M. Campercholi
D. Castaño
J. P. Díaz Varela

Quasivarieties and Congruence Permutability of Łukasiewicz Implication Algebras

Abstract. In this paper we study some questions concerning Łukasiewicz implication algebras. In particular, we show that every subquasivariety of Łukasiewicz implication algebras is, in fact, a variety. We also derive some characterizations of congruence permutable algebras. The starting point for these results is a representation of finite Łukasiewicz implication algebras as upwardly-closed subsets in direct products of MV-chains.

Keywords: Łukasiewicz implication algebras, quasivarieties, congruence permutability.

In the first section we present the definition and some basic facts about Łukasiewicz implication algebras which we use throughout the article. Section 2 begins with a representation of finite Łukasiewicz implication algebras as upwardly-closed subsets in direct products of finite MV-chains. In this section we also develop some consequences of this representation; namely, we characterize congruences and show that every homomorphic image is, in fact, a retract.

In Section 3 we study quasivarieties. We deal first with the locally finite case by means of critical algebras and then use a representation of free Łukasiewicz implication algebras given in [6] to extend these results to the general case.

In the last two sections we concentrate on giving characterizations for congruence permutability. First we show that congruence permutability is equivalent to the existence of meets for every pair of elements. In addition, we prove a Nachbin-like theorem characterizing congruence permutable finite algebras as those algebras which do not have certain algebras as quotients.

1. Preliminaries

Łukasiewicz implication algebras are the algebraic counterpart of the implicational fragment of super-Łukasiewicz logic (see [13, 14]). In fact, they are

Special Issue: Algebras Related to Non-classical Logic
Edited by **Manuel Abad and Alejandro Petrovich**

 © Springer 2011

the class of all $\{\to, 1\}$-subreducts of Wajsberg algebras (Wajsberg algebras are term-wise equivalent to Chang's MV-algebras and bounded commutative BCK-algebras [5, 9, 15]). They are also called C-algebras in [13, 14] and Łukasiewicz residuation algebras by J. Berman and W. J. Blok in [3].

A ***Łukasiewicz implication algebra*** is an algebra $\mathbf{A} = \langle A, \to, 1\rangle$ of type $\langle 2, 0\rangle$ that satisfies the equations:

(Ł1) $1 \to x \approx x$,

(Ł2) $(x \to y) \to ((y \to z) \to (x \to z)) \approx 1$,

(Ł3) $(x \to y) \to y \approx (y \to x) \to x$,

(Ł4) $(x \to y) \to (y \to x) \approx y \to x$.

We denote by $\mathbb{L}$ the variety of all Łukasiewicz implication algebras. The following properties are satisfied by any Łukasiewicz implication algebra:

(Ł5) $x \to x \approx 1$,

(Ł6) $x \to 1 \approx 1$,

(Ł7) if $x \to y \approx 1$ and $y \to x \approx 1$, then $x \approx y$,

(Ł8) $x \to (y \to x) \approx 1$,

(Ł9) $x \to (y \to z) \approx y \to (x \to z)$.

If $\mathbf{A} \in \mathbb{L}$ then the relation $a \leq b$ if and only if $a \to b = 1$ is a partial order on A, called the *natural order of* $\mathbf{A}$, with 1 as its greatest element. The join operation $x \vee y$ is given by the term $(x \to y) \to y$ and if $c \in A$, then the polynomial $p(x, y, c) = ((x \to c) \vee (y \to c)) \to c$ is such that $p(a, b, c) = a \wedge b = \inf\{a, b\}$ for $a, b \geq c$. The lattice operations satisfy the following properties:

(Ł10) $(x \to y) \vee (y \to x) \approx 1$,

(Ł11) $(x \vee y) \to z \approx (x \to z) \wedge (y \to z)$,

(Ł12) $z \to (x \vee y) \approx (z \to x) \vee (z \to y)$,

and if for $a, b \in A$ the meet $a \wedge b$ exists, then for any $c \in A$,

(Ł13) $(a \wedge b) \to c \approx (a \to c) \vee (b \to c)$,

(Ł14) $c \to (a \wedge b) \approx (c \to a) \wedge (c \to b)$.

