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Principal and Boolean Congruences on *IKt*-Algebras

Abstract. The *IKt*-algebras were introduced in the paper *An algebraic axiomatization of the Ewald's intuitionistic tense logic* by the first and third author. In this paper, our main interest is to investigate the principal and Boolean congruences on *IKt*-algebras. In order to do this we take into account a topological duality for these algebras obtained in Figallo et al. (Stud Log 105(4):673–701, 2017). Furthermore, we characterize Boolean and principal *IKt*-congruences and we show that Boolean *IKt*-congruence are principal *IKt*-congruences. Also, bearing in mind the above results, we obtain that Boolean *IKt*-congruences are commutative, regular and uniform. Finally, we characterize the principal *IKt*-congruences in the case that the *IKt*-algebra is linear and complete whose prime filters are complete and also the case that it is linear and finite. This allowed us to establish that the intersection of two principal *IKt*-congruences on these algebras is a principal one and also to determine necessary and sufficient conditions so that a principal *IKt*-congruence is a Boolean one on theses algebras.

Keywords: Heyting algebras, Tense operators, IKt-algebras, Boolean congruences, Principal congruences.

1. Introduction and Preliminaries

This paper is organized as follows: in Section 1, we summarize the principal notions and results on *IKt*-algebras ([17,27]), in particular the topological duality for these algebras obtained in [28]. In Section 2, our attention is focused on Boolean *IKt*-congruences. Firstly, we characterize them by means of certain closed and open subsets of the associated space. These results allow us to prove that the Boolean *IKt*-congruences are commutative and they are the congruences associated with filters generated by a Boolean and *IKt*-element of the algebra, and so they are regular and uniform. In Section 3, we characterize the closed and increasing *IKt*-subsets of the dual

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space associated with an *IKt*-algebra which determine the principal *IKt*-congruences. This last result enables us to prove that every Boolean *IKt*-congruence is a principal *IKt*-congruence. Furthermore, whenever the *IKt*-algebra is linear and complete and its prime filters are complete or linear and finite, we obtain the filters which determine principal congruences. This last results allows to show that the intersection of two principal *IKt*-congruences on these algebras is a principal one and also to determine necessary and sufficient conditions so that a principal *IKt*-congruence is a Boolean one on theses algebras.

In this paper, we take for granted the concepts and results on distributive lattices, Heyting algebras, category theory, universal algebra and Priestley duality. To obtain more information about these topics, we direct the reader to the bibliography indicated in [1,6,7,31,34–36]. However, in order to simplify reading, in this section we will summarize the fundamental concepts we use.

If X is a poset (i.e. partially ordered set) and $Y \subseteq X$, then we will denote by $\downarrow Y$ ($\uparrow Y$) the set of all $x \in X$ such that $x \leq y$ ($y \leq x$) for some $y \in Y$. If $x \in X$ we shall denote by $\downarrow x$ ($\uparrow x$) instead of $\downarrow \{x\}$ ($\uparrow \{x\}$).

Let X, Y be sets. Given a relation $R \subseteq X \times Y$, for each $Z \subseteq X, R(Z)$ will denote the image of Z by R. If $Z = \{x\}$, we will write $R(x)$ instead of $R(\{x\})$. Moreover, for each $V \subseteq Y$, $R^{-1}(V)$ will denote the inverse image of V by R, i.e. $R^{-1}(V) = \{x \in X : R(x) \cap V \neq \emptyset\}$. If $V = \{y\}$, we will write $R^{-1}(y)$ instead of $R^{-1}(\{y\})$.

On the other hand, it is known that propositional logics, both classical or non-classical ones, do not incorporate the dimension of time. To obtain a tense logic, we enrich the given propositional logic by new unary operators (called tense operators) which are usually denoted by G, H, F and P. Study of tense operators has originated in 1980's [5,8]. Recall that for a classical propositional calculus represented by means of a Boolean algebra $\mathcal{B} = \langle B, \vee, \wedge, \neg, 0, 1\rangle$, tense operators were axiomatized [16] by the following axioms:

$$G(1) = 1,\ H(1) = 1,$$
$$G(x \wedge y) = G(x) \wedge G(y),\ H(x \wedge y) = H(x) \wedge H(y),$$
$$x \leq GP(x),\ x \leq HF(x),$$

where $P(x) = \neg H(\neg x)$ and $F(x) = \neg G(\neg x)$.

In recent years tense operators have been considered by different authors for various classes of algebras. Some contributions in this area have been the papers by Diaconescu and Georgescu [16], Botur et al [3], Chiriţă [14,15], Chajda [9], Chajda and Kolařik [13], Figallo and Pelaitay [20,24,26,27], Chajda and Paseka [12], Botur and Paseka [4,33], Menni and Smith [32], Dzik et al [17]. In particular, intuitionistic tense logic IKt was introduced by Ewald [19] by extending the language of intuitionistic propositional logic with the unary operators P (it was the case), F (it will be the case), H (it has always been the case) and G (it will always be the case). The Hilbert-style axiomatization of IKt can be found in [19][p. 171]. It is well-known that the axiomatization of Ewald is not minimal because several axioms can be deduced from the other axioms. Besides, in contrast to classical tense logic, F and P cannot be defined in terms of G and H (see [17,24]). In [27], Figallo and Pelaitay introduced the variety **IKt** of *IKt*-algebras and proved that the IKt system has *IKt*-algebras as algebraic counterpart. These algebras were defined as we will describe below.

Let us recall that an algebra $\mathcal{A} = \langle A, \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra if the following conditions hold for all x, $y \in A$:

(H1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a lattice with 0, 1,

(H2) $x \wedge (x \to y) = x \wedge y$,

(H3) $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$,

(H4) $(x \wedge y) \to x = 1$.

A Heyting algebra $\mathcal{A} = \langle A, \vee, \wedge, \to, 0, 1\rangle$ is linear or an L-Heyting algebra if satisfies the followig property for all x, $y \in A$:

(L1) $(x \to y) \vee (y \to x) = 1$.

In a Heyting algebra A, the following properties hold for all $a, b, c \in A$:

(H5) If $a \leq b$ then $b \to c \leq a \to c$,

(H6) $a \leq a^{**}$, where where $a^* := a \to 0$,

(H7) $a = a^{**}$ (i.e. a is a regular element of A), iff $a = b^*$ for some $b \in A$,

(H8) $\mathcal{B}(A) \subseteq \mathcal{R}(A)$, where $\mathcal{B}(A) = \{a \in A : a$ is a boolean element of $A\}$, $\mathcal{R}(A) = \{a \in A : a^{**} = a\}$.

If A is an L-Heyting algebra, then:

(L2) $\mathcal{B}(A) = \mathcal{R}(A)$,

(L3) $a^{**} \in \mathcal{B}(A)$ for all $a \in A$.

Definition 1.1. Let $\mathcal{A} = \langle A, \vee, \wedge, \to, 0, 1\rangle$ be a Heyting algebra and let G, H, F and P be unary operations on A satisfying:

(t1) $G(1) = 1$ and $H(1) = 1$,

(t2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,

(t3) $x \leq GP(x)$ and $x \leq HF(x)$,

(t4) $F(0) = 0$ and $P(0) = 0$,

(t5) $F(x \vee y) = F(x) \vee F(y)$ and $P(x \vee y) = P(x) \vee P(y)$,

(t6) $PG(x) \leq x$ and $FH(x) \leq x$,

(t7) $F(x \to y) \leq G(x) \to F(y)$ and $P(x \to y) \leq H(x) \to P(y)$.

Then the algebra $(\mathcal{A}, G, H, F, P)$ will be called an *IKt*-algebra and G, H, F and P will be called tense operators.

Remark 1.1. Note that (t1), (t2), (t4) and (t5) can be replaced by the assumption that G, H, F and P are order-preserving (see [8]).

Definition 1.2. An *IKt*-algebra $(\mathcal{A}, G, H, F, P)$ will be called a linear *IKt*-algebra if $\mathcal{A}$ is a linear Heyting algebra.

Independently, in [17] (see also [32]), two algebraic models of the IKt system were obtained in terms of Heyting algebras expanded with two Galois connections verifying the Dunn's axioms, in one case, and the Fisher–Servi's axioms, in another case. It is not difficult to check that all three algebraic models of the IKt system are equivalent. In [28], we developed a topological duality for *IKt*-algebras taking into account the results established by Dzik, Järvinen and Kondo in [17].

In order to determine this duality, we introduced a topological category whose objects and their corresponding morphisms will be described below.

Definition 1.3. An *IKt*-space is a system $(X, \leq, R)$, where

(i) $(X, \leq)$ is a Heyting space, specifically, $(X, \leq)$ is a Priestley space such that the downset $\downarrow U$ is clopen (closed and open) for every clopen U in X. Alternatively, $(X, \leq)$ is a Priestley space such that the downset $\downarrow U$ is open for every open set U.

(ii) R is a binary relation on X and R^{-1} is the converse of R such that the following conditions are satisfied:

(tS1) for each $x \in X$, $R(x)$ and $R^{-1}(x)$ are closed subsets of X,

(tS2) for each $x \in X$, $R(x) = \downarrow R(x) \cap R(\uparrow x)$,

(tS3) for each $x \in X$, $R(\uparrow x)$ and $R^{-1}(\uparrow x)$ are increasing and closed subsets of X,

(tS4) $G_R(U), H_{R^{-1}}(U), F_R(U), P_{R^{-1}}(U) \in D(X)$ for each $U \in D(X)$, where $G_R, H_{R^{-1}}, F_R$ and $P_{R^{-1}}$ are operators on $\mathcal{P}(X)$ defined

for all $Y \subseteq X$ as follows:
$G_R(U) = \{x \in X \mid\ R(\uparrow x) \subseteq U\}$,
$H_{R^{-1}}(U) = \{x \in X \mid\ R^{-1}(\uparrow x) \subseteq U\}$,
$F_R(U) = \{x \in X \mid\ R(x) \cap U \neq \emptyset\}$ and
$P_{R^{-1}}(U) = \{x \in X \mid\ R^{-1}(x) \cap U \neq \emptyset\}$.

and $D(X)$ is the set of all increasing and clopen subsets of X.

