

# THE $K$ -THEORY OF TORIC VARIETIES

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ABSTRACT. Recent advances in computational techniques for  $K$ -theory allow us to describe the  $K$ -theory of toric varieties in a new way.

## 1. INTRODUCTION

In this paper, we revisit the  $K$ -theory of toric varieties, using the new perspective afforded by the recent papers [H], [CHSW], [CHW]. These papers provide an important new technique for computations in the  $K$ -theory of singular algebraic varieties over fields of characteristic 0, in terms of their homotopy  $K$ -theory and cohomological data. For such a variety  $X$ , the homotopy fiber of  $K(X) \rightarrow KH(X)$  coincides up to homotopy with the analogous relative term in the comparison between cyclic homology and its  $cdh$ -fibrant replacement (the result of imposing descent for the  $cdh$  topology). Here,  $KH(X)$  denotes the homotopy  $K$ -theory of  $X$ ; see [11].

The  $KH$ -theory of an affine toric variety is just the  $K$ -theory of a Laurent polynomial ring, and is well understood. Even when  $X$  is a non-affine toric variety,  $KH_*(X)$  is tractable; we show in Proposition 5.6 that it is a summand of  $K_*(X)$ . This allows us to give a short proof in Proposition 5.8 that  $K_0(X) = \mathbb{Z}$  for affine  $X$ .

Thus the problem of understanding  $K_*(X)$  is reduced to understanding the cyclic homology of  $X$  and its  $cdh$ -fibrant version. Because toric varieties admit resolutions of singularities that are formed in a purely combinatorial manner, it turns out this is indeed a more accessible problem.

The main goal of this paper is to use these new techniques to give a streamlined approach to two of Gubeladze's results concerning the  $K$ -theory of toric varieties: Examples of toric varieties with "huge" Grothendieck groups [5] and the "Nilpotence Theorem" [6]. Our proof of this theorem is considerably shorter than the original. On the other hand, our approach and Gubeladze's are cousins in the sense that they have a common ancestor: Cortiñas' proof of the KABI conjecture [1].

Since varieties are locally smooth in the  $cdh$ -topology, it is not surprising that the  $cdh$ -fibrant version of cyclic homology is strongly related to the  $cdh$  cohomology of the sheaf  $\Omega^p$  of Kahler differentials. Theorem 4.1 below shows that the  $cdh$  cohomology of  $\Omega^p$  is computed, for a toric variety  $X$ , by the Zariski cohomology of Danilov's differentials  $\tilde{\Omega}_X^p$ . Since the global sections of  $\Omega_X^p$  and  $\tilde{\Omega}_X^p$  can be computed

Merkurjev-Panin? I am still not sure why Chuck is worried about Merkurjev-Panin. I am fairly sure they only concern themselves with smooth toric varieties. –MW

too much?

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explicitly for toric varieties, we are therefore able to find easily examples of toric varieties with huge Grothendieck groups; see Example 5.12.

Gubeladze’s Nilpotence Theorem (stated and proven in Theorem 6.9 below) asserts, roughly speaking, that after inverting the action of “dilations”, the  $K$ -theory of a toric variety becomes homotopy invariant. Our Theorem 6.6 shows that, after inverting the action of dilations, the global sections of  $\tilde{\Omega}_X^q$  agree with the Hochschild homology groups  $HH_q(X)$ . By the technique of [CHSW], this quickly leads to our new proof of Gubeladze’s theorem.

*Notation.* Throughout this paper, we will adhere to the following notation. Let  $N$  be a free abelian group of rank  $n < \infty$  and let  $M = N^* = \text{Hom}(N, \mathbb{Z})$ . Define  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$  and  $M_{\mathbb{R}} = \text{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R}) \cong M \otimes_{\mathbb{Z}} \mathbb{R}$ . For  $m \in M_{\mathbb{R}}, n \in N_{\mathbb{R}}$ , let  $\langle m, n \rangle$  denote the value of  $m$  at  $n$ . Finally, let  $k$  denote a field of characteristic 0.

## 2. REVIEW OF TORIC VARIETIES

The material in this section may be found in standard texts, such as [F] or [D].

A *strongly convex rational cone* in  $N_{\mathbb{R}}$  is a subset  $\sigma \subset N_{\mathbb{R}}$  that is a cone spanned by finitely many vectors in  $N$  and that contains no lines. That is,  $\sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_k$  for some  $v_1, \dots, v_k \in N \subset N_{\mathbb{R}}$  and whenever both  $u$  and  $-u$  belong to  $\sigma$ , we must have  $u = 0$ . Given such a cone  $\sigma$ , let  $\sigma^{\vee} \subset M_{\mathbb{R}}$  denote the dual cone, defined to consist of those  $m \in M_{\mathbb{R}}$  such that  $\langle m, - \rangle \geq 0$  on  $\sigma$ . Note that  $\sigma^{\vee} \cap M$  is the abelian monoid (under addition of functions) of linear functions with integer coefficients on  $N_{\mathbb{R}}$  whose restrictions to  $\sigma$  are nowhere negative. A *face* of  $\sigma$  is a subset  $\tau$  of the form

$$\boxed{\text{sigma-m}} \quad (2.1) \quad \sigma(m) = \{n \in \sigma \mid \langle m, n \rangle = 0\}$$

for some  $m \in \sigma^{\vee}$ . Observe that a face of strongly convex rational cone is again a strongly convex rational cone. We write  $\tau \prec \sigma$  to indicate that  $\tau$  is a face of  $\sigma$ .

Recall that  $k$  denotes a field of characteristic zero. The affine toric  $k$ -variety associated to a strongly convex rational cone  $\sigma$  is  $U_{\sigma} = \text{Spec } k[\sigma^{\vee} \cap M]$ . We write elements of the monoid ring  $k[\sigma^{\vee} \cap M]$  as  $k$ -linear combinations of the set of formal symbols  $\{\chi^m \mid m \in \sigma^{\vee} \cap M\}$ , so that multiplication in this ring is given on this  $k$ -basis by  $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ .

A *fan* in  $N_{\mathbb{R}}$  is a finite collection of strongly convex rational cones in  $N_{\mathbb{R}}$  such that (1) any face of a cone in  $\Delta$  is again in  $\Delta$  and (2) the intersection of any two cones in  $\Delta$  is a face of each. If  $\tau$  is a face of  $\sigma$ , then  $U_{\tau} \rightarrow U_{\sigma}$  is an open immersion, because the evident map  $k[\sigma^{\vee} \cap M] \rightarrow k[\tau^{\vee} \cap M]$  is given by inverting a finite number of the  $\chi^m$ . It follows that for any fan  $\Delta$ , we may form a scheme  $X(\Delta)$  by patching together the affine schemes  $U_{\sigma}$  corresponding to cones  $\sigma$  along the open subschemes associated to their intersections. We call  $X(\Delta)$  the *toric variety* associated to  $\Delta$ .

*Orbits.* We write  $T_N = \text{Spec } k[M]$  for the  $n$ -dimensional torus associated to  $N$ . Observe that  $T_N$  acts on each  $U_{\sigma}$  — equivalently, the ring  $k[\sigma^{\vee} \cap M]$  is naturally  $M$ -graded with weight  $m$  part being  $k \cdot \chi^m$ , if  $m \in \sigma^{\vee}$ , and 0 if  $m \notin \sigma^{\vee}$ . Since these actions are compatible, the torus  $T_N$  acts on  $X(\Delta)$ .

The orbits of this action are tori, and are in 1–1 correspondence with the the cones of  $\Delta$ ; thus  $X(\Delta)$  is the disjoint union of the orbits  $T_{\tau} = \text{orb}(\tau)$  corresponding to the  $\tau \in \Delta$ . To describe the orbit  $T_{\tau}$ , let  $\mathbb{Z}(\tau \cap N)$  denote the subgroup of  $N$  generated by  $\tau \cap N$ , and let  $\overline{N}$  be the free abelian group  $N/\mathbb{Z}(\tau \cap N)$ . Then

$T_\tau \cong T_{\overline{N}}$ . Note that the orbit corresponding to the minimal cone  $\{0\}$  is the dense open  $\text{orb}(0) = U_0$ , and is naturally isomorphic to  $T_N$ .

We write  $V_\Delta(\sigma)$  for the closure of  $\text{orb}(\sigma)$  in  $X(\Delta)$ . The orbits in  $V_\Delta(\sigma)$  are indexed by the *star* of  $\sigma$ ,  $\text{Star}_\Delta(\sigma)$ , defined as the set of cones in  $\Delta$  containing  $\sigma$ :

$$V_\Delta(\sigma) = \coprod_{\sigma \prec \tau} \text{orb}(\tau).$$

Each orbit-closure  $V_\Delta(\sigma)$  has the structure of a toric variety. To see this, let  $\overline{N} = N/\mathbb{Z}(\sigma \cap N)$ . Then  $\{\bar{\epsilon} \mid \sigma \prec \epsilon\}$  forms a fan in  $\overline{N}_\mathbb{R}$ , and the corresponding toric variety is  $V_\Delta(\sigma)$ . The torus  $T_{\overline{N}}$  is a quotient of  $T_N$  and the inclusion  $V_\Delta(\sigma) \subset X(\Delta)$  is  $T_N$ -equivariant, the action of  $T_N$  on  $V_\Delta(\sigma)$  being induced by the quotient map  $T_N \rightarrow T_{\overline{N}}$ . In the case where  $\Delta$  has a single maximal cone  $\epsilon$ , so that  $\sigma$  is a face of  $\epsilon$ , we have

$$V_\Delta(\sigma) = \text{Spec } k[\epsilon^\vee \cap M \cap \sigma^\perp],$$

and the closed immersion  $V_\Delta(\sigma) \hookrightarrow U_\epsilon$  is given by the ring surjection

$$\text{Spec } k[\epsilon^\vee \cap M] \rightarrow \text{Spec } k[\epsilon^\vee \cap M \cap \sigma^\perp]$$

sending  $\chi^m$  to 0, if  $m \notin \sigma^\perp$ , and to  $\chi^m$ , if  $m \in \sigma^\perp$ .