Moreover, if $\mathbf{A} = \langle A, \to, 1\rangle$ is a Łukasiewicz implication algebra and $c \in A$, then $\mathbf{A}_c = \langle [c) = \{a \in A : c \leq a\}, \to_c, \neg_c, c, 1\rangle$ becomes a Wajsberg algebra defining

- $\neg_c x := x \to c$,
- $x \to_c y := x \to y$.

Łukasiewicz implication algebras are congruence 1-regular. For each congruence relation θ on an algebra $\mathbf{A} \in \mathbb{L}$, $1/\theta$ is an *implicative filter*, i.e., it contains 1 and if $a, a \to b \in 1/\theta$, then $b \in 1/\theta$ (modus ponens); in particular, $1/\theta$ is upwardly-closed with respect to the natural order. Conversely, for any implicative filter F of $\mathbf{A}$ the relation

$$\theta_F = \{(a, b) \in A^2 : a \to b, b \to a \in F\}$$

is a congruence on $\mathbf{A}$ such that $F = 1/\theta_F$. In fact, the correspondence $\theta \mapsto 1/\theta$ gives an order isomorphism from the family of all congruence relations on $\mathbf{A}$ onto the family of all implicative filters of $\mathbf{A}$, ordered by inclusion. For this reason, we often write $\mathbf{A}/F$ instead of $\mathbf{A}/\theta_F$. We usually write $a \equiv_F b$ instead of $(a, b) \in \theta_F$.

It is easily shown that if $a, b \in F$ and the meet of a and b exists, then it must lie in F as well. Note also that since any implicative filter F contains 1 and is closed by $\to$, it is the universe of a subalgebra $\mathbf{F}$ of $\mathbf{A}$.

The following result about implicative filters is needed in Section 5. We include it here for completeness.

PROPOSITION 1.1. *Let $\mathbf{A}$ be a Łukasiewicz implication algebra and F an implicative filter of $\mathbf{A}$. The following conditions are equivalent:*

(i) *$\mathbf{A}/F$ is a chain,*

(ii) *for every $x, y \in A$, either $x \to y \in F$ or $y \to x \in F$,*

(iii) *for every $x, y \in A$, if $x \vee y \in F$, then $x \in F$ or $y \in F$.*

PROOF. The only nontrivial implication is $(iii) \Rightarrow (i)$. Assume (iii) and consider $\overline{x}, \overline{y} \in A/F$. By (Ł10), we have that $(\overline{x} \to \overline{y}) \vee (\overline{y} \to \overline{x}) = \overline{1}$, so $(x \to y) \vee (y \to x) \in F$. Hence either $x \to y \in F$ or $y \to x \in F$, so $\overline{x} \leq \overline{y}$ or $\overline{y} \leq \overline{x}$. ■

An implicative filter satisfying the conditions in the previous proposition is called a **prime filter**.

PROPOSITION 1.2. *Given a Łukasiewicz implication algebra $\mathbf{A}$ and two elements $x_1, x_2 \in A$, $Fg(x_1) \cap Fg(x_2) = Fg(x_1 \vee x_2)$, where $Fg(x)$ stands for the implicative filter generated by x.*

Every subdirectly irreducible algebra in $\mathbb{L}$ is linearly ordered relative to the natural order. Linearly ordered Łukasiewicz implication algebras are called $\mathbb{L}$-*chains*. Finite $\mathbb{L}$-chains are the $\{\to, 1\}$-reducts of finite MV-chains and we denote them by $\mathbf{Ł}_n$, $n \geq 1$. The universe of $\mathbf{Ł}_n$ is the set of rational numbers $Ł_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$, and for each $a, b \in Ł_n$, $a \to b = \min(1, 1 - a + b)$. Another important $\mathbb{L}$-chain is the $\{\to, 1\}$-reduct of Chang's algebra (see [5, p. 474]):

$$\mathbf{C}_\omega = \{(0, y) : y \in \mathbb{N}\} \cup \{(1, -y) : y \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of non-negative integers and

$$(x, y) \to (z, u) = \begin{cases} (1, 0) & \text{if } z > x, \\ (1, \min(0, u - y)) & \text{if } z = x, \\ (1 - x + z, u - y) & \text{otherwise.} \end{cases}$$

