Let $\underset{\sim}{K} = V(\bigcup_{n \geq 2} C_n)$. This is the variety called by A. Monteiro the variety of totally linear symmetric Heyting algebras.

Let $\underset{\sim}{P} = V(\bigcup_{n \geq 1} C_{2n})$.

It is clear that $\underset{\sim}{P} \subseteq \underset{\sim}{K}$. Furthermore, $\underset{\sim}{P} \neq \underset{\sim}{K}$. Indeed, for $A \in \mathbf{Si}_{\text{fin}}(\underset{\sim}{P})$, A is isomorphic to C_{2n}, and then, for odd t, there isn't $A \in \mathbf{Si}_{\text{fin}}(\underset{\sim}{P})$, such that $C_t \in \mathbf{S}(A)$. Thus, $\underset{\sim}{P} \subset \underset{\sim}{K}$.

Therefore, $\mathscr{J}(\Lambda(\mathscr{SH}_L))$ is the poset indicated in fig. 1.

Theorem 7.3. *$\underset{\sim}{K}$, $\underset{\sim}{P}$ and $\mathscr{SH}_L$ are join-irreducible in $\Lambda(\mathscr{L})$.*

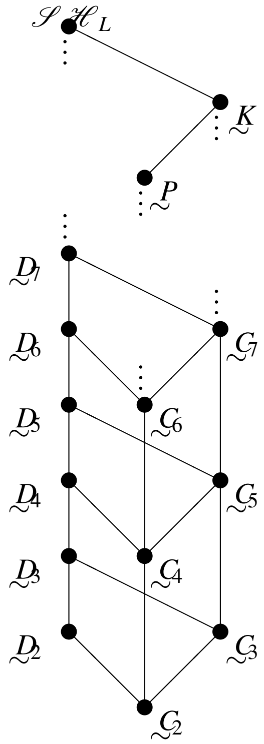


Fig. 1

Observe that from the definitions of $\underset{\sim}{K}$, $\underset{\sim}{P}$ and $\mathscr{S}\mathscr{H}_L$, it follows that they are not completely join-irreducible. Furthermore, they are not finitely generated. The following theorem can be found in [8].

Theorem 7.4. *$\underset{\sim}{K}$, $\underset{\sim}{P}$ and $\mathscr{S}\mathscr{H}_L$ are the unique join-irreducible varieties that are not finitely generated. Every variety $V \in \Lambda(\mathscr{S}\mathscr{H}_L)$ is a join of finitely many varieties in $\mathscr{J}(\Lambda(\mathscr{S}\mathscr{H}_L))$.*

Now we will find equational bases for each subvariety of $\mathscr{S}\mathscr{H}_L$.

Consider the terms $\gamma_{\underset{\sim}{K}}(x) = \neg x \Rightarrow \sim x$ and $\gamma_{\underset{\sim}{P}}(x) = \neg \sim (\sim x \Rightarrow x) \Rightarrow \neg\neg \sim (x \Rightarrow \sim x)$. In [2] it is proved that

Theorem 7.5. [21]. *The equation $\gamma_{\underset{\sim}{K}}(x) = 1$ characterizes the variety $\underset{\sim}{K}$ within $\mathscr{S}\mathscr{H}_L$, and the equation $\gamma_{\underset{\sim}{P}}(x) = 1$ characterizes the variety $\underset{\sim}{P}$ within $\underset{\sim}{K}$.*

As we pointed out in Section 1, the variety I_n is generated by $D_n \times D_{n-1}$. Nevertheless, we proved that D_{n-1} is a subalgebra of D_n, thus I_n is the variety generated by D_n, that is, the variety I_n is the variety $\underset{\sim}{D}_n$, and consequently, the Ivo Thomas identity $\gamma_n(x_0, \ldots, x_{n-1}) = 1$ determines the variety $\underset{\sim}{D}_n$, for $n \geq 2$.

The variety that A. Monteiro called $\underset{\sim}{I}\underset{\sim}{K}_n$ is defined as the subvariety of $\mathscr{S}\mathscr{H}_L$ characterized by the identities $\gamma_n(x_0,\ldots,x_{n-1}) = 1$ and $\gamma_{\underset{\sim}{K}}(x) = 1$. Monteiro proved ([21], p. 152, Th. 1.1) that $\underset{\sim}{I}\underset{\sim}{K}_n = V(C_n) = \underset{\sim}{C}_n$, for n odd, and $\underset{\sim}{I}\underset{\sim}{K}_n = V(C_n \times C_{n-1})$, for n even. Then we have:

Theorem 7.6. *The equations $\gamma_n(x_0,\ldots,x_{n-1}) = 1$ and $\gamma_{\underset{\sim}{K}}(x) = 1$ determine the variety $\underset{\sim}{C}_n$ for n odd, and the equations $\gamma_n(x_0,\ldots,x_{n-1}) = 1$, $\gamma_{\underset{\sim}{K}}(x) = 1$ and $\gamma_{\underset{\sim}{P}}(x) = 1$ determine the variety $\underset{\sim}{C}_n$ for n even.*

7.2. **The linearity in $\mathscr{S}\mathscr{C}$.** The structure of the lattice of subvarieties of the variety of linear closure algebras has been extensively studied by Blok in [6]. In this variety, the image of the closure operator is a linear Heyting algebra, and hence the subdirectly irreducible algebras are those closure algebras such that the image of the operator is a chain with a dual atom. More precisely we have the following theorem proved by Blok in [6].

Theorem 7.7. (W. Blok)

(i) *The equation*

$$(Q(x) \to Q(y)) \vee (Q(y) \to Q(x)) = Q(-Q(x) \vee Q(y)) \vee Q(-Q(y) \vee Q(x)) = 1,$$

determines the variety $\mathscr{C}_L$ in $\mathscr{C}$.

(ii) $\mathbf{Si}(\mathscr{C}_L) = \{A \in \mathscr{C} : Q(A)$ *is a chain with a dual atom*$\}$.

(iii) $\mathscr{C}_L = V(\mathbf{Si}(\mathscr{C}_L))$.

(iv) $\mathscr{C}_L^n = V(\{A \in \mathscr{C} : Q(A) \cong C_n,$ *and A is finite*$\}$.

(v) $\mathscr{C}_L = V\left(\bigcup_{n\geq 1} \mathscr{C}_L^n\right)$.

(vi) $\mathscr{C}_L^1 \subset \mathscr{C}_L^2 \subset \ldots \subset \mathscr{C}_L$.

Observe that the class $\mathscr{C}_L^1$ is the class of monadic Boolean algebras and the class $\mathscr{C}_L^2$ is the class $\mathscr{C}_T$ of three-valued closure algebras investigated by M. Abad and J. P. Díaz Varela in [1] and [10]. Also observe that the finite subdirectly irreducible algebras in $\mathscr{C}_L$ are those closure algebras in which the set of open elements is isomorphic to a chain C_n, for some n. Let $\mathbf{B}_{k_1,k_2,\ldots,k_s}$ denote the subdirectly irreducible closure algebra with $n = \sum_{i=1}^{s} k_i$ atoms and $Q(\mathbf{B}_{k_1,k_2,\ldots,k_s}) = C_s = \{0 = a_0 < a_1 < \ldots < a_{s-1} < a_s = 1\}$, where the intervals $[a_i, a_{i+1}]$ have k_{i+1} atoms for $i = 0,\ldots,s-1$, that is, there are k_1 atoms of $\mathbf{B}_{k_1,k_2,\ldots,k_s}$ preceding $a_1,\ldots$, there are $\sum_{i=1}^{j} k_i$ atoms preceding $a_j, \ldots$. Thus, the finite subdirectly irreducible algebras in $\mathscr{C}_L$ are the algebras $\mathbf{B}_{k_1,k_2,\ldots,k_s}$. In addition, these algebras generate the variety $\mathscr{C}_L$, that is, $\mathscr{C}_L$ is generated by its finite members.

Let us consider the following subvarieties:

$$V_{k_1,k_2,\ldots,k_s} = V\left(\mathbf{B}_{k_1,k_2,\ldots,k_s}\right).$$

In order to give a characterization of the ordering for the subvarieties $V_{k_1,k_2,\ldots,k_s}$, ww consider the set $I\!N_f$ of all finite sequences of positive integers. Given $X = x_1,\ldots,x_s \in I\!N_f$ we will abbreviate V_X for $V_{x_1,\ldots,x_s}$ and $\mathbf{B}_X$ for $\mathbf{B}_{x_1,\ldots,x_s}$. For $X, Y \in I\!N_f$, we have that $V_X \leq V_Y$ if and only if $\mathbf{B}_X \in \mathbf{HS}(\mathbf{B}_Y)$. Hence we can define an ordering in $I\!N_f$ by means of

$$X \leq_N Y \iff \mathbf{B}_X \in \mathbf{HS}(\mathbf{B}_Y).$$

The mapping $\varphi : \mathbf{Si_{fin}}(\mathscr{C}_L) \longrightarrow I\!N_f$ defined by $\varphi(\mathbf{B}_X) = X$ is clearly an order isomorphism.

