Topological methods for a nonlinear elliptic system with nonlinear boundary conditions

Pablo Amster

Departamento de Matemática,
Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires,
Ciudad Universitaria,
Pabellón I, (1428) Buenos Aires, Argentina

Consejo Nacional de Investigaciones Científicas
y Técnicas (CONICET), Argentina
E-mail: pamster@dm.uba.ar

Abstract: A nonlinear elliptic second order system with a nonlinear boundary condition is studied. We prove an existence result under a Hartman type condition. Moreover, assuming appropriate conditions of Nirenberg type, we prove the existence of solutions applying topological degree theory.

Keywords: Leray-Schauder degree; nonlinear elliptic systems; nonlinear boundary conditions.

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Biographical notes: Pablo Amster is a Professor at the University of Buenos Aires (Argentina) and Researcher at Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET – Argentina). He has published over 60 papers in international journals, most of them in the field of topological methods in differential equations.

1 Introduction

This work is devoted to the study of the nonlinear second order system of elliptic equations for a vector function $u : \overline{\Omega} \to \mathbb{R}^N$ given by

$$\Delta u = f(x, u) \quad \text{in} \quad \Omega, \tag{1}$$

subject to the following nonlinear boundary condition:

$$\frac{\partial u}{\partial \nu} = g(x, u) \quad \text{on} \quad \partial\Omega. \tag{2}$$

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Here $\Omega \subset \mathbb{R}^d$ is a bounded smooth domain, and ν denotes the unit outward normal vector field on $\partial\Omega$. Nonlinear boundary conditions of this kind appear for example when the problem of finding extremals for the best constant in the Sobolev trace inequality is considered (Fernández Bonder and Rossi, 2001). On the other hand, if $d = 1$, the scalar problem $N = 1$ may be regarded as a second order analogue of a fourth order equation that models a beam with a nonlinear elastic foundation $g : \mathbb{R} \to \mathbb{R}$ acting at the extremities (Grossinho and Ma, 1994; Rebelo and Sanchez, 1995). In this case, if $\Omega = (0, T)$ the nonlinear condition (2) reads:

$$u'(0) = -g(u(0)), \quad u'(T) = g(u(T)).$$

Using topological methods, we shall prove the existence of solutions of the above problem (1)–(2) under a variant of the so-called Hartman condition, which has been firstly used by Hartman (1960) for the Dirichlet problem. Later, it was employed in Knobloch (1971) for the periodic problem, under a Lipschitz assumption on the nonlinear term. Further extensions can be found in Gaines and Mawhin (1997), and Mawhin and Ureña (2002).

Here, we adapt the mentioned results for the nonlinear boundary conditions (2). Our proof relies on the maximum principle and the unique solvability of the associated linear Robin problem.

Moreover, in Section 3 we obtain a second existence result assuming appropriate conditions of Nirenberg type (Nirenberg, 1971). For the scalar case, a Landesman-Lazer type condition can be deduced from our results (see e.g., Mawhin (2000)). This kind of conditions have been obtained in Martínez and Rossi (2003) for a p-Laplacian scalar equation under nonlinear boundary conditions, using variational methods. In contrast with these results, the arguments below can be extended for a more general equation $\Delta u = f(x, u, Du)$, which has non-variational structure.

For simplicity, we shall assume that the functions $f : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $g : \partial\Omega \times \mathbb{R}^N \to \mathbb{R}^N$ are continuous.

2 Hartman type conditions

In this section we prove the existence of solutions of equations (1)–(2) under the following Hartman type condition for some $R > 0$:

$$\langle f(\cdot, u), u\rangle \geq 0 \geq \langle g(\cdot, u), u\rangle \quad \text{for} \quad u \in \mathbb{R}^N, \quad |u| = R. \tag{3}$$

In particular, if $g = 0$ we retrieve the standard (non-strict) Hartman condition, for a Neumann boundary value problem.

Theorem 2.1: *Assume that condition* (3) *holds. Then problem* (1)–(2) *admits at least one classical solution u, with* $\|u\|_{C(\overline{\Omega}, \mathbb{R}^N)} \leq R$.