The set $Ł_\omega = \{(1, -y) : y \in \mathbb{N}\}$ is the only maximal implicative filter of $\mathbf{C}_\omega$, and $\mathbf{C}_\omega / Ł_\omega \cong \mathbf{Ł}_1$. Its associated subalgebra $\mathbf{Ł}_\omega$ is not finitely generated, and any infinite subalgebra of $\mathbf{Ł}_\omega$ is isomorphic to a copy of it. Moreover, every non-trivial finite subalgebra of $\mathbf{Ł}_\omega$ is isomorphic to $\mathbf{Ł}_n$, for some $n \geq 1$. In addition, $\mathbf{C}_\omega$ and all $\mathbf{Ł}_n$, $n \geq 1$, are two-generated and every non-trivial finitely generated subalgebra of $\mathbf{Ł}_\omega$ is isomorphic to $\mathbf{Ł}_n$, for some $n \geq 1$. In particular, $\mathbf{Ł}_n$ is isomorphic to a subalgebra of $\mathbf{Ł}_m$ if and only if $n \leq m$, and every infinite $\mathbb{L}$-chain contains a copy of $\mathbf{Ł}_n$ for all $n \geq 1$ (see [14]). Finally, it is easy to see that any simple algebra in $\mathbb{L}$ is isomorphic to $\mathbf{Ł}_\alpha$ for some $\alpha \in \omega \cup \{\omega\}$ (again, see [14]).

The lattice of all subvarieties of $\mathbb{L}$ was described in [14], and it is an $(\omega + 1)$-chain:

$$\mathbb{T} \subsetneq V(\mathbf{Ł}_1) \subsetneq \ldots \subsetneq V(\mathbf{Ł}_n) \subsetneq \ldots \subsetneq V(\mathbf{Ł}_\omega) = V(\mathbf{C}_\omega) = \mathbb{L},$$

where $V(\mathbf{A})$ denotes the variety generated by an algebra $\mathbf{A}$ and $\mathbb{T}$ stands for the trivial variety. Observe that $V(\mathbf{Ł}_1)$ is the variety of all implication algebras, also known as Tarski algebras (see [1, 2, 7]).

2. Finite Łukasiewicz Implication Algebras

We give here a representation for finite Łukasiewicz implication algebras that is useful to get an insight of these algebras. In the following theorem and in the sequel we refer to the set of complemented elements of a lattice $\mathbf{B}$ as the *Boolean skeleton* of $\mathbf{B}$.

THEOREM 2.1. *Given a finite Łukasiewicz implication algebra* $\mathbf{A}$*, there exists a finite MV-algebra* $\mathbf{B}$ *(which is a direct product of* $\mathbf{Ł}_k$*'s) such that* A *is an upwardly-closed subset in* $\mathbf{B}$ *and such that every element in* B *is a meet of elements in* A*. Moreover,* A *contains the coatoms of the Boolean skeleton of* $\mathbf{B}$*.*

PROOF. Let $\mathbf{A}$ be a subdirect product of $\mathbf{Ł}_{k_1}, \ldots, \mathbf{Ł}_{k_n}$. Since each $\mathbf{Ł}_{k_i}$ is simple, $\mathbf{Ł}_{k_i} \cong \mathbf{A}/M_i$ for some maximal implicative filter M_i in $\mathbf{A}$. We must also have that $\bigcap_i M_i = \{1\}$. In addition, we may assume that for each j, $\bigcap_{i \neq j} M_i \neq \{1\}$, for otherwise we could erase the j-th factor in the subdirect product.

In what follows, we identify $\mathbf{A}$ with its image in the product $\prod_{i=1}^n \mathbf{Ł}_{k_i}$. Looking at this product as an MV-algebra, we can define the set B consisting of those elements in $\prod_{i=1}^n \mathbf{Ł}_{k_i}$ which are meets of elements in A. Let $\pi_i : \mathbf{A} \to \mathbf{Ł}_{k_i}$ denote the projection of $\mathbf{A}$ onto $\mathbf{Ł}_{k_i}$. Since π_i is surjective, there exists $a_i \in A$ such that $\pi_i(a_i) = 0$. Hence $\bigwedge_i a_i = 0$ and so $0 \in B$. Moreover, if $\bigwedge_i a_i, \bigwedge_j a'_j \in B$,

$$\bigwedge_i a_i \to \bigwedge_j a'_j = \bigwedge_j \left(\bigvee_i (a_i \to a'_j) \right) \in B.$$

This shows that B is closed under $\to$ and, consequently, also under $\neg x := x \to 0$ and $x \oplus y = \neg x \to y$. Hence B is an MV-subuniverse of $\prod \mathbf{Ł}_{k_i}$.