The following theorem gives us a useful characterization of the order relation defined on $\mathbb{N}_f$. Let $X,Y \in \mathbb{N}_f$ with $X = x_1,\ldots,x_k$ and $Y = y_1,\ldots,y_l$.

Theorem 7.8. (W. Blok) *$X \leq_N Y$ if and only if there exist $1 = i_1 < i_2 < \ldots < i_k \leq l$ such that $x_j \leq y_{i_j}$ for $j = 1,2,\ldots,k$.*

In [9] an equation is given for each subvariety of the variety

$$\mathscr{C}_L^n = V(\{\mathbf{B}_X, \text{ where } X \text{ is a } k\text{−tuple}, \ k \leq n\})$$

Consider now the variety $\mathscr{S}\mathscr{C}_L$ of linear closure symmetric algebras. This variety consists of those symmetric closure algebras that satisfy the linearity condition on the set of its open elements, that is,

$$(Q(x) \to Q(y)) \vee (Q(x) \to Q(y)) = 1.$$

This variety is the symmetric counterpart of linear closure algebras. Observe that if $A \in \mathscr{S}\mathscr{C}_L$ then $Q(A)$ is a linear symmetric Heyting algebra, so the results of the previous section will be useful.

Since the variety of linear closure algebras is not locally finite then $\mathscr{S}\mathscr{C}_L$ is not locally finite. Of course, it is congruence-distributive.

The proof of the following theorem is similar to that of its analogous for linear closure algebras (see [6]).

Theorem 7.9. *Any subvariety V of $\mathscr{S}\mathscr{C}_L$ is generated by its finite members.*

7.3. **Finitely generated subvarieties of $\mathscr{S}\mathscr{C}_L^n$.** Now we will study the subvarieties of $\mathscr{S}\mathscr{C}_L^n$, where $\mathscr{S}\mathscr{C}_L^n$ is the subvariety of $\mathscr{S}\mathscr{C}_L$ characterized by

$$(CLn) \qquad \bigvee_{i=0}^{n}(Q(x_i) \to Q(x_{i+1})) = \bigvee_{i=0}^{n} Q(-Q(x_i) \vee Q(x_{i+1})) = 1.$$

Observe that if $A \in \mathbf{Si}_{fin}(\mathscr{S}\mathscr{C}_L)$, then $Q(A) \in \mathbf{Si}_{fin}(\mathscr{S}\mathscr{H}_L)$ and then $Q(A)$ is isomorphic to a subalgebra of D_n, for some n. From this we can deduce that $\mathscr{S}\mathscr{C}_L^n$ is locally finite. Indeed, let $A \in \mathbf{Si}(\mathscr{S}\mathscr{C}_L^n)$, A finitely generated. Let $g_1,g_2,\ldots,g_m$ be generators of A. Then

$$A = [g_1,g_2,\ldots,g_m]_{\mathscr{S}\mathscr{C}} = [g_1,g_2,\ldots,g_n,\ Q(A)]_{\mathscr{S}\mathscr{B}}.$$

But $|Q(A)| \leq n^2$, so $|A| \leq 2^{2^{2(m+n^2)}}$. Hence A is finite. Consequently we can conclude that $\mathscr{S}\mathscr{C}_L^n$ is locally finite, and thus, generated by its finite members.

In the following we will study the ordering between the finitely generated subvarieties. To this end we need to characterize the simple algebras generating finitely generated join-irreducible subvarieties.

If $Q(A) \cong C_t$, $(t \leq n)$, then $A \cong_{\mathscr{C}} \mathbf{B}_X$, with $X = (x_1,x_2,\ldots,x_t)$. In addition, if t is even, there exists $a \in Q(A)$ such that $\sim a = a$. Thus $x_1 = x_t$, $x_2 = x_{t-1}$, and in general, if $i \leq t/2$, $x_i = x_{t-i+1}$. If t is odd then for $i \leq \frac{t+1}{2}$, we have $x_i = x_{t-i+1}$, as $\sim (a_1] = [a_{t-1})$, and in general, $\sim (a_i] = [a_{t-i})$ where $Q(A) = \{0 = a_0 < a_1 < \ldots < a_{t-1} < a_t = 1\}$. Observe also that $T = \sim -$, when restricted to the atoms of A, $T : At(A) \to At(A)$, is a bijection, by means of which the set $A_{x_i} = At(a_i) \setminus At(a_{i-1})$ of atoms that precede a_i and do not precede a_{i-1} corresponds to the set $A_{x_i}^{-} = At(-a_{t-i+1}) \setminus At(-a_{t-i})$. If t is even then the atoms of A are non-invariant by T, as $T(A_{x_i}) \cap A_{x_i} = \emptyset$.

After these considerations we will analyze the structure of the finite subdirectly irreducible closure algebras of $\mathscr{SC}^n_L$. Let $A \in \mathbf{Si_{fin}}(\mathscr{SC}^n_L)$. We consider three cases.

I. $Q(A) \cong D_n$. In this case there exists $a \in B(Q(A)) \setminus \{0,1\}$ such that $A \cong_{\mathscr{C}} (a] \times [a)$. Observe in addition that $(a] \cong \mathbf{B}_X$ with $X \in \mathbf{N}_f$, $X = (x_1, x_2, \ldots, x_{n-1}, x_n)$, and that $[a) \cong \mathbf{B}_{X^*}$ with $X^* \in \mathbf{N}_f$ and $X^* = (x_n, x_{n-1}, \ldots, x_2, x_1)$. Taking $\sim_d (x,y) = (d^{-1}(y), d(x))$, where $d : \mathbf{B}_X \to \mathbf{B}_{X^*}$ is the application such that $x \leq y$ if and only if $d(y) \leq d(x)$, we state that $A \cong_{\mathscr{SC}} (\mathbf{B}_X \times \mathbf{B}_{X^*}, \sim_d)$. Indeed, it is clear that d is a negation over $\mathbf{B}_X \times \mathbf{B}_{X^*}$. Let us see that every negation that can be defined on $\mathbf{B}_X \times \mathbf{B}_{X^*}$ that maps $a \longrightarrow a$ equals d, up to isomorphism.

Let $(\mathbf{B}_X \times \mathbf{B}_{X^*}, \sim_1)$ and $(\mathbf{B}_X \times \mathbf{B}_{X^*}, \sim_2)$, two structures of symmetric closure algebras with the same universe $\mathbf{B}_X \times \mathbf{B}_{X^*}$. Let $T_1 = \sim_1 - = - \sim_1$ and $T_2 = \sim_2 - = - \sim_2$. Since $T_i(a) = -a$ with $i = 1,2$ and $a \in B(Q(A)) \setminus \{0,1\}$, it follows that all atoms are non-invariant, that is, both structures are isomorphic as symmetric Boolean algebras, by means of an isomorphism α such that $\alpha T_1 = T_2$. So $\alpha = T_2 T_1^{-1} = T_2 T_1$ as $T_1^2 = I_d$. In addition

$$\alpha{\upharpoonright}_{Q(A)}(a_i, a_j) = T_1 T_2 (a_i, a_j) = \sim_1 \sim_2 (a_i, a_j).$$

Since the negation of $Q(A)$ is the negation of D_n, as $\sim a = a$, then in $Q(A)$, $\sim_1 = \sim_2$. Then

$$\alpha{\upharpoonright}_{Q(A)}(a_i, a_j) = (a_i, a_j),$$

that is, $\alpha{\upharpoonright}_{Q(A)} = Id_{Q(A)}$. Hence α is an $\mathscr{SC}$−isomorphism. Thus the structure is unique up to isomorphism. We then denote

$$D^n_X \cong (\mathbf{B}_X \times \mathbf{B}_{X^*}, \sim_d),$$

where $(n+1)^2 = |Q(A)|$.

II. $Q(A) \cong C_n$, n even. Then there exists $a \in C_n$ such that $\sim a = a$. So every element is non-invariant, and consequently, as in the previous case, there exists a unique structure of symmetric closure algebras, up to isomorphism. Observe that $A \cong_{\mathscr{C}} \mathbf{B}_X$ and from the action of $\sim$, $x_i = x_{n-i+1}$, for all $i = 1, \ldots, \frac{n}{2}$.