DEFINITION 1.4. An *IKt*-function from an *IKt*-space $(X_1, \leq_1, R_1)$ into another one, $(X_2, \leq_2, R_2)$, is a function $f : X_1 \longrightarrow X_2$, which satisfies the following conditions:

(pf) for all $x \in X_1$, $z \in X_2$ such that $f(x) \leq_2 z$, there is $y \in X_1$ such that $x \leq_1 y$ and $f(y) = z$ (i.e. f is a p-function),

(tf1) $f(R_1(x)) \subseteq R_2(f(x))$,

(tf2) $R_2(\uparrow f(x)) = f(R_1(\uparrow x))$,

(tf3) ${R_2}^{-1}(\uparrow f(x)) = f({R_1}^{-1}(\uparrow x))$,
for all $x \in X$.

Then, we proved that the category ***IKtS*** that has *IKt*-spaces as objects and *IKt*-functions as morphisms is naturally equivalent to the dual category of the category ***IKtA*** of *IKt*-algebras and *IKt*-homomorphisms.

Next we will describe some results of the above duality with the aim of fixing the notation we are about to use in this paper.

First, we defined a contravariant functor from ***IKtS*** onto ***IKtA*** in the following way:

(A1) Let $(X, \leq, R)$ be an *IKt*-space. Then,

$$\Psi(X) = (D(X), \rightarrow, G_R, H_{R^{-1}}, F_R, P_{R^{-1}})$$

is an *IKt*-algebra, where $D(X)$ is the lattice of all clopen and increasing subsets of X, $G_R, H_{R^{-1}}, F_R, P_{R^{-1}}$ are defined as in (tS4) and

$$U \rightarrow V = \{x \in X : \uparrow x \cap U \subseteq V\} \text{ for all } U,\, V \in D(X),$$

or equivalently

$$U \rightarrow V = X \backslash (\downarrow U \cap (X \backslash V)) = V \cup (X \backslash \downarrow U) \text{ for all } U,\, V \in D(X).$$

(A2) Let $f : (X_1, \leq_1, R_1) \longrightarrow (X_2, \leq_2, R_2)$ be a morphism of *IKt*-spaces. Then, $\Psi(f) : D(X_2) \longrightarrow D(X_1)$, defined by $\Psi(f)(U) = f^{-1}(U)$ for all $U \in D(X_2)$, is an *IKt*-homomorphism.

To achieve our goal we needed to define a contravariant functor from ***IKtA*** to ***IKtS***, as indicated below:

(A3) Let $(\mathcal{A}, G, H, F, P)$ be an *IKt*-algebra and let $(\mathfrak{X}(A), \subseteq)$ be the Heyting space associated with $\mathcal{A}$. Then, $\Phi(A) = (\mathfrak{X}(A), \subseteq, R^A)$ is an *IKt*-space, where for all $S, T \in \mathfrak{X}(A)$,

$$(S, T) \in R^A \iff G^{-1}(S) \subseteq T \subseteq F^{-1}(S), \tag{I}$$

or equivalently,

$$(S, T) \in R^A \iff H^{-1}(T) \subseteq S \subseteq P^{-1}(T). \tag{II}$$

(A4) Let $(\mathcal{A}_1, G_1, H_1, F_1, P_1)$ and $(\mathcal{A}_2, G_2, H_2, F_2, P_2)$ be two *IKt*-algebras and $h : A_1 \longrightarrow A_2$ be an *IKt*-homomorphism. Then, the map $\Phi(h) : X(A_2) \longrightarrow X(A_1)$, defined by $\Phi(h)(S) = h^{-1}(S)$ for all $S \in X(A_2)$, is an *IKt*-function.

Besides, we proved the following necessary results for the duality:

(A5) Let $(\mathcal{A}, G, H, F, P)$ be an *IKt*-algebra and let $(\mathfrak{X}(A), \subseteq, R^A)$ be the *IKt*-space associated with $\mathcal{A}$. Then, $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$ defined by

$$\sigma_A(a) = \{S \in \mathfrak{X}(A) : a \in S\}, \tag{III}$$

is an *IKt*-isomorphism.

(A6) Let $(X, \leq, R)$ be an *IKt*-space. Then, $\varepsilon_X : X \longrightarrow X(D(X))$, defined by

$$\varepsilon_X(x) = \{U \in D(X) : x \in U\}, \tag{IV}$$

is an isomorphism in the category ***IKtS***.

Then, from the above results and using the usual procedures we proved that the categories ***IKtS*** and ***IKtA*** are dually equivalent.

In addition, this duality allowed us to to characterize the *IKt*-congruences on these algebras for which we introduced the following notion:

Definition 1.5. Let $(X, \leq, R)$ be an *IKt*-space. A subset Y of X is an *IKt*-subset if it satisfies the following conditions for all $y, z \in X$:

(IKt1) if $y \in Y$ and $z \in R(\uparrow y)$, then, there is $w \in Y$ such that $w \in R(\uparrow y)$ and $w \leq z$,

(IKt2) if $y \in Y$ and $z \in R^{-1}(\uparrow y)$, then, there is $v \in Y$ such that $v \in R^{-1}(\uparrow y)$ and $v \leq z$.

The notion of an increasing *IKt*-subset of an *IKt*-space has several equivalent formulations, which were useful in the characterization of simple and subdirectly irreducible algebras and also they will be used later.

PROPOSITION 1.1. *Let $(X, \leq, R)$ be an IKt-space. If Y is an increasing subset of X, then, the following conditions are equivalent:*

(i) *Y is an IKt-subset,*

(ii) *$Y = G_R(Y) \cap Y \cap H_{R^{-1}}(Y)$, where $G_R(Y) = \{x \in X : R(\uparrow x) \subseteq Y\}$, and $H_{R^{-1}}(U) = \{x \in X : R^{-1}(\uparrow x) \subseteq Y\}$,*

(iii) *for all $y \in Y$, the following conditions are satisfied:*

(IKt3) *$R(\uparrow y) \subseteq Y$,*

(IKt4) *$R^{-1}(\uparrow y) \subseteq Y$.*

The closed and increasing *IKt*-subsets of the *IKt*-space associated with an *IKt*-algebra perform a fundamental roll in the characterization of the *IKt*-congruences on these algebras as we will indicate in what follows.

THEOREM 1.6. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra, and $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. Then, the lattice $\mathcal{C}_{IT}(\mathfrak{X}(A))$ of closed and increasing IKt-subsets of $\mathfrak{X}(A)$ is isomorphic to the dual lattice $Con_{IKt}(\mathcal{A})$ of IKt-congruences on $\mathcal{A}$, and the isomorphism is the function Θ_{IT} defined by the prescription,*

$$\Theta_{IT}(Y) = \{(a, b) \in A \times A : \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\}, \tag{V}$$

for all $Y \in \mathcal{C}_{IT}(\mathfrak{X}(A))$.

The characterization of increasing *IKt*-subsets given in Proposition 1.1 suggested us to introduce the following definitions:

Let $(X, \leq, R)$ be an *IKt*-space and let $d_X : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, defined for all $Z \in \mathcal{P}(X)$, by:

$$d_X(Z) = G_R(Z) \cap Z \cap H_{R^{-1}}(Z), \tag{VI}$$

For each $n \in \omega$, let $d_X^n : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, defined for all $Z \in \mathcal{P}(X)$, by:

$$d_X^0(Z) = Z, \; d_X^{n+1}(Z) = d_X(d_X^n(Z)). \tag{VII}$$

By using the above functions d_X, d_X^n, $n \in \omega$, we obtained another equivalent formulation of the notion of increasing *IKt*-subset of an *IKt*-space as we will indicate below:

LEMMA 1.1. *Let $(X, \leq, R)$ be an IKt-space. If Y is an increasing IKt-subset of X, then the following conditions are equivalent:*

(i) *Y is an IKt-subset,*

(ii) *$Y = d_X^n(Y)$ for all $n \in \omega$,*

(iii) *$Y = \bigcap_{n\in\omega} d_X^n(Y)$.*

Then, we considered the restrictions to $D(X)$ of the maps d_X and d_X^n, $n \in \omega$, defined in VI and VII respectively, and as a consequence of the duality for *IKt*-algebras we defined the functions $d : A \longrightarrow A$ and $d^n : A \longrightarrow A$, $n \in \omega$ by the prescriptions:

$$d(a) = G(a) \wedge a \wedge H(a),\ d^0(a) = a, \tag{VIII}$$

$$d^{n+1}(a) = d(d^n(a)), \tag{IX}$$

for all $a \in A$, respectively. It should be noted that these operators were previously defined in [30] for tense Boolean algebras and in [16] for tense MV-algebras. Then we determined the following properties of the *IKt*-algebras:

(A6) Let $(\mathcal{A}, G, H, F, P)$ be an *IKt*-algebra. Then, for all $a \in A$, the following conditions are equivalent:

(i) $a = d(a)$,

(ii) $a = d^n(a)$ for all $n \in \omega$,

(iii) $a = \bigwedge_{n\in\omega} d^n(a)$,

besides if $(\mathcal{A}, G, H, F, P)$ is a complete *IKt*-algebra,

(iv) $a = \bigwedge_{n\in\omega} d^n(b)$ for some $b \in A$.

(A7) ([24, 28]) Let $(\mathcal{A}, G, H, F, P)$ be an *IKt*-algebra. Then, for all $n \in \omega$ and for all $a, b \in A$, the following conditions are satisfied:

(d1) $d^n(1) = 1$ and $d^n(0) = 0$,

(d2) $d^{n+1}(a) \leq d^n(a)$,

(d3) $d^n(a \wedge b) = d^n(a) \wedge d^n(b)$,

(d4) $a \leq b$ implies $d^n(a) \leq d^n(b)$,

(d5) $d^n(a) \leq a$,

(d6) $d^{n+1}(a) \leq G(d^n(a))$ and $d^{n+1}(a) \leq H(d^n(a))$.

