Tense operators on De Morgan algebras

ALDO V. FIGALLO* and GUSTAVO PELAITAY†, *Departamento de Matemática, Universidad Nacional del Sur, 8000 Bahía Blanca and Instituto de Ciencias Básicas, Universidad Nacional de San Juan, 5400 San Juan, Argentina*

Abstract

The purpose of this article is to investigate the variety of algebras, which we call tense De Morgan algebras, as a natural generalization of tense algebras developed in Burges (1984, *Handbook of Philosophical Logic*, vol. II, pp. 89–139) (see also, Kowalski (1998, *Rep. Math. Logic*, 32, 53–95)). It is worth mentioning that the latter is a proper subvariety of the first one, as it is shown in a simple example. Our main interest is the representation theory for these classes of algebras.

Keywords: De Morgan algebras, tense operators, frames, duality theory.

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1 Introduction

Propositional logics usually do not incorporate the dimension of time; consequently, in order to obtain a tense logic, we enrich a propositional logic by the addition of new unary operators (or connectives) which are usually denoted by G, H, F and P. We can define F and P by means of G and H as follows: $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$, where $\neg x$ denotes negation of the proposition x.

Tense algebras are algebraic structures corresponding to the propositional tense logic (see [5, 21]). An algebra $\langle A, \vee, \wedge, \neg, G, H, 0, 1\rangle$ is a tense algebra if $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and G, H are unary operators on A which satisfy these axioms:

$$G(1) = 1,\ H(1) = 1,$$

$$G(x \wedge y) = G(x) \wedge G(y),\ H(x \wedge y) = H(x) \wedge H(y),$$

$$x \leq GP(x),\ x \leq HF(x),$$

where $P(x) = \neg H(\neg x)$ and $F(x) = \neg G(\neg x)$.

In the last few years tense operators have been considered by different authors for various classes of algebras. Some contributions in this area have been the papers by Botur *et al.* [4], Chajda [6], Chiriţă [7, 8], Diaconescu and Georgescu [10] and Figallo *et al.* [12–16].

In the present article we consider tense operators on De Morgan algebras. De Morgan algebras are related to logic and have been studied by several authors (see [1, 3, 9, 20, 22]). In particular, they are related with a four-valued logic developed by Belnap in [2].

The article is organized as follows. In Section 1 we summarize the main definitions and results needed in this article. In Section 2 we define the variety of tense De Morgan algebras and introduce some examples. In Section 3 we study the representation of tense De Morgan algebras. In Section 4 we describe a topological duality for tense De Morgan algebras. Finally, in Section 5, we characterize the congruences of these algebras in terms of the mentioned duality and certain closed subsets of the space associated with them.

*E-mail: avfigallo@gmail.com
†E-mail: gpelaitay@gmail.com

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2 Preliminaries

Given a relational structure $\langle X, \leq \rangle$ where $X \neq \emptyset$ and $\leq$ is a reflexive and transitive binary relation on X (i.e. a quasi-order set), we will denote by $(Y]$ ($[Y)$) the set $\{x \in X : \exists y \in Y\, x \leq y\}$ ($[Y) = \{x \in X : \exists y \in Y\, y \leq x\}$), for any $Y \subseteq X$.

A set $Y \subseteq X$ is increasing if $Y = [Y)$, and it is decreasing if $Y = (Y]$. The distributive lattice of all increasing subsets of $\langle X, \leq \rangle$ will be denoted by $\mathcal{P}_i(X)$ and the power set of X by $\mathcal{P}(X)$. If Y is a subset of X, then Y^c will denote the set-theoretical complement of Y, i.e. $Y^c = X \setminus Y$. Let R and Q be binary relations defined on a set X. The composition of R with Q is denoted by $R \circ Q$. Let X, Y be sets. Given a relation $R \subseteq X \times Y$, for each $Z \subseteq X$, $R(Z)$ we will denote the image of Z by R. If $Z = \{x\}$, we will write $R(x)$ instead of $R(\{x\})$. Moreover, for each $V \subseteq Y$, $R^{-1}(V)$ we will denote the inverse image of V by R, i.e. $R^{-1}(V) = \{x \in X : R(x) \cap V \neq \emptyset\}$. If $V = \{y\}$, we will write $R^{-1}(y)$ instead of $R^{-1}(\{y\})$.

An algebra $\langle A, \vee, \wedge, \sim, 0, 1 \rangle$ is a De Morgan algebra if $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice and $\sim$ is a unary operation on A satisfying the following identities:

1. $\sim(x \vee y) = \sim x \wedge \sim y$;
2. $\sim\sim x = x$; and
3. $\sim 0 = 1$.

In what follows a De Morgan algebra $\langle A, \vee, \wedge, \sim, 0, 1 \rangle$ will be denoted briefly by $(A, \sim)$.