III. $Q(A) \cong C_n$, n odd. As in the previous case, $A \cong_{\mathscr{C}} \mathbf{B}_X$ and $x_i = x_{n-i+1}$, for all $i = 1, \ldots, \frac{n+1}{2}$. Let $a_0 = 0, a_1, \ldots, a_n$ the open elements of A. Observe that if $b \in At(a_{\frac{n-1}{2}})$, then $T(b) = - \sim b \leq -a_{\frac{n+1}{2}}$. Since $At(a_{\frac{n-1}{2}}) \cap At(-a_{\frac{n+1}{2}}) = \emptyset$, it follows that $T(b) \neq b$. So if $b \in At(a_{\frac{n-1}{2}})$, b in non-invariant. In the same way it can be proved that if $b \in At(-a_{\frac{n+1}{2}})$, b is non-invariant. Hence, if b is an invariant atom, $b \in At(a_{\frac{n+1}{2}}) \setminus At(a_{\frac{n-1}{2}})$. Thus there can be at most $x_{\frac{n+1}{2}}$ invariant atoms in A. In addition, if we fix a negation with its invariant atoms and its non-invariant atoms and with the structure of closure algebra of $\mathbf{B}_X$, there exists a unique structure of symmetric closure algebra for A. So there exist so many structures of closure algebras as negations that can be defined on $\mathbf{B}_X$, such that

- $\sim (a_i) = a_{n-i}$.
- $- \sim : At(a_{\frac{n-1}{2}}) \to At(-a_{\frac{n+1}{2}})$ is a bijection.

Then we have that if $x_{\frac{n+1}{2}} = 2s + t$ with $2s$ non-invariant atoms and t invariant atoms, then the notation C^{2s+t}_X describes a unique structure of algebra A, up to isomorphism.

We will denote $\mathscr{U}^n_X = V(D^n_X)$, $\mathscr{V}^n_X = V(C_X)$, $\mathscr{V}^n_{X,2s+t} = V(C^{2s+t}_X)$. Also $\frac{X}{2}^+ = (x_1, \ldots, x_{\frac{n-1}{2}})$ if n is odd and $\frac{X}{2}^+ = (x_1, \ldots, x_{\frac{n}{2}})$ if n is even.

The following lemmas will be important to obtain the ordering on the algebras C_X^{2s+t}.

Lemma 7.10. *Let $x \in C_X$. Then $x \wedge Q(x) \wedge Q(\sim x) \in [0, a_{\frac{n}{2}}]$ if n is even and $x \wedge Q(x) \wedge Q(\sim x) \in [0, a_{\frac{n-1}{2}}]$ if n is odd.*

Proof. Let n be even. If $Q(x) = a_{\frac{n}{2}}$, there is nothing to prove. If $Q(x) > a_{\frac{n}{2}}$, then $Q(x) \geq a_{\frac{n}{2}+1}$, so $Q(\sim x) \leq a_{\frac{n}{2}}$. Hence $x \wedge Q(x) \wedge Q(\sim x) \leq a_{\frac{n}{2}}$. The case n odd is similar. □

By the previous lemma, it is immediate that if $x \in C_X$ then $\sim (Q(x) \wedge Q(\sim x)) = \sim Q(x) \vee \sim Q(\sim x) \in [a_{\frac{n}{2}}, 1]$ if n is even and $\sim (Q(x) \wedge Q(\sim x)) = \sim Q(x) \vee \sim Q(\sim x) \in [a_{\frac{n+1}{2}}, 1]$ if n is odd.

Lemma 7.11. *Let $x \in C_X^n$, with n even. Then*

1. $x = Q(x)$, $x \wedge \sim x = x$ and $x \vee (Q(y) \wedge Q(\sim y)) = x$ *for all $y \in C_X^n$, if and only if* $x = a_{\frac{n}{2}}$.
2. $x = Q(x)$, $x \vee \sim x = x$ *and* $x \wedge (\sim Q(y) \vee \sim Q(\sim y)) = x$ *for all $y \in C_X^n$, if and only if* $x = a_{\frac{n}{2}+1}$.

Proof. If $x = Q(x)$, then x is open, that is, $x = a_i$. Then $\sim x = a_{n-i}$. Since $x \leq \sim x$ then $i \leq n - i$ and $i \leq \frac{n}{2}$. From the hypothesis, $x \vee (Q(a_{\frac{n}{2}}) \wedge Q(\sim a_{\frac{n}{2}})) = x \vee a_{\frac{n}{2}} = x$. Thus, $x \geq a_{\frac{n}{2}}$. So $x = a_{\frac{n}{2}}$. Conversely if $x = a_{\frac{n}{2}}$, it is easy to check that these properties hold. The rest of the proof is analogous. □

Lemma 7.12. *Let $x \in C_X^n$, with n odd. If $Q(x) = Q(\sim x)$ and $x \vee (Q(y) \wedge Q(\sim y)) = x$ for all $y \in C_X^n$, then $x \in (a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}})$.*

Proof. $\Leftarrow$) If $x \in (a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}})$, it is easy to see that $\sim x \in (a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}})$. So $Q(x) = Q(\sim x) = a_{\frac{n-1}{2}}$. In addition, $x > a_{\frac{n-1}{2}}$, and by the previous lemma $Q(y) \wedge Q(\sim y) \leq a_{\frac{n-1}{2}}$. Hence $x \vee (Q(y) \wedge Q(\sim y)) = x$.
$\Rightarrow$) If $x \vee Q(y) \wedge Q(\sim y) = x$ for all $y \in C_X^n$, then $x \vee (Q(a_{\frac{n-1}{2}}) \wedge Q(\sim a_{\frac{n-1}{2}})) = x$, and consequently $x \vee (a_{\frac{n-1}{2}} \wedge a_{\frac{n+1}{2}}) = x \vee a_{\frac{n-1}{2}} = x$, that is, $x \geq a_{\frac{n-1}{2}}$. Since $Q(x) = Q(\sim x)$, x is not open. So $x > a_{\frac{n-1}{2}}$. If $Q(x) \neq a_{\frac{n-1}{2}}$ then $x > a_{\frac{n+1}{2}}$ is not open and $\sim x < \sim a_{\frac{n+1}{2}} = a_{\frac{n-1}{2}}$. Therefore $Q(\sim x) < a_{\frac{n-1}{2}} < a_{\frac{n+1}{2}} \leq Q(x)$. So $Q(x) \neq Q(\sim x)$, a contradiction. Hence $Q(x) = Q(\sim x) = a_{\frac{n-1}{2}}$. Thus $x > a_{\frac{n-1}{2}}$ and thus $a_{\frac{n-1}{2}} < \sim x < a_{\frac{n+1}{2}}$. From this, $a_{\frac{n-1}{2}} < x < a_{\frac{n+1}{2}}$, which concludes the proof. □

After this description of subdirectly irreducible (simple) algebras, we will provide an order for the join-irreducible finitely generated varieties. First observe that $\mathscr{U}_X^n \not\leq \mathscr{V}_X^n$ for any X and n, being that the algebras C_X satisfy the Kleene condition $\gamma_K(Q(x)) = 1$ for its open elements and D_X^n does not. Similarly, $\mathscr{V}_X^n \not\leq \mathscr{V}_X^m$ for $n-1$ odd and $m-1$ even, since $\mathscr{V}_X^m$ satisfies the condition $\gamma_P(Q(x)) = 1$ for its open elements and $\mathscr{V}_X^n$ does not.

The ordering for $J_{fin}(\Lambda(\mathscr{SC}_L))$ is given by the following theorems.

Theorem 7.13. *Let n, m be odd. Then $\mathscr{V}_{X,2k+l}^n \leq \mathscr{V}_{Y,2s+t}^m$ if and only if $\frac{X}{2}^+ \leq_{\mathbf{N_f}} \frac{Y}{2}^+$, $x_{\frac{n+1}{2}} \leq y_{\frac{m+1}{2}}$, $k \leq s$ and $k + l \leq s + t$.*

Proof. $\Rightarrow$) If $\mathscr{V}^n_{X,2k+l} \le \mathscr{V}^m_{Y,2s+t}$ there exists an $\mathscr{S}\mathscr{C}$-embedding $i : C_X^{2k+l} \to C_Y^{2s+t}$. Observe that i is a $\mathscr{C}$-embedding, and in addition $i|Q(C_X^{2k+l})$ is an $\mathscr{S}\mathscr{H}$-embedding. If $a_0 = 0, a_1, \ldots, a_n$ are the open elements of C_X^{2k+l} and $b_0 = 0, b_1, \ldots, b_m$ are the open elements of C_Y^{2s+t} then

$$i(Q(x) \wedge Q(\sim x)) = Q(i(x)) \wedge Q(\sim (i(x))) \le b_{\frac{n-1}{2}}.$$