(d7) If $\bigwedge_{i\in I} a_i$ exists then:

(i) $\bigwedge_{i\in I} G(a_i)$ exists and $\bigwedge_{i\in I} G(a_i) = G(\bigwedge_{i\in I} a_i)$,

(ii) $\bigwedge_{i\in I} H(a_i)$ exists and $\bigwedge_{i\in I} H(a_i) = H(\bigwedge_{i\in I} a_i)$,

(iii) $\bigwedge_{i\in I} d(a_i)$ exists and $\bigwedge_{i\in I} d(a_i) = d(\bigwedge_{i\in I} a_i)$.

In what follows if $(\mathcal{A}, G, H, F, P)$ is an *IKt*-algebra, we will denoted by $C(\mathcal{A}) = \{a \in A : d(a) = a\} = \left\{a \in A : a = \bigwedge_{n\in\omega} d^n(a)\right\}$, and the elements of $C(\mathcal{A})$ will be called *IKt*-elements of $\mathcal{A}$. It should be mentioned that $\langle C(\mathcal{A}), \wedge, \vee, \rightarrow, 0, 1\rangle$ is a Heyting algebra (see [28, Lemma 3.5]).

In the *IKt*-algebras, the following particular case of filters is defined:

Definition 1.7. Let $(\mathcal{A}, G, H, F, P)$ be an *IKt*-algebra. A filter S of $\mathcal{A}$ is an *IKt*-filter iff for all $a \in S$, $d(a) \in S$, or equivalently $d^n(a) \in S$ for all $n \in \omega$.

The duality of *IKt*-algebras provide an isomorphism between the lattice $\mathcal{F}_{IKt}(\mathcal{A})$ of all *IKt*-filters of an *IKt*-algebra $\mathcal{A}$ and the dual lattice of the lattice $\mathcal{C}_{IT}(\mathfrak{X}(A))$ of all closed and increasing *IKt*-subsets of $\mathfrak{X}(A)$. Under this isomorphism, any $S \in \mathcal{F}_{IKt}(\mathcal{A})$ corresponds to the increasing and closed *IKt*-subset

$$Y_S = \{Q \in \mathfrak{X}(A) : S \subseteq Q\} = \bigcap_{a\in S} \sigma_A(a), \tag{X}$$

and any $Y \in \mathcal{C}_{IT}(\mathfrak{X}(A))$ corresponds to the filter

$$S_Y = \bigcap_{Q\in Y} Q = \{a \in A : Y \subseteq \sigma_A(a)\}, \tag{XI}$$

and $\Theta(S) = \Theta_{IT}(Y_S)$, $\Theta_{IT}(Y) = \Theta(S_Y)$, where $\Theta_{IT}(Y)$ and $\Theta_{IT}(Y_S)$ are defined as in Theorem 1.6 and $\Theta(S)$ and $\Theta(S_Y)$ are the congruence associated with the *IKt*-filters S and S_Y, respectively.

A direct consequence of these last results and Theorem 1.6 is the fact that there exists a lattice isomorphism between the lattice of all *IKt*-filters of an *IKt*-algebra $\mathcal{A}$ and the lattice of all *IKt*-congruences on $\mathcal{A}$. Under that isomorphism, any *IKt*-congruence θ on $\mathcal{A}$ corresponds to the *IKt*-filter

$$S_\theta = \{a \in A : (a, 1) \in \theta\} \tag{XII}$$

and any filter S of $\mathcal{A}$ corresponds to the *IKt*-congruence

$$\begin{aligned}\theta_S &= \{(a, b) \in A \times A : (a \rightarrow b) \wedge (b \rightarrow a) \in S\}\\ &= \{(a, b) \in A \times A : \text{ there is } s \in S \text{ such that } a \wedge s = b \wedge s\}.\end{aligned} \tag{XIII}$$

And so, we obtained another characterization of the lattice of the *IKt*-congruences.

Proposition 1.2. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then, the following conditions are equivalent for all $\varphi \subseteq A \times A$:*

(i) $\varphi \in Con_{IKt}(\mathcal{A})$,

(ii) *there is* $S \in \mathcal{F}_{IKt}(\mathcal{A})$ *such that* $\varphi = \Theta(S)$, *where* $\Theta(S)$ *is the lattice congruence associated with the filter* S.

Finally, we will emphasize the following property of Priestley spaces and so, *IKt*-spaces which will be quite useful in order to characterize the principal congruences on *IKt*-algebras.

(A8) C is a convex clopen of X if and only if there are $U, V \in D(X)$ such that $V \subseteq U$ and $C = U \backslash V$, where a subset Y of X is convex if $Y = \downarrow Y \cap \uparrow Y$, or equivalently if $x, y \in Y$ and $x \leq z \leq y$ imply $z \in Y$.

2. Boolean Congruences on *IKt*-Algebras

Next, our attention is focus on determine the Boolean congruences on *IKt*-algebras bearing in mind the topological duality for them established in Section 1. In order to do this, we will start obtaining a new characterization of the congruences on *IKt*-algebras, which will be fundamental to reach our goal.

THEOREM 2.1. *Let* $(\mathcal{A}, G, H, F, P)$ *be an IKt-algebra,* $\mathcal{O}_{CIT}(\mathfrak{X}(A))$ *be the lattice of the open subsets of* $\mathfrak{X}(A)$ *whose complements are increasing IKt-subsets of* $\mathfrak{X}(A)$ *and* $Con_{IKt}(\mathcal{A})$ *be lattice of the IKt-congruences on* $\mathcal{A}$. *Then* $\mathcal{O}_{CIT}(\mathfrak{X}(A))$ *and* $Con_{IKt}(\mathcal{A})$ *are isomorphic and the isomorphism*

$$\Theta_{OT} : \mathcal{O}_{CIT}(\mathfrak{X}(A)) \longrightarrow Con_{IKt}(\mathcal{A}),$$

is defined by:

$$\begin{aligned}\Theta_{OT}(Y) &= \left\{(a,b) \in A^2 : \sigma_A(a) \cap (\mathfrak{X}(A)\backslash Y) = \sigma_A(b) \cap (\mathfrak{X}(A)\backslash Y)\right\} \\ &= \left\{(a,b) \in A^2 : (\sigma_A(b) \triangle \sigma_A(a)) \subseteq Y\right\}.\end{aligned}$$

PROOF. It is a direct consequence of Theorem 1.6, bearing in mind that there is a one-to-one correspondence between the closed and open subsets of $\mathfrak{X}(A)$ and the fact that $\sigma_A(a) \cap Y = \sigma_A(b) \cap Y$ if and only if $(\sigma_A(b) \triangle \sigma_A(a)) \subseteq \mathfrak{X}(A)\backslash Y$. ■

The lattice of the increasing, decreasing, closed and open *IKt*-subsets of the *IKt*-space associated with an *IKt*-algebra perform a fundamental roll in the characterization of the Boolean *IKt*-congruences on these algebras, as we will show next.

LEMMA 2.1. *Let* $(X, \leq, R)$ *be an IKt-space. If* Y *is an increasing and decreasing IKt-subset of* X*, then* $X \backslash Y$ *is an increasing and decreasing IKt-subset of* X*.*

PROOF. From the hypothesis we have that $X \backslash Y$ is an increasing and decreasing subset of X. Suppose that there is (1) $x \in X \backslash Y$ such that $x \notin G_R(X \backslash Y)$. Then $R(\uparrow x) \cap Y \neq \emptyset$, from which it follows that there is (2) $y \in Y$ such that $y \in R(\uparrow x)$, which allows us to assert that there is (3) $z \in \uparrow x$, such that $(z, y) \in R$. Since $y \in \uparrow y$, then from the last assertion we obtain that $z \in R^{-1}(\uparrow y)$. From the hypothesis, Y is an increasing *IKt*-subset of X, then from (2) and Proposition 1.1, we infer that (4) $z \in Y$. On the other hand, from (1), (3) and the fact that $X \backslash Y$ is an increasing subset of X it results that $z \in X \backslash Y$, which contradicts (4). Therefore $X \backslash Y \subseteq G_R(X \backslash Y)$. The inclusion $X \backslash Y \subseteq H_{R^{-1}}(X \backslash Y)$ follows similarly and so, $X \backslash Y = (X \backslash Y) \cap G_R(X \backslash Y) \cap H_{R^{-1}}(X \backslash Y)$. Taking into account this last assertion, the fact that $X \backslash Y$ is an increasing subset of X and Proposition 1.1, we conclude that $X \backslash Y$ is an *IKt*-subset of X. ■

COROLLARY 2.2. *Let* $(X, \leq, R)$ *be an IKt-space,* $B(D(X))$ *be the Boolean algebra of the complemented elements of the lattice* $D(X)$ *and let* $C(X) = \{U \in D(X) : U = d_X(U)\}$*. Then, for all* $U \in B(D(X))) \cap C(D(X))$*,* $X \backslash U \in B(D(X)) \cap C(D(X))$ *and* $d_X(X \backslash U) = X \backslash d_X(U)$*.*

PROOF. It follows immediately that $X \backslash U \in B(D(X))$ for all $U \in B(D(X)) \cap C(D(X))$. Besides, from Lemma 1.1 we have that U is an increasing and decreasing *IKt*-subset of X, from which we infer by Lemma 2.1 that $X \backslash U$ is an *IKt*-subset of X and so $X \backslash U \in B(D(X)) \cap C(D(X))$. Then, taking into account that U, $X \backslash U \in C(D(X))$, we obtain that $d_X(X \backslash U) = X \backslash U = X \backslash d_X(U)$. ■

COROLLARY 2.3. *Let* $(\mathcal{A}, G, H, F, P)$ *be an IKt-algebra,* $B(\mathcal{A})$ *be the Boolean algebra of the complemented elements of* $\mathcal{A}$ *and* $C(\mathcal{A}) = \{a \in A : a = d(a)\}$*. Then for all* $a \in B(\mathcal{A}) \cap C(\mathcal{A})$*,* $-a \in B(\mathcal{A}) \cap C(\mathcal{A})$ *and* $d(-a) = -d(a)$*, where* $-x$ *is the Boolean complement of* x *for all* $x \in B(\mathcal{A})$*.*

PROOF. : It is a direct consequence of Corollary 2.2 and the fact that the map $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$ is an *IKt*-isomorphism. ■

In what follows we will denote by $Con_{bIKt}(\mathcal{A})$ the lattice of all Boolean *IKt*-congruences on $\mathcal{A}$, and we will denote by $\mathcal{CO}_{IDT}(\mathfrak{X}(A))$ the lattice of all increasing, decreasing and clopen *IKt*-subset of $\mathfrak{X}(A)$.