A De Morgan frame (see [11]) is a structure $\langle X, \leq, g \rangle$, where $\langle X, \leq \rangle$ is a quasi-order and $g : X \to X$ is a function such that $g(g(x)) = x$, and if $x \leq y$, then $g(y) \leq g(x)$, for all $x, y \in X$. Let A be a De Morgan algebra and let $X(A)$ be the set of all prime filters of A. It is known that $\langle X(A), \subseteq, g_A \rangle$ is a De Morgan frame, where $g_A : X(A) \to X(A)$ is the involution defined by

$$g_A(S) = \{x \in A : \sim x \notin S\}, \quad \text{for all } S \in X(A). \tag{1}$$

Moreover, if $\langle X, \leq, g \rangle$ is a De Morgan frame, then $\langle \mathcal{P}_i(X), \cup, \cap, \sim_g, \emptyset, X \rangle$ is a De Morgan algebra, where $\sim_g : \mathcal{P}_i(X) \to \mathcal{P}_i(X)$ is defined by

$$\sim_g U = X \setminus g(U), \quad \text{for all } U \in \mathcal{P}_i(X). \tag{2}$$

Even though the theory of Priestley spaces and its relation to bounded distributive lattices are well known (see [17–19]), we will recall some definitions and results with the purpose of fixing the notations used in this article.

Let us recall that a Priestley space (or P-space) is a compact totally disconnected ordered topological space. If X is a P-space and $D(X)$ is the family of increasing, closed and open subsets of a P-space X, then $\langle D(X), \cap, \cup, \emptyset, X \rangle$ is a bounded distributive lattice.

On the other hand, let A be a bounded distributive lattice and $X(A)$ the set of all prime filters of A. Then $X(A)$, ordered by set inclusion and with the topology which has as a sub-basis the sets $\sigma_A(a) = \{S \in X(A) : a \in S\}$ and $X(A) \setminus \sigma_A(a)$, for each $a \in A$, is a P-space. In addition the mapping $\sigma_A : A \to D(X(A))$ is a lattice isomorphism. Besides, if X is a P-space, the mapping $\varepsilon_X : X \to X(D(X))$ defined by $\varepsilon_X(x) = \{U \in D(X) : x \in U\}$, is a homeomorphism and an order isomorphism.

If we denote by $\mathcal{L}$ the category of bounded distributive lattices and their corresponding homomorphisms and by $\mathcal{P}$ the category of P-spaces and the continuous increasing mappings (or P-functions), then there exists a duality between both categories by defining the contravariant functors $\Psi : \mathcal{P} \to \mathcal{L}$

and $\Phi : \mathcal{L} \to \mathcal{P}$ as follows:

(P1) For each P-space X, $\Psi(X) = D(X)$, and for every P-function $f : X_1 \to X_2$, $\Psi(f)(U) = f^{-1}(U)$ for all $U \in D(X_2)$.
(P2) For each bounded distributive lattice A, $\Phi(A) = X(A)$, and for every bounded lattice homomorphism $h : A_1 \to A_2$, $\Phi(h)(S) = h^{-1}(S)$ for all $S \in X(A_2)$.

On the other hand, Priestley proved that, if A is a bounded distributive lattice and Y is a closed subset of $X(A)$, then

(P3) $\Theta(Y) = \{(a,b) \in A \times A : \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\}$ is a congruence on A and the correspondence $Y \mapsto \Theta(Y)$ is an anti-isomorphism from the lattice of all closed sets of $X(A)$ onto the lattice of all congruences on A.

In 1977, Cornish and Fowler [9] extended the Priestley duality to De Morgan algebras considering De Morgan spaces (or mP-spaces) as pairs (X, g), where X is a P-space and g is an involutive homeomorphism of X and an anti-isomorphism. They also defined the mP-functions from an mP-space (X_1, g_1) into another, (X_2, g_2), as continuous and increasing functions (P-functions) f from X_1 into X_2, which satisfy the additional condition $f \circ g_1 = g_2 \circ f$.

In addition, these authors introduced the notion of an involutive set in an mP-space (X, g) as a subset Y of X such that $Y = g(Y)$ and characterized the congruences of a De Morgan algebra $(A, \sim)$ by means of the family $\mathcal{C}_I(X(A))$ of involutive closed subsets of $X(A)$. To achieve this result, they proved that the function Θ_I from $\mathcal{C}_I(X(A))$ onto the family $Con_M(A)$ of congruences on A defined as in (P3) is a lattice anti-isomorphism.

3 Tense De Morgan algebras

Let $(A, \sim)$ be a De Morgan algebra and $G, H : A \to A$ two unary operators. We define the operators F and P by $F(x) = \sim G(\sim x)$ and $P(x) = \sim H(\sim x)$, for any $x \in A$.

Definition 3.1
An algebra $(A, \sim, G, H)$ is a tense De Morgan algebra if $(A, \sim)$ is a De Morgan algebra and G and H are two unary operations on A such that:

(T1) $G(1) = 1$ and $H(1) = 1$;
(T2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$;
(T3) $x \leq GP(x)$ and $x \leq HF(x)$; and
(T4) $G(x \vee y) \leq G(x) \vee F(y)$ and $H(x \vee y) \leq H(x) \vee P(y)$.

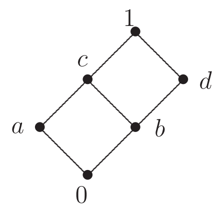
We will denote these algebras by $(A, \sim, G, H)$ or simply by A if no confusion may arise.

Example 3.2
It is easy to see that every tense algebra is also a tense De Morgan algebra.