We claim that $B = \prod Ł_{k_i}$. Note first that B contains the elements $c_i = (1, \ldots, 1, 0, 1, \ldots, 1)$ with 0 in the i-th position and 1 everywhere else. Indeed, for each i, $\bigcap_{j \neq i} M_j \neq \{1\}$, so there is some $d_i \in \bigcap_{j \neq i} M_j$, $d_i \neq 1$. But $d_i \notin M_i$, thus $d_i = (1, \ldots, 1, x, 1, \ldots, 1)$ where $x \neq 1$ occupies the i-th entry. Recall that in $\mathbf{Ł}_{k_i}$ we have $x^{k_i} = 0$ if $x \neq 1$. Hence, $d_i^{k_i} = c_i$. This shows that c_i belongs to B, since d_i does.

Consider now an arbitrary element $b \in \prod Ł_{k_i}$, $b = (b_1, \ldots, b_n)$. Since $\pi_i : A \to Ł_{k_i}$ is surjective, there exists $a_i \in A$ such that $\pi_i(a_i) = b_i$. So $(1, \ldots, 1, b_i, 1, \ldots, 1) = a_i \vee c_i \in B$, where b_i is in the i-th entry. Thus $b = \bigwedge (a_i \vee c_i) \in B$. This shows that $B = \prod Ł_{k_i}$.

We prove now that A is upwardly-closed in B. Let $a \in A$ and $b \in B$ with $a \leq b$. We have $b = \bigwedge a_i$, $a_i \in A$. Since $a \leq a_i$ for every i, the meet of $\{a_i\}$ belongs to A.

We have already noted that there are $a_i \in A$ such that $\pi_i(a_i) = 0$. Since A is upwardly-closed and $c_i \geq a_i$, $c_i \in A$, where $c_i = (1, \ldots, 1, 0, 1, \ldots, 1)$, with 0 in the i-th position. Thus A contains the coatoms of the Boolean skeleton of B. ■

REMARK 2.2. Observe that the MV-extension where $\mathbf{A}$ is embedded is unique in the following sense: if $\mathbf{B}_1$ and $\mathbf{B}_2$ are to such extensions, then there exists an isomorphism $\varphi : \mathbf{B}_1 \to \mathbf{B}_2$ such that φ is the identity map on A. Indeed, since A is upwardly-closed and contains the coatoms of the Boolean skeleton of $\mathbf{B}_1$ and $\mathbf{B}_2$, the meet-irreducible elements of $\mathbf{A}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ coincide. Hence, the assertion follows immediately.

We continue to use the notation of the last theorem throughout the remainder of this section, that is, $\mathbf{A}$ denotes a *finite* Łukasiewicz implication algebra embedded as an upwardly-closed subset in an MV-algebra, which is a finite product of $\mathbf{Ł}_k$'s. Moreover, A contains $C = \{c_1, \dots, c_n\}$, the set of coatoms of the Boolean skeleton of the MV-algebra.

We now characterize congruences on $\mathbf{A}$ by means of subsets of C. Note first the following property of congruences.

PROPOSITION 2.3. *Let F be an implicative filter in a Łukasiewicz implication algebra $\mathbf{A}$. If $a \equiv_F b$, $c \equiv_F d$ and $a \wedge c$, $b \wedge d$ both exist, then $a \wedge c \equiv_F b \wedge d$.*

PROOF. Note that

$$\begin{aligned}(a \wedge c) \to (b \wedge d) &= ((a \wedge c) \to b) \wedge ((a \wedge c) \to d)\\ &= ((a \to b) \vee (c \to b)) \wedge ((a \to d) \vee (c \to d)).\end{aligned}$$

Since $a \to b, c \to d \in F$ and F is upwardly-closed and closed under $\wedge$ (where $\wedge$ is defined), we get that $(a \wedge c) \to (b \wedge d) \in F$. Analogously, $(b \wedge d) \to (a \wedge c)$ also lies in F. ■

PROPOSITION 2.4. *The mapping $F \mapsto C \cap F$ gives a lattice isomorphism between the lattice of implicative filters of $\mathbf{A}$ and the power set of C. The corresponding inverse map is given by $U \mapsto A \cap [\bigwedge U)$, for $U \subseteq C$, where $[x) = \{a \in A : a \geq x\}$.*

PROOF. Let h be the mapping given by $h(F) = C \cap F$ for any implicative filter F. We claim that $F = A \cap [\bigwedge h(F))$.