PROPOSITION 2.1. *Let* $(\mathcal{A}, G, H, F, P)$ *be an IKt-algebra and let* $\mathfrak{X}(A)$ *be the IKt-space associated with* $\mathcal{A}$*. Then for each* $Y \subseteq \mathfrak{X}(A)$ *holds:*

(i) $\Theta_{IT}(Y)$ *is a Boolean IKt-congruence on* $\mathcal{A}$ *if and only if* Y *is an increasing, decreasing and clopen IKt-subset of* $\mathfrak{X}(A)$*, where* $\Theta_{IT}(Y)$ *is defined as in* Theorem 1.6.

(ii) $\Theta_{OT}(Y)$ *is a Boolean IKt-congruence on* $\mathcal{A}$ *if and only if* Y *is an increasing, decreasing and clopen IKt-subset of* $\mathfrak{X}(A)$*, where* $\Theta_{OT}(Y)$ *is defined as in* Theorem 2.1.

Proof. The statements (i) and (ii) are direct consequences of Theorems 1.6 and 2.1 and Lemma 2.1, respectively. Indeed, let $\mathcal{CO}_{IDT}(\mathfrak{X}(A))$ be the lattice of all increasing, decreasing and clopen *IKt*-subset of $\mathfrak{X}(A)$.
(i): Let $\Theta_{IT}(Y) \in Con_{bIKt}(\mathcal{A})$, then there is $\varphi \in Con_{bIKt}(\mathcal{A})$ such that $\Theta_{IT}(Y) \wedge \varphi = \{(a,a) : a \in A\}$ and $\Theta_{IT}(Y) \vee \varphi = A \times A$. Besides, from Theorem 1.6, we have that there is $Z \in \mathcal{O}_{CIT}(\mathfrak{X}(A))$ such that $\varphi = \Theta_{IT}(Z)$. Therefore, from these last assertions and Theorem 1.6, we obtain that $Z = \mathfrak{X}(A)\backslash Y$ and so, $Y \in \mathcal{CO}_{IDT}(\mathfrak{X}(A))$. On the other hand, let $Y \in \mathcal{CO}_{IDT}(\mathfrak{X}(A))$. Then from Lemma 2.1, $\mathfrak{X}(A)\backslash Y \in \mathcal{CO}_{IDT}(\mathfrak{X}(A))$, and so, from Theorem 1.6, we obtain that $\Theta_{IT}(Y)$ and $\Theta_{IT}(\mathfrak{X}(A)\backslash Y)$ are *IKt*-congruences on $\mathcal{A}$. Besides, from Theorem 1.6, we get that $\Theta_{IT}(Y) \wedge \Theta_{IT}(\mathfrak{X}(A)\backslash Y) = \Theta_{IT}(Y \cup (\mathfrak{X}(A)\backslash Y)) = \Theta_{IT}(\mathfrak{X}(A)) = A \times A$ and $\Theta_{IT}(Y) \vee \Theta_{IT}(\mathfrak{X}(A)\backslash Y) = \Theta_{IT}(Y \cap (\mathfrak{X}(A)\backslash Y) = \{(a,a) : a \in A\}$, from which it follows that $\Theta_{IT}(Y) \in Con_{bIKt}(\mathcal{A})$.
(ii): Let $\Theta_{OT}(Y) \in Con_{bIKt}(\mathcal{A})$, then there is $\varphi \in Con_{bIKt}(\mathcal{A})$ such that $\Theta_{OT}(Y) \wedge \varphi = \{(a,a) : a \in A\}$ and $\Theta_{OT}(Y) \vee \phi = A \times A$. Besides, from Theorem 2.1, we have there is $Z \in \mathcal{O}_{CIT}(\mathfrak{X}(A))$ such that $\varphi = \Theta_{OT}(Z)$. Therefore, from the last assertions and Theorem 2.1, we infer that $Z = \mathfrak{X}(A)\backslash Y$ and so, $Y \in \mathcal{CO}_{IDT}\mathfrak{X}(A))$. On the other hand, let $Y \in \mathcal{CO}_{IDT}(\mathfrak{X}(A))$. Then, from Lemma 2.1, we have that $\mathfrak{X}(A)\backslash Y \in \mathcal{CO}_{IDT}(\mathfrak{X}(A))$, and therefore, by Theorem 2.1, we obtain that $\Theta_{OT}(Y)$ and $\Theta_{OT}(\mathfrak{X}(A)\backslash Y)$ are *IKt*-congruences on $\mathcal{A}$. Also, from Theorem 2.1, we get that $\Theta_{OT}(Y) \wedge \Theta_{OT}(\mathfrak{X}(A) \backslash Y) = \Theta_{OT}(Y \cap (\mathfrak{X}(A)\backslash Y)) = \Theta_{OT}(\emptyset) = \{(a,a) : a \in A\}$, and $\Theta_{OT}(Y) \vee \Theta_{OT}(\mathfrak{X}(A)\backslash Y) = \Theta_{OT}(Y \cup (\mathfrak{X}(A)\backslash Y)) = \Theta_{OT}(\mathfrak{X}(A)) = A \times A$, and so it follows that $\Theta_{OT}(Y) \in Con_{bIKt}(\mathcal{A})$. ■

The above results allow us to obtain the description of Boolean *IKt*-congruences we were looking for.

Theorem 2.4. *Let* $(\mathcal{A}, G, H, F, P)$ *be an IKt-algebra and let* $(\mathfrak{X}(A), \subseteq, R^A)$ *be the IKt-space associated with A. Then, the lattice* $\mathcal{CO}_{IKt}(\mathfrak{X}(A))$ *of all increasing, decreasing and clopen IKt-subsets of* $\mathfrak{X}(A)$*, is isomorphic to the lattice (dual of the lattice)* $Con_{bIKt}(\mathcal{A})$ *of the Boolean IKt-congruence on* $\mathcal{A}$*,* *where the isomorphism* Θ_{COT} (Θ_{CIT}) *is the restriction of the isomorphism* Θ_{OT} (Θ_{IT}) *to* $\mathcal{CO}_{IDT}(\mathfrak{X}(A))$, *defined in* Theorem 2.1 (Theorem 1.6).

Proof. From Theorem 2.1 and Proposition 2.1 it follows that $\Theta_{COT} = \Theta_{OT}|_{\mathcal{CO}_{IDT}(\mathfrak{X}(A))}$ is an isomorphism between the lattice $\mathcal{CO}_{IDT}(\mathfrak{X}(A))$ and the lattice $Con_{bIKt}(\mathcal{A})$. In addition, from Theorem 1.6 and Proposition 2.1 we infer that $\Theta_{CIT} = \Theta_{IT}|_{\mathcal{CO}_{IDT}(\mathfrak{X}(A))}$ is an isomorphism between the lattice $\mathcal{CO}_{IDT}(\mathfrak{X}(A))$ and the dual lattice of $Con_{bIKt}(\mathcal{A})$. ■

Corollary 2.5. *Let* $(\mathcal{A}, G, H, F, P)$ *be an IKt-algebra. Then, the Boolean IKt-congruences on* $\mathcal{A}$ *are commutative.*

Proof. Let $\varphi_1, \varphi_2 \in Con_{bIKt}(\mathcal{A})$. Then, by Theorem 2.4 there are increasing, decreasing and clopen *IKt*-subsets Y_1, Y_2 of $\mathfrak{X}(A)$ such that $\Theta_{OCT}(Y_1) = \varphi_1$ and $\Theta_{OCT}(Y_2) = \varphi_2$. Suppose that $(x, y) \in \varphi_2 \circ \varphi_1$. Hence, there is $z \in A$ such that $(x, z) \in \varphi_1$ and $(z, y) \in \varphi_2$ and so, from Theorem 2.4 we obtain that $\sigma_A(x) \cap Y_1 = \sigma_A(z) \cap Y_1$ and $\sigma_A(y) \cap Y_2 = \sigma_A(z) \cap Y_2$. These statements imply that $\sigma_A(x) \cap (Y_1 \cap Y_2) = \sigma_A(y) \cap (Y_1 \cap Y_2)$. On the other hand, since $\mathcal{CO}_{IKt}(\mathfrak{X}(A)) \subseteq D(\mathfrak{X}(A))$, we have that $(\sigma_A(x) \cap (Y_1 \cap Y_2)) \cup (\sigma_A(x) \cap (Y_2 \backslash Y_1)) \cup (\sigma_A(y) \cap (Y_1 \backslash Y_2)) \in D(\mathfrak{X}(A))$, and so $w = \sigma_A^{-1}((\sigma_A(x) \cap (Y_1 \cap Y_2)) \cup (\sigma_A(x) \cap (Y_2 \backslash Y_1)) \cup (\sigma_A(y) \cap (Y_1 \backslash Y_2))) \in A$. Furthermore, we have that $\sigma_A(x) \cap Y_2 = \sigma_A(w) \cap Y_2$ and $\sigma_A(w) \cap Y_1 = \sigma_A(y) \cap Y_1$, hence $(x, w) \in \varphi_2$ and $(w, y) \in \varphi_1$. Therefore, $(x, y) \in \varphi_1 \circ \varphi_2$ from which we conclude that $\varphi_2 \circ \varphi_1 \subseteq \varphi_1 \circ \varphi_2$. The other inclusion follows similarly. ■

Next, we will give another characterization of the Boolean congruences on an *IKt*-algebra, which will be useful in order to determine some properties of them.