Example 3.3
There are two extreme examples of tense operators on a De Morgan algebra $(A, \sim)$. Define $G = H$, such that $G(1) = 1$ and $G(x) = 0$ for $x \neq 1$. We can check that $(A, \sim, G, H)$ is a tense De Morgan algebra. Another example of tense operators are identical mappings, i.e. $G(x) = x = H(x)$ for all $x \in A$.

We will indicate an example of tense De Morgan algebra which is not a tense algebra.

EXAMPLE 3.4
Let us consider the De Morgan algebra $(\{0,a,b,c,d,1\},\sim)$, which is described as follows:



x	$\sim x$
0	1
a	d
b	c
c	b
d	a
1	0

Define operators G, H by the table

x	$G(x)$	$H(x)$
0	a	0
a	a	0
b	a	b
c	c	b
d	a	1
1	1	1

Then, $(\{0,a,b,c,d,1\},\sim,G,H)$ is a tense De Morgan algebra.

PROPOSITION 3.5
The following properties hold in any tense De Morgan algebra $(A,\sim,G,H)$:

(T5) $x\leq y$ implies $G(x)\leq G(y)$ and $H(x)\leq H(y)$;
(T6) $x\leq y$ implies $P(x)\leq P(y)$ and $F(x)\leq F(y)$;
(T7) $F(0)=0$ and $P(0)=0$;
(T8) $F(x\vee y)=F(x)\vee F(y)$ and $P(x\vee y)=P(x)\vee P(y)$;
(T9) $FH(x)\leq x$ and $PG(x)\leq x$; and
(T10) $G(x)\wedge F(y)\leq F(x\wedge y)$ and $H(x)\wedge P(y)\leq P(x\wedge y)$.

PROOF. It is routine. ∎

4 Representation by sets

In this section we will show a representation theorem for tense De Morgan algebras in terms of tense De Morgan algebras of sets, using the well-known representation theorem for De Morgan algebras.

Let X be a set, R be a binary relation on X and R^{-1} the converse of R. We define four operators on $\mathcal{P}(X)$ as follows:

$$G_R(U)=\{x\in X:R(x)\subseteq U\},\quad H_{R^{-1}}(U)=\{x\in X:R^{-1}(x)\subseteq U\}.$$

$$F_R(U)=\{x\in X:R(x)\cap U\neq\emptyset\},\quad P_{R^{-1}}(U)=\{x\in X:R^{-1}(x)\cap U\neq\emptyset\}.$$

In general, the De Morgan algebra $(\mathcal{P}_i(X), \sim_g)$ is not closed under the operators introduced above. In the next result we will give conditions on the relation R and its converse for that $\mathcal{P}_i(X)$ be closed under these operations.

PROPOSITION 4.1
Let $\langle X, \leq \rangle$ be a poset, R a binary relation on X, and R^{-1} the converse of R. Then

(i) $(\leq \circ R) \subseteq (R \circ \leq)$ if and only if $G_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$.
(ii) $(\leq \circ R^{-1}) \subseteq (R^{-1} \circ \leq)$ if and only if $H_{R^{-1}}(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$.
(iii) $(\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$ if and only if $F_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$.
(iv) $(\leq^{-1} \circ R^{-1}) \subseteq (R^{-1} \circ \leq^{-1})$ if and only if $P_{R^{-1}}(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$.

PROOF. We will only prove (i). Let us show that, if $U \in \mathcal{P}_i(X)$, then $G_R(U) \in \mathcal{P}_i(X)$. Let $x \leq y$ and $R(x) \subseteq U$. Let $z \in X$, such that $(y,z) \in R$. Then, there exists $w \in X$ such that $(x,w) \in R$ and $w \leq z$. As $R(x) \subseteq U$, $w \in U$; since U is increasing, $z \in U$. Conversely, suppose that $G_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$. Let $x,y,z \in X$ such that $x \leq y$ and $(y,z) \in R$. Let us consider the increasing set $[R(x))$. As $R(x) \subseteq [R(x))$, we have that $x \in G_R([R(x)))$. Then, $y \in G_R([R(x)))$. Thus, $z \in [R(x))$, i.e. there exists $w \in X$ such that $(x,w) \in R$ and $w \leq z$, i.e. $(x,z) \in (R \circ \leq)$. ∎

DEFINITION 4.2
A structure $\mathcal{F} = \langle X, \leq, g, R, R^{-1} \rangle$ is a tense De Morgan frame if $\langle X, \leq, g \rangle$ is a De Morgan frame, R is a binary relation on X, and R^{-1} is the converse of R such that:

(F1) $(\leq \circ R) \subseteq (R \circ \leq)$;
(F2) $(\leq \circ R^{-1}) \subseteq (R^{-1} \circ \leq)$; and
(F3) $(x,y) \in R$ implies $(g(x), g(y)) \in R$.

REMARK 4.3
In every tense De Morgan frame $\mathcal{F} = \langle X, \leq, g, R, R^{-1} \rangle$ the following conditions are satisfied:

(F4) $(\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$; and
(F5) $(\leq^{-1} \circ R^{-1}) \subseteq (R^{-1} \circ \leq^{-1})$.