Proof: We shall apply a fixed point argument on the space $C(\overline{\Omega}, \mathbb{R}^N)$ equipped with the supremum norm.

Define the function $P : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$Pu = \begin{cases} u & \text{if } |u| \le R \\ R\dfrac{u}{|u|} & \text{otherwise,} \end{cases}$$

and for fixed $\lambda, \mu > 0$ consider the Robin problem given by

$$\begin{cases} \Delta u - \lambda u = f(x, Pu(x)) - \lambda Pu(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \mu u = g(x, Pu(x)) + \mu Pu(x) & \text{on } \partial\Omega. \end{cases} \tag{4}$$

As the right hand side terms are both bounded, a straightforward application of Schauder's Theorem shows that equation (4) has at least one solution $u \in C(\overline{\Omega}, \mathbb{R}^N)$. We shall see that $|u(x)| \le R$ for every $x \in \overline{\Omega}$, which implies that u is a solution of the original problem.

Indeed, let us define the function $\phi(x) = |u(x)|^2$, then

$$\frac{\partial \phi}{\partial x_j} = 2\sum_{i=1}^N u_i \frac{\partial u_i}{\partial x_j},$$

and

$$\Delta\phi = 2\left(\langle \Delta u, u\rangle + \sum_{i=1}^N |\nabla u_i|^2\right).$$

Let x_0 be the absolute maximum of ϕ over $\overline{\Omega}$, and suppose that $|u(x_0)| > R$. If $x_0 \in \Omega$, then $\nabla\phi(x_0) = 0$ and

$$\begin{aligned} \Delta\phi(x_0) &\ge 2\langle f(x_0, Pu(x_0)), u(x_0)\rangle + 2\lambda\langle u(x_0) - Pu(x_0), u(x_0)\rangle \\ &= 2\frac{|u(x_0)|}{R}\langle f(x_0, Pu(x_0)), Pu(x_0)\rangle + 2\lambda|u(x_0)|(|u(x_0)| - R) > 0, \end{aligned}$$

a contradiction.

Next, assume that $x_0 \in \partial\Omega$. As $|u(x)| \le |u(x_0)|$ for $x \in \Omega$, then

$$\begin{aligned} 0 \le \frac{\partial\phi}{\partial\nu}(x_0) &= 2\left\langle \frac{\partial u}{\partial \nu}(x_0), u(x_0)\right\rangle \\ &= 2\langle g(x_0, Pu(x_0)), u(x_0)\rangle - 2\mu\langle u(x_0) - Pu(x_0), u(x_0)\rangle \\ &= 2\frac{|u(x_0)|}{R}\langle g(x_0, Pu(x_0)), Pu(x_0)\rangle - 2\mu|u(x_0)|(|u(x_0)| - R) < 0. \end{aligned}$$

This new contradiction proves that $\|u\|_{C(\overline{\Omega},\mathbb{R}^N)} \le R$, and so completes the proof. □

Remark 2.2: It follows from the previous proof that the solution belongs to the space $C^2(\Omega) \cap C^1(\overline{\Omega})$.

3 An extension of a result by Nirenberg

In this section, we prove the existence of solutions of equations (1)–(2) under Nirenberg type conditions. We recall that, in Nirenberg's original work (Nirenberg, 1971) for $f(x,u) = p(x) - \tilde{f}(u)$ and linear boundary conditions, the assumption is made that the radial limits

$$\tilde{f}_v := \lim_{s\to+\infty} \tilde{f}(sv)$$

exist uniformly on $v \in S^{N-1}$, the unit sphere of $\mathbb{R}^N$. In this case, if $\tilde{f}_v \neq \bar{p} := \frac{1}{|\Omega|}\int_\Omega p(x)dx$ for any $v \in S^{N-1}$, existence of solutions can be proved when the degree of the mapping

$$v \mapsto \frac{\tilde{f}_v - \bar{p}}{|\tilde{f}_v - \bar{p}|}$$

is different from zero.