Lemma 2.2. *Let* $(\mathcal{A}, G, H, F, P)$ *be an IKt-algebra. Then* $\Theta(\uparrow a)$ *is an IKt-congruence on* $\mathcal{A}$ *for all* $a \in C(\mathcal{A})$, *where* $\uparrow a$ *is the principal filter generated by* a.

Proof. From the hypothesis we have that $a = d(a)$ and so $\uparrow a$ is an *IKt*-filter of $\mathcal{A}$. Then by Proposition 1.2, we conclude the proof. ■

Proposition 2.2. *Let* $(\mathcal{A}, G, H, F, P)$ *be an IKt-algebra and let* $(\mathfrak{X}(A), \subseteq, R^A)$ *be the IKt-space associated with* A. *Then, for all* $Y \subseteq \mathfrak{X}(A)$, *the following conditions are equivalent:*

(i) $Y \in \mathcal{CO}_{IDT}(\mathfrak{X}(A))$,

(ii) *there is* $a \in C(\mathcal{A}) \cap B(\mathcal{A})$ *such that* $Y = \sigma_A(a)$.

Proof. (i) ⇒ (ii): From the hypothesis, we have that $Y \in D(\mathfrak{X}(A))$ and $\mathfrak{X}(A)\backslash Y \in D(\mathfrak{X}(A))$. Hence, there exists $a \in B(\mathcal{A})$ such that (1) $Y = \sigma_A(a)$. Then, from the hypothesis and Lemma 1.1, we obtain that $Y = d_{\mathfrak{X}(A)}(\sigma_A(a))$. Taking into account that σ_A is an *IKt*-isomorphism it follows that $Y = \sigma_A(d(a))$ and therefore, by (1) we can assert that $a = d(a)$. This statement implies that $a \in C(\mathcal{A})$, from which we conclude that $a \in B(\mathcal{A}) \cap C(\mathcal{A})$.

(ii) ⇒ (i): Since $a \in B(\mathcal{A})$ and $Y = \sigma_A(a)$ it follows that $Y \in D(\mathfrak{X}(A))$ and $\mathfrak{X}(A)\backslash Y \in D(\mathfrak{X}(A))$. Besides, $a \in C(\mathcal{A})$, which implies that $\sigma_A(a) = \sigma_A(d(a)) = d_{\mathfrak{X}(A)}\sigma_A(a) = d_{\mathfrak{X}(A)}(Y)$. Therefore, $d_{\mathfrak{X}(A)}(Y) = Y$, which completes the proof by Lemma 1.1. ■

Corollary 2.6. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then, the Boolean algebras $C(\mathcal{A}) \cap B(\mathcal{A})$ and $Con_{bIKt}(\mathcal{A})$ are isomorphic and therefore, $|Con_{bIKt}(\mathcal{A})| = |C(\mathcal{A}) \cap B(\mathcal{A})|$, where $|Z|$ denotes the cardinality of the set Z.*

Proof. It is a direct consequence of Theorem 2.4 and Proposition 2.2. ■

Proposition 2.3. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then for all binary relation $\varphi \subseteq A \times A$, the following conditions are equivalent:*

(i) *φ is a Boolean IKt-congruence on $\mathcal{A}$,*

(ii) *there is $a \in C(\mathcal{A}) \cap B(\mathcal{A})$ such that $\varphi = \Theta(\uparrow a)$,*

(iii) *there is $a \in A$ such that $\bigwedge\limits_{n\in\omega} d^n(a) \in C(\mathcal{A}) \cap B(\mathcal{A})$ and*

$$\varphi = \Theta\Big(\uparrow \bigwedge_{n\in\omega} d^n(a)\Big),$$

(iv) *there is a principal IKt-filter S of $\mathcal{A}$, generated by a Boolean IKt-element of $\mathcal{A}$, such that $\varphi = \Theta(S)$.*

Proof. (i) ⇒ (ii): From the hypothesis and Theorem 2.4 there is $Y \in \mathcal{CO}_{IKt}(\mathfrak{X}(A))$ such that $\varphi = \Theta_{IT}(Y)$. Besides, from Proposition 2.2 there is $a \in C(\mathcal{A}) \cap B(\mathcal{A})$ such that $Y = \sigma_A(a)$. Therefore, for all $b, c \in A$ we have that $(b, c) \in \Theta_{IT}(Y)$ if and only if $\sigma_A(b) \cap \sigma_A(a) = \sigma_A(c) \cap \sigma_A(a)$. Hence, taking into account that the map σ_A is an isomorphism, it follows that $(b, c) \in \Theta_{IT}(Y)$ if and only if $b \wedge a = c \wedge a$. From this last assertion we conclude that $\varphi = \Theta([a))$.

(ii) ⇒ (i): Since $a \in C(\mathcal{A}) \cap B(\mathcal{A})$, then from Proposition 2.2, we infer that $\sigma_A(a)$ is an increasing, decreasing and clopen *IKt*-subset of $\mathfrak{X}(A)$, and so by Theorem 2.4, $\Theta_{IT}(\sigma_A(a))$ is a Boolean *IKt*-congruence on $\mathcal{A}$. Besides, taking into account that $\Theta(\uparrow a) = \Theta_{IT}(\sigma_A(a))$, the proof is complete.

(ii) $\Leftrightarrow$ (iii): It is a direct consequence of (A6).

(ii) $\Leftrightarrow$ (iv): It follows from the fact that for all $a \in C(\mathcal{A})$, $\uparrow a$ is a principal *IKt*-filter of $\mathcal{A}$. ■

Corollary 2.7. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then, for any $a \in C(\mathcal{A}) \cap B(\mathcal{A})$, $\Theta(\uparrow a)$ and $\Theta(\uparrow -a)$ are Boolean IKt-congruences on $\mathcal{A}$.*

Proof. From Corollary 2.3, we have that $-a \in C(\mathcal{A}) \cap B(\mathcal{A})$ for any $a \in C(\mathcal{A}) \cap B(\mathcal{A})$. From this last statement and Proposition 2.3 it follows that $\Theta(\uparrow a)$ and $\Theta(\uparrow -a)$ are Boolean *IKt*-congruences on $\mathcal{A}$. ■

Corollary 2.8. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then, the Boolean IKt-congruences on $\mathcal{A}$ are regular and uniform.*

Proof. Let φ be a Boolean *IKt*-congruence on A. Then, by Proposition 2.3 there is $a \in C(\mathcal{A}) \cap B(\mathcal{A})$ such that $\varphi = \Theta([a))$. Furthermore, for each $b \in A$ we have that $[b]_\varphi = \{(b \wedge a) \vee c : c \in \downarrow -a\}$, where $[b]_\varphi$ stands for the equivalence class of b modulo φ. From this last assertion we infer that $[0]_\varphi = \downarrow -a$ and therefore, $[b]_\varphi = \{(b \wedge a) \vee c : c \in [0]_\varphi\}$ and $|[b]_\varphi| = |[0]_\varphi|$ for all $b \in A$, which allows us to conclude the proof. ■

3. Principal Congruences on *IKt*-Algebras

In this section our first objective is to characterize the principal *IKt*-congruences on an *IKt*-algebra by means of certain closed and increasing *IKt*-subsets of its associated *IKt*-space.

From now on, if $\mathcal{A}$ is an *IKt*-algebra and $a, b \in A$, we will denote by $\Theta(a, b)$ the principal *IKt*-congruence generated by (a, b). Besides, we will consider $\Theta(a, b)$, with $a \leq b$. This condition does not remove generality of the problem because for all $a, b \in A$, $\Theta(a, b) = \Theta(a \wedge b, a \vee b)$.

To reach our objective we begin by characterizing those closed and increasing *IKt*-subsets of the *IKt*-space associated with an *IKt*-algebra that correspond to principal *IKt*-congruences of this algebra under the duality. For this we consider the following definition:

Definition 3.1. Let $(X, \leq, R)$ be an *IKt*-space and let $\mathcal{F}$ be a family of subsets of X. If M is a subset of X, then $Y \in \mathcal{F}$ is maximally disjoint from M with respect to $\mathcal{F}$ if and only if $Y \cap M = \emptyset$ and $Z \subseteq Y$ for all $Z \in \mathcal{F}$ such that $Z \cap M = \emptyset$.

Remark 3.1. Let $(X, \leq, R)$ be an *IKt*-space and let $\mathcal{F}$ be a family of subsets of X. If M is a subset of X, then $Y \in \mathcal{F}$ is maximally disjoint from M with respect to $\mathcal{F}$ if and only if $Y \cap M = \emptyset$ and $Z \cap M \neq \emptyset$ for all $Z \in \mathcal{F}$ such that $Z \supset Y$.