PROPOSITION 4.4
Let $\mathcal{F} = \langle X, \leq, g, R, R^{-1} \rangle$ be a tense De Morgan frame. Then,

(i) for all $U \in \mathcal{P}_i(X)$, $F_R(U) = \sim_g G_R(\sim_g U)$; and
(ii) for all $U \in \mathcal{P}_i(X)$, $P_{R^{-1}}(U) = \sim_g H_{R^{-1}}(\sim_g U)$.

PROOF. We will prove only (i). Let $x \in F_R(U)$ and suppose that $x \notin \sim_g G_R(\sim_g U)$, i.e. $R(g(x)) \subseteq \sim_g U$. Then, there is $z \in U$ such that $(x,z) \in R$. By (F3) we have that $(g(x), g(z)) \in R$. Then, we can deduce that $g(z) \in \sim_g U$, which is a contradiction. Therefore, $x \in \sim_g G_R(\sim_g U)$. Conversely, suppose that $x \in \sim_g G_R(\sim_g U)$. Thus, $g(x) \notin G_R(\sim_g U)$, i.e. $R(g(x)) \nsubseteq \sim_g U$. From the last assertion, there is $z \in R(g(x))$ such that $g(z) \in U$. Thus, by (F3), we deduce that $g(z) \in R(x) \cap U$. Therefore, $x \in F_R(U)$. ∎

PROPOSITION 4.5
If $\mathcal{F} = \langle X, \leq, g, R, R^{-1} \rangle$ is a tense De Morgan frame, then $A(\mathcal{F}) = (\mathcal{P}_i(X), \sim_g, G_R, H_{R^{-1}})$ is a tense De Morgan algebra.

PROOF. Taking into account Proposition 4.1, Remark 4.3 and Proposition 4.4, we only have to prove (T4). Indeed: let $x \in G_R(U \cup V)$ and suppose that $x \notin G_R(U)$. Then, there exists $y \in X$ such that $(x,y) \in R$ and $y \notin U$. Since, $y \in U \cup V$, we have that $y \in V$. Therefore, $x \in F_R(V)$. In a similar way we can prove $H_{R^{-1}}(U \cup V) \subseteq H_{R^{-1}}(U) \cup P_{R^{-1}}(V)$. ∎

Definition 4.6
Let $(A,\sim,G,H)$ be a tense De Morgan algebra. We will define two binary relations R_G^A and R_H^A on $X(A)$ as follows:

$$(S,T)\in R_G^A \text{ if and only if } G^{-1}(S)\subseteq T\subseteq F^{-1}(S),$$

$$(S,T)\in R_H^A \text{ if and only if } H^{-1}(S)\subseteq T\subseteq P^{-1}(S).$$

The following Proposition is fundamental for the proof to Proposition 4.9.

Proposition 4.7
Let $(A,\sim,G,H)$ be a tense De Morgan algebra. Then the following conditions are equivalents:

(i) $(S,T)\in R_G^A$; and
(ii) $(T,S)\in R_H^A$.

Proof. (i)$\Rightarrow$(ii). Let S,T be two prime filters of A such that $G^{-1}(S)\subseteq T\subseteq F^{-1}(S)$ and let us suppose that $H(x)\in T$. Thus, $FH(x)\in S$. From the preceding assertion and (T9), we have that $x\in S$. Then, $H^{-1}(T)\subseteq S$. On the other hand, suppose that $z\in S$. By (T3), $GP(z)\in S$ and, since $G^{-1}(S)\subseteq T$, we have that $P(z)\in T$. Therefore, $S\subseteq P^{-1}(T)$. The converse implication is similar. ∎

Remark 4.8
From Proposition 4.7 we have that R_H^A is the converse of R_G^A.

Proposition 4.9
Let $(A,\sim,G,H)$ be a tense De Morgan algebra. Then, $\mathcal{F}(A)=\langle X(A),\subseteq,g_A,R_G^A,R_H^A\rangle$ is a tense De Morgan frame.

Proof. Taking into account Remark 4.8, we only have to prove (F1), (F2) and (F3).
(F1): Let S,T,D be three prime filters of A such that $S\subseteq D$ and $G^{-1}(D)\subseteq T\subseteq F^{-1}(D)$. Let us consider the ideal

$$(T^c\cup F^{-1}(S)^c]$$

and prove that

$$G^{-1}(S)\cap(T^c\cup F^{-1}(S)^c]=\emptyset. \tag{3}$$

Suppose that (3) is not fulfilled. Then there exist elements $x\in G^{-1}(S)$, $y\in T^c$ and $z\notin F^{-1}(S)$ such that $x\leq y\vee z$. So, by (T5) and (T4), we have that $G(x)\leq G(y\vee z)\leq G(y)\vee F(z)$. Since $G(x)\in S$ and $F(z)\notin S$, we deduce that $G(y)\in S$, which is impossible. Then by the Birkhoff–Stone theorem there is a prime filter Z such that

$$G^{-1}(S)\subseteq Z \text{ and } (T^c\cup F^{-1}(S)^c)\cap Z=\emptyset$$

Therefore, $G^{-1}(S)\subseteq Z\subseteq F^{-1}(S)$ and $Z\subseteq T$., i.e. $(S,T)\in(R_G^A\circ\subseteq)$.

(F2): It is proved in a similar way to (F1).