Indeed, let $f \in F$. If $f = 1$ there is nothing to prove. Suppose that $f \neq 1$. Then $f = \bigwedge_{i \in I} f_i$ with $f_i \neq 1$ and $f_i = (1, \dots, 1, x_i, 1, \dots, 1) \geq c_i$. Here x_i is of the form $\frac{r}{k}$ with $0 \leq r < k$. If $r = 0$, $f_i = c_i \in F$. If $1 \leq r < k$, we get $f_i \to (1, \dots, 1, \frac{r-1}{k}, 1, \dots, 1) = (1, \dots, 1, \frac{k-1}{k}, 1, \dots, 1) \in F$ so $(1, \dots, 1, \frac{r-1}{k}, 1, \dots, 1) \in F$. Applying this procedure as many times as necessary we get that $c_i \in F$.

Finally, $\bigwedge h(F) \leq \bigwedge_{i\in I} c_i \leq \bigwedge_{i\in I} f_i = f$. This shows that $f \in A \cap [\bigwedge h(F))$.

Conversely, let $a \in A \cap [\bigwedge h(F))$. We have that $a \geq \bigwedge_{c\in C\cap F} c$. We also know that $a = \bigwedge_{i\in I} a_i$ where each a_i is greater than or equal to some $c \in C$.

If $c \leq a_i$ for some $c \in C \cap F$, then $a_i \in F$. Now suppose $c \not\leq a_i$ for all $c \in C \cap F$. Then there is some $c_0 \in C \setminus F$ with $c_0 \leq a_i$. Since $c_0 \in C \setminus F$, $\neg c_0 \leq c$ for every $c \in C \cap F$. Hence

$$\neg c_0 \leq \bigwedge_{c\in C\cap F} c \leq a_i.$$

Since $c_0 \leq a_i$ and $\neg c_0 \leq a_i$, it follows that $a_i = 1 \in F$.

This shows that $a_i \in F$ for every $i \in I$. Since $a \in A$ and F is $\wedge$-closed (where $\wedge$ is defined), $a \in F$.

We now show that given $U \subseteq C$, we have $C \cap A \cap [\bigwedge U) = U$. Since $C \subseteq A$, we must show that $C \cap [\bigwedge U) = U$.

Let $c \in C \cap [\bigwedge U)$. Then $c \geq \bigwedge_{u\in U} u$. If $c \notin U$, $\neg u \leq c$ for every $u \in U$. Hence

$$\neg \bigwedge_{u\in U} u = \bigvee_{u\in U} \neg u \leq c.$$

Therefore

$$c \geq \neg \bigwedge_{u\in U} u \vee \bigwedge_{u\in U} u = 1,$$

a contradiction. So $c \in U$. The converse is immediate.

This proves that h is a bijection. We show now that h preserves the partial ordering.

If $F_1 \subseteq F_2$, it is clear that $h(F_1) \subseteq h(F_2)$. Conversely, if $U_1 \subseteq U_2$, $U_i \subseteq C$, then $\bigwedge U_2 \leq \bigwedge U_1$. So $[\bigwedge U_1) \subseteq [\bigwedge U_2)$ and $A \cap [\bigwedge U_1) \subseteq A \cap [\bigwedge U_2)$. ■

Let D denote the set of meet-irreducible elements of A, that is, $D = [C) \setminus \{1\} \subseteq A$. For $a \in A$, let $D_a = \{x \in D : x \geq a\}$. Given an implicative filter F, let $D_F = D \cap F$.