It is useful to regard the open complement of  $V_\Delta(\sigma)$  in  $X(\Delta)$  as the toric variety corresponding to the largest sub-fan of  $\Delta$  in  $N_\mathbb{R}$  that does not contain  $\tau$ .

Every toric variety is normal, but need not be smooth. A toric variety  $X(\Delta)$  is smooth if and only if, for every cone  $\sigma$  in the fan  $\Delta$ , the minimal lattice points along the 1-dimensional faces (rays) of  $\sigma$  form part of a  $\mathbb{Z}$ -basis of  $N$ . In particular, in order for  $X(\Delta)$  to be smooth, the set of rays of each cone must be  $\mathbb{R}$ -linearly independent (such a cone is said to be *simplicial*).

*Resolution of Singularities.* We will need a detailed description of resolutions of singularities for toric varieties, which we now recall from [F]. If  $v \in N$  is contained in one (or more) of the cones of  $\Delta$ , one may subdivide  $\Delta$  by the ray  $\rho = \mathbb{R}_{\geq 0}v$  through  $v$  to form a new fan  $\Delta'$  in  $N_\mathbb{R}$  as follows: If  $\tau \in \Delta$  does not contain  $\rho$ , then  $\tau$  is also a cone of  $\Delta'$ . For each cone  $\tau \in \Delta$  containing  $\rho$  and for each face  $\nu$  of  $\tau$  not containing  $\rho$ ,  $\Delta'$  contains the cone spanned by  $\rho$  and  $\nu$ :

$$\tilde{\nu} := \nu + \mathbb{R}_{\geq 0}\rho.$$

Finally,  $\rho$  itself belongs to  $\Delta'$ . Thus if  $\sigma \in \Delta$  is the minimal cone of  $\Delta$  containing  $\rho$ , then  $\Delta'$  is the disjoint union of  $\Delta \setminus \text{Star}_\Delta(\sigma)$  and  $\text{Star}_{\Delta'}(\rho)$ .

There is a map of toric varieties  $X' = X(\Delta') \rightarrow X = X(\Delta)$  and it is proper, birational, and equivariant with respect to the action of the torus  $T_N$ . Starting with any toric variety  $X(\Delta)$ , one can arrive at a desingularization of  $X(\Delta)$  by performing a finite number of subdivisions of this type.

Suppose  $\Delta'$  is the fan obtained by subdividing  $\Delta$  by inserting a ray  $\rho$ , and let  $\sigma \in \Delta$  be the minimal cone in  $\Delta$  containing  $\rho$ . Then the description of the orbit-closures given above makes it clear that

$$\begin{array}{ccc} V' = V_{\Delta'}(\rho) & \xrightarrow{i'} & X(\Delta') = X' \\ \downarrow & & \downarrow \pi \\ V = V_\Delta(\sigma) & \xrightarrow{i} & X(\Delta) = X \end{array} \quad (2.2)$$

is an abstract blow-up square. That is, this a pull-back square in which the horizontal arrows are closed immersions and the map on open complements is an isomorphism:

$$X(\Delta') \setminus V_{\Delta'}(\rho) \xrightarrow{\cong} X(\Delta) \setminus V_{\Delta}(\sigma).$$

As with any abstract blow-up, the maps  $\{X(\Delta') \rightarrow X(\Delta), V_{\Delta}(\sigma) \rightarrow X(\Delta)\}$  form a covering for the *cdh*-topology. Recall that the torus  $T_N$  acts on each variety in the above square and each map in this square is  $T_N$ -equivariant.

### 3. DANILOV'S SHEAVES $\tilde{\Omega}^p$

**tOmega**

In this section, we introduce the coherent sheaves  $\tilde{\Omega}_X^p$ , first defined by Danilov [D, 4.2]. We will see in the next section that their Zariski cohomology groups turn out to give the *cdh*-cohomology groups of  $\Omega^p$ .

Given a fan  $\Delta$ , let  $\Delta(1)$  denote the collection of rays in  $\Delta$ ; the 1-skeleton of  $\Delta$  is the fan  $\Delta(1) \cup \{0\}$  and its toric variety  $X^{(1)}$  lies in the smooth locus of  $X(\Delta)$ .

**Definition 3.1.** For a toric  $k$ -variety  $X = X(\Delta)$  defined by a fan  $\Delta$  in  $N_{\mathbb{R}}$ , we define  $\tilde{\Omega}_X^p$  to be the coherent sheaf on  $X$  fitting into the exact sequence

$$0 \rightarrow \tilde{\Omega}_X^p \rightarrow \mathcal{O}_X \otimes_{\mathbb{Z}} \wedge^p(M) \xrightarrow{\delta} \bigoplus_{\rho \in \Delta(1)} \mathcal{O}_{V_{\Delta}(\rho)} \otimes_{\mathbb{Z}} \wedge^{p-1}(M \cap \rho^{\perp}).$$

The component of the map  $\delta$  indexed by  $\rho$  sends  $f \otimes (m_1 \wedge \cdots \wedge m_p)$  in  $\mathcal{O}_X \otimes \wedge_{\mathbb{Z}}^p(M)$  to

$$i^*(f) \otimes \left( \sum_i (-1)^i \langle m_i, n_{\rho} \rangle m_1 \wedge \cdots \wedge \hat{m}_i \wedge \cdots \wedge m_p \right)$$

where  $i : V_{\Delta}(\rho) \hookrightarrow X$  is the canonical closed immersion, and  $n_{\rho} \in N$  is the minimal lattice point on  $\rho$ . By convention,  $\tilde{\Omega}_X^0 = \mathcal{O}_X$ .

On the affine  $U_{\sigma}$ , the ring  $\mathcal{O}(U_{\sigma})$  is  $M$ -graded, so the sections of  $\mathcal{O}_X \otimes \wedge^p M$  are  $M$ -graded with  $\wedge^p M$  in weight 0; the weight  $m$  summand is  $k \cdot \chi^m \otimes \wedge^p M$  if  $m \in \sigma^{\vee}$ . Since  $\delta$  is graded, it follows that each  $\tilde{\Omega}_X^p(U_{\sigma})$  is  $M$ -graded.

**log-poles**

*Remark 3.2.* Sections of  $\tilde{\Omega}_X^1$  may be considered as differential forms on  $X$ , with  $1 \otimes m$  corresponding to the form  $d \log(\chi^m) = d\chi^m / \chi^m$ . On a nonsingular cone  $\sigma$ , we may identify  $\mathcal{O} \otimes \wedge^p M$  with the locally free sheaf  $\Omega^p(\log D)$  of differentials with logarithmic poles along  $D = \cup V(\rho)$ . This identifies the map  $\delta$  with the residue map, so we have  $\Omega^p|_{U_{\sigma}} \cong \tilde{\Omega}^p|_{U_{\sigma}}$ .

As shown by Danilov [D, 4.3], the sheaf  $\tilde{\Omega}_X^p$  is naturally isomorphic to  $j_*(\Omega_U^p)$ , where  $j : U \hookrightarrow X$  is the immersion of the open subscheme  $U$  of smooth points of  $X$ . Applying Remark 3.2 to  $X^{(1)} \hookrightarrow U$ , we see that  $\tilde{\Omega}_X^p = j_*^{(1)}(\Omega_{X^{(1)}}^p)$  where  $j^{(1)} : X^{(1)} \hookrightarrow X$  is the evident open immersion.

We will need an explicit description of the  $M$ -grading on  $\tilde{\Omega}^1$ , or rather on the module of sections  $\tilde{\Omega}^1(U_{\sigma})$  over an affine toric variety  $U_{\sigma}$ . (See [D, 4.2.3].) When  $m \in \sigma^{\vee} \cap M$ , its weight  $m$  summand is the subspace  $\tilde{\Omega}^1(U_{\sigma})_m = k \cdot \chi^m \otimes (M \cap \sigma(m)^{\perp})$  of the weight  $m$  summand  $k \cdot \chi^m \otimes \wedge^p M$  of  $\mathcal{O}(U_{\sigma}) \otimes \wedge^p M$ . Here  $\sigma(m)^{\perp}$  is the orthogonal complement of the face  $\sigma(m)$  of  $\sigma$  defined in (2.1) by the vanishing of

$m$ : For  $m \notin \sigma^\vee$ ,  $\tilde{\Omega}^1(U_\sigma)_m = 0$  because  $\mathcal{O}(U_\sigma)_m = 0$ . More generally, we have for  $m \in M$  and  $p \geq 0$

$$\boxed{\text{E1}} \quad (3.3) \quad \tilde{\Omega}_X^p(U_\sigma)_m = \begin{cases} k \cdot \chi^m \otimes \wedge^p(M \cap \sigma(m)^\perp) & \text{if } m \in \sigma^\vee \\ 0 & \text{if } m \notin \sigma^\vee. \end{cases}$$

It is instructive to compare (3.3) to the analogous formula for  $\Omega^p(U_\sigma)$  and  $HH_p(U_\sigma)$ , which are graded by the submonoid  $\sigma^\vee \cap M$  of  $M$ . There is a natural map from the module  $\Omega_X^p$  of Kähler differentials to  $\tilde{\Omega}_X^p$ . On  $U_\sigma$  it is the  $M$ -graded map induced by the  $M$ -graded map  $\Omega^p(U_\sigma) \rightarrow \mathcal{O}(U_\sigma) \otimes \wedge^p(M)$  defined by:

$$\boxed{\text{E3}} \quad (3.4) \quad \chi^{m_0} d\chi^{m_1} \wedge \cdots \wedge d\chi^{m_p} \mapsto \chi^m \otimes (m_1 \wedge \cdots \wedge m_p), \quad m = \sum m_i.$$

Recall that the orbit-closure  $V(\tau)$  for the face  $\tau$  is  $\text{Spec}(k[\sigma^\vee \cap M \cap \tau^\perp])$ .