(F3): Let S,T be two prime filters such that $G^{-1}(S)\subseteq T\subseteq F^{-1}(S)$. Suppose that $x\in G^{-1}(g_A(S))$. So, $\sim G(x)=F(\sim x)\notin S$, and since $T\subseteq F^{-1}(S)$, we have that $\sim x\notin T$, i.e. $x\in g_A(T)$. On the other hand, suppose that $z\in g_A(T)$ and $F(z)\notin g_A(S)$. So, $\sim F(z)=G(\sim z)\in S$, and since $G^{-1}(S)\subseteq T$, we have that

$\sim z \in T$, i.e. $z \notin g_A(T)$, which is a contradiction. Thus, $F(z) \in g_A(S)$. Therefore, $G^{-1}(g_A(S)) \subseteq g_A(T) \subseteq F^{-1}(g_A(S))$.

■

´ The following result is necessary for the proof to Theorem 4.11.

LEMMA 4.10
Let $(A, \sim, G, H)$ be a tense De Morgan algebra and let $S \in X(A)$ and $a \in A$. Then,

(i) $G(a) \notin S$ if and only if there exists $T \in X(A)$ such that $(S, T) \in R_G^A$ and $a \notin T$; and
(ii) $H(a) \notin S$ if and only if there exists $Z \in X(A)$ such that $(S, Z) \in R_H^A$ and $a \notin Z$.

PROOF. Suppose that $G(a) \notin S$. Let us consider the ideal $(\{a\} \cup F^{-1}(S)^c]$, and we will prove that

$$G^{-1}(S) \cap (\{a\} \cup F^{-1}(S)^c] = \emptyset. \tag{4}$$

Suppose the opposite. Then there exist elements $b \in G^{-1}(S)$ and $c \in F^{-1}(S)^c$ such that $b \leq a \vee c$. From (T5) and (T4) we have that

$$G(b) \leq G(a \vee c) \leq G(a) \vee F(c).$$

From this assertion we deduce that $F(c) \in S$, which is a contradiction. Thus (4) is verified.
Therefore from the Birkhoff–Stone theorem there is a prime filter T such that

$$G^{-1}(S) \subseteq T \text{ and } (\{a\} \cup F^{-1}(S)^c) \cap T = \emptyset.$$

Then, $(S, T) \in R_G^A$ and $a \notin T$. The other implication is easy. In a similar way we can prove (ii). ■

Let $(A, \sim, G, H)$ be a tense De Morgan algebra and let us consider its associated frame $\mathcal{F}(A) = \langle X(A), \subseteq, g_A, R_G^A, R_H^A \rangle$. The structure

$$A(\mathcal{F}(A)) = (\mathcal{P}_i(X(A)), \sim_{g_A}, G_{R_G^A}, H_{R_H^A})$$

is a tense De Morgan algebra.
Let A be a De Morgan algebra. For each $a \in A$ consider the set

$$\sigma_A(a) = \{S \in X(A) : a \in S\}$$

and the family of sets

$$\beta(A) = \{\sigma_A(a) : a \in A\}.$$

It is known that the structure

$$(\beta(A), \sim_{g_A})$$

is a De Morgan subalgebra of the De Morgan algebra

$$\mathcal{P}_i(X(A)) = (\mathcal{P}_i(X(A)), \sim_{g_A}).$$

The assignment $\sigma_A : A \to \mathcal{P}_i(X(A))$ is an injective homomorphism of De Morgan algebras whose range is the set $\beta(A)$. Moreover, this result can be extended to tense De Morgan algebras as follows.

THEOREM 4.11
For every tense De Morgan algebra $(A,\sim,G,H)$ the algebra

$$\beta(A)=(\beta(A),\sim_{g_A},G_{R_G^A},H_{R_H^A})$$

is a subalgebra of $A(\mathcal{F}(A))$ isomorphic to A by means of the assignment $\sigma_A : A \to \beta(A)$.

PROOF. It is easy to check that $\beta(A)$ is a subalgebra of $A(\mathcal{F}(A))$ and that the mapping σ_A is injective. We will prove that σ_A is a homomorphism of tense De Morgan algebras. We need only to see that $\sigma_A(G(a))=G_{R_G^A}(\sigma_A(a))$ and $\sigma_A(H(a))=H_{R_H^A}(\sigma_A(a))$. Now we will prove the inclusion $G_{R_G^A}(\sigma_A(a))\subseteq \sigma_A(G(a))$. Let us take a prime filter S such that $G(a)\notin S$. From Lemma 4.10, there exists $T\in X(A)$ such that $(S,T)\in R_G^A$ and $a\notin T$. Then, we have that $R_G^A(S)\nsubseteq \sigma_A(a)$. Thus $S\notin G_{R_G^A}(\sigma_A(a))$. The inclusion $\sigma_A(G(a))\subseteq G_{R_G^A}(\sigma_A(a))$ is immediate. The equality $\sigma_A(H(a))=H_{R_H^A}(\sigma_A(a))$ follows by a similar argument using (ii) of Lemma 4.10. ∎

5 Duality for tense De Morgan algebras

In this section, we will indicate a duality for tense De Morgan algebras taking into account the results established by Cornish and Fowler in [9].