In the present work, these assumptions are relaxed. More precisely, we shall assume that for each $v \in S^{N-1}$ uniform (radial) *upper* limits of the bounded functions f and g exist, but only on a neighbourhood of v, and some specified directions. Then, under an appropriate degree condition, we shall prove a generalisation of Nirenberg's result for problem (1)–(2). A result of this kind has been obtained in Amster and De Nápoli (2007) for a one-dimensional p-Laplacian equation under periodic conditions.

Condition (L): There exists a family $\{(U_j, w_j)\}_{j=1,\dots,K}$ where U_j is an open subset of S^{N-1} and $w_j \in S^{N-1}$, such that $\{U_j\}$ covers S^{N-1} and the limits

$$\limsup_{s\to+\infty}\langle f(x,su), w_j\rangle := \bar{f}_{u,j}(x) \tag{5}$$

and

$$\liminf_{s\to+\infty}\langle g(x,su), w_j\rangle := \underline{g}_{u,j}(x) \tag{6}$$

exist uniformly for $u \in U_j$.

Remark 3.1: In particular, if the limits

$$\lim_{s\to+\infty} f(\cdot, sv) \quad \text{and} \quad \lim_{s\to+\infty} g(\cdot, sv)$$

exist uniformly on $v \in S^{N-1}$, then condition (L) holds. Indeed, it suffices to consider any family $\{(U_j, w_j)\}_{j=1,\dots,K}$ such that $\{U_j\}$ is an open covering of S^{N-1} and $w_j \in S^{N-1}$.

Theorem 3.2: *Let f and g be bounded, and assume that Condition (L) holds. Further, assume that:*

1 *For each $u \in S^{N-1}$ there exists $j \in \{1,\dots,K\}$ such that $u \in U_j$ and*

$$\int_\Omega \bar{f}_{u,j}(x)dx < \int_{\partial\Omega} \underline{g}_{u,j}(x)dS.$$

2 $deg_B(\phi, B_r, 0) \neq 0$ *for all* $r > 0$ *large enough, where* $\phi : \mathbb{R}^N \to \mathbb{R}^N$ *is given by:*

$$\phi(u) = \int_\Omega f(x,u)dx - \int_{\partial\Omega} g(x,u)dS$$

and $B_r \subset \mathbb{R}^N$ *is the ball of radius* r *centred at* 0.

Then problem (1)–(2) *admits at least one solution.*

For a proof of Theorem 3.2, we shall apply Leray-Schauder topological degree. Let us prove first a continuation theorem associated to our problem:

Theorem 3.3: *Assume that the following conditions hold:*

1 *There exists* $R > 0$ *such that for each* $0 < \lambda \leq 1$ *the problem*

$$\begin{cases} \Delta u = \lambda f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda g(x,u) & \text{on } \partial\Omega \end{cases} \tag{7}$$

has no solution $u \in H^2(\Omega, \mathbb{R}^N)$ *with* $\|u\|_{H^1} = R$.

2 $deg_B(\phi, B_r, 0) \neq 0$, *with* ϕ *as before and* $r = \frac{R}{|\Omega|^{1/2}}$.

Then (1)–(2) *admits at least one solution* $u \in H^2(\Omega, \mathbb{R}^N)$ *with* $\|u\|_{H^1} < R$.

Proof: For $(\varphi, \xi) \in L^2(\Omega, \mathbb{R}^N) \times H^{1/2}(\partial\Omega, \mathbb{R}^N)$ define the constant $Q(\varphi, \xi) \in \mathbb{R}^N$ given by:

$$Q(\varphi, \xi) = \frac{1}{|\Omega| + |\partial\Omega|} \left(\int_\Omega \varphi(x)dx - \int_{\partial\Omega} \xi(x)dS \right).$$