HH-m **Lemma 3.5.** *For each  $m \in \sigma^\vee \cap M$ , let  $V = V(\sigma(m))$  denote the orbit-closure for the face  $\sigma(m)$  of  $\sigma$ . Then the closed immersion  $V \subset \sigma$  induces an isomorphism  $HH_*(U_\sigma)_m \cong HH_*(V)_m$ . In particular, for all  $p$ :*

$$\Omega_X^p(U_\sigma)_m = \Omega^p(V)_m$$

*Proof.* For convenience, let us set  $A = \sigma^\vee \cap M$  and  $B = A \cap \sigma(m)^\perp$ , so that  $U_\sigma = \text{Spec}(k[A])$  and  $V(\sigma(m)) = \text{Spec}(k[B])$ . The immersion  $V \subset U_\sigma$  corresponds to a surjection  $k[A] \rightarrow k[B]$ , which is split by the evident inclusion  $\iota: k[B] \rightarrow k[A]$ . Hence  $HH_*(k[B])$  is a summand of  $HH_*(k[A])$ , and it suffices to show that  $\iota$  induces a surjection on the weight  $m$  summand of the complex for Hochschild homology.

Now the degree  $p$  part of the Hochschild complex for  $k[A]$  is  $k[A]^{\otimes p+1}$ , so the weight  $m$  summand has a basis consisting of the  $\chi^{u_0} \otimes \chi^{u_1} \cdots \otimes \chi^{u_p}$  where  $u_i \in A$  and  $\sum u_i = m$ . If  $n \in \sigma(m)$ , then  $\langle u_i, n \rangle \geq 0$  and  $\sum_i \langle u_i, n \rangle = \langle m, n \rangle = 0$ . This forces each  $\langle u_i, n \rangle = 0$ , i.e.,  $u_i \in B$ . Hence  $k[B]_m^{\otimes p+1} = k[A]_m^{\otimes p+1}$ , as claimed.  $\square$

orbitBU **Lemma 3.6.** *Every orbit blow-up square (2.2) determines a distinguished triangle on  $X_{Zar}$  of the form*

$$\tilde{\Omega}_X^p \rightarrow \pi_* \tilde{\Omega}_{X'}^p \oplus i_* \tilde{\Omega}_{V'}^p \rightarrow \pi_* i'_* \tilde{\Omega}_{V'}^p \rightarrow \tilde{\Omega}_X^p[1],$$

and hence a long exact sequence of Zariski cohomology groups:

$$\cdots \rightarrow H^q(X, \tilde{\Omega}^p) \rightarrow H^q(X', \tilde{\Omega}^p) \oplus H^q(V, \tilde{\Omega}^p) \rightarrow H^q(V', \tilde{\Omega}^p) \rightarrow H^{q+1}(X, \tilde{\Omega}^p) \rightarrow \cdots$$

*Proof.* We have short exact sequences of coherent sheaves

$$0 \rightarrow \tilde{\Omega}_{(X,V)}^p \rightarrow \tilde{\Omega}_X^p \rightarrow i_* \tilde{\Omega}_V^p \rightarrow 0$$

on  $X$ , and  $0 \rightarrow \tilde{\Omega}_{(X',V')}^p \rightarrow \tilde{\Omega}_{X'}^p \rightarrow i_* \tilde{\Omega}_{V'}^p \rightarrow 0$  on  $X'$ . Applying  $\mathbb{R}\pi_*$  to the latter yields a morphism of distinguished triangles

$$\begin{array}{ccccc} \tilde{\Omega}_{(X,V)}^p & \longrightarrow & \tilde{\Omega}_X^p & \longrightarrow & i_* \tilde{\Omega}_V^p \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}\pi_* \tilde{\Omega}_{(X',V')}^p & \longrightarrow & \mathbb{R}\pi_* \tilde{\Omega}_{X'}^p & \longrightarrow & \mathbb{R}\pi_* i_* \tilde{\Omega}_{V'}^p \end{array}$$

Danilov proved in [D2, Prop 1.8] that the left vertical map is a quasi-isomorphism, i.e., that  $\mathbb{R}^j \pi_* \tilde{\Omega}_{(X',V')}^p = 0$  for  $j > 0$ , and  $\tilde{\Omega}_{(X,V)}^p \xrightarrow{\simeq} \pi_* \tilde{\Omega}_{(X',V')}^p$ . The distinguished triangle follows from this in a standard way.  $\square$

**ratsing**

*Remark 3.7.* Danilov [D, 8.5.1] proved that if  $\pi : X' \rightarrow X$  is a morphism of toric varieties resulting from a subdivision of the fan, then  $\mathcal{O}_X \xrightarrow{\cong} \mathbb{R}\pi_* \mathcal{O}_{X'}$ , i.e.,  $\pi_* \mathcal{O}_{X'} = \mathcal{O}_X$  and  $R^i \pi_* \mathcal{O}_{X'} = 0$  for  $i > 0$ . This proves that toric varieties have (at most) rational singularities.

#### 4. THE $cdh$ -COHOMOLOGY OF $\Omega^p$ FOR TORIC VARIETIES

In this short section, we prove Theorem 4.1, that Danilov's sheaves compute the  $cdh$ -cohomology groups  $H_{cdh}^*(X, \Omega^p)$  for toric varieties.

**Thm1**

**Theorem 4.1.** *Let  $X$  be an arbitrary toric  $k$ -variety. There is an isomorphism*

$$H_{Zar}^*(X, \tilde{\Omega}_X^p) \cong H_{cdh}^*(X, \Omega^p)$$

for all  $p$ , natural for morphisms of toric varieties and for the closed embedding of an orbit-closure of  $X$  into  $X$ .

**Example 4.2.** The case  $* = 0$  of Theorem 4.1 is that  $\tilde{\Omega}^0(X) \cong H_{cdh}^0(X, \Omega^0)$ . This is equivalent to Danilov's calculation [D2, 1.5] that in (2.2),  $\tilde{\Omega}_X^p \xrightarrow{\cong} \pi_* \tilde{\Omega}_{X'}^p$ , for all  $p$ .

For the proof, we recall that  $H_{cdh}^*(X, \Omega^p)$  is just the Zariski hypercohomology of the complex  $\mathbb{R}a_* a^* \Omega^p|_X$ , where  $a : (Sch/k)_{cdh} \rightarrow (Sch/k)_{Zar}$  is the morphism of sites and  $|_X$  denotes the restriction from the big Zariski site  $(Sch/k)_{Zar}$  to  $X_{Zar}$ . The following Mayer-Vietoris lemma is an immediate consequence of [8, 12.1].

Does this go here?

**RaRpi**

**Lemma 4.3.** *For every  $cdh$  sheaf  $\mathcal{F}$ ,  $\mathbb{R}a_* \mathcal{F}|_X \cong \mathbb{R}\pi_*(\mathbb{R}a_* \mathcal{F}|_{Y_\bullet})$ .*

Recall that we can resolve the singularities of a toric variety via equivariant blow-up squares of the form (2.2). Iterating the orbit blow-up operations described in (2.2), as in [2, 6.2.5] we can find a smooth toric  $cdh$ -hypercover  $\pi : Y_\bullet \rightarrow X$ .

*Proof of Theorem 4.1.* As in [2, 5.2.6], Lemma 3.6 implies that the maps  $\tilde{\Omega}_X^p \rightarrow \mathbb{R}\pi_* \tilde{\Omega}_{Y_\bullet}^p$  are quasi-isomorphisms. By Remark 3.2, the maps  $\Omega_{Y_\bullet}^p \rightarrow \tilde{\Omega}_{Y_\bullet}^p$  are isomorphisms. Hence we have quasi-isomorphisms of complexes of Zariski sheaves:

$$\mathbb{R}\pi_* \Omega_{Y_\bullet}^p \xrightarrow{\cong} \mathbb{R}\pi_* \tilde{\Omega}_{Y_\bullet}^p \xleftarrow{\cong} \tilde{\Omega}_X^p.$$

Now by [CHW, 2.5], we have  $\Omega_{Y_n}^p \cong \mathbb{R}a_* a^* \Omega^p|_{Y_n}$ . Applying Lemma 4.3 to  $\mathcal{F} = a^* \Omega^p$  yields:

$$\mathbb{R}a_* a^* \Omega^p|_X \xrightarrow{\cong} \mathbb{R}\pi_*(\mathbb{R}a_* a^* \Omega^p|_{Y_\bullet}) \cong \mathbb{R}\pi_* \Omega_{Y_\bullet}^p.$$

Applying  $H_{Zar}^*(X, -)$  yields  $H_{cdh}^*(X, \Omega^p) \xrightarrow{\cong} H_{Zar}^*(Y_\bullet, \Omega^p) \cong H_{Zar}^*(X, \tilde{\Omega}^p)$ , an isomorphism which is natural in the pair  $Y_\bullet \rightarrow X$ . As any two smooth toric hypercovers have a common refinement, the isomorphism  $\tilde{\Omega}_X^p \simeq \mathbb{R}a_* a^* \Omega^p|_X$  in the derived category is independent of  $Y_\bullet$ . The asserted naturality follows.  $\square$