DEFINITION 5.1
A tense De Morgan space (or tmP-space) is a system (X,g,R,R^{-1}) where (X,g) is an mP-space, R is a binary relation on X and R^{-1} the converse of R such that:

(S1) For each $U\in D(X)$ it holds that $G_R(U)$, $H_{R^{-1}}(U)\in D(X)$, where $G_R(U)$, $H_{R^{-1}}(U)$ are defined as in Section 3.
(S2) $(x,y)\in R$ implies $(g(x),g(y))\in R$.
(S3) For each $x\in X$, $R(x)$ is a closed set.
(S4) For each $x\in X$, $R(x)=(R(x)]\cap[R(x))$.

DEFINITION 5.2
A tmP-function from a tmP-space (X_1,g_1,R_1,R_1^{-1}) into another one, (X_2,g_2,R_2,R_2^{-1}), is a mP-function $f:X_1\to X_2$ which satisfies the following conditions:

(r1) $(x,y)\in R_1$ implies $(f(x),f(y))\in R_2$ for $x,y\in X_1$;
(r2) if $(f(x),y)\in R_2$, then there is an element $z\in X$ such that $(x,z)\in R_1$ and $f(z)\leq y$; and
(r3) if $(y,f(x))\in R_2$, then there is an element $z\in X_1$ such that $(z,x)\in R_1$ and $f(z)\leq y$.

Next we will show that the category **tmPS** of tmP-spaces and tmP-functions is dually equivalent to the category **TDMA** of tense De Morgan algebras and tense De Morgan homomorphisms.

LEMMA 5.3
Let (X,g,R,R^{-1}) be a tmP-space. Then, $(D(X),\sim_g,G_R,H_{R^{-1}})$ is a tense De Morgan algebra, where for all $U\in D(X)$, $\sim_g U$ is the set defined in (2).

PROOF. The definition of G_R and $H_{R^{-1}}$ are consequences of (S1). Conditions (T3) and (T4) give us the equation (S2). Finally, (T1) and (T2) are consequences of the definitions of G_R and $H_{R^{-1}}$. ∎

LEMMA 5.4
Let $f:(X_1,g_1,R_1,R_1^{-1})\to(X_2,g_2,R_2,R_2^{-1})$ be a morphism of tmP-spaces. Then $\Psi(f):D(X_2)\to D(X_1)$ defined as in (P1) is a tense De Morgan morphism.

PROOF. We only prove that $f^{-1}(G_{R_2}(U))=G_{R_1}(f^{-1}(U))$. The inclusion $f^{-1}(G_{R_2}(U))\subseteq G_{R_1}(f^{-1}(U))$ follows by (r1). Let $x\in G_{R_1}(f^{-1}(U))$ and let $y\in X_2$ such that $(f(x),y)\in R_2$. By (r2), there exists $z\in X_1$ such that $(x,z)\in R_1$ and $f(z)\leq y$. Since $z\in R_1(x)\subseteq f^{-1}(U)$, $y\in U$. Thus $f(x)\in G_{R_2}(U)$, and consequently $G_{R_1}(f^{-1}(U))\subseteq f^{-1}(G_{R_2}(U))$.

■

The previous two lemmas show that Ψ is a contravariant functor from **TDMA** to **tmPS**.
Let $R^{D(X)}$ be the relation defined in $X(D(X))$ by means of the operator G_R, i.e.

$$(\varepsilon_X(x),\varepsilon_X(y))\in R^{D(X)}\Leftrightarrow G_R^{-1}(\varepsilon_X(x))\subseteq\varepsilon_X(y)\subseteq F_R^{-1}(\varepsilon_X(x)).$$

PROPOSITION 5.5
Let (X,g,R,R^{-1}) be a tmP-space. Then the following conditions are equivalent:

(i) for all $x\in X$, $R(x)$ is closed and $R(x)=(R(x)]\cap[R(x))$.
(ii) for all $x\in X$, $R(x)$ is compact and $R(x)=(R(x)]\cap[R(x))$.
(iii) for every $x,y\in X$, if $(\varepsilon_X(x),\varepsilon_X(y))\in R^{D(X)}$ then $(x,y)\in R$.

PROOF. (i)$\Rightarrow$(ii) It is immediate. (ii)$\Rightarrow$(iii). Suppose that $(\varepsilon_X(x),\varepsilon_X(y))\in R^{D(X)}$ and $y\notin R(x)=(R(x)]\cap[R(x))$. If $y\notin[R(x))$, then for all $z\in R(x)$, $z\not\leq y$. As X is a Priestley space, for each $z\in R(x)$ there exists $U_z\in D(X)$ such that $z\in U_z$ and $y\notin U_z$. Then, $R(x)\subseteq\bigcup_{x\in R(x)}U_z$ and $y\notin\bigcup_{z\in R(x)}U_z$. Since $R(x)$ is compact, $R(x)\subseteq U$ for some $U\in D(X)$ such that $y\notin U$. But as $(\varepsilon_X(x),\varepsilon_X(y))\in R^{D(X)}$ and $x\in G_R(U)$, then $y\in U$, which is impossible. If $y\notin(R(x)]$, by a similar argument, we obtain also a contradiction. (iii)$\Rightarrow$(i). We prove that $(R(x)]\cap[R(x))\subseteq R(x)$; the other inclusion always holds. Suppose that $y\in(R(x)]\cap[R(x))$. Then there exists $z_1,z_2\in X$ such that