Moreover, if $Q(\varphi, \xi) = 0$ define $K_\lambda(\varphi, \xi)$ as the unique solution u of the problem

$$\begin{cases} \Delta u = \lambda\varphi & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda\xi & \text{on } \partial\Omega \\ \bar{u} = 0, \end{cases}$$

where, as before, $\bar{u}$ denotes the average of u given by $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)dx$. Next, for $0 \leq \lambda \leq 1$ define the operator $T_\lambda : H^1(\Omega, \mathbb{R}^N) \to H^1(\Omega, \mathbb{R}^N)$ given by

$$T_\lambda u = \bar{u} + Q_u + K_\lambda(f(\cdot, u) - Q_u, g(\cdot, u|_{\partial\Omega}) + Q_u),$$

where $Q_u := Q(f(\cdot, u), g(\cdot, u|_{\partial\Omega}))$. From the definition of Q,

$$\int_\Omega (f(x,u) - Q_u)dx = \int_{\partial\Omega} (g(x,u) + Q_u)dS;$$

thus, T_λ is well defined. Furthermore, a standard argument proves that T_λ is compact.

Suppose that $u = T_\lambda u$, then taking average at both sides it follows that $Q_u = 0$. If $\lambda = 0$, then $u = \bar{u} \in \mathbb{R}^N$, and $\int_\Omega f(\cdot, u) = \int_{\partial\Omega} g(\cdot, u)$; hence $|u| \neq r$, or equivalently $\|u\|_{H^1} \neq R$.

On the other hand, if $\lambda > 0$, then $u = \bar{u} + K_\lambda(f(\cdot, u), g(\cdot, u|_{\partial\Omega}))$. Then u is a solution of equation (7), which implies that $\|u\|_{H^1} \neq R$. Thus, $deg_{LS}(I - T_\lambda, B_R(0), 0)$ is well defined, and by homotopy invariance and the definition of the Leray-Schauder degree, we have:

$$deg_{LS}(I - T_1, B_R(0), 0) = deg_{LS}(I - T_0, B_R(0), 0) = deg_B((I - T_0)|_{\mathbb{R}^N}, B_r, 0).$$

As

$$(I - T_0)|_{\mathbb{R}^N}(u) = -Q_u = -(|\Omega| + |\partial\Omega|)\phi(u),$$

we deduce that $deg_{LS}(I - T_1, B_R(0), 0) \neq 0$. Hence T_1 has a fixed point in $B_R(0) \subset H^1(\Omega, \mathbb{R}^N)$, which is a solution of the problem. □

Proof of Theorem 3.2: In order to verify the conditions of the previous continuation theorem, we shall obtain a priori bounds for the solutions of problem (7).

Suppose there exists a sequence $\{u_n\}$ such that $\|u_n\|_{H^1} \to \infty$ and

$$\begin{cases} \Delta u_n = \lambda_n f(x, u_n) & \text{in } \Omega \\ \dfrac{\partial u_n}{\partial \nu} = \lambda_n g(x, u_n) & \text{on } \partial\Omega \end{cases}$$

with $0 < \lambda_n \leq 1$. Multiplying by $u_n - \bar{u}_n$ and integrating by parts yields the following inequality:

$$\|\nabla u_n\|_{L^2}^2 \leq \|u_n - \bar{u}_n\|_{L^2}.\|\Delta u_n\|_{L^2} + \int_{\partial\Omega} \left\langle u_n - \bar{u}_n, \frac{\partial u_n}{\partial \nu} \right\rangle dS.$$

Furthermore, as

$$\|\Delta u_n\|_{L^2} \leq \|f(\cdot, u_n)\|_{L^2} \leq C,$$

and

$$\left\| \frac{\partial u_n}{\partial \nu} \right\|_{L^2(\partial\Omega)} \leq \|g(\cdot, u_n)\|_{L^2(\partial\Omega)} \leq C$$

for some constant C, from the Poincaré-Wirtinger and the trace inequalities we deduce that the sequence $\{\|\nabla u_n\|_{L^2}\}$ is bounded.