Now recall that every variety is locally smooth for the  $cdh$  topology. Hence the Hochschild homology sheaf  $HH_n$  has  $a^* HH_n \cong a^* \Omega^n$ . We write  $\mathbb{H}_{cdh}(X, HH)$  for  $\mathbb{R}a_* a^*$  applied to the Hochschild complex, and  $\mathbb{H}_{cdh}(X, HH^{(t)})$  for its summand in Hodge weight  $t$ . We write the Zariski hypercohomology of these complexes as  $\mathbb{H}_{cdh}^*(X, HH)$  and  $\mathbb{H}_{cdh}^*(X, HH^{(t)})$ , respectively. By [CHW, 2.2],  $\mathbb{H}_{cdh}(X, HH^{(t)}) \cong \mathbb{R}a_* a^* \Omega^t[t]$ . Hence Theorem 4.1 translates into the following language:

**HH/k**

**Corollary 4.4.** *For every toric variety  $X$ ,  $\mathbb{H}_{cdh}^n(X, HH^{(t)}) \cong H_{Zar}^{t+n}(X, \tilde{\Omega}_X^t)$ , and*

$$\mathbb{H}_{cdh}^n(X, HH) \cong \bigoplus_{t \geq 0} H_{Zar}^{t+n}(X, \tilde{\Omega}_X^t).$$

Probably not crystal clear to non-experts. The basic issue: 3.6 deals with ‘‘cubical’’ resolutions, whereas the conclusion you are drawing from it concerns simplicial ones. Perhaps there is a reference that would help? –MW  
How is [HodgeIII]? –CW

The Hochschild homology in 4.4 is taken over any field  $k$  of characteristic zero. Since every toric variety  $X = X_k$  over  $k$  is obtained by base-change from a toric variety  $X_{\mathbb{Q}}$  over the ground field  $\mathbb{Q}$ , flat base-change yields  $\Omega_{X/k}^* \cong \Omega_{X_{\mathbb{Q}}/\mathbb{Q}}^* \otimes_{\mathbb{Q}} k$ , and the Künneth formula yields  $\Omega_{X/\mathbb{Q}}^* = \Omega_{X_{\mathbb{Q}}/k}^* \otimes_{\mathbb{Q}} \Omega_{k/\mathbb{Q}}^* = \Omega_{X/k}^* \otimes_k \Omega_{k/\mathbb{Q}}^*$ . Similar formulas hold for  $HH_*(X/\mathbb{Q})$  and hence for  $\mathbb{H}_{\text{cdh}}^*(X, HH(-/\mathbb{Q}))$ .

We define  $\tilde{\Omega}_{X/\mathbb{Q}}^t$  to be  $j_*\Omega_{X/\mathbb{Q}}^t$ . The above remarks imply that  $\tilde{\Omega}_X^t \cong \tilde{\Omega}_{X_{\mathbb{Q}}/\mathbb{Q}}^t \otimes_{\mathbb{Q}} k$ , and that there is also a Künneth formula  $\tilde{\Omega}_{X/\mathbb{Q}}^* \cong \tilde{\Omega}_X^* \otimes_k \Omega_{k/\mathbb{Q}}^*$ . Hence we have the following variant of the previous corollary.

HH/ $\mathbb{Q}$ 

**Corollary 4.5.** *For every toric  $k$ -variety  $X$ ,*

$$\mathbb{H}_{\text{cdh}}^n(X, HH^{(t)}(-/\mathbb{Q})) \cong H_{\text{Zar}}^{t+n}(X, \tilde{\Omega}_{X/\mathbb{Q}}^t) \cong \bigoplus_{i+j=t} H_{\text{Zar}}^{t+n}(X, \tilde{\Omega}_X^i) \otimes_k \Omega_{k/\mathbb{Q}}^j,$$

and

$$\mathbb{H}_{\text{cdh}}^n(X, HH(-/\mathbb{Q})) \cong \bigoplus_{t \geq 0} H_{\text{Zar}}^{t+n}(X, \tilde{\Omega}_{X/\mathbb{Q}}^t).$$

## 5. $K$ -THEORY AND CYCLIC HOMOLOGY OF TORIC VARIETIES

Recall from section 3 that  $\tilde{\Omega}_X^p$  has both a combinatorial definition, and an interpretation as  $j_*\Omega_U^p$  where  $j : U \hookrightarrow X$  is the inclusion of the smooth locus. In this section, we study the exterior differentiation map  $d : \tilde{\Omega}_X^p \rightarrow \tilde{\Omega}_X^{p+1}$  which arises as the pushforward of the de Rham differential  $d : \Omega_U^p \rightarrow \Omega_U^{p+1}$ . The following combinatorial description of this map is useful.

**Lemma 5.1.** ([D, 4.4]) *The map  $d : \tilde{\Omega}_X^p \rightarrow \tilde{\Omega}_X^{p+1}$  induced by exterior differentiation  $d : \Omega_U^p \rightarrow \Omega_U^{p+1}$  is the  $M$ -graded map which in weight  $m$  is  $k\chi^m \otimes (m_1 \wedge \cdots) \mapsto k\chi^m \otimes (m \wedge m_1 \wedge \cdots)$ . That is, it is induced by:*

$$(\mathcal{O}_X(U_\sigma)_m \otimes_{\mathbb{Z}} \wedge^p M) \cong \wedge^p M \xrightarrow{m \wedge \cdot} \wedge^{p+1} M \cong (\mathcal{O}_X(U_\sigma)_m \otimes_{\mathbb{Z}} \wedge^{p+1} M).$$

Pushing forward the de Rham complex  $\Omega_U^*$ , we see that the  $\tilde{\Omega}_X^p$ 's fit together to form a “log de Rham” complex  $\tilde{\Omega}_X^*$  on  $X$ . There is a natural map  $\Omega_X^* \rightarrow \tilde{\Omega}_X^*$  of complexes, which is an isomorphism on the smooth locus of  $X$ . Similarly, pushing forward the de Rham complex  $\Omega_{U/\mathbb{Q}}^*$  from the smooth locus to  $X$ , we obtain a log de Rham complex  $\tilde{\Omega}_{X/\mathbb{Q}}^*$ .

As in [CHSW] and [CHW],  $\mathbb{H}_{\text{cdh}}(X, HC)$  denotes  $\mathbb{R}a_*a^*$  applied to the cyclic homology cochain complex, and  $\mathbb{H}_{\text{cdh}}(X, HC^{(t)})$  is its summand in Hodge weight  $t$ . The Zariski hypercohomology of these complexes is written as  $\mathbb{H}_{\text{cdh}}^*(X, HC)$  and  $\mathbb{H}_{\text{cdh}}^*(X, HC^{(t)})$ , respectively, and is called the *cdh-fibrant cyclic homology* of  $X$ .

By [CHW, 2.2],  $\mathbb{H}_{\text{cdh}}(X, HC^{(t)}) \cong \mathbb{R}a_*a^*\Omega^{\leq t}[2t]$ , where  $\Omega^{\leq t}$  denotes the brutal truncation of the de Rham complex. Similarly, we write  $\tilde{\Omega}_X^{\leq t}$  for the brutal truncation of the Danilov complex  $\tilde{\Omega}_X^*$ . By Theorem 4.1,  $\mathbb{H}_{\text{cdh}}(X, HC^{(t)}) \cong \tilde{\Omega}_X^{\leq t}[2t]$ .

As with Hochschild homology, the cyclic homology in the above paragraph is taken over  $k$ . As in the previous section, we may also consider cyclic homology taken over the ground field  $\mathbb{Q}$ , and we also have  $\mathbb{H}_{\text{cdh}}(X, HC^{(t)}(-/\mathbb{Q})) \cong \mathbb{R}a_*a^*\Omega_{/\mathbb{Q}}^{\leq t}[2t]$ , again by [CHW, 2.2]. Again by Theorem 4.1, we have an isomorphism in the derived category:  $\mathbb{R}a_*a^*\Omega_{/\mathbb{Q}}^{\leq t} \cong \tilde{\Omega}_{X/\mathbb{Q}}^{\leq t}$ . Concatenating these identifications, we have:

cdhHC

**Proposition 5.2.** *If  $X$  is a toric  $k$ -variety, the cdh-fibrant cyclic homology is given by the formula:*

$$\mathbb{H}_{\text{cdh}}^{-n}(X, HC) \cong \bigoplus_{t \geq 0} H_{\text{Zar}}^{2t-n}(X, \tilde{\Omega}_X^{\leq t}).$$

and

$$\mathbb{H}_{\text{cdh}}^{-n}(X, HC(-/\mathbb{Q})) \cong \bigoplus_{t \geq 0} H_{\text{Zar}}^{2t-n}(X, \tilde{\Omega}_{X/\mathbb{Q}}^{\leq t}).$$

HC0-cdh

**Example 5.3.** The case  $t = 0$  of 5.2 yields the formula

$$HC_n^{(0)}(X) = H_{\text{Zar}}^{-n}(X, \mathcal{O}) \xrightarrow{\cong} H_{\text{cdh}}^{-n}(X, \mathcal{O}) = \mathbb{H}_{\text{cdh}}^{-n}(X, HC^{(0)}).$$

This illustrates the interconnections between the case  $p = 0$  of Theorem 4.1, Danilov's calculation in Remark 3.7, and the convention that  $\tilde{\Omega}_X^0 = \mathcal{O}_X$ .