$$z_1\leq y\leq z_2 \text{ and } z_1,z_2\in R(x).$$

By assumption, $(\varepsilon_X(x),\varepsilon_X(z_1))\in R^{D(X)}$, $(\varepsilon_X(x),\varepsilon_X(z_2))\in R^{D(X)}$ and also $\varepsilon_X(z_1)\subseteq\varepsilon_X(y)\subseteq\varepsilon_X(z_2)$. It is easy to check that $(\varepsilon_X(x),\varepsilon_X(y))\in R^{D(X)}$. It follows from the assumption (iii) that $(x,y)\in R$. Therefore $y\in R(x)$. We now prove that $R(x)$ is closed. Suppose that $y\in Cl(R(x))$ and $y\notin R(x)$, where $Cl(R(x))$ denote the closure of the set $R(x)$. By the asumption (iii), we have that $(\varepsilon_X(x),\varepsilon_X(y))\notin R^{D(X)}$. So, there exists $U\in D(X)$ such that $G_R(U)\in\varepsilon_X(x)$ and $U\notin\varepsilon_X(y)$, or there exists $V\in D(X)$ such that $V\in\varepsilon_X(y)$ and $F_R(V)\notin\varepsilon_X(x)$. In the first case $R(x)\subseteq U$ and $y\in U^c$. It follows that $R(x)\cap U^c=\emptyset$ and $y\in U^c$, which is impossible, because $y\in Cl(R(x))$. In the other case, $y\in V$ and $R(x)\cap V=\emptyset$, which is a contradiction. Therefore, $R(x)$ is a closed subset of X.

■

LEMMA 5.6
Let $(A,\sim,G,H)$ be a tense De Morgan algebra. Then $(X(A),\subseteq,g_A,R_G^A,R_H^A)$ is a tmP-space, where g_A, R_G^A and R_H^A are defined as in (1) and Definition 4.6, respectively.

PROOF. We will only prove (S3). Let us check that $R_G^A(S)$ is closed for $S\in X(A)$. Suppose that $T\notin R_G^A(S)$. Then by definition of the relation R_G^A, there exists $x\in G^{-1}(S)$ such that $x\notin T$ or there exists $y\in T$ such that $y\notin F^{-1}(S)$. In the first case $R_G^A(S)\subseteq\sigma_A(x)$ and in the second case $R_G^A(S)\subseteq X(A)\setminus\sigma_A(y)$.

■

LEMMA 5.7
Let $h:A_1\to A_2$ be a tense De Morgan morphism. Then, $\Phi(h):X(A_2)\to X(A_1)$ defined as in (P2) is a morphism of tmP-spaces.

PROOF. We will prove (r1) and (r2); (r3) can be proved in a similar way to (r2). Let $S,T\in X(A_2)$. Let us prove that if $(S,T)\in R_G^{A_2}$, then $(h^{-1}(S),h^{-1}(T))\in R_G^{A_1}$. Suppose that $a\in G^{-1}(h^{-1}(S))$. Then we have that $h(G(a))=G(h(a))\in S$, from which it follows that $h(a)\in T$, i.e. $a\in h^{-1}(T)$. In a similar way we can prove $h^{-1}(T)\subseteq F^{-1}(h^{-1}(S))$. Now suppose that $(h^{-1}(S),T)\in R_G^{A_1}$. Let us prove that there exists $Z\in X(A_2)$ such that $(S,Z)\in R_G^{A_2}$ and $h^{-1}(Z)\subseteq T$. First we will prove the existence of the filter Z. The proof will be consists of two parts:

1. We will prove the existence of a prime filter $D\subseteq A_2$, such that $G^{-1}(S)\subseteq D$ and $h^{-1}(D)\subseteq T$.
2. We will prove the existence of a prime filter $Z\subseteq A_2$, such that $G^{-1}(S)\subseteq Z\subseteq F^{-1}(S)$ and $Z\subseteq D$.

From 1 and 2, the existence of the prime filter Z follows, which fulfills the condition will be established.

By hypothesis we have that $G^{-1}(h^{-1}(S))\subseteq T\subseteq F^{-1}(h^{-1}(S))$. Now let us prove that

$$G^{-1}(S)\cap(h(T^c)]=\emptyset. \tag{5}$$

Suppose that (5) not satisfied. Consequently there exist elements $a\in G^{-1}(S)$, $b\in T^c$, such that $a\leq h(b)$. Then, $G(a)\leq G(h(a))\in S$. From this it follows that, $b\in G^{-1}(h^{-1}(S))\subseteq T$, which is a contradiction. Then (5) holds. By the Birkhoff–Stone theorem, there is a prime filter D, such that

$$G^{-1}(S)\subseteq D \text{ and } h^{-1}(D)\subseteq T.$$

Now we will prove the second part. For this purpose let us first prove that

$$G^{-1}(S)\cap(D^c\cup F^{-1}(S)^c]=\emptyset. \tag{6}$$

Suppose the opposite. Then, there exist elements $a\in G^{-1}(S)$, $b\in D^c$ and $c\in F^{-1}(S)^c$ such that

$$a\leq b\vee c.$$

Then, we have

$$G(a)\leq G(b\vee c)\leq G(b)\vee F(c)\in S.$$

Since $b\notin D$ and $G^{-1}(S)\subseteq D$, $G(b)\notin S$. Therefore, $F(c)\in S$, which is a contradiction. Then, (6) holds. By applying the Birkhoff–Stone theorem, we can make sure that there is a prime filter Z such that

$$G^{-1}(S)\subseteq Z\subseteq F^{-1}(S) \text{ and } Z\subseteq D.$$

As $h^{-1}(Z)\subseteq h^{-1}(D)\subseteq T$, we have proved the existence of the prime filter Z. ∎