Observe that, as a partially ordered subset, D consists of mutually incomparable chains. The minimal elements in D are the coatoms of the Boolean skeleton of **A**, that is, C. Moreover, given an implicative filter F on **A**, $F \cap D$ consists of some of those chains of meet-irreducibles. In particular $F \cap D$ contains precisely those chains whose minimal elements are in $F \cap C$. Note that $F \cap D = F \cap C = \emptyset$ if and only if $F = \{1\}$.

The following proposition gives a characterization of the congruence relation associated with an implicative filter in terms of meet-irreducible elements.

Proposition 2.5. *Let F be an implicative filter in $\mathbf{A}$ and $a, b \in A$. Then $a \equiv_F b$ if and only if $D_a \setminus D_F = D_b \setminus D_F$.*

Proof. First suppose $a \equiv_F b$, so $a \to b, b \to a \in F$. Let $x \in D_a \setminus D_F$. We have $a \leq x$, so $b \to a \leq b \to x$ and then $b \to x \in F$. Since x es meet-irreducible, $b \to x$ is either meet-irreducible or 1.

Assume $b \to x$ is meet-irreducible. As $b \to x \in F$, we get $x \in F$ (since x lies in the same chain of meet-irreducibles as $b \to x$), a contradiction. Hence $b \to x = 1$ and $b \leq x$, i.e., $x \in D_b \setminus D_F$.

This shows that $D_a \setminus D_F \subseteq D_b \setminus D_F$. The reverse inclusion follows analogously.

Now assume $D_a \setminus D_F = D_b \setminus D_F$ holds. Write $a = \bigwedge D_a$, $b = \bigwedge D_b$. Deleting those meet-irreducibles which are in F, we get $a \equiv_F \bigwedge D_a \setminus D_F$ and $b \equiv_F \bigwedge D_b \setminus D_F$. It follows immediately that $a \equiv_F b$. ∎

As a result of Proposition 2.4, we get a useful fact about finite Łukasiewicz implication algebras. This allows us to determine the critical algebras in the next section.

Proposition 2.6. *Every homomorphic image of $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{A}$. In symbols, $H(\mathbf{A}) \subseteq IS(\mathbf{A})$. Moreover, every homomorphic image of $\mathbf{A}$ is a retract of $\mathbf{A}$.*

Proof. Let F be a filter in $\mathbf{A}$ and let U be the subset of C which determines F, that is, $U = F \cap C$. Without loss of generality we may assume that $U = \{c_1, \ldots, c_t\}$ with $t \leq n$. Consider the subuniverse of $\mathbf{A}$ given by $S = \{x \in A : \pi_i(x) = 1, 1 \leq i \leq t\}$ and define $h : A \to S$ by $h(a_1, \ldots, a_n) = (1, \ldots, 1, a_{t+1}, \ldots, a_n)$ where there is a 1 in each of the first t entries. Recalling that $F = A \cap [\bigwedge U)$, it is easily seen that $\mathbf{A}/F \cong \mathbf{S}$. Moreover, since $h(x) = x$ for any $x \in S$, h is a retraction. ∎

3. Quasivarieties

In this section we study quasivarieties in $\mathbb{L}$. In fact, we show that every quasivariety of Łukasiewicz implication algebras is a variety. We show this first in the locally finite case, that is, within the varieties $V(\mathbf{Ł}_n)$. Then we extend the result to the whole variety.

A *finite* algebra $\mathbf{A}$ is called **critical** if it does not belong to the quasivariety generated by its proper subalgebras. The importance of these algebras is exemplified in the next theorem. We reproduce a proof of this fact given by J. Gispert and A. Torrens in [11].

THEOREM 3.1. *Every locally finite quasivariety is generated by its critical algebras.*

PROOF. Let K be a locally finite quasivariety and let $\mathbf{A} \in K$. Consider the family F of finitely generated subalgebras of $\mathbf{A}$. It is known that $\mathbf{A}$ can be embedded in an ultraproduct of members of F (see [4, Chapter V, Theorem 2.14]). Therefore $\mathbf{A} \in ISP_U(F)$. Since K is locally finite we have that $\mathbf{A} \in ISP_U(\{\mathbf{B} \leq \mathbf{A} : \mathbf{B} \text{ is finite}\}) \subseteq Q(K_{fin})$, where K_{fin} denotes the family of finite algebras in K. This shows that $K = Q(K_{fin})$.