PROPOSITION 3.1. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra, $a, b \in A$ such that $a \leq b$ and let $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. If Y is an increasing and closed IKt-subset of $\mathfrak{X}(A)$, then the following conditions are equivalent:*

(i) $\Theta_{IT}(Y) = \Theta(a, b)$,

(ii) *Y is maximally disjoint from $\sigma_A(b) \backslash \sigma_A(a)$ with respect to $\mathcal{C}_{IT}(\mathfrak{X}(A))$.*

PROOF. (i) $\Rightarrow$ (ii): From the hypothesis and the definition of $\Theta_{IT}(Y)$ given in Theorem 1.6, we have that $(\sigma_A(b) \backslash \sigma_A(a)) \cap Y = \emptyset$. On the other hand, if $Q \in \mathcal{C}_{IT}(\mathfrak{X}(A))$ and $(\sigma_A(b) \backslash \sigma_A(a)) \cap Q = \emptyset$, then by Theorem 1.6, $\Theta_{IT}(Q)$ is an *IKt*-congruence on A and $(a, b) \in \Theta_{IT}(Q)$. Since $\Theta_{IT}(Y) = \Theta(a, b)$, then $\Theta_{IT}(Y) \subseteq \Theta_{IT}(Q)$ and by Theorem 1.6 it follows that $Q \subseteq Y$.
(ii) $\Rightarrow$ (i): Let $Y \in \mathcal{C}_{IT}(\mathfrak{X}(A))$ such that (1) Y is maximally disjoint from $\sigma_A(b) \backslash \sigma_A(a)$ with respect to $\mathcal{C}_{IT}(\mathfrak{X}(A))$, where $a, b \in A$, $a \leq b$. By virtue Theorem 1.6, $\Theta_{IT}(Y)$ is an *IKt*-congruence on $\mathcal{A}$, and by Theorem 1.6, we have that $(a, b) \in \Theta_{IT}(Y)$. On the other hand, if ϑ is an *IKt*-congruence on $\mathcal{A}$ such that $(a, b) \in \vartheta$, then by Theorem 1.6, it follows that there exists (2) $Q \in \mathcal{C}_{IT}(\mathfrak{X}(A))$ such that $\vartheta = \Theta_{IT}(Q)$, and by Theorem 1.6, we obtain that (3) $(\sigma_A(b) \backslash \sigma_A(a)) \cap Q = \emptyset$. From (1), (2) and (3), we infer that $Q \subseteq Y$. From Theorem 1.6, we conclude that $\Theta_{IT}(Y) \subseteq \vartheta$ and therefore $\Theta_{IT}(Y) = \Theta(a, b)$. ■

PROPOSITION 3.2. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra, and let $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. If Y is an increasing and closed IKt-subset of $\mathfrak{X}(A)$, then the following conditions are equivalent:*

(i) *$\Theta_{IT}(Y)$ is a principal IKt-congruence on $\mathcal{A}$,*

(ii) *there exists a clopen and convex subset C of $\mathfrak{X}(A)$ such that Y is maximally disjoint from C with respect to $\mathcal{C}_{IT}(\mathfrak{X}(A))$.*

PROOF. (i) $\Rightarrow$ (ii): It follows from Proposition 3.1, considering $C = \sigma_A(b) \backslash \sigma_A(a)$.
(ii) $\Rightarrow$ (i): From the hypothesis (ii) and (A8), we infer that there exist U, $V \in D(\mathfrak{X}(A))$ such that $V \subseteq U$ and Y is maximally disjoint from $U \backslash V$ with respect to $\mathcal{C}_{IT}(\mathfrak{X}(A))$. From the fact that $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$ is a lattice isomorphism, we obtain that $V = \sigma_A(a)$ and $U = \sigma_A(b)$, with $a, b \in A$, $a \leq b$, and so by Proposition 3.1, we conclude that $\Theta_{IT}(Y) = \Theta(a, b)$. ■

REMARK 3.2. Note that, in Proposition 3.2, the clopen and convex subset C is not necessarily unique.

Corollary 3.2. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra and $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. Then every Boolean IKt-congruence on $\mathcal{A}$ is a principal IKt-congruence on $\mathcal{A}$.*

Proof. Let φ be a Boolean *IKt*-congruence on $\mathcal{A}$. Then, Theorem 2.4 implies that there is $Y \in \mathcal{CO}_{IKt}(\mathfrak{X}(A))$ such that $\varphi = \Theta_{IT}(Y)$. Furthermore, the fact that $\mathfrak{X}(A)\backslash Y \in \mathcal{CO}_{IKt}(\mathfrak{X}(A))$ implies that $\mathfrak{X}(A)\backslash Y$ is a clopen and convex subset of $\mathfrak{X}(A)$. Since Y is maximally disjoint from $\mathfrak{X}(A)\backslash Y$ with respect to $\mathcal{C}_{IT}(\mathfrak{X}(A))$, we conclude, by Proposition 3.2, that φ is a principal *IKt*-congruence on $\mathcal{A}$. ■

Corollary 3.3. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra and let $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. If Y is an increasing IKt-subset of $\mathfrak{X}(A)$ such that $\mathfrak{X}(A)\backslash Y$ is clopen and convex subset of $\mathfrak{X}(A)$, then $\Theta_{IT}(Y)$ is a principal IKt-congruence on $\mathcal{A}$.*

Proof. It follows immediately from Proposition 3.2, considering $C = \mathfrak{X}(A)\backslash Y$. ■

Lemma 3.1. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra, $a, b \in A$ such that $a \leq b$ and let $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. If S is an IKt-filter of $\mathcal{A}$, then the following conditions are equivalent:*

(i) $\Theta(S) = \Theta(a, b)$,

(ii) *Y_S is maximally disjoint from $\sigma_A(b)\backslash\sigma_A(a)$ with respect to $\mathcal{C}_{IT}(\mathfrak{X}(A))$.*

Proof. It is a direct consequence of Proposition 3.1 and the fact that for all $S \in \mathcal{F}_{IKt}(\mathcal{A})$, $Y_S \in \mathcal{C}_{IT}(\mathfrak{X}(A))$ and $\Theta(S) = \Theta_{IT}(Y_S)$. ■

Proposition 3.3. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra and $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. If S is an IKt-filter of $\mathcal{A}$, then the following conditions are equivalent:*

(i) *$\Theta(S)$ is a principal IKt-congruence on $\mathcal{A}$,*

(ii) *there exists a clopen and convex subset C of $\mathfrak{X}(A)$ such that Y_S is maximally disjoint from C with respect to $\mathcal{C}_{IT}(\mathfrak{X}(A))$.*

Proof. (i) ⇒ (ii): It follows from Lemma 3.1, considering $C = \sigma_A(b)\backslash\sigma_A(a)$ if $\Theta(S) = \Theta(a, b)$, with a, $b \in A$, $a \leq b$.
(ii) ⇒ (i): From the hypothesis (ii) and Lemma 3.1, we infer that there are $U, V \in D(\mathfrak{X}(A))$ such that $V \subseteq U$ and Y_S is maximally disjoint from $U\backslash V$ with respect to $\mathcal{C}_{IT}(\mathfrak{X}(A))$. From the fact that $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$ is a lattice isomorphism, we obtain that $V = \sigma_A(a)$ and $U = \sigma_A(b)$, with a, $b \in A$, $a \leq b$, and so by Lemma 3.1, we get that $\Theta(S) = \Theta(a, b)$. ■

Corollary 3.4. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra and $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. If S is a principal IKt-filter of $\mathcal{A}$ generated by some $a \in C(\mathcal{A})$, then $\Theta(S)$ is a principal IKt-congruence on $\mathcal{A}$.*

Proof. Let $S = [a)$ for some $a \in C(\mathcal{A})$. Taking into account that $a = d(a)$ and $Y_S = \sigma_A(a)$, it follows that $Y_S \in \mathcal{C}_{IT}(\mathfrak{X}(A))$. Besides, since $\mathfrak{X}(A)\backslash\sigma_A(a)$ is a clopen and convex subset of $\mathfrak{X}(A)$, then by Proposition 3.3 the proof is complete considering $C = \mathfrak{X}(A)\backslash\sigma_A(a)$. ■

Remark 3.3. Note that Corollary 3.4 is also a consequence of Corollary 3.3 and the fact that $\Theta(S) = \Theta_{IT}(Y_S)$.

Corollary 3.5. *Let $(\mathcal{A}, G, H, F, P)$ be an IKt-algebra. Then every IKt-congruence on $\mathcal{A}$ associated with a principal IKt-filter of $\mathcal{A}$ is a principal IKt-congruence on $\mathcal{A}$.*

Proof. Let $\varphi = \Theta(S)$, where $S = [a)$ is an *IKt*-filter of $\mathcal{A}$. From this last statement, we infer that $d(a) \in [a)$, from which it follows by the property (d5) that $a = d(a)$ and so $a \in C(\mathcal{A})$. Then, by Corollary 3.4, we conclude that φ a principal *IKt*-congruence on $\mathcal{A}$. ■

Corollary 3.6. *Let $(\mathcal{A}, G, H, F, P)$ be a finite IKt-algebra. Then every IKt-congruence on $\mathcal{A}$ is a principal IKt-congruence, and therefore the intersection of two principal IKt-congruences on $\mathcal{A}$ is a principal one.*

Proof. It is a direct consequence of Proposition 1.2, Corollary 3.5 and the fact that every *IKt*-filter of a finite *IKt*-algebra is a principal filter. ■

Corollary 3.7. *Let $(\mathcal{A}, G, H, F, P)$ be a complete IKt-algebra. Then every IKt-congruence on $\mathcal{A}$ associated with a complete IKt-filter of $\mathcal{A}$ is a principal IKt-congruence on $\mathcal{A}$.*

Proof. It is a direct consequence of Corollary 3.5 and the fact that every complete *IKt*-filter of $\mathcal{A}$ is a principal filter. ■

Corollary 3.8. *Let $(\mathcal{A}, G, H, F, P)$ be a complete IKt-algebra such that every IKt-filter of $\mathcal{A}$ is complete. Then every IKt-congruence on $\mathcal{A}$ is a principal IKt-congruence, and therefore the intersection of two principal IKt-congruences on $\mathcal{A}$ is a principal one.*

Proof. It is a direct consequence of Proposition 1.2 and Corollary 3.7. ■

Next, we will characterize the principal *IKt*-congruences on linear and complete *IKt*-algebras whose filters are complete and on linear and finite *IKt*-algebras. First, we will show a property of the dual algebra of a Heyting space whose dual Heyting algebra is linear, which will be useful for this characterization.