These calculations tell us about the algebraic  $K$ -theory of toric varieties, via the following translation of [CHW, 1.6] into the present language.

 $\mathcal{F}$  or  $\mathcal{F}_{HC}$ ? -CW

**Definition 5.4.** Let  $\mathcal{F}_{HC}[1]$  denote the mapping cone complex of  $HC(-/\mathbb{Q}) \rightarrow \mathbb{R}a_* a^* HC(-/\mathbb{Q})$ ; the indexing we use is such that there is a long exact sequence:

$$\cdots \rightarrow H^{-n}(X, \mathcal{F}_{HC}) \rightarrow HC_n(X/\mathbb{Q}) \rightarrow \mathbb{H}_{\text{cdh}}^{-n}(X, HC(-/\mathbb{Q})) \rightarrow \cdots$$

FKFHC

**Theorem 5.5.** ([CHW, 1.6]) *For every  $X$  in  $\text{Sch}/k$ , there is a long exact sequence*

$$\cdots \rightarrow KH_{n+1}(X) \rightarrow H_{\text{Zar}}^{-n}(X, \mathcal{F}_{HC}[1]) \rightarrow K_n(X) \rightarrow KH_n(X) \rightarrow \cdots$$

For toric varieties, the sequence (5.5) splits:

Willies

**Proposition 5.6.** *For every toric variety  $X$ ,  $K_*(X) \rightarrow KH_*(X)$  is a split surjection. Hence*

$$K_n(X) \cong KH_n(X) \oplus H_{\text{Zar}}^{-n}(X, \mathcal{F}_{HC}[1]).$$

*Proof.* For each affine cone  $\sigma$ ,  $M(\sigma) := M \cap \sigma^\perp$  is a free abelian monoid, so  $T_\sigma = \text{Spec}(k[M(\sigma)])$  is a torus. We first claim that the inclusion  $i_\sigma : k[M(\sigma)] \hookrightarrow k[M \cap \sigma^\vee]$ , or  $U_\sigma \rightarrow T_\sigma$ , induces an isomorphism on  $KH$ -theory, *i.e.*,

eq:KHT

$$(5.7a) \quad K(T_\sigma) \xrightarrow{\cong} KH(T_\sigma) \xrightarrow{\cong} KH(U_\sigma).$$

Since (5.7a) factors  $K(T_\sigma) \rightarrow K(U_\sigma) \rightarrow KH(U_\sigma)$ , this proves the lemma for  $U_\sigma$ .

Because  $T_\sigma$  is regular, the first map is an isomorphism. For a suitable rational  $n \in \sigma$ , evaluation at  $n$  is a monoid map from  $M \cap \sigma^\vee$  to  $\mathbb{N}$  with kernel  $M(\sigma)$ . This gives  $k[M \cap \sigma^\vee]$  the structure of an  $\mathbb{N}$ -graded algebra with  $k[M(\sigma)]$  in degree zero. Hence  $i_\sigma$  induces an isomorphism  $KH(k[M(\sigma)]) \cong KH(k[M \cap \sigma^\vee])$ , as claimed.

If  $\tau$  is a face of  $\sigma$ , we have a commutative diagram

$$\begin{array}{ccc} k[M(\sigma)] & \longrightarrow & k[M \cap \sigma^\vee] \\ \text{into} \downarrow & & \downarrow \text{into} \\ k[M(\tau)] & \longrightarrow & k[M \cap \tau^\vee]. \end{array}$$

Thus the isomorphism in (5.7a) is natural in  $\sigma$ , for  $\sigma$  a face of a fan  $\Delta$ , and so is the splitting of  $K(U_\sigma) \rightarrow KH(U_\sigma)$ . Since  $K(X)$  is the homotopy limit over  $\Delta$  of the  $K(U_\sigma)$ , and similarly for  $KH(X)$ , the homotopy limit of the splittings provides a splitting of the map  $K(X) \rightarrow KH(X)$ .  $\square$

The remark is not needed at all in the proof, but maybe it makes things clearer

*Remark 5.7.1.* The proof amounts to the observation that there is an algebraic homotopy from  $U_\sigma$  onto its smallest orbit  $\text{orb}(\sigma)$ , and that this homotopy is natural with respect to face inclusions.



The sequence (5.5) is compatible with the decomposition arising from the Adams operations;  $K_*^{(i)}(X)$  and  $KH_*^{(i)}(X)$  fit into a long exact sequence with  $\mathcal{F}_{HC}^{(i-1)}$ . For example, it is immediate from Example 5.3 that  $\mathcal{F}_{HC}^{(0)}(X)$  is acyclic, proving that  $K_*^{(1)}(X) \cong KH_*^{(1)}(X)$  for toric varieties. The case  $* = 0$ , which is a well known assertion about the Picard group of normal varieties, has the following extension:

**K0** **Proposition 5.8.** *If  $X = U_\sigma$  is an affine toric  $k$ -variety, then  $K_0(X) = \mathbb{Z}$ .*

*Proof.* Note that the coordinate ring of  $U_\sigma$  is graded, so  $KH_0(X) = \mathbb{Z}$ . By 5.5, we need to show that  $H^0(X, \mathcal{F}_{HC}) = 0$ . Since  $HC_{-1}(X) = 0$ , we are reduced to proving that the map

$$HC_0(X) \rightarrow \mathbb{H}_{\text{cdh}}^0(X, HC)$$

is onto. By 5.2, the target of this map is  $\bigoplus_{t \geq 0} H_{\text{Zar}}^{2t}(X, \tilde{\Omega}_{\mathbb{Q}}^{\leq t})$ . Since  $X$  is affine, we have  $H_{\text{Zar}}^{2t}(X, \tilde{\Omega}_{\mathbb{Q}}^{\leq t}) = 0$  for all  $t > 0$ . Finally, when  $t = 0$  we have

$$H_{\text{Zar}}^0(X, \tilde{\Omega}_{\mathbb{Q}}^{\leq 0}) = H_{\text{Zar}}^0(X, \mathcal{O}_X) = HC_0(X). \quad \square$$

*Remark 5.9.* A much better version of this Corollary was proven years ago by Gubeladze [3]: For a PID  $R$ , every finitely projective module over  $R[A]$ , where  $A$  is a semi-normal, abelian, cancellative monoid without non-trivial units, is free. This was extended to the case where  $R$  is regular by Swan [S].

Of course, the dictionary coming from [CHW] via 5.5 also allows us to say something about the higher  $K$ -theory of toric varieties. Let  $K_n^{(i)}(X)$  denote the weight  $i$  part of  $K_n(X) \otimes \mathbb{Q}$  with respect to the Adams operations, *i.e.*, the eigenspace where  $\psi^k = k^i$  for all  $k$ . We adopt the parallel notation  $KH_n^{(i)}(X)$  for the weight  $i$  part of  $KH_n(X)$ .

The absolute cotangent sheaf  $\mathbb{L}_X$  of  $X/\mathbb{Q}$  has  $\mathbb{L}_X^{\geq 0} = \Omega_{X/\mathbb{Q}}^1$  and  $H^{1-n}(X, \mathbb{L}_X) = HH_n^{(1)}(X/\mathbb{Q})$ ; see [12, 8.8.9]. There is a natural map  $\mathbb{L}_X \rightarrow \Omega_{X/\mathbb{Q}}^1 \rightarrow \tilde{\Omega}_{X/\mathbb{Q}}^1$ .

**cor3** **Corollary 5.10.** *For any toric  $k$ -variety  $X$ , we have a distinguished triangle*

$$\mathcal{F}_{HC}^{(1)} \rightarrow \mathbb{L}_X \rightarrow \tilde{\Omega}_{X/\mathbb{Q}}^1 \rightarrow \mathcal{F}_{HC}^{(1)}[1],$$

*and hence an isomorphism  $K_q^{(2)}(X) \cong KH_q^{(2)}(X) \oplus H_{\text{Zar}}^{2-q}(X, \mathbb{L}_X \rightarrow \tilde{\Omega}_{X/\mathbb{Q}}^1)$ .*

*Proof.* The Zariski sheaf  $HC^{(1)}$  is the mapping cone of  $\mathcal{O} \rightarrow \mathbb{L}_X$ ; see [12, 9.8.18]. Since  $\mathbb{R}a_*(a^*\mathcal{O})|_X = \mathcal{O}_X$  by Remark 3.7, and  $\mathbb{H}_{\text{cdh}}(X, HC^{(1)}) \simeq (\mathcal{O} \rightarrow \tilde{\Omega}_X^1)[2]$  by 5.2, it follows that the mapping cone  $\mathcal{F}_{HC}^{(1)}$  of  $HC^{(1)} \rightarrow \mathbb{H}_{\text{cdh}}(X, HC^{(1)})$  is also the mapping cone of  $\mathbb{L}_X \rightarrow \tilde{\Omega}_X^1$ . This proves the first assertion; the second assertion follows from this, Proposition 5.6 and [CHW, 1.6], which refines Theorem 5.5.  $\square$

The techniques of [CHW] allow us to find examples of toric varieties with “huge”  $K_0$  and  $K_1$  groups, in the spirit of Gubeladze [5]. Our toric varieties will have quotient singularities because all the cones will be simplices; see [F].