That is, $i|[0, a_{\frac{n-1}{2}}] : [0, a_{\frac{n-1}{2}}] \to [0, b_{\frac{n-1}{2}}]$ is a $\mathscr{C}$-embedding. So $\frac{X}{2}^+ \le_{\mathbf{N_f}} \frac{Y}{2}^+$. Also, as i is an $\mathscr{S}\mathscr{C}$-morphism, the images of the elements satisfying the equations of Lemma 7.12 satisfy the same equations. Then $i((a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}})) \subseteq (b_{\frac{n-1}{2}}, b_{\frac{n+1}{2}})$. If $(a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}) = \emptyset$ the case is trivial. Suppose $(a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}) \ne \emptyset$. If $z \in (a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}})$ then $Q(z) = a_{\frac{n-1}{2}}$ and $i(a_{\frac{n-1}{2}}) = i(Q(z)) = Q(i(z)) = b_{\frac{n-1}{2}}$. So $i(a_{\frac{n+1}{2}}) = i(\sim a_{\frac{n-1}{2}}) = \sim b_{\frac{n-1}{2}} = b_{\frac{n+1}{2}}$. Then $i|[a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}] : [a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}] \to [b_{\frac{n-1}{2}}, b_{\frac{n+1}{2}}]$ is an $\mathscr{S}\mathscr{C}-$embedding and, by the previous section, $x_{\frac{n+1}{2}} \le y_{\frac{m+1}{2}}$, $k \le s$ y $k + l \le s + t$.

$\Leftarrow$) Let $\frac{X}{2}^+ \le_{\mathbf{N_f}} \frac{Y}{2}^+$ and let P_1 the partition associated to a $\mathscr{C}$-subalgebra isomorpphic to $[0, a_{\frac{n-1}{2}}]$ in the $\mathscr{C}-$algebra $[0, b_{\frac{n-1}{2}}]$ (with the $\mathscr{C}-$structure inherited from C_Y^{2s+t}). Observe that $T(P_1) = \sim -(P_1)$ generates a partition in $At(C_Y^{2s+t}) \setminus At([0, b_{\frac{n+1}{2}}]) = At([0, -b_{\frac{n+1}{2}}])$. If $x_{\frac{n-1}{2}} \le y_{\frac{m-1}{2}}$, $k \le s$ and $k + l \le s + t$, then there exists a partition P_2 of the atoms of $At([0, b_{\frac{n+1}{2}}]) \setminus At([0, b_{\frac{n-1}{2}}])$ that generates a $\mathscr{S}\mathscr{C}$-subalgebra isomorphic to $[a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}]$ in $[b_{\frac{n-1}{2}}, b_{\frac{n+1}{2}}]$, (recall that there exists a bijection between the atoms of $[b_{\frac{n-1}{2}}, b_{\frac{n+1}{2}}]$, and $At([0, b_{\frac{n+1}{2}}]) \setminus At([0, b_{\frac{n-1}{2}}])$). Then, it is easy to see that $P = P_1 \cup T(P_1) \cup P_2$ is a partition that generates a symmetric closure subalgebra isomorphic to C_X^{2k+l}. □

Theorem 7.14. *Let n, m be even. Then $\mathscr{V}^n_X \le \mathscr{V}^m_Y$ if anf only if $\frac{X}{2}^+ \le_{\mathbf{N_f}} \frac{Y}{2}^+$.*

Consider the term $\gamma_{D^n_Y} = [(\gamma_Y \leftrightarrow 1) \vee (T(\gamma_Y) \leftrightarrow 1)] \wedge [(\gamma_{Y^*} \leftrightarrow 1) \vee (T(\gamma_{Y^*}) \leftrightarrow 1)]$

Lemma 7.15. *D^n_Y satisfies the equation $= 1$, where $\gamma_Y = 1$ is the equation that characterizes $\mathscr{C}_Y$, the variety of linear closure algebras with $Y = (y_1, \ldots, y_r)$. (See* [9, 10]*).*

Proof. Recall that $D^n_Y \cong_{\mathscr{C}} C_Y \times C_{Y^*}$. Let $(x_i, y_i)_{i=1}^r \in C_Y \times C_{Y^*}$, with r as needed. Then

$$\gamma_Y((x_1, y_1), \ldots, (x_r, y_r)) = (\gamma_Y(x_1, \ldots, x_r), \gamma_Y(y_1, \ldots, y_r)) = (1, a).$$

Then $(\gamma_Y \leftrightarrow 1) = (1, a) \leftrightarrow (1, 1) = (1, c)$ with c open. In addition

$$(T(\gamma_Y) \leftrightarrow 1) = T((1, a)) \leftrightarrow (1, 1) = \sim -(1, a) \leftrightarrow (1, 1) = \sim (0, -a) \leftrightarrow (1, 1) =$$

$$(d^{-1}(-a), d(0)) \leftrightarrow (1, 1) = (a', 1) \leftrightarrow (1, 1) = (c', 1).$$

So

$$(\gamma_Y \leftrightarrow 1) \vee (T(\gamma_Y) \leftrightarrow 1) = (1, c) \vee (c', 1) = (1, 1).$$

In a similar way it can be proved that

$$(\gamma_{Y^*} \leftrightarrow 1) \vee (T(\gamma_{Y^*}) \leftrightarrow 1) = 1,$$

which concludes the proof. □

Theorem 7.16. *$\mathscr{U}^n_X \le \mathscr{U}^m_Y$ if and only if $X \le_{\mathbf{N_f}} Y$ and $X^* \le_{\mathbf{N_f}} Y^*$ or $X^* \le_{\mathbf{N_f}} Y$ and $X \le_{\mathbf{N_f}} Y^*$.*

Proof. $\Rightarrow$) Let $i : D^n_X \to D^m_Y$, $a, -a \in B(D^n_X) \setminus \{0,1\}$, and $b, -b \in B(D^n_Y) \setminus \{0,1\}$. Then $i(a) = b$ and $i(-a) = -b$ or $i(a) = -b$ and $i(-a) = b$. In the first case, $i|[0,a] : [0,a] \to [0,b]$ and $i|[0,-a] : [0,-a] \to [0,-b]$ are $\mathscr{C}$-embeddings and so $X \leq Y$ y $X^* \leq Y^*$. In the other case, $i|[0,a] : [0,a] \to [0,-b]$ and $i|[0,-a] : [0,-a] \to [0,b]$ are $\mathscr{C}$-embeddings and so $X \leq Y^*$ y $X^* \leq Y$.
$\Leftarrow$) If $X \leq Y$ then $\mathbf{B}_X$ is a subalgebra of $\mathbf{B}_Y$. Then there exists a $\mathscr{C}-$embedding $i : \mathbf{B}_X \to \mathbf{B}_Y$. In addition, $\mathbf{B}_{X^*} \cong_{\mathscr{C}} di(\mathbf{B}_X) \in \mathbf{S}_{\mathscr{C}}(\mathbf{B}_{Y^*})$. Thus $D^n_X \cong_{\mathscr{SC}} i(\mathbf{B}_X) \times d\ i(\mathbf{B}_X) \in \mathbf{S}_{\mathscr{SC}}(D^n_Y)$. □

Theorem 7.17. *Let n be odd. Then $\mathscr{V}^n_X \leq \mathscr{U}^m_Y$ if and only if $X \leq_{\mathbf{N_f}} Y$ and $X \leq_{\mathbf{N_f}} Y^*$.*

Proof. $\Rightarrow$) By Lemma 7.15, C^n_X satisfies the equation $\gamma_{D^n_Y} = 1$. Then, as $Q(C^n_X) \cong C_n$, $\gamma_Y = 1$ or $T(\gamma_Y) = 1$. But $TT(\gamma_Y) = \gamma_Y = T(1) = 1$. Hence $\gamma_Y = 1$ in C^n_X, and consequently $X \leq_{\mathbf{N_f}} Y$. The other part of the equation proves that $X \leq Y^*$.
$\Leftarrow$) Suposse that $X \leq_{\mathbf{N_f}} Y$ and $X \leq_{\mathbf{N_f}} Y^*$. Recall that $X = X^*$. Then $X^* \leq_{\mathbf{N_f}} Y^*$. Let $A \cong_{\mathscr{SC}} (C^n_X \times C^n_X, \sim)$ with $\sim (x,y) = (\sim y, \sim x)$. Then $A \cong_{\mathscr{SC}} D^n_X$, and by the previous theorem, $A \in \mathbf{S}_{\mathscr{SC}}(D^n_Y)$. The $\mathscr{SC}-$subalgebra of A, $D = \{(x,x) \in A\}$ is isomorphic to C^n_X, and hence $C^n_X \in \mathbf{S}_{\mathscr{SC}}(D^n_Y)$. □

Observe that the proof of the previous theorem does not make use of the fact that n is even, so the theorem is also true for n odd. So we have:

Theorem 7.18. *Let n be odd. Then $\mathscr{V}^n_{X,2s+t} \leq \mathscr{U}^m_Y$ if and only if $X \leq_{\mathbf{N_f}} Y$ and $X \leq_{\mathbf{N_f}} Y^*$.*

Lemma 7.19. *Let n be even. Then $\mathscr{V}^1_{2s+t} \leq \mathscr{V}^n_Y$ if and only if $2s+t \leq y_1$.*

Theorem 7.20. *Let n be odd and m be even and let $\frac{X}{2}^+ + 1 = (x_1, x_2, \ldots, x_{\frac{n-1}{2}}, x_{\frac{n+1}{2}})$. Then $\mathscr{V}^n_{X,2s+t} \leq \mathscr{V}^m_Y$ if and only if $\frac{X}{2}^+ + 1 \leq_{\mathbf{N_f}} \frac{Y}{2}^+$.*

Proof. $\Rightarrow$) If $\mathscr{V}^n_{X,2s+t} \leq_{\mathscr{SC}} \mathscr{V}^m_Y$ then we have an embedding $i : C^n_{X,2s+t} \to C^m_Y$. Since in 7.13, if $a_0 = 0, a_1, \ldots, a_n$ are the open elements of C^{2k+l}_X and $b_0 = 0, b_1, \ldots, b_m$ are the open elements of C_Y, then

$$i(Q(x) \wedge Q(\sim x)) = Q(i(x)) \wedge Q(\sim (i(x))) \leq b_{\frac{m}{2}},$$

that is, $i|[0, a_{\frac{n-1}{2}}] : [0, a_{\frac{n-1}{2}}] \to [0, b_{\frac{m}{2}}]$ is a $\mathscr{C}$-embedding. Then $\frac{X}{2}^+ \leq_{\mathbf{N_f}} \frac{Y}{2}^+$. In addition, $i(a_{\frac{n-1}{2}}) = b_h$. Observe that $h \neq \frac{m}{2}$, as $\sim b_{\frac{m}{2}} = b_{\frac{m}{2}}$, and this does not occur with $a_{\frac{n-1}{2}}$, and $h \not> \frac{m}{2}$ since otherwise $b_{m-h} \leq b_h$ and $i(a_{\frac{n+1}{2}}) < i(a_{\frac{n-1}{2}})$, a contradiction. So $h < \frac{m}{2}$. Then $i|[a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}] : [a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}] \to [b_h, b_{n-h}]$ is an $\mathscr{SC}-$enbedding, and by the previoues lemma, $x_{\frac{n+1}{2}} \leq y_{h+1}$, $h \leq \frac{m}{2} - 1$. Thus $\frac{X}{2}^+ + 1 \leq_{\mathbf{N_f}} \frac{Y}{2}^+$.
The converse is similar to Theorem 7.13. □

Then we have given the ordering for each subvariety of $J_{fin}(\Lambda(\mathscr{SH}))$.

7.4. **Infinite Subvarieties of $\mathscr{SC}^n_L$.** We recall the ordering for infinite subvarieties of linear closure algebras. These results can be seen in [9].

Consider the following $s-$tuples

$$\overline{X} = (x_1, x_2, \ldots, x_s),$$

where there exists a (nonempty) subset of indexes $i_1 < i_2 < \ldots < i_{t^*} \leq s$ such that $x_{i_j} = *$ and another (possibly empty) $h_1 < h_2 < \ldots < h_t \leq s$ such that $x_{h_l} = k_{h_l}$, with $k_{h_l} \in I\!N$, $t^* + t = s$.

By way of example, for $s = 5$, $\overline{X} = (3, *, *, 2, *)$ is a $5-$tuple.

Let χ_s denote the set of all such $s-$tuples, and consider the following subvarieties:

$$U_s^{\overline{X}} = V\left(\{\mathbf{B}_{r_1,\ldots,r_s} : r_{i_j} \geq 1,\ j = 1,\ldots,t^*,\ r_{h_l} = k_{h_l},\ l = 1,\ldots,t\}\right).$$

Observe that $U_2^{(*,*)} = \mathscr{C}_L^2 = \mathscr{C}_T$, and in general, $\mathscr{C}_L^n = U_n^{\overline{X}}$, where $\overline{X} = (\underbrace{*, *, \ldots, *}_{n\ times})$.

The following propositions prove that these varieties are join irreducible and they are the unique non finitely generated join-irreducible subvarieties in $\mathscr{C}_L^n$.

Proposition 7.21. *([9]) The varieties of the form $U_s^{\overline{X}}$, with $s \leq n$, are join-irreducible in $\Lambda(\mathscr{C}_L^n)$.*

Let χ_t^d denote those $t-$tuples $\overline{X}$ of the set χ_t, satisfying the condition $x_i = x_{t-i+1}$. Observe that this only holds for the even case or the odd case with $x_{\frac{n+1}{2}} = *$. It remains the case $x_{\frac{n+1}{2}} = *_h$ and $x_{\frac{n+1}{2}} = 2s + t$ (odd case).

Recall that the ordering in χ_t, (see [9]) is $\overline{Y}_s \leq^* \overline{X}_t$ if and only if

- $s \leq t$
- There exist $1 = i_1 < i_2 < \ldots < i_k \leq l$ such that $y_j \leq^* x_{i_j}$ for $j = 1, 2, \ldots, k$, where $y_j \leq^* x_{i_j}$
  - y_j and x_{i_j} are positive integers and $y_j \leq x_{i_j}$,
  - y_j is a positive integer and $x_{i_j} = *$,
  - $y_j = *$ and $x_{i_j} = *$,

  (observe that if x_{i_j} is a positive integer and $y_i = *$, then $y_j \not\leq^* x_{i_j}$).

For the symmetric case we define an order in χ_t, in the following way.

- For s, t even, $\overline{\frac{Y_s}{2}}^+ = y_1, \ldots, y_{\frac{s}{2}}$, and $\overline{\frac{X_t}{2}}^+ = x_1, \ldots, x_{\frac{t}{2}}$. Then we put $\overline{Y}_s \leq \overline{X}_t$ if and only if
$$\overline{\frac{Y_s}{2}}^+ \leq \overline{\frac{X_t}{2}}^+.$$
- For the case t, s odd, we have new n-tuples, those that represent the non-invariant elements of $y_{\frac{s+1}{2}}$ (see the previous section). Let us put $\overline{\frac{Y_s}{2}}^+ = y_1, \ldots, y_{\frac{s-1}{2}}$, and $\overline{\frac{X_t}{2}}^+ = y_1, \ldots, y_{\frac{s-1}{2}}$. Nor $y_{\frac{s+1}{2}}$ can take any of these three values: $*$, $2s + t$ and $*_h$ whis is the one that corresponds to the monadic variety S_h. Then the ordering is $\overline{Y}_s \leq \overline{X}_t$ if and only if
$$\overline{\frac{Y_s}{2}}^+ \leq \overline{\frac{X_t}{2}}^+,$$
and
  - $y_{\frac{s+1}{2}} = * \leq x_{\frac{t+1}{2}} = *$.
  - $y_{\frac{s+1}{2}} = *_h \leq x_{\frac{t+1}{2}} = *_k$ or $x_{\frac{t+1}{2}} = *$, if $h \leq k$.
  - $y_{\frac{s+1}{2}} = 2h + l \leq x_{\frac{t+1}{2}} = *_k$ or $x_{\frac{t+1}{2}} = *$, if $h \leq k$.
  - $y_{\frac{s+1}{2}} = 2h + l \leq x_{\frac{t+1}{2}} = 2k + r$, if $h \leq k$, $h + l \leq k + r$.

- The case s odd, t even, $\frac{\overline{Y_s}}{2}^+ + 1 = y_1, \ldots, y_{\frac{s-1}{2}}, y_{\frac{s+1}{2}}$ and $\frac{\overline{X_t}}{2}^+ = y_1, \ldots, y_{\frac{s-1}{2}}$. Observe that $*_h < *$. Then the ordering is $\overline{Y}_s \leq \overline{X}_t$ if and only if
$$\frac{\overline{Y_s}}{2}^+ + 1 \leq \frac{\overline{X_t}}{2}^+ ,$$

To simplify the notation when introducing the irreducible infinite varieties, we order the $n-$tuples X of $\mathbf{N}_f^d$ and the $n-$tuples $\overline{Y}$ of χ_t^d, by means of $X_s \leq_* \overline{Y}_t$ if and only if there exist $1 = i_1 < i_2 < \ldots < i_k \leq \frac{t}{2}$ such that $x_j \leq y_{i_j}$ for $j = 1, 2, \ldots, \frac{s}{2}$, where $x_j \leq y_{i_j}$ if x_j is a positive integer and y_{i_j} is equal to $*$ or a greater positive integer, and for the case s, t odd we put $x_{\frac{s+1}{2}} \leq y_{\frac{t+1}{2}}$ with the ordering

- $x_{\frac{s+1}{2}}$ a positive integer and $y_{\frac{t+1}{2}} = *$.
- $x_{\frac{s+1}{2}} = 2h + l \leq y_{\frac{t+1}{2}} = *_k$, if $h \leq k$.
- $x_{\frac{s+1}{2}} = 2h + l \leq y_{\frac{t+1}{2}} = 2k + r$, if $h \leq k$, $h + l \leq k + r$.