Let $\mathbf{A} \in K_{fin}$. We claim that $Q(\mathbf{A}) = Q(\{\mathbf{B} \leq \mathbf{A} : \mathbf{B} \text{ is critical}\})$. We proceed by induction on the cardinal of A. If $|A| = 1$, then $\mathbf{A}$ is critical and the statement becomes obvious. Now assume that $|A| = n$. If $\mathbf{A}$ is critical, the statement again follows immediately. On the other hand, if $\mathbf{A}$ is not critical, then $Q(\mathbf{A}) = Q(\{\mathbf{B} \leq \mathbf{A} : \mathbf{B} \neq \mathbf{A}\})$. For any $\mathbf{B} \leq \mathbf{A}$, $\mathbf{B} \neq \mathbf{A}$, we have that $|B| < n$ and, by inductive hypothesis, $Q(\mathbf{B}) = Q(\{\mathbf{C} \leq \mathbf{B} : \mathbf{C} \text{ is critical}\})$. Putting everything together we get that $Q(\mathbf{A}) = Q(\{\mathbf{C} \leq \mathbf{A} : \mathbf{C} \text{ is critical}\})$, as was to be shown.

We finally get

$$K = Q(K_{fin}) = Q(\{\mathbf{B} : \mathbf{B} \text{ is critical}, \mathbf{B} \leq \mathbf{A}, \mathbf{A} \in K_{fin}\}) = Q(K_{crit}) \subseteq K,$$

where K_{crit} is the family of critical algebras in K. ∎

We now determine all critical Łukasiewicz implication algebras.

THEOREM 3.2. *Let* $\mathbf{A}$ *be a Łukasiewicz implication algebra.* $\mathbf{A}$ *is critical if and only if* $\mathbf{A} \cong \boldsymbol{Ł}_k$ *for some* $k \in \mathbb{N}$.

PROOF. Let $\mathbf{A}$ be a finite Łukasiewicz implication algebra. Since $\mathbf{A}$ is finite, $\mathbf{A} \in ISP(\mathbf{Ł}_{i_1}, \ldots, \mathbf{Ł}_{i_r})$ for some $i_1, \ldots, i_r \in \mathbb{N}$, where each $\mathbf{Ł}_{i_j}$ is a homomorphic image of $\mathbf{A}$ and, by Corollary 2.6, isomorphic to a subalgebra of $\mathbf{A}$.

If $\mathbf{A}$ is not isomorphic to any $\mathbf{Ł}_k$, $k \in \mathbb{N}$, then $\mathbf{A}$ belongs to the quasivariety generated by its proper subalgebras, that is, $\mathbf{A}$ is not critical.

Conversely, if $\mathbf{A} \cong \mathbf{Ł}_k$, its proper subalgebras are $\mathbf{Ł}_i$, $1 \leq i \leq k-1$. Since $\mathbf{A} \notin V(\mathbf{Ł}_{k-1})$, it follows that $\mathbf{A}$ is critical. ∎

COROLLARY 3.3. $V(\boldsymbol{Ł}_k) = Q(\boldsymbol{Ł}_k)$ *for every* $k \in \mathbb{N}$.

PROOF. By Theorem 3.1, $V(\mathbf{Ł}_k)$ is generated as quasivariety by its critical algebras. But the only critical algebras in $V(\mathbf{Ł}_k)$ are $\mathbf{Ł}_1, \ldots, \mathbf{Ł}_k$ and they are all subalgebras of $\mathbf{Ł}_k$. Therefore $V(\mathbf{Ł}_k) = Q(\mathbf{Ł}_k)$. ∎

Corollary 3.4. *Every subquasivariety of* $V(\mathbf{Ł}_k)$, $k \in \mathbb{N}$, *is a variety.*

Proof. Let Q be a subquasivariety of $V(\mathbf{Ł}_k)$. By Theorem 3.1, Q is generated as quasivariety by its critical members which are $\mathbf{Ł}_1, \mathbf{Ł}_2, \dots, \mathbf{Ł}_r$ for some $r \leq k$. Therefore $Q = Q(\mathbf{Ł}_r) = V(\mathbf{Ł}_r)$, which is a variety. ∎

We would like to extend Corollary 3.4 to the variety $\mathbb{L}$ of Łukasiewicz implication algebras. In [10] it is shown that the variety of MV-algebras has the Finite Embeddability Property (FEP, for short). Since Łukasiewicz implication algebras are the subreducts of MV-algebras, the following also holds.