LEMMA 3.2. *Let $(X, \leq)$ be Heyting space such that the associated Heyting algebra $D(X)$ is linear. Then, for all $U \in D(X)$:*

(i) $\nabla U :=\downarrow U \in B(D(X))$, *where $B(D(X))$ is the Boolean algebra of the Boolean elements of $D(X)$,*

(ii) ∇U *is the least element of $B_U = \{V \in B(D(X)) : U \subseteq V\}$.*

PROOF. (i): Let $U \in D(X)$. Since X is a Heyting space, then (1) $\downarrow U$ is closed and open. It is immediate that (2) $\downarrow U$ is decreasing. Taking into account that $D(X)$ is a linear Heyting algebra and the fact that $\downarrow U$ is an decreasing and closed subset of X, we obtain that (3) $\downarrow U$ is increasing. Indeed, suppose that there exists $y \in X$ such that $x \leq y$ for some $x \in\downarrow U$ and $y \notin\downarrow U$. Since $\downarrow U$ is closed and consequently it is compact, then from the last assertion and (2) we infer that there is $V \in D(X)$ such that $y \in V$ and $V\cap \downarrow U = \emptyset$. Hence $U\cap V = \emptyset$, from which it follows that $\{x \in X :\uparrow x \cap U \subseteq V\} = \emptyset$ and $\{x \in X :\uparrow x \cap V \subseteq U\} = \emptyset$, and so $(U \to V) \cup (V \to U) = \emptyset$, which contradicts that $D(X)$ is a linear Heyting algebra. Therefore $\downarrow U$ is increasing. Finally, the statements (1), (2) and (3) and the fact that $B(D(X))$ is the Boolean algebra of all increasing, decreasing, closed and open subsets of X, allow us to assert that $\nabla U :=\downarrow U \in B(D(X))$.
(ii): It is immediate that $U \subseteq \nabla U$. Besides, let $V \in B(D(X))$, such that $U \subseteq V$, then $\downarrow U \subseteq\downarrow V$. And so, from this last assertion and the fact that $V =\downarrow V$ the proof is complete. ■

LEMMA 3.3. *Let $\mathcal{A}$ be a linear Heyting algebra and $B(\mathcal{A})$ be the Boolean algebra of the Boolean elements of $\mathcal{A}$. Then for all $a \in A$, the set $B_a := \{b \in B(\mathcal{A}) : a \leq b\}$ has least element and it will be denoted by ∇a.*

PROOF. For each $a \in A$, let $a^* := a \to 0$. Assume that $a \leq b$, $b \in B(\mathcal{A})$. Then, from property (H5) of Heyting algebras we infer that $b^* \leq a^*$ and hence $a^{**} \leq b^{**}$. From this last assertion, properties (H6) and (H8) and the fact that $b \in B(\mathcal{A})$, we obtain that $a \leq a^{**} \leq b$. Besides, since A is a linear Heyting algebra, then from property (L3) we have that $a^{**} \in B(\mathcal{A})$, and so we conclude that $\nabla a =: a^{**}$ is the least element of B_a. ■

COROLLARY 3.9. *Let $(X, \leq)$ be Heyting space such that the associated Heyting algebra $D(X)$ is linear. Then, for all $U \in D(X)$, $\downarrow U = U^{**} = \nabla U$, where $U^* := U \to \emptyset = \{x \in U :\uparrow x \cap U = \emptyset\}$.*

PROOF. It is a direct consequence of Lemmas 3.2 and 3.3 and the fact that $\langle D(X), \cup, \cap, \to, X, \emptyset\rangle$ is a Heyting algebra, where for all $U, V \in D(X)$,
$U \to V := \{x \in X :\uparrow x \cap U \subseteq V\}$. ■

Bearing in mind the above results, our next task is to obtain an another characterization of the principal congruences on complete and linear *IKt*-algebras whose prime filters are complete, which will enable us to determine the generator of the *IKt*-filter that corresponds to each principal *IKt*-congruence of this algebra.

PROPOSITION 3.4. *Let $(\mathcal{A}, G, H, F, P)$ be a linear IKt-algebra and $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. Then, for all $a, b \in A$, $a \leq b$,*

$$\mathfrak{X}(A) \backslash \downarrow (\sigma_A(b) \backslash \sigma_A(a)) = \sigma_A(-\nabla b \vee e), \text{ for some } e \in A,\ a \leq e \leq \nabla a,$$

where $-x$ is the Boolean complement of x for all $x \in B(\mathcal{A})$ and

(i) $e = a$ *if* $\downarrow \sigma_A(a) \backslash \sigma_A(a) \subseteq \downarrow (\sigma_A(b) \backslash \sigma_A(a))$,

(ii) $e = \nabla a$ *if* $(\downarrow \sigma_A(a) \backslash \sigma_A(a)) \cap \downarrow (\sigma_A(b) \backslash \sigma_A(a)) = \emptyset$,

(iii) $a < e < \nabla a$, *in another case.*

PROOF. Since $\mathfrak{X}(A)$ is a Heyting space and for all $a, b \in A$, $a \leq b$, the set $\sigma_A(b) \backslash \sigma_A(a)$ is a clopen subset of $\mathfrak{X}(A)$, it follows that $\downarrow (\sigma_A(b) \backslash \sigma_A(a))$ is a clopen and decreasing subset of $\mathfrak{X}(A)$ and so, $\mathfrak{X}(A) \backslash \downarrow (\sigma_A(b) \backslash \sigma_A(a))$ is a clopen and increasing subset of $\mathfrak{X}(A)$. Then, there is $c \in A$ such that $\sigma_A(c) = \mathfrak{X}(A) \backslash \downarrow (\sigma_A(b) \backslash \sigma_A(a))$. In what follows we determine this element $c \in A$. By virtue of hypothesis and the fact that $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$ is an *IKt*-isomorphism, we have that $\sigma_A(a) \subseteq \sigma_A(b)$, and so we obtain that $\sigma_A(b) = (\sigma_A(b) \backslash \sigma_A(a)) \cup \sigma_A(a)$, from which it follows that:

$$\begin{aligned}(1)\ \downarrow \sigma_A(b) &= \downarrow (\sigma_A(b) \backslash \sigma_A(a)) \cup \downarrow \sigma_A(a) \\ &= \downarrow (\sigma_A(b) \backslash \sigma_A(a)) \cup (\downarrow \sigma_A(a) \backslash \downarrow (\sigma_A(b) \backslash \sigma_A(a))),\end{aligned}$$

and one of the conditions holds:

(a) $\downarrow \sigma_A(a) \backslash \sigma_A(a) \subseteq \downarrow (\sigma_A(b) \backslash \sigma_A(a))$,

(b) $(\downarrow \sigma_A(a) \backslash \sigma_A(a)) \cap \downarrow (\sigma_A(b) \backslash \sigma_A(a)) = \emptyset$,

(c) $\downarrow \sigma_A(a) \backslash \sigma_A(a) \not\subseteq \downarrow (\sigma_A(b) \backslash \sigma_A(a))$ and
$(\downarrow \sigma_A(a) \backslash \sigma_A(a)) \cap \downarrow (\sigma_A(b) \backslash \sigma_A(a)) \neq \emptyset$.

(i): Suppose that the condition (a) holds. Then, from (1) we infer that

$$\downarrow \sigma_A(b) = \downarrow (\sigma_A(b) \backslash \sigma_A(a)) \cup \sigma_A(a).$$

Therefore, from this last statement and taking into account that

$$\downarrow (\sigma_A(b) \backslash \sigma_A(a)) \cap \sigma_A(a) = \emptyset,$$

and the fact that the map σ_A is an *IKt*-isomorphism and A is a linear *IKt*-algebra, we obtain that

$$\begin{aligned}\mathfrak{X}(A)\backslash \downarrow (\sigma_A(b)\backslash\sigma_A(a)) &= (\mathfrak{X}(A)\backslash\nabla\sigma_A(b)) \cup \sigma_A(a)\\ &= \sigma_A(-\nabla b \vee a).\end{aligned}$$

(ii): From the condition (b) and (1) we get that

$$\begin{aligned}&\downarrow \sigma_A(b) = \downarrow (\sigma_A(b)\backslash\sigma_A(a)) \cup \downarrow \sigma_A(a),\\ &\downarrow (\sigma_A(b)\backslash\sigma_A(a)) \cap \downarrow \sigma_A(a) = \emptyset.\end{aligned}$$

From these last assertions and taking into account that the map σ_A is an *IKt*-isomorphism and A is a linear *IKt*-algebra, we infer that

$$\begin{aligned}\mathfrak{X}(A)\backslash \downarrow (\sigma_A(b)\backslash\sigma_A(a)) &= (\mathfrak{X}(A)\backslash\sigma_A(\nabla b)) \cup \nabla\sigma_A(a)\\ &= \sigma_A(-\nabla b \vee \nabla a).\end{aligned}$$

(iii): From the conditions stated in (c), we infer that

$$(2)\ \sigma_A(a) \subset \downarrow \sigma_A(a)\backslash \downarrow (\sigma_A(b)\backslash\sigma_A(a)) \subset \downarrow \sigma_A(a).$$

Besides, from (1) we obtain that

$$(3)\ \mathfrak{X}(A)\backslash \downarrow (\sigma_A(b)\backslash\sigma_A(a)) = (\mathfrak{X}(A)\backslash \downarrow \sigma_A(b)) \cup (\downarrow \sigma_A(a)\backslash \downarrow (\sigma_A(b)\backslash\sigma_A(a))).$$

Since $\downarrow (\sigma_A(b)\backslash\sigma_A(a))$ is a clopen and decreasing subset of $\mathfrak{X}(A)$, and $\downarrow \sigma_A(a) \in D(\mathfrak{X}(A))$, it follows that

$$\downarrow \sigma_A(a)\backslash \downarrow (\sigma_A(b)\backslash\sigma_A(a)) \in D(\mathfrak{X}(A)),$$

and therefore there is $e \in A$ such that

$$(4)\ \sigma_A(e) = \downarrow \sigma_A(a)\backslash \downarrow (\sigma_A(b)\backslash\sigma_A(a)).$$

Finally, from the assertions (2), (3) and (4), we conclude that

$$\begin{aligned}\mathfrak{X}(A)\backslash \downarrow (\sigma_A(b)\backslash\sigma_A(a)) &= (\mathfrak{X}(A)\backslash\sigma_A(\nabla b)) \cup \sigma_A(e)\\ &= \sigma_A(-\nabla b \vee e),\end{aligned}$$

for some $e \in A$, $a < e < \nabla a$. ■

PROPOSITION 3.5. *Let $(\mathcal{A}, G, H, F, P)$ be a complete and linear IKt-algebra whose prime filters are complete, or a finite and linear IKt-algebra and let $\mathfrak{X}(A)$ be the IKt-space associated with $\mathcal{A}$. If $a, b \in A$, $a \leq b$ and $Y \in \mathcal{C}_{IT}(\mathfrak{X}(A))$, then the following conditions are equivalent:*

(i) $\Theta_{IT}(Y) = \Theta(a, b)$,

(ii) $Y = \sigma_A\Big(\bigwedge_{n\in\omega} d^n(-\nabla b \vee e)\Big)$, *for some* $e \in A$, $a \leq e \leq \nabla a$, *where*
$e = a$ *if* $\downarrow \sigma_A(a)\backslash\sigma_A(a) \subseteq \downarrow (\sigma_A(b)\backslash\sigma_A(a))$,
$e = \nabla a$ *if* $(\downarrow \sigma_A(a)\backslash\sigma_A(a)) \cap \downarrow (\sigma_A(b)\backslash\sigma_A(a)) = \emptyset$,
$a < e < \nabla a$, *in another case.*

Proof. (i) $\Rightarrow$ (ii): From the hypothesis (i) and Lemma 3.1, we can assert that
(1) Y is maximally disjoint from $\sigma_A(b)\backslash\sigma_A(a)$ with respect to $\mathcal{C}_{IT}(\mathfrak{X}(A))$.