We denote the ordering in χ_t^d by $\leq^*$. In a similar way as for linear closure algebras we construct the following subvarieties: If t is even
$$P_t^{\overline{X}} = V\left(\bigcup_{Y \leq \overline{X}} C_Y^s\right), \quad P_t = V\left(\bigcup_{s \leq t, s = odd} C_X^s\right)$$
and is t is odd
$$K_t^{\overline{X}} = V\left(\bigcup_{Y \leq_* \overline{X}} C_Y^s\right), \quad K_t = V\left(\bigcup_{s,\ s \leq t} C_X^s\right).$$
For the case of algebras in which their open elements asr isomorphic to D_n for some n, we define
$$I_t^{\overline{X}} = V\left(\bigcup_{Y \leq_* \overline{X}} D_Y^s\right), \quad I_t = V\left(\bigcup_{s \leq t} D_Y^s\right).$$
Observe that $I_t = \mathscr{SC}_L^t$.

>From the results of the previous section and the beginning of this section, and making use of the ordering for finite join-irreducible varieties, the following theorem can be proved.

Theorem 7.22. *The varieties $P_s^{\overline{X}}$, P_s, $K_t^{\overline{X}}$, K_t, $I_t^{\overline{X}}$, I_t are the unique infinite join-irreducible varieties in $\Lambda(\mathscr{SC}_L)$.*

Theorem 7.23. *Every variety is a finite join of join-irreducible varieties in $\Lambda(\mathscr{SC}_L)$.*

7.5. **Equational Bases in $\mathscr{SC}_L^n$.** Our purpose is to determine equations characterizing each join-irreducible variety, and then, equations characterizing a finite join of join-irreducible varieties. By Theorem 7.23 we will have equations for every subvariety in $\Lambda(\mathscr{SC}_L^n)$.

Observe that if $x \neq y$, $x, y \in A$ and $x, y \in Q(A) \cong C_n$, then, if $x = a_i$, $y = a_j$ and $x \leq y$, we have that
$$x \leftrightarrow y = Q(-x \vee y) \wedge Q(-x \vee y) = (a_i \to a_j) \wedge (a_j \to a_i) = 1 \wedge a_i = a_i \wedge a_j = a_i.$$
This simple observation will be useful for the proof of the following theorems.

Consider the following terms:

$$A_1(z_0, z_1, \ldots, z_{\frac{n+1}{2}}) = \bigwedge_{i=1}^{x_{\frac{n+1}{2}}} Q(\sim z_i) \leftrightarrow Q(z_i),$$

$$A_2(z_0, z_1, \ldots, z_{\frac{n+1}{2}}) = \bigwedge_{i=1}^{x_{\frac{n+1}{2}}} z_i \vee (Q(z_0) \wedge Q(\sim z_0)) \leftrightarrow z_i,$$

$$A_3(z_0, z_1, \ldots, z_{\frac{n+1}{2}}) = \bigwedge_{i \neq j,\ j,i=1}^{x_{\frac{n+1}{2}}} Q(z_i \wedge z_j) \leftrightarrow (z_i \wedge z_j),$$

$$A_4(z_0, z_1, \ldots, z_{\frac{n+1}{2}}) = Q\left(\bigvee_{i=1}^{x_{\frac{n+1}{2}}} z_i\right) \leftrightarrow \left(\bigvee_{i=1}^{x_{\frac{n+1}{2}}} z_i\right),$$

and

$$\gamma\gamma_{x_{\frac{n+1}{2}}}(z_0, z_1, \ldots, z_{\frac{n+1}{2}}) = (A_1 \wedge A_2 \wedge A_3) \rightarrow A_4$$

Theorem 7.24. $C_X^{n,2s+t}$ *satisfies the equation.* $\gamma\gamma_{x_{\frac{n+1}{2}}}(z_0, z_1, \ldots, z_{\frac{n+1}{2}}) = 1$.

Proof. We have the following cases:

1. Suppose that $A_1 \neq 1$. Then there exists z_i such that $Q(\sim z_i) \neq Q(z_i)$. Then $Q(\sim z_i) \leftrightarrow Q(z_i) \leq Q(\sim z_i) \wedge Q(z_i) \leq a_{\frac{n-1}{2}}$. So $\bigwedge_{i=1}^{3} A_i \leq (Q(\sim z_i) \wedge Q(z_i)) \leq Q(z_i) \leq Q(\bigvee_{i=1}^{x_{\frac{n+1}{2}}} z_i) \leq A_4$, and consequently $\gamma\gamma_{x_{\frac{n+1}{2}}} = (\bigwedge_{i=1}^{3} A_i) \rightarrow A_4 = 1$.
2. Suppose that $A_2 \neq 1$. Then there exist z_i and z_0 such that $z_i \vee (Q(z_0) \wedge Q(\sim z_0) \neq z_i$. Then $A_2 \leq Q(z_i)$, and arguing as in the previous case, we have that $\gamma\gamma_{x_{\frac{n+1}{2}}} = 1$.
3. Suppose that $A_3 \neq 1$. Then there exist z_i and z_j such that $Q(z_i \wedge z_j) \neq (z_i \wedge z_j)$. Thus $A_3 \leq Q(z_i \wedge z_j)$ and so $A_1 \wedge A_2 \wedge A_3 \leq A_4$, and $\gamma\gamma_{x_{\frac{n+1}{2}}} = 1$.
4. Suppose that $A_1 = A_2 = 1$. Then, by Lemma 7.11 we have that $z_i \in (a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}})$ and if $A_3 = 1$ then the elements z_i are distinct atoms (necessarily all the atoms) of $[a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}]$. Then
$$Q(\bigvee_{i=1}^{x_{\frac{n+1}{2}}} z_i) = (\bigvee_{i=1}^{x_{\frac{n+1}{2}}} z_i),$$
and hence $\gamma\gamma_{x_{\frac{n+1}{2}}} = 1$.

□

Theorem 7.25. *Let m be odd. Then* $C_Y^{m,2k+l}$ *satisfies* $\gamma\gamma_{x_{\frac{n+1}{2}}} = 1$ *if and only if* $y_{\frac{m+1}{2}} \leq x_{\frac{n+1}{2}}$.

Proof. $\Rightarrow$) Suppose that $y_{\frac{m+1}{2}} > x_{\frac{m+1}{2}}$. Let $b_1, \ldots, b_m$ the open elements of $C_Y^{m,2k+l}$ and $z_1, \ldots, z_{\frac{m+1}{2}}$, the atoms of $[b_{\frac{m-1}{2}}, b_{\frac{m+1}{2}}]$. Let $\{z_1, \ldots, z_{\frac{n+1}{2}}\}$ be a proper subset of atoms of $[b_{\frac{m-1}{2}}, b_{\frac{m+1}{2}}]$ and $z_0 = b_{\frac{n-1}{2}}$. Then it is easy to check that $A_1 = A_2 = A_3 = 1$. But $Q(\bigvee_{i=1}^{x_{\frac{n+1}{2}}} z_i) = b_{\frac{n-1}{2}} \neq (\bigvee_{i=1}^{x_{\frac{n+1}{2}}} z_i)$, and thus $A_4 \neq 1$. So $\gamma\gamma_{x_{\frac{n+1}{2}}} \neq 1$. A contradiction.