**hugeK1** **Example 5.11.** Let  $N = \mathbb{Z}^3$ , and let us to agree to write elements of  $N$  as column vectors and elements of  $M \cong \mathbb{Z}^3$  as row vectors. Define  $\tau$  to be the cone in the  $xy$ -

plane of  $N_{\mathbb{R}} = \mathbb{R}^3$  spanned by the vectors  $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $e_1 + 2e_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ . Then  $U_\tau$  is a

singular, affine toric  $k$ -variety. In fact,  $U_\tau = \text{Spec}(k[X, Y, Z]/(YZ - X^2)[T, T^{-1}])$ , where  $X = \chi^{(1,0,0)}$ ,  $Y = \chi^{(0,1,0)}$ ,  $Z = \chi^{(2,-1,0)}$  and  $T^{\pm 1} = \chi^{(0,0,\pm 1)}$ . This is because  $\tau^\vee \cap M$  is generated by the vectors  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(2, -1, 0)$  and  $(0, 0, \pm 1)$ .

Let  $m \in M$  be the vector  $(1, 0, 0)$ . Its face is  $\tau(m) = \{0\}$ , so  $\tau(m)^\perp = M$ . We see from (3.3) that  $\tilde{\Omega}^1(U_\tau)_m = k \cdot X \otimes M \cong k^3$ . The forms  $dX$ ,  $X dY/Y$  and  $X dT/T$  form a basis. On the other hand,  $\Omega^1(U_\tau)_m$  is the  $k$ -vector space spanned by  $\chi^u d(\chi^v)$  with  $u, v \in \tau^\vee \cap M$  satisfying  $u + v = m$ . It is easy to see that the only  $u, v \in \tau^\vee \cap M$  satisfying  $u + v = (1, 0, 0)$  are when  $u, v$  is  $\{(0, 0, -j), (1, 0, j)\}$ . Thus  $\Omega^1(U_\tau)_m$  is the 2-dimensional vector space spanned by  $dX$  and  $X dT/T$ . It follows that  $\Omega^1(U_\tau) \rightarrow \tilde{\Omega}^1(U_\tau)$  is not onto in weight  $m$ .

Similar reasoning shows that for  $m = (1, 0, c)$  we also have  $\tilde{\Omega}^1(U_\tau)_m \cong k^3$  on  $T^c dX$ ,  $T^c X dY/Y$  and  $T^{c-1} X dT$ , and that  $\tilde{\Omega}^1(U_\tau)_m = \Omega^1(U_\tau)$  for all other  $m$ . (It is useful to use the fact that  $\Omega^1(U_\tau)$  is a submodule of  $\tilde{\Omega}^1(U_\tau)$  by [9].) Thus  $\tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau) \cong k[T, T^{-1}]$ . By the Künneth formula,

$$\text{coker}\{\Omega^1(U_\tau/\mathbb{Q}) \rightarrow \tilde{\Omega}^1(U_\tau/\mathbb{Q})\} \cong \tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau).$$

As in Proposition 5.6, it is easy to see that  $KH_*(U_\tau) \cong K_*(k[T, T^{-1}])$ . Hence 5.10 implies that  $K_1^{(2)}(U_\tau)$  is isomorphic to a nonzero  $k$ -vector space:

$$K_1^{(2)}(U_\tau) \cong H_{\text{Zar}}^1(U_\tau, \Omega^1 \rightarrow \tilde{\Omega}^1) \cong \tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau) \cong k[T, T^{-1}].$$

huge

**Example 5.12.** We now extend the  $\tau$  of Example 5.11 to form a fan  $\Delta$  consisting of two 3-dimensional cones  $\sigma_1, \sigma_2$  (together with all of their faces) such that  $\sigma_1 \cap \sigma_2 = \tau$ . Specifically, let  $\sigma_1$  and  $\sigma_2$  be spanned by the two edges of  $\tau$  together with

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix},$$

respectively. Let  $X = X(\Delta)$ , so  $X = U_{\sigma_1} \cup U_{\sigma_2}$  and  $U_\tau = U_{\sigma_1} \cap U_{\sigma_2}$ . It follows from 5.6 that  $KH_0(X) = \mathbb{Z} \oplus \mathbb{Z}$  and that

$$K_0(X) \cong \mathbb{Z}^2 \oplus H_{\text{Zar}}^1(X, \mathcal{F}_{HC}).$$

We will show that the right-hand term is nonzero; since it is a  $k$ -vector space, it will follow that  $K_0(X)$  contains additive group underlying a non-zero  $k$ -vector space. Taking  $k$  to be uncountable, for example  $k = \mathbb{C}$ , we see  $K_0(X)$  is uncountable.

Because the singular locus of  $X$  is 1-dimensional,  $H^n(X, \mathbb{L}_X) = H^n(X, \Omega_X^1)$  for  $n > 0$ . By Corollary 5.10,

$$K_0^{(2)}(X) = H_{\text{Zar}}^1(X, \mathcal{F}_{HC}) = H_{\text{Zar}}^2(X, \Omega^1 \rightarrow \tilde{\Omega}^1).$$

From the Mayer-Vietoris sequence for the given cover of  $X$ , and Proposition 5.8, we see that there is an exact sequence

$$\tilde{\Omega}^1(U_{\sigma_1})/\Omega^1(U_{\sigma_1}) \oplus \tilde{\Omega}^1(U_{\sigma_2})/\Omega^1(U_{\sigma_2}) \rightarrow \tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau) \rightarrow K_0^{(2)}(X) \rightarrow 0.$$

By Example 5.11, it suffices to consider the forms  $T^c X dY/Y$  in weight  $m = (1, 0, c)$ ,  $c \in \mathbb{Z}$ . Note that  $\tau(m) = \{0\}$ . If  $c > 0$  then  $m \in \sigma_1^\vee$  and the element  $\chi^m dY/Y \in \tilde{\Omega}^1(U_{\sigma_1})$  maps to  $T^c X dY/Y \in \tilde{\Omega}^1(U_\tau)$ . If  $c < 0$  then  $m \in \sigma_2^\vee$  and the element  $\chi^m dY/Y \in \tilde{\Omega}^1(U_{\sigma_2})$  maps to  $T^c X dY/Y \in \tilde{\Omega}^1(U_\tau)$ .

We are left with the form  $X dY/Y$  in weight  $m = (1, 0, 0)$ . Since  $m \notin \sigma_i^\vee$  for  $i = 1, 2$ , we have  $\tilde{\Omega}^1(U_{\sigma_1})_m = \tilde{\Omega}^1(U_{\sigma_2})_m = 0$ . This proves the claim that

$$K_0^{(2)}(X) \cong \tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau)_m \cong k.$$

As in Gubeladze's example of toric varieties with "huge" Grothendieck groups, we can further extend  $\Delta$  to obtain a complete fan consisting of simplicial cones  $\overline{\Delta}$ , so that  $\overline{X} = X(\overline{\Delta})$  is a projective closure of  $X$  and such that  $Y = X(\overline{\Delta} - \Delta)$  is smooth. Since  $Y$  and  $X$  form an open cover of  $\overline{X}$ , we see that  $K_0(\overline{X})$  also contains the additive group underlying a non-zero  $k$ -vector space.

## 6. GUBELADZE'S NILPOTENCE THEOREM

The main goal of this section is to give a new proof of Gubeladze's "Nilpotence Theorem" [6] for the  $K$ -theory of toric varieties. This is done in Theorem 6.9.

For a toric variety  $X = X(\Delta)$  with  $\Delta$  a fan in  $N_{\mathbb{R}}$  and integer  $c \in \mathbb{N}$ , define  $\theta_c : X(\Delta) \rightarrow X(\Delta)$  to be the endomorphism of toric varieties induced by the endomorphism of the lattice  $N$  given by multiplication by  $c$ . If  $\sigma \subset N_{\mathbb{R}}$  is a cone, the map  $\theta_c : U_\sigma \rightarrow U_\sigma$  of affine toric  $k$ -varieties is induced by the ring endomorphism of  $k[\sigma^\vee \cap M]$  that sends  $\chi^m$  to  $\chi^{cm}$ . That is, this is the map that raises all monomials to the  $c$ -th power. Observe that if  $k = \mathbb{F}_p$  and  $c = p$ , this is precisely the Frobenius endomorphism, and it useful to think of  $\theta_c$  as a generalization of Frobenius that exists in the category of toric varieties.

Fix a sequence  $\mathbf{c} = (c_1, c_2, \dots)$  of integers with  $c_i \geq 2$  for all  $i$ . If  $F$  is a contravariant functor from toric varieties to abelian groups, we let

$$F(X)^\mathbf{c} = \varinjlim \left( F(X) \xrightarrow{\theta_{c_1}^*} F(X) \xrightarrow{\theta_{c_2}^*} \dots \right).$$

Gubeladze's Nilpotence Theorem asserts that the natural map  $K_*(X) \rightarrow KH_*(X)$  induces an isomorphism  $K_*(X)^\mathbf{c} \rightarrow KH_*(X)^\mathbf{c}$  for any toric variety  $X$ . Our proof of this theorem involves computing  $HH_q(X)^\mathbf{c}$  where  $HH_*$  denotes Hochschild homology.

Fix a cone  $\sigma$ . As in the proof of Lemma 3.5, the chain complex defining the Hochschild homology of  $k[\sigma^\vee \cap M]$  is  $\sigma^\vee \cap M$ -graded with the weight of  $\chi^{m_0} \otimes \dots \otimes \chi^{m_p}$  defined to be  $m_0 + \dots + m_p$ , and the Hochschild homology groups of  $U_\sigma$  are  $\sigma^\vee \cap M$ -graded  $k[\sigma^\vee \cap M]$ -modules. A fortiori, they are  $M$ -graded, with zero in weight  $m$  if  $m \notin \sigma^\vee$ . Since  $\theta_c(\chi^{m_0} \otimes \dots) = \chi^{cm_0} \otimes \dots$ ,  $\theta_c$  sends the weight  $m$  summand to the weight  $cm$  summand.