The two previous lemmas show that Φ is a contravariant functor from **tmPS** to **TDMA**.
Now, we will see that these categories are dually equivalent to each other.

PROPOSITION 5.8
Let (X,g,R,R^{-1}) be a tmP-space; then, $\varepsilon_X : X\to X(D(X))$ is an isomorphism of tmP-spaces.

PROOF. It follows from the results established in [9] and Proposition 5.5. ∎

PROPOSITION 5.9
Let $(A,\sim,G,H)$ be a tense De Morgan algebra; then, $\sigma_A : A \to D(X(A))$ is a tense De Morgan isomorphism.

PROOF. It follows from the results established in [9] and Lemma 4.10. ∎

The following theorem follows from Propositions 5.8 and 5.9.

THEOREM 5.10
Functors ε_X and σ_A establish a dual equivalence between the categories **TDMA** and **tmPS**.

6 Application of the duality

In this section we will use the duality described in Section 4 to characterize the congruence lattice $Con(A)$ of a tense De Morgan algebra A.

DEFINITION 6.1
Let (X,g,R,R^{-1}) be a tmP-space. An involutive closed subset Y of X is a tmP-subset of X if it satisfies the following conditions for $u,v \in X$:

(ts1) if $(v,u) \in R$ and $u \in Y$, then there exists, $w \in Y$ such that $(w,u) \in R$ and $w \le v$.
(ts2) if $(u,v) \in R$ and $u \in Y$, then there exists, $z \in Y$ such that $(u,z) \in R$ and $z \le v$.

LEMMA 6.2
Let $(A,\sim,G,H)$ be a tense De Morgan algebra and Y, a tmP-subset of $X(A)$. Then $\Theta(Y)$ is a tense De Morgan congruence, where $\Theta(Y)$ is defined as in (P3).

PROOF. We will only prove that $\Theta(Y)$ preserves G and H. Let $(a,b) \in \Theta(Y)$ and suppose that $S \in G_{R^A_G}(\sigma_A(a)) \cap Y$. Hence, $(R^A_G)^{-1}(S) \subseteq \sigma_A(a)$ and $S \in Y$. Suppose that $T \in (R^A_G)^{-1}(S)$. Consequently, from (ts1) there is $W \in Y$, such that $W \subseteq T$ and $W \in (R^A_G)^{-1}(S)$. This last assertion allows us to infer that $W \in \sigma_A(a)$, from which we conclude $W \in \sigma_A(b) \cap Y$. Therefore, since $W \subseteq T$, we have that $T \in \sigma_A(b)$; hence, $S \in G_{R^A_G}(\sigma_A(b)) \cap Y$. Then, $G_{R^A_G}(\sigma_A(a)) \cap Y \subseteq G_{R^A_G}(\sigma_A(b)) \cap Y$. The other inclusion is proved in a similar way. Analogously, $\Theta(Y)$ preserves H. ∎

LEMMA 6.3
Let $(A,\sim,G,H)$ be a tense De Morgan algebra and $\theta \in Con(A)$. If $q : A \to A/\theta$ is the natural epimorphism, then $Y = \{\Phi(q)(S) : S \in X(A/\theta)\}$ is a tmP-subset, where Φ is defined as in (P2).

PROOF. Since $Con_M(A)$ is a sublattice of $Con(A)$ we have that $Y = \{\Phi(q)(S) : S \in X(A/\theta)\}$ is an involutive closed subset of $X(A)$ and $\theta = \Theta(Y)$ (see [9]). Besides, from Lemma 5.7 we have that $\Phi(q)$ is a tmP-function. In addition, Y is a tmP-subset of $X(A)$. Indeed, let $(V,U) \in R^A_G$ and $U \in Y$. From the last assertion, there is $S \in X(A/\theta)$ such that $\Phi(q)(S) = U$. Then, $(V,\Phi(q)(S)) \in R^A_G$. As a consequence, we have from (r2) that there is $T \in X(A/\theta)$ such that $(T,S) \in R^{A/\theta}_G$ and $\Phi(q)(T) \subseteq V$. Therefore, we have from (r1) that $\Phi(q)(T) \subseteq (R^A_G)^{-1}(U)$. (ts2) can be proved in a similar way. ∎

From Lemmas 6.2 and 6.3, we obtain Theorem 6.4.

THEOREM 6.4
Let $(A,\sim,G,H)$ be a tense De Morgan algebra and $(X(A),g_A,R^A_G,R^A_H)$, the associated tmP-space of A. Then, there is an anti-isomorphisms between $Con(A)$ and the lattice of all tmP-subset of $X(A)$.

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