Moreover, if $p > d$, from standard estimates and the boundedness of $\frac{\partial u_n}{\partial \nu}$ on $\partial\Omega$ we deduce that $\|u_n - \bar{u}_n\|_{W^{1,p}} \leq c_1 + c_2\|\Delta u_n\|_{L^p}$ for some constants c_1 and c_2. As f is bounded, from the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ we conclude that $\|u_n - \bar{u}_n\|_{L^\infty}$ is bounded, and hence $|\bar{u}_n| \to \infty$. Passing to a subsequence if necessary, we may assume that $\frac{\bar{u}_n}{|\bar{u}_n|} \to u$ for some $u \in S^{N-1}$. Set $z_n := \frac{u_n}{|u_n|}$, then

$$\left| z_n(x) - \frac{\bar{u}_n}{|\bar{u}_n|} \right| \le \frac{|(|\bar{u}_n| - |u_n(x)|) \cdot u_n(x) + |u_n(x)|(u_n(x) - \bar{u}_n)|}{|u_n(x)| \cdot |\bar{u}_n|} \le \frac{2|u_n(x) - \bar{u}_n|}{|\bar{u}_n|},$$

and it follows that z_n also converges to u, uniformly in x. From assumption 1, we may fix j such that

$$\int_\Omega \bar{f}_{u,j}(x)dx < \int_{\partial\Omega} \underline{g}_{u,j}(x)dS.$$

Integrating the equation, by the divergence theorem we obtain:

$$\lambda_n \int_\Omega f(x, u_n)dx = \lambda_n \int_{\partial\Omega} g(x, u_n)dS,$$

and by Fatou's Lemma we deduce that

$$\int_\Omega \limsup_{n\to\infty} \langle f(x, |u_n|z_n), w_j\rangle dx \ge \int_{\partial\Omega} \liminf_{n\to\infty} \langle g(x, |u_n|z_n), w_j\rangle dS.$$

For $x \in \Omega$ and $\varepsilon > 0$, fix $s_0 > 0$ and a sequence $s_n \to +\infty$ such that

$$\begin{aligned} &\langle f(x, sv), w_j\rangle < \bar{f}_{v,j}(x) + \frac{\varepsilon}{4} \quad s \ge s_0, \\ &\langle f(x, s_n v), w_j\rangle < \bar{f}_{v,j}(x) - \frac{\varepsilon}{4} \quad n \in \mathbb{N} \end{aligned}$$

for every $v \in U_j$. Fix $s \in \{s_n\}_{n\in\mathbb{N}}$ such that $s \ge s_0$, and $\delta > 0$ such that if $|v - u| < \delta$, then

$$|\langle f(x, sv), w_j\rangle - \langle f(x, su), w_j\rangle| < \frac{\varepsilon}{4},$$

whence

$$\begin{aligned} |\bar{f}_{v,j}(x) - \bar{f}_{u,j}(x)| \le |\bar{f}_{v,j}(x) - \langle f(x, sv), w_j\rangle| + |\langle f(x, sv), w_j\rangle - \langle f(x, su), w_j\rangle| \\ + |\langle f(x, su), w_j\rangle - \bar{f}_{u,j}(x)| < \frac{3\varepsilon}{4}. \end{aligned}$$

Thus, if n_0 is such that $|u_n(x)| \ge s_0$ and $|z_n(x) - u| < \delta$ for $n \ge n_0$, we conclude that

$$\begin{aligned} \langle f(x, |u_n|z_n), w_j\rangle = (\langle f(x, |u_n|z_n), w_j\rangle - \bar{f}_{z_n,j}(x)) \\ + (\bar{f}_{z_n,j}(x) - \bar{f}_{u,j}(x)) + \bar{f}_{u,j}(x) < \bar{f}_{u,j}(x) + \varepsilon. \end{aligned}$$

In the same way, if $x \in \partial\Omega$ then

$$\langle g(x, |u_n|z_n), w_j\rangle \ge \underline{g}_{u,j}(x) - \varepsilon$$

for n large enough, and hence

$$\int_{\Omega} \bar{f}_{u,j}(x) \geq \int_{\partial\Omega} \underline{g}_{u,j}(x),$$

a contradiction. □

Next, we present an example for which conditions of Theorem 3.2 hold.