The Hochschild homology of a non-affine variety is defined by taking Zariski hypercohomology of the sheafification of the complex defined just as in the definition of  $HH_*(R)$ , but with  $\mathcal{O}_X \otimes_k \dots \otimes_k \mathcal{O}_X$  in place of  $R \otimes_k \dots \otimes_k R$  (see [10, 4.1]).

For a toric variety  $X = X(\Delta)$ , we may compute  $HH_*(X)$  as follows: Let  $\sigma_1, \dots, \sigma_m$  denote the maximal cones in the fan  $\Delta$ . For each  $1 \leq i_0 \leq \dots \leq i_p \leq m$ , we may form the complex defining the Hochschild homology of the affine toric variety  $U_{\sigma_{i_0} \cap \dots \cap \sigma_{i_p}}$ . We then assemble these into a bicomplex in the usual Čech manner and take the homology of the associated total complex.

**Mgraded**

**Lemma 6.1.** *For any toric variety  $X = X(\Delta)$ , the groups  $HH_*(X)$  have a natural  $M$ -grading, and the endomorphism  $\theta_c$  maps the weight  $m$  summand to the weight  $cm$  summand.*

*Proof.* We have seen that the Hochschild complexes forming the columns of the bi-complex are  $M$ -graded. Since the ring maps are all  $M$ -graded, the Čech differentials are also  $M$ -graded. Since  $HH_*(X)$  is the homology of an  $M$ -graded bicomplex, it is  $M$ -graded. Since the map  $\theta_c$  sends the weight  $m$  subcomplex to the weight  $cm$  subcomplex, it has the same effect on homology.  $\square$

**CechSS**

*Remark 6.1.1.* This construction implies that the Čech spectral sequence is  $M$ -graded:

$$E_{pq}^1 = \bigoplus_{i_0 < \dots < i_p} HH_q(U_{\sigma_{i_0} \cap \dots \cap \sigma_{i_p}}) \Rightarrow HH_{q-p}(X).$$

**invert-m**

**Lemma 6.2.** *Set  $A = \sigma^\vee \cap M$ . If  $m \in A$  lies on no proper face of  $\sigma^\vee$ , then  $A + \langle -m \rangle = M$ , and  $k[A][\chi^{-m}] = k[M]$ .*

*Proof.* Since  $k[A][\chi^{-m}] = k[A + \langle -m \rangle]$ , it suffices to prove the first assertion, *i.e.*, that every  $t \in M$  is of the form  $a - Nm$  for some positive integer  $N$ . Fix a nonzero  $n \in N$ . The assumption that  $m$  lies on no proper face of  $\sigma^\vee$  implies that  $\langle m, n \rangle > 0$ . Hence  $\langle t + Nm, n \rangle > 0$  for  $n \gg 0$ . Since  $\sigma \cap N$  is finitely generated, it follows that  $t + Nm \in A$  for  $N \gg 0$ , as claimed.  $\square$

**lem:scale**

**Lemma 6.3.** *The map  $\theta_c : \Omega^q(U_\sigma)_m \rightarrow \Omega^q(U_\sigma)_{cm}$  is multiplication by  $c^q \chi^{(c-1)m}$ .*

*Proof.* When  $\sum u_i = m$ ,  $\theta_c$  takes  $\omega = \chi^{u_0} d\chi^{u_1} \wedge \dots \wedge d\chi^{u_q}$  to  $c^q \chi^m \omega$ .  $\square$

**t-scale**

*Remark 6.3.1.* The same proof shows that  $\theta_c : \tilde{\Omega}^q(U_\sigma)_m \rightarrow \tilde{\Omega}^q(U_\sigma)_{cm}$  is multiplication by  $c^q \chi^{(c-1)m}$ . By (3.3), this is an isomorphism for all  $c \neq 0$ .

**cOmega**

**Proposition 6.4.** *For any toric  $k$ -variety  $X$ , the natural maps (3.4) induce isomorphisms, for all  $q$ :*

$$\Omega^q(X)^c \rightarrow \tilde{\Omega}^q(X)^c$$

*Proof.* We may assume  $X = U_\sigma$ , so that  $\Omega^q(X) = \Omega_{k[A]}^q$  for  $A = \sigma^\vee \cap M$ . It suffices to check that the map is an isomorphism in each weight  $m \in M_c$ ; without loss of generality, one may assume  $m \in M$ . By Lemma 3.5,  $(\Omega_{k[A]}^q)_m \cong (\Omega_{k[B]}^q)_m$ , where  $B = A \cap \sigma(m)^\perp$ . By Lemma 6.3,  $\theta_c$  coincides with multiplication by  $c^q \chi^{(c-1)m}$  both as a map  $(\Omega_{k[A]}^q)_m \rightarrow (\Omega_{k[A]}^q)_{cm}$  and as a map  $(\Omega_{k[B]}^q)_m \rightarrow (\Omega_{k[B]}^q)_{cm}$ . Hence the group

$$\Omega^q(X)_m^c = \varinjlim \left( (\Omega_{k[A]}^q)_m \xrightarrow{\theta_{c_1}} (\Omega_{k[A]}^q)_{c_1 m} \xrightarrow{\theta_{c_2}} \dots \right)$$

is the weight  $m$  part of the localization of  $\Omega_{k[B]}^q$  at  $\chi^m$ , *i.e.*, of  $\Omega^q(k[B][\chi^{-m}])$ . By construction,  $m$  is not on any proper face of  $\sigma(m)^\vee \cap \sigma(m)^\perp$ . By Lemma 6.2,

$$\Omega^q(k[B][\chi^{-m}])_m \cong \Omega^q(k[B + \langle -m \rangle])_m = \Omega^q(k[T])_m, \quad T = M \cap \sigma(m)^\perp.$$

Since  $T$  is a free abelian group,  $(\Omega_{k[T]}^q)_m \cong \wedge^q(T)$ . Now recall that by Remark 6.3.1 and (3.3) we also have

$$(\tilde{\Omega}_{k[T]}^q)_m^c \cong \tilde{\Omega}^q(U_\sigma)_m^c = \tilde{\Omega}^q(U_\sigma)_m \cong k \cdot \chi^m \otimes \wedge^q(T),$$

The map  $(\Omega_{k[T]}^q)_m \rightarrow (\tilde{\Omega}_{k[T]}^q)_m$  is given by (3.4), and it is an isomorphism by inspection.  $\square$

add details?

In order to prove an analogous result for Hochschild homology, we need to briefly review the decomposition of Hochschild homology into summands given by the (higher) André-Quillen homology groups. For more details, we refer the reader to [7, 3.5] or [12, 8.8].

For a commutative  $k$ -algebra  $R$ , one forms a simplicial polynomial  $k$ -algebra  $R_\bullet$  and a simplicial ring map  $R_\bullet \rightarrow R$  which is a homotopy equivalence on underlying simplicial sets. The (higher) cotangent complex  $\mathbb{L}_{X/k}^{(q)}$  is defined to be the simplicial  $R$ -module  $R \otimes_{R_\bullet} \Omega_{R_\bullet}^q$ , and the André-Quillen homology groups of  $R$  are defined to be  $D_p^{(q)}(R) = H_p(\mathbb{L}_{X/k}^{(q)})$ . The  $R$ -modules  $D_p^{(q)}(R)$  are independent up to isomorphism of the choices made. In general, there is a natural spectral sequence of  $R$ -modules

$$D_p^{(q)}(R) \implies HH_{p+q}(R)$$

and a natural  $R$ -module isomorphism  $D_0^{(q)}(R) \cong \Omega_{R/k}^q$ . Since we are assuming  $\text{char}(k) = 0$ , this spectral sequence degenerates to give a natural decomposition of  $R$ -modules

$$HH_n(R) \cong \bigoplus_{p+q=n} D_p^{(q)}(R) = \Omega_{R/k}^q \oplus \bigoplus_{p+q=n, p>0} D_p^{(q)}(R).$$

Since the André-Quillen homology groups are functorial for ring maps, the endomorphisms  $\theta_{c_i}$  preserve this decomposition.