$\Leftarrow$) We have already seen in the proof of the previous theorem that if A_1, A_2, A_3 are distinct from 1, then the equation holds. Observe that if $y_{\frac{m-1}{2}} < x_{\frac{m-1}{2}}$ then A_1 or A_2 or A_3 are distinct from 1, and consequently the equation holds. If $y_{\frac{m-1}{2}} = x_{\frac{m-1}{2}}$ and there exist $z_1,\ldots,z_{\frac{n+1}{2}}$, such that $A_1,A_2,A_3 = 1$, then $z_1,\ldots,z_{\frac{n+1}{2}}$ are the atoms of $[b_{\frac{m-1}{2}}, b_{\frac{m+1}{2}}]$. Then $A_4 = 1$ and the equation holds. □

As we have already proved, the elements of the form $x \wedge A(x_0)$ with $A(x_0) = Q(x_0) \wedge Q(\sim x_0)$ in $C_X^{n,2s+t}$ belong to the interval $[0, a_{\frac{n-1}{2}}]$ in the odd case, and to $[0, a_{\frac{n}{2}}]$ in the even case. We can algebrize the interval $[0, A(x_0)]$ with $\mathscr{C}$-terms and we can put $A = ([0,A(x_0)], \wedge_A, \vee_A, -_A, Q_A, 1_A, 0_A)$, where the $A-$operations are defined: $\wedge_A = \wedge$, $\vee_A = \vee$, $-_A(x) = -x \wedge A(x_0)$, $Q_A = Q$, $1_A = A(x_0)$, $0_A = 0$. With these operations it is easy to see that $[0,A(x_0)]$ is a closure algebra. Observe that $[0,A(x_0)] = [0,a_i]$ with $i \leq \frac{n}{2}$ or $i \leq \frac{n-1}{2}$, and $[0, A(a_{\frac{n-1}{2}})] = [0, a_{\frac{n-1}{2}}]$, for the odd case and $[0, A(a_{\frac{n}{2}})] = [0, a_{\frac{n}{2}}]$, for the even case. So $[0,A(x_0)] \in \mathbf{S}(\mathbf{B}_{\frac{X}{2}^+})$ for every x_0. Thus $[0,A(x_0)]$ satisfies tha equation

$$\gamma_{\frac{X}{2}^+}^{A(x_0)}(x_0, x_1, \ldots, x_r) = 1,$$

where $\gamma_{\frac{X}{2}^+}^{A(x_0)}$ is the equation $\gamma_{\frac{X}{2}^+}$ that characterizes the subvariety of closure algebras $C_{\frac{X}{2}^+}$ (ver [Tesis][UMA]), where the $\mathscr{C}$-operations are replaced by the $A-$operations. Observe that if $C_Y^{m,2k+l}$ satisfies this equation, then $[0, b_{\frac{m-1}{2}}]$, satisfies $\gamma_{\frac{X}{2}^+}$, hence $\frac{Y}{2}^+ \leq_{\mathbf{N}_f} \frac{X}{2}^+$. So we have proved the following theorem.

Theorem 7.26. *$C_X^{n,2s+t}$ satisfies the equation $\gamma_{\frac{X}{2}^+}^{A(x_0)} = 1$. Moreover if $C_Y^{m,2k+l}$ satisfies this equation then $\gamma_{\frac{Y}{2}^+} \leq_{\mathbf{N}_f} \gamma_{\frac{X}{2}^+}$.*

As we have already seen in Lemma 7.12 the elements $x \in C_X^{2s+t,n}$ that satisfy $Q(x) = Q(\sim x)$ and $x \vee (Q(y) \wedge Q(\sim y)) = x$ for all $y \in C_X^{2s+t,n}$, are those that belong to $(a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}})$. In addition, $Q(x) = a_{\frac{n-1}{2}}$ and $\sim Q(x) = \sim a_{\frac{n-1}{2}} = a_{\frac{n+1}{2}}$. In this way we can define, for every x satisfying this equation, in the interval $M = [Q(x), \sim Q(x)]$ a structure of symmetric closure algebra if we write $(M, \wedge_M, \vee_M, -_M, Q_M, \sim_M, 1_M, 0_M)$ where the operations are defined by $\wedge_M = \wedge$, $\vee_M = \vee$, $Q_M = Q$, $\sim_M = \sim$, $1_M = \sim Q(x)$, $0_M = Q(x)$, and $-_M(y) = (-y \wedge \sim Q(x)) \vee Q(x)$. In this way we have $T_M(y) = -_M \sim_M (y) = (- \sim y \wedge \sim Q(x)) \vee Q(x) = (T(y) \wedge \sim Q(x)) \vee Q(x)$. Observe that c is an invariant atom of $C_X^{2s+t,n}$ if and only if $c \vee a_{\frac{n-1}{2}}$ is an T_M-invariant atom of M. Indeed, if c is $T-$invariant then

$$T_M(c \vee a_{\frac{n-1}{2}}) = (T(c \vee a_{\frac{n-1}{2}}) \wedge \sim Q(x)) \vee Q(x) = ((c \vee -a_{\frac{n+1}{2}}) \wedge a_{\frac{n+1}{2}}) \vee a_{\frac{n-1}{2}} =$$
$$(c \wedge a_{\frac{n+1}{2}}) \vee a_{\frac{n-1}{2}} = c \vee a_{\frac{n-1}{2}}.$$

Conversely if $c \vee a_{\frac{n-1}{2}}$ is T_M-invariant then

$$c \vee a_{\frac{n-1}{2}} = T_M(c \vee a_{\frac{n-1}{2}}) = (T(c) \wedge a_{\frac{n+1}{2}}) \vee a_{\frac{n-1}{2}} = T(c) \vee a_{\frac{n-1}{2}}$$

and $T(c) \wedge -a_{\frac{n-1}{2}} = c$. As $T(c)$ is an atom, then $T(c) = c$. Hence (M, T_M) is a symmetric monadic algebra isomorphic to $\mathbf{C}_{2s,t}$ where $2s$ is the number of non–invariant atoms and t is the number of invariant atoms of M.

By these remarks and the results obtained for monadic algebras we have the following theorem.

Theorem 7.27. $C_X^{n,2s+t}$ *satisfies the equation*

$$\beta\beta_{2s}(z_0, z_1, \ldots, z_{2s+1}) =$$

$$\Bigg[\underbrace{\left(\bigwedge_{i=1}^{2s}(Q(\sim z_i) \leftrightarrow Q(z_i))\right)}_{A_1} \wedge \underbrace{\left(\bigwedge_{i=1}^{2s}(z_i \vee (Q(z_0) \wedge Q(\sim z_0)) \leftrightarrow z_i)\right)}_{A_2} \wedge$$

$$\underbrace{\left(\bigwedge_{i=1}^{2s} Q(z_i \wedge z_j) \leftrightarrow (z_i \wedge z_j)\right)}_{A_3} \wedge \underbrace{\left(\bigvee_{j=1}^{s}\bigwedge_{i=1}^{s}(T_{M(z_j)}(z_{2i-1}) \leftrightarrow z_{2i})\right)}_{A_4}\Bigg] \rightarrow$$

$$\Bigg[\underbrace{\bigvee_{i=1}^{s} T_{M(z_i)}\left(\bigvee_{i=1}^{2s+1} z_i\right) \leftrightarrow \left(\bigvee_{i=1}^{2s+1} z_i\right)}_{A_5}\Bigg] = 1.$$

Proof.

1. Suppose that $A_1 \neq 1$. Then, as in the proof of Theorem 7.24, there exists z_i such that $\bigwedge_{i=1}^{4} A_i \leq Q(z_i)$. But $T_{M(z_i)}\left(\bigvee_{i=1}^{2s+1} z_i\right) = \left(\left(\bigvee_{i=1}^{2s+1} z_i\right) \wedge \sim Q(z_i)\right) \vee Q(z_i) \geq Q(z_i)$, that is, $\bigwedge_{i=1}^{4} A_i \leq Q(z_i) \leq A_5$. So $\beta\beta_{2s} = 1$.
2. If $A_2 \neq 1$ or $A_3 \neq 1$ the proof is similar.
3. Suppose that $A_1 = A_2 = A_3 = 1$. Then $z_i \in (a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}})$, and thus $T_{M(z_i)} = T_M$. Suppose that there exists z_{2i-1} such that $T_M(z_{2i-1}) \neq z_{2i}$. Then $T_{M(z_i)}(z_{2i-1}) \leftrightarrow z_{2i}$ is an open element less than $Q(z_{2i})$, and arguing as in the first item, $\bigwedge_{i=1}^{4} A_i \leq Q(z_{2i}) \leq A_5$. Hence $\beta\beta_{2s} = 1$.
4. In the case that $\bigwedge_{i=1}^{4} A_i = 1$, the elements z_i for $i = 1, \ldots, 2s$ are all distinct and non-invariant. Thus, as for monadic algebras, we have that $\bigvee_{i=1}^{s} T_{M(z_i)}\left(\bigvee_{i=1}^{2s+1} z_i\right) = \left(\bigvee_{i=1}^{2s+1} z_i\right)$, and consequently $A_5 = 1$. Then $\beta\beta_{2s} = 1$.

□

In a similar way it can be proved the following.