If $\mathcal{A}$ is a complete *IKt*-algebra whose prime filters are complete, since $\mathcal{A}$ is isomorphic to $D(\mathfrak{X}(A))$, it follows that $D(\mathfrak{X}(A))$ is a complete *IKt*-algebra and for all $U \in D(\mathfrak{X}(A))$,

$$(2)\ \bigwedge_{n\in\omega} d^n_{\mathfrak{X}(A)}U = \bigwedge_{n\in\omega} d^n_{\mathfrak{X}(A)}\sigma_A(a) = \sigma_A\Big(\bigwedge_{n\in\omega} d^n(a)\Big) = \bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}\sigma_A(a) = \bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}U.$$

On the other hand, if $\mathcal{A}$ is a finite *IKt*-algebra, then $D(\mathfrak{X}(A))$ is a finite *IKt*-algebra and so for all $U \in D(\mathfrak{X}(A))$, there is $n_0 \in \omega$, such that $d^n_{\mathfrak{X}(A)}(U) = d^{n_0}_{\mathfrak{X}(A)}(U)$ for all $n \in \omega$, $n_0 \leq n$, from which it follows that

$$(3)\ \bigwedge_{n\in\omega} d^n_{\mathfrak{X}(A)}U = d^{n_0}_{\mathfrak{X}(A)}(U) = \bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}U.$$

From (2) and (3), we have that for all a, b, $e \in A$, $a \leq b$, $a \leq e \leq \nabla a$,

$$\bigwedge_{n\in\omega} d^n_{\mathfrak{X}(A)}\sigma_A(-\nabla b \vee e) \in \mathcal{C}(D(\mathfrak{X}(A))),$$

and so

$$(4)\ \bigwedge_{n\in\omega} d^n_{\mathfrak{X}(A)}\sigma_A(-\nabla b \vee e) \in \mathcal{C}_{IT}(\mathfrak{X}(A)).$$

In addition, since by the property (d2) of the *IKt*-algebras,

$$\bigwedge_{n\in\omega} d^n_{\mathfrak{X}(A)}\sigma_A(-\nabla b \vee e) \subseteq \sigma_A(-\nabla b \vee e),$$

and by Proposition 3.4,

$$\sigma_A(-\nabla b \vee e)\ \cap\ (\sigma_A(b)\backslash\sigma_A(a)) = \emptyset,$$

it follows that

$$(5)\ \bigwedge_{n\in\omega} d^n_{\mathfrak{X}(A)}\sigma_A(-\nabla b \vee e)\ \cap\ (\sigma_A(b)\backslash\sigma_A(a)) = \emptyset.$$

Now, let (6) $W \in \mathcal{C}_{IT}(\mathfrak{X}(A))$ such that (7) $W \cap (\sigma_A(b)\backslash\sigma_A(a)) = \emptyset$. Since W is increasing, then from (7) it results that

$$(8)\, W \cap \downarrow (\sigma_A(b)\backslash\sigma_A(a)) = \emptyset.$$

Besides, from Proposition 3.4, we have that

$$\mathfrak{X}(A)\backslash \downarrow (\sigma_A(b)\backslash\sigma_A(a)) = \sigma_A(-\nabla b \vee e).$$

Then, from (8) and this last statement, it follows that

$$W \subseteq \sigma_A(-\nabla b \vee e),$$

and consequently, from Lemma 1.1, the assertions (2) and (3), and the fact that σ_A is an *IKt*-isomorphism, we get that

$$\begin{aligned}(9)\, W &= \bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(W) \subseteq \bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}\sigma_A\,(-\nabla b \vee e)\\ &= \bigwedge_{n\in\omega} d^n_{\mathfrak{X}(A)}\sigma_A(-\nabla b \vee e) = \sigma_A\Big(\bigwedge_{n\in\omega} d^n(-\nabla b \vee e)\Big).\end{aligned}$$

From (4), (5), (6), (7) and (9), we conclude that $\sigma_A\Big(\bigwedge_{n\in\omega} d^n(-\nabla b \vee e)\Big)$ is maximally disjoint from $\sigma_A(b)\backslash\sigma_A(a)$ with respect to $\mathcal{C}_{IT}(\mathfrak{X}(A))$, and so by (1) the proof is complete. ∎

Corollary 3.10. *Let $(\mathcal{A}, G, H, F, P)$ be a complete and linear IKt-algebra whose prime filters are complete, or particularly a finite and linear IKt-algebra. If $a, b \in A$, $a \leq b$ then the following conditions are equivalent:*

(i) $\varphi = \Theta(a, b)$,

(ii) $\varphi = \Theta\Big(\uparrow \bigwedge_{n\in\omega} d^n(-\nabla b \vee e)\Big)$, *for some $e \in A$, $a \leq e \leq \nabla a$, where*
$e = a$ *if* $\downarrow \sigma_A(a)\backslash\sigma_A(a) \subseteq \downarrow (\sigma_A(b)\backslash\sigma_A(a))$,
$e = \nabla a$ *if* $(\downarrow \sigma_A(a)\backslash\sigma_A(a)) \cap \downarrow (\sigma_A(b)\backslash\sigma_A(a)) = \emptyset$,
$a < e < \nabla a$, *in another case.*

Proof. From Theorem 1.6 we have that $\varphi = \Theta(a, b)$ if and only if there exists $Y \in \mathcal{C}_{IT}(\mathfrak{X}(A))$ such that $\Theta_{IT}(Y) = \varphi = \Theta(a, b)$, where $\mathfrak{X}(A)$ is the *IKt*-space associated with $\mathcal{A}$. Then, from this last assertion, Proposition 3.5 and bearing in mind that for all $a \in C(A)$, $\sigma_A(a) \in \mathcal{C}_{IT}(\mathfrak{X}(A))$ and $\Theta_{IT}(\sigma_A(a)) = \Theta(\uparrow a)$, we conclude the proof. ∎

Corollary 3.11. *Let $(\mathcal{A}, G, H, F, P)$ be a complete and linear IKt-algebra whose prime filters are complete, or particularly a finite and linear IKt-algebra. If $a, b \in A$, $a \leq b$, then the following conditions are equivalent:*

(i) $\Theta(a,b)$ *is a Boolean IKt-congruence,*

(ii) $\Theta(a,b) = \Theta\Big(\uparrow \bigwedge\limits_{n\in\omega} d^n(-\nabla b \vee e)\Big)$ *such that* $\bigwedge\limits_{n\in\omega} d^n(-\nabla b \vee e) \in B(\mathcal{A})$, *for some* $e \in A$, $a \leq e \leq \nabla a$.

PROOF. It is a direct consequence of Corollaries 3.2 and 3.10. ■

COROLLARY 3.12. *Let* $(\mathcal{A}, G, H, F, P)$ *be a complete and linear IKt-algebra whose prime filters are complete, or particularly a finite and linear IKt-algebra. If* $a, b \in A$, $\nabla a, \nabla b \in C(\mathcal{A})$, $a \leq b$ *and* $\downarrow \sigma_A(a)\backslash\sigma_A(a) \subseteq\downarrow (\sigma_A(b)\backslash\sigma_A(a))$, *then* $\Theta(a,b)$ *is a Boolean IKt-congruence and* $\Theta(a,b) = \Theta\left(\uparrow(-\nabla b \vee \nabla a)\right)$.

PROOF. Taking into account the fact that $\nabla a, \nabla b \in C(\mathcal{A})$ and Corollary 3.3, we have that $\nabla a, \nabla b \in B(\mathcal{A}) \cap C(\mathcal{A})$. Then from Corollary 2.3, we infer that $-\nabla b \vee \nabla a \in B(\mathcal{A}) \cap C(\mathcal{A})$, and therefore from the statement (A6) in Section 1, we obtain that (1) $-\nabla b \vee \nabla a = \bigwedge\limits_{n\in\omega} d^n(-\nabla b \vee \nabla a)$. Hence, we can assert that (2) $\bigwedge\limits_{n\in\omega} d^n(-\nabla b \vee \nabla a) \in B(\mathcal{A}) \cap C(\mathcal{A})$. Besides, since $\downarrow \sigma_A(a)\backslash\sigma_A(a) \subseteq\downarrow (\sigma_A(b)\backslash\sigma_A(a))$, from Corollary 3.10 it follows that $\Theta(a,b) = \Theta\Big(\uparrow \bigwedge\limits_{n\in\omega} d^n(-\nabla b \vee \nabla a)\Big)$. From (2), this last assertion and Corollary 3.11, we conclude that $\Theta(a,b)$ is a Boolean *IKt*-congruence and by (1) the proof is complete. ■

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