Example 3.4: In order to illustrate Theorem 3.2, we shall consider a system which uncouples at infinity, namely

$$\begin{cases} \Delta u_i = \theta_i(u_i) + \dfrac{V_i(x,u)}{1+|u_i|} := f_i(x,u), \\ \dfrac{\partial u_i}{\partial \nu} = \omega_i(u_i) + \dfrac{W_i(x,u)}{1+|u_i|} := g_i(x,u) \end{cases}$$

where $\theta_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $V_i : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ $\omega_i : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, $W_i : \partial\Omega \times \mathbb{R}^N \to \mathbb{R}$ are bounded. Further, assume that

$$\limsup_{s\to+\infty} \theta_i(s) - \liminf_{s\to+\infty} \omega_i(s) < 0 < \liminf_{s\to-\infty} \theta_i(s) - \limsup_{s\to-\infty} \omega_i(s). \tag{8}$$

It is worth noticing that Nirenberg's Theorem does not apply for this case, since radial limits of the functions f and g do not necessarily exist in those directions $v \in S^{N-1}$ such that $v_i = 0$ for some i.

However, conditions of Theorem 3.2 hold. Indeed, we may set $\{w_1, \ldots, w_N\}$ as the canonical basis of $\mathbb{R}^N$. For fixed $u \in S^{N-1}$, there exists j such that $u_j \neq 0$, and then $|v_j| \geq c$ for some constant $c > 0$ and v in an appropriate connected neighbourhood V_u of u. If $u_j > 0$, then

$$\limsup_{s\to+\infty} \langle f(x,sv), w_j\rangle = \limsup_{s\to+\infty} f_j(x,sv) = \limsup_{s\to+\infty} \theta_j(s)$$

uniformly on V_u. If $u_j < 0$, then

$$\limsup_{s\to+\infty} \langle f(x,sv), w_j\rangle = -\liminf_{s\to-\infty} f_j(x,-sv) = -\liminf_{s\to-\infty} \theta_j(s)$$

uniformly on V_u. Thus, compactness of S^{N-1} and an analogous argument for g prove that condition (L) is satisfied. Moreover, by equation (8) condition 1 of Theorem 3.2 holds.

Finally, the second condition of Theorem 3.2 is verified by considering the homotopy $H : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ given by

$$H(t,u) = t\phi(u) - (1-t)u.$$

If $H(t_n, u_n) = 0$ for some $t_n \in [0,1]$ and $|u_n| \to \infty$, then taking a subsequence we may assume that $|(u_n)_i| \to \infty$ and $t_n \to t \in [0,1]$. Suppose for example that $(u_n)_i \to +\infty$, then

$$0 = H_i(t_n, u_n) = t_n\phi_i(u_n) - (1-t_n)(u_n)_i.$$

Hence

$$t_n\left(\int_\Omega f_i(x,u_n)dx - \int_{\partial\Omega} g_i(x,u_n)dS\right) = (1-t_n)(u_n)_i \geq 0,$$

and from Fatou's Lemma a contradiction yields. A similar argument for the case $(u_n)_i \to -\infty$ shows that H does not vanish on $[0,1]\times\partial B_R$ for $R \gg 0$. Thus,

$$deg(\phi, B_R, 0) = (-1)^N deg(Id, B_R, 0) = \pm 1,$$

and all the assumptions of Theorem 3.2 are fulfilled.

Remark 3.5: As it was mentioned in the introduction, Theorem 3.2 can be extended for the system

$$\Delta u = f(x,u,Du),$$

which is non-variational.

However, some obvious changes are required: firstly, the limits in equation (5) now depend on a third variable $V \in \mathbb{R}^{N\times N}$; thus, it is needed to replace the assumption by

$$\limsup_{s\to+\infty}\langle f(x,su,V), w_j\rangle := \bar{f}_{u,j}(x)$$

uniformly for $u \in U_j$ and $|V| \leq M$, where the constant M is defined from the a priori L^∞-bounds for Du. On the other hand, also the definition of ϕ must be changed to:

$$\phi(u) = \int_\Omega f(x,u,0)dx - \int_{\partial\Omega} g(x,u)dS.$$

Further extensions could be obtained also when f is unbounded, as far as one is able to get a priori bounds for Du. For example, this is straightforward if f has sublinear growth in V, although a more general result (e.g., subquadratic growth) could be expected.

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