Theorem 7.28. $C_X^{n,2s+t}$ *satisfies the equation*

$$\beta\beta_{s+t}(z_0, z_1, \ldots, z_{s+t}) =$$

$$\left(\bigwedge_{i=1}^{s+t}(Q(\sim z_i) \leftrightarrow Q(z_i))\right) \wedge \left(\bigwedge_{i=1}^{s+t}(z_i \vee (Q(z_0) \wedge Q(\sim z_0)) \leftrightarrow z_i)\right) \wedge$$

$$\left(\bigwedge_{i\neq j,\ j,i=1}^{s+t} Q(z_i \wedge z_j) \leftrightarrow (z_i \wedge z_j)\right) \wedge \left(\bigvee_{j=1}^{s}\bigwedge_{i=1}^{t}(T_{M(z_j)}(z_i) \leftrightarrow z_i)\right)\Bigg] \rightarrow$$

$$\left[\left(\bigvee_{i=1}^{s+t} z_i\right) \leftrightarrow \left(\bigvee_{i=1}^{s+t} \sim Q(z_i)\right)\right] = 1.$$

As a consequence of the previous theorems we have the following.

Theorem 7.29. *For n odd, the following equations form an equational basis for $V^n_{X,2s+t}$:* $\gamma_K(Q(x_0)) \wedge \gamma_P(Q(x_1)) = 1$, $\beta\beta_{2s} = 1$, $\beta\beta_{s+t} = 1$, $\gamma\gamma_{x_{\frac{n-1}{2}}} = 1$, $\gamma^{A(x_0)}_{\frac{X}{2}^+} = 1$.

Theorem 7.30. *For n even, the following equations form an equational basis for V^n_X :* $\gamma_K(Q(x_0)) = 1$, $\gamma^{A(x_0)}_{\frac{X}{2}^+} = 1$.

$$\gamma\gamma_X^{\sim Q(x_0)}(x_1,\dots,x_r) = [(x_0 \leftrightarrow x_0 \vee (Q(y)\wedge Q(\sim y))) \wedge (Q(x_0) \leftrightarrow Q(\sim x_0))]$$
$$\to \left[\left(\gamma^{\sim Q(x_0)}_{\frac{X}{2}^+} \leftrightarrow 1\right) \vee \sim Q(x_0)\right] = 1.$$

Proof. We know that C^n_X satisfies the two first equations. Let $x_0 \in C^n_X$ and suppose that $x_0 = x_0 \vee (Q(y)\wedge Q(\sim y))$. Then $x_0 \geq a_{\frac{n}{2}}$ and $Q(\sim x_0) \leq \sim x_0 \leq a_{\frac{n}{2}} \leq Q(x_0)$. Then if $Q(\sim x_0) = Q(x_0)$ we have that $\sim x_0 = x_0 = Q(\sim x_0) = Q(x_0) = a_{\frac{n}{2}}$. So the third equation holds. If A is subdirectly irreducible and satisfies these equations then $Q(A)$ is a chain, $A \cong C^m_Y$ and $\frac{Y}{2}^+ \leq_{\mathbf{N}} \frac{X}{2}^+$. If m is even, $C^m_Y \in \mathbf{S}(C^n_X)$. If m is odd, then if the third equation holds, it holds for $x_0 = b_{\frac{m-1}{2}}$. So $\gamma^{\sim Q(x_0)}_{\frac{X}{2}^+} = 1$, which implies that $\frac{Y}{2}^+ + 1 \leq_{\mathbf{N}} \frac{X}{2}^+$, and from the previous section, $C^m_Y \in \mathbf{S}(C^n_X)$. □

Observe that all these varieties are within $\mathscr{SC}^n_L$ and consequently they satisfy que equations characterizing this variety.

Theorem 7.31. *The variety U^n_Y is characterized by the following equation*

$$\gamma_{D^n_Y} = \left[\underbrace{(\gamma_Y \leftrightarrow 1) \vee (T(\gamma_Y) \leftrightarrow 1)}_{A_1}\right] \wedge \left[\underbrace{(\gamma_{Y^*} \leftrightarrow 1) \vee (T(\gamma_{Y^*}) \leftrightarrow 1)}_{A_2}\right] = 1.$$

Now we give an equational basis for each infinitely generated join irreducible subvariety. As for the finitely generated varieties, we use the equations of the infinitely generated varieties in $\mathscr{C}^n_L$ (ver [9]).

Observe that in the odd case, we can consider three different types of varieties $P^{\overline{X}}_s$, according the value of $x_{\frac{s+1}{2}}$.

1. $P^{\overline{X},h}_s$, where $x_{\frac{s+1}{2}} = *_h$, that is, if $C_X \in P^{\overline{X},h}_s$, then $[a_{\frac{s-1}{2}}, a_{\frac{s+1}{2}}] \in S_h$, where S_h is the variety in troduced in [9].
2. $P^{\overline{X},2h+t}_s$, where $x_{\frac{s+1}{2}} = 2h+t$, that is, if $C_X \in P^{\overline{X},2h+t}_s$, then $[a_{\frac{s-1}{2}}, a_{\frac{s+1}{2}}] \in \mathbf{S}(B_{2s+t})$.
3. $P^{\overline{X},*}_s$, where $x_{\frac{s+1}{2}} = *$, that is, if $C_X \in P^{\overline{X},h}_s$, then $[a_{\frac{s-1}{2}}, a_{\frac{s+1}{2}}] \in S$. S is the variety introduced in [9].

In the following theorem, we give an equational characterization of these varieties.

Theorem 7.32. (1) *An equational base for the variety P_s is :*
$\gamma_K(Q(x_0)) \wedge \gamma_P(Q(x_1)) = 1$, $\gamma_{I_s}(Q(x_1),\dots,Q(x_s)) = 1$.
(2) *An equational base for the variety $P^{\overline{X},*}_s$ is:*
$\gamma_K(Q(x_0)) \wedge \gamma_P(Q(x_1)) = 1$, $\gamma_{I_s}(Q(x_1),\dots,Q(x_s)) = 1$, $\gamma^{A(x_0)}_{\frac{X}{2}^+} = 1$.

(3) *An equational base for the variety* $P_s^{\overline{X},h}$ *is:*
$\gamma_K(Q(x_0)) \wedge \gamma_P(Q(x_1)) = 1$, $\gamma_{I_s}(Q(x_1),\ldots,Q(x_s)) = 1$. $\gamma_{\frac{\overline{X}^+}{2}}^{A(x_0)} = 1$. $\beta\beta_{2h} = 1$.

(4) *An equational base for the variety* $P_s^{\overline{X},2h+t}$ *is:* $\gamma_K(Q(x_0)) \wedge \gamma_P(Q(x_1)) = 1$, $\gamma_{I_s}(Q(x_1),$ $\ldots,Q(x_s)) = 1$, $\gamma_{\frac{\overline{X}^+}{2}}^{A(x_0)} = 1$ $\beta\beta_{2h} = 1$, $\beta\beta_{h+t} = 1$, $\gamma\gamma_{x_{\frac{n+1}{2}}} = 1$,

(5) *An equational base for the variety* K_s *is:*
$\gamma_K(Q(x_0)) = 1$, $\gamma_{I_s}(Q(x_1),\ldots,Q(x_n)) = 1$.

(6) *An equational base for the variety* $K_s^{\overline{X}}$ *is:*
$\gamma_K(Q(x_0))(Q(x_1)) = 1$, $\gamma_{I_s}(Q(x_1),\ldots,Q(x_s)) = 1$, $\gamma_{\frac{\overline{X}^+}{2}}^{A(x_0)} = 1$, $\gamma\gamma_{\overline{X}}^{\sim Q(x_0)} = 1$.

(7) *An equational base for the variety* $I_s^{\overline{X}}$ *is:*
$\gamma_{I_s}(Q(x_1),\ldots,Q(x_s)) = 1$,

$$[(\gamma_{\overline{X}} \leftrightarrow 1) \vee (T(\gamma_{\overline{X}}) \leftrightarrow 1)] \wedge [(\gamma_{\overline{X^*}} \leftrightarrow 1) \vee (T(\gamma_{\overline{X^*}}) \leftrightarrow 1)] = 1.$$

(8) *An equational base for the variety* I_s *is:*
$\gamma_{I_s}(Q(x_1),\ldots,Q(x_s)) = 1$.

As in the case of symmetric Heyting algebras the following theorem can be proved for join reducible varieties.

Theorem 7.33. *Let* $V \in \Lambda(\mathscr{SC}_L)$, $V = \bigvee_{i=1}^{n} V_i$ *with* $V_i \in J(\Lambda(\mathscr{SC}_L))$. *Then*

$$\gamma_V(x_1,\ldots,x_{r_V}) = \bigvee_{i=1}^{n} \left(\gamma_{V_i}(x_1^i,\ldots,x_{r_i}^i) \leftrightarrow 1\right) \wedge (\sim \gamma_{V_i}(x_1,\ldots,x_{r_i}) \leftrightarrow 0) = 1$$

is an equational basis for V *en* $\mathscr{SC}_L$.

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