AQscale

**Lemma 6.5.** *Let  $U_\sigma$  be an affine toric variety. Then the  $D_p^q(U_\sigma)$  are  $M$ -graded modules and, for every  $m \in \sigma^\vee \cap M$ , the map  $\theta_c : D_p^q(U_\sigma)_m \rightarrow D_p^q(U_\sigma)_{cm}$  is multiplication by  $c^q \chi^{(c-1)m}$ .*

*Proof.* Let  $A = \sigma^\vee \cap M$  and form a simplicial resolution of  $A$  by free abelian monoids  $A_\bullet \rightarrow A$ . That is,  $A_\bullet$  is a simplicial abelian monoid which in each degree is free abelian and the map of simplicial abelian monoids  $A_\bullet \rightarrow A$  is a homotopy equivalence. This is possible by the same basic cotriple resolution used to form simplicial free resolutions of  $k$ -algebras (see [12, 8.6]). For functorial reasons,  $k[A_\bullet] \rightarrow k[A]$  is a free simplicial resolution of  $k[A]$ . We therefore have

$$D_p^{(q)}(k[A]) = H_p(k[A]) \otimes_{k[A_\bullet]} \Omega_{k[A_\bullet]}^q.$$

For each  $n$ , the ring  $k[A_n]$  is  $M$ -graded by the maps  $\delta_n : A_n \rightarrow A \subset M$ . Thus the simplicial ring  $k[A_\bullet]$  is also  $M$ -graded and the map  $k[A_\bullet] \rightarrow k[A]$  of simplicial rings preserves this grading. It follows that  $k[A] \otimes_{k[A_\bullet]} \Omega_{k[A_\bullet]}^q$  is naturally  $M$ -graded, where the weight of  $\chi^{u_0} \otimes d(\chi^{u_1}) \wedge \cdots \wedge d(\chi^{u_q})$  is  $u_0 + \delta_n(u_1) + \cdots + \delta_n(u_q)$ , for any  $u_0 \in A$  and  $u_1, \dots, u_q \in A_n$ . Hence  $D_p^{(q)}(k[A])$  is an  $M$ -graded  $k[A]$ -module, and it is clear that, for any positive integer  $c$ , the endomorphism  $\theta_c$  of  $D_p^{(q)}(k[A])$  maps the weight  $m$  summand to the weight  $cm$  summand. To prove that the map

$$\theta_c : D_p^{(q)}(k[A])_m \rightarrow D_p^{(q)}(k[A])_{cm}$$

coincides with multiplication by  $c^q \chi^{(c-1)m}$ , it suffices to prove the analogous assertion for the  $M$ -graded  $k[A]$ -modules  $k[A] \otimes_{k[A_n]} \Omega_{k[A_n]}^q$ . The proof of this is exactly like the proof of Lemma 6.3, using  $\omega = \chi^{u_0} \otimes d\chi^{u_1} \wedge \cdots \wedge d\chi^{u_q}$ .  $\square$

Thm2

**Theorem 6.6.** *For any toric  $k$ -variety  $X$ , the natural maps*

$$\Omega^q(X)^c \rightarrow HH_q(X)^c$$

*are isomorphisms, for all  $q$ .*

Do we need this? Probably not, but I have not checked carefully. In any case, such a decomposition exists and is nice to know about. -MW

8/4 D The way this equation was before is correct. Indeed, this one makes no sense: what does  $H_p(k[A])$  even mean? -MW

*Proof.* By the spectral sequence in 6.1.1, we may assume that  $X$  is affine, say of the form  $X = U_\sigma$  for some cone  $\sigma$ . Setting  $A = \sigma^\vee \cap M$ , the coordinate ring of  $X$  is  $k[A]$ . To establish the isomorphism  $\Omega^p(U_\sigma)^\mathfrak{c} \cong HH_p(U_\sigma)^\mathfrak{c}$  it suffices to prove that

$$D_p^{(q)}(k[\sigma^\vee \cap M])^\mathfrak{c} = 0$$

for all  $p > 0$ . As in the proof of Proposition 6.4, it suffices to fix an arbitrary  $m \in M$  and show that the weight  $m$  part vanishes. By Lemma 3.5,  $D_p^{(q)}(k[A])_m \cong D_p^{(q)}(k[B])_m$ , where  $B = A \cap \sigma(m)^\perp$ . By Lemma 6.5,  $\theta_c$  coincides with multiplication by  $c^q \chi^{(c-1)m}$  both as a map  $D_p^{(q)}(k[A])_m \rightarrow D_p^{(q)}(k[A])_{cm}$  and as a map  $D_p^{(q)}(k[B])_m \rightarrow D_p^{(q)}(k[B])_{cm}$ . Hence the weight  $m$  summand

$$D_p^{(q)}(X)_m^\mathfrak{c} = \varinjlim \left( D_p^{(q)}(k[A])_m \xrightarrow{\theta_{c_1}} D_p^{(q)}(k[A])_{c_1 m} \xrightarrow{\theta_{c_2}} \dots \right)$$

is the weight  $m$  part of the localization of  $D_p^{(q)}(k[B])$  at  $\chi^m$ , *i.e.*, of  $D_p^{(q)}(k[B][\chi^{-m}])$ . Recall that  $\sigma(m) \subset \sigma$  denotes the face of  $\sigma$  (possibly just the origin) on which  $m = 0$ . By Lemma 6.2,

$$D_p^{(q)}(k[B][\chi^{-m}])_m \cong D_p^{(q)}(k[B + \langle -m \rangle])_m = D_p^{(q)}(k[T])_m, \quad T = M \cap \sigma(m)^\perp.$$

Since  $T = M \cap \sigma(m)^\perp$  is a free abelian group, we have

$$D_p^{(q)}(k[B][\frac{1}{\chi^m}]) = D_p^{(q)}(k[T]) = 0$$

for all  $p > 0$ . This proves that  $D_p^{(q)}(k[A])^\mathfrak{c} = 0$  for all  $p > 0$ , proving the theorem.  $\square$

**cor2** **Corollary 6.7.** *For any field  $k$  of characteristic 0 and any toric  $k$ -variety  $X$ , we have a natural isomorphism for all  $n$ :*

$$HH_n(X/\mathbb{Q})^\mathfrak{c} \xrightarrow{\cong} \mathbb{H}_{cdh}^{-n}(X, HH(-/\mathbb{Q}))^\mathfrak{c}.$$

The right hand side in 6.7 denotes Hochschild homology with *cdh* descent imposed (and localized by  $\mathfrak{c}$ ). (On both sides, we take Hochschild homology over  $\mathbb{Q}$ .)

*Proof.* Let us write  $X_\mathbb{Q}$  for the model of  $X$  defined over the rationals and  $X_k = X$  for the model over  $k$ . We have  $X_k = X_\mathbb{Q} \times_{\text{Spec } \mathbb{Q}} \text{Spec } k$ .

The natural map

$$HH_n(X_{k/k})^\mathfrak{c} \longrightarrow \mathbb{H}_{cdh}^{-n}(X, HH)^\mathfrak{c}$$

is an isomorphism. Since both sides satisfy Zariski descent, this is an immediate consequence Theorem 4.1 and Theorem 6.6. The Künneth formula for Hochschild homology, described before Corollary 4.5, gives

$$HH_*(X/\mathbb{Q})^\mathfrak{c} \cong HH_*(X_\mathbb{Q}/\mathbb{Q})^\mathfrak{c} \otimes_{\mathbb{Q}} \Omega_{k/\mathbb{Q}}^*.$$

In particular, one gets long exact sequences for  $HH_*(-/\mathbb{Q})^\mathfrak{c}$  associated to abstract blow-ups of toric  $k$ -varieties. Since the map

$$HH_n(X_k/\mathbb{Q})^\mathfrak{c} \cong \mathbb{H}_{cdh}^{-n}(X, HH(-/\mathbb{Q}))^\mathfrak{c}$$

is an isomorphism whenever  $X$  is smooth by [CHW, 2.4], the result holds by induction and the five-lemma.  $\square$

**Corollary 6.8.** *For any field  $k$  of characteristic 0 and any toric  $k$ -variety  $X$ , and all  $n$ , we have*

$$HC_n(X/\mathbb{Q})^\epsilon \cong \mathbb{H}_{cdh}^{-n}(X, HC(-/\mathbb{Q}))^\epsilon.$$

*Proof.* There is a map from the SBI sequence for  $HH$  and  $HC$  to the SBI sequence for its  $cdh$ -fibrant variant. Applying the exact functor  $(-)^{\epsilon}$  yields a similar map of long exact sequences, every third term of which is the isomorphism of Corollary 6.7. The result now follows by induction on  $n$ , since all complexes are cohomologically bounded above.  $\square$

Better or worse than the old proof?

**Thm3** **Theorem 6.9.** *For any field  $k$  of characteristic 0 and any toric  $k$ -variety  $X$ , we have*

$$K_*(X)^\epsilon \cong KH_*(X)^\epsilon.$$

*Proof.* Since  $(-)^{\epsilon}$  is exact, it suffices by Theorem 5.5 to show that  $H_{Zar}^*(X, \mathcal{F}_{HC})^\epsilon$  vanishes. Again because  $(-)^{\epsilon}$  is exact, we have a long exact sequence

$$\cdots \rightarrow H_{Zar}^n(X, \mathcal{F}_{HC})^\epsilon \rightarrow HC_{-n}(X/\mathbb{Q})^\epsilon \rightarrow \mathbb{H}_{cdh}^n(X, HC(-/\mathbb{Q}))^\epsilon$$

The desired vanishing follows from the previous corollary.  $\square$

*Remark 6.10.* In other words, we have given a new proof of (a case of) Gubeladze’s “Nilpotence Theorem.” For say  $X = U_\sigma$  is an affine toric variety, and assume  $\sigma \subset N_{\mathbb{R}}$  is  $n$ -dimensional (where  $N = \mathbb{Z}^n$ ), so that the monoid  $\sigma^\vee \cap M$  has no non-trivial units. Then  $k[\sigma^\vee \cap M]$  is  $\mathbb{N}$ -graded with weight 0 part being  $k$ . Thus  $KH(X) \simeq K(\text{Spec } k)$ , so that the map  $K_*(X) \rightarrow KH_*(X) \cong K_*(\text{Spec } k)$  is a split surjection. Gubeladze’s “Nilpotence Theorem” asserts that we have that  $(K_*(X)/K_*(\text{Spec } k))^\epsilon = 0$ , and this is a special case of the above theorem.

8/4 K

This entire remark needs rewriting! Why only a “case”? Answer: Gubeladze’s Theorem is more general than this in that we proves  $K_*(k[A]) = KH_*(k[A])$  for any cancellative abelian monoid  $A$  without non-trivial units.  $A = \sigma^\vee \cap M$  is an example of such a monoid, but not the only example, since this example of  $A$  is “normal” but a general such  $A$  need not be. It’s possible (I have not looked in a while) that Gubeladze reduces to this case early on in his proof. –MW

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