Theorem 3.5. *The variety* $\mathbb{L}$ *has the FEP.*

Corollary 3.6. $\mathbb{L}$ *is generated as quasivariety by its finite members.*

Corollary 3.7. $\mathbb{L} = Q(\{\mathbf{Ł}_k : k \in \mathbb{N}\})$.

In [6], free Łukasiewicz implication algebras are characterized. In particular, the free Łukasiewicz implication algebra over two generators is computed. We reproduce the construction here.

Let $\mathbf{Free}_{\mathbb{MV}}(1)$ be the free MV-algebra over one generator, that is, the MV-algebra of McNaughton functions over the real interval $[0, 1]$. Let

$$\mathbf{M}_2 = \mathbf{Free}_{\mathbb{MV}}(1) \times \mathbf{Free}_{\mathbb{MV}}(1).$$

We say that $(f_1, f_2) \in M_2$ is *compatible* if $f_1(0) = f_2(0)$. Let $\mathbf{M}_2^c$ be the MV-subalgebra of compatible pairs. Let $x_1 = (x, 0)$ and $x_2 = (0, x)$, where x is the free generator of $\mathbf{Free}_{\mathbb{MV}}(1)$. For $i = 1, 2$, we consider the upwardly-closed sets $[x_i) = \{(f_1, f_2) \in M_2^c : x_i \leq (f_1, f_2)\}$. Then

$$\mathbf{Free}_{\mathbb{L}}(2) \cong [x_1) \cup [x_2).$$

Proposition 3.8. $\mathbf{Free}_{\mathbb{L}}(2)$ *has a subalgebra isomorphic to* $\mathbf{Ł}_k$ *for every* $k \in \mathbb{N}$.

Proof. $\mathbf{Ł}_1$ is a subalgebra of any non-trivial Łukasiewicz implication algebra. For each $k \geq 2$ we consider the following McNaughton function

$$f_k(x) = \begin{cases} 1 - x & \text{for } 0 \leq x \leq \frac{1}{k}, \\ (k-1)x & \text{for } \frac{1}{k} \leq x \leq \frac{1}{k-1}, \\ 1 & \text{for } \frac{1}{k-1} \leq x \leq 1. \end{cases}$$

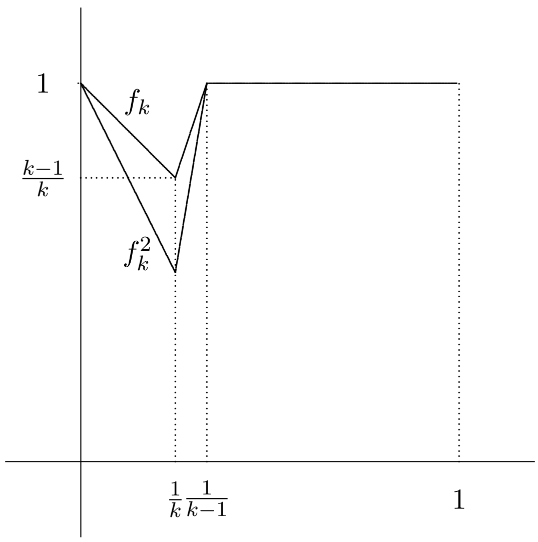

Figure 1. McNaughton functions f_k

It is easily seen that

$$f_k^r(x) = \begin{cases} 1 - rx & \text{for } 0 \leq x \leq \frac{1}{k}, \\ r(k-1)x - r + 1 & \text{for } \frac{1}{k} \leq x \leq \frac{1}{k-1}, \\ 1 & \text{for } \frac{1}{k-1} \leq x \leq 1, \end{cases}$$

for $1 \leq r \leq k$. See Figure 1.

We claim that $\{1, f_k, f_k^2, \ldots, f_k^k\}$ is an implicative subalgebra of $\mathbf{Free}_{\mathbb{MV}}(1)$ isomorphic to $\mathbf{Ł}_k$. Indeed, from Figure 1 it is clear that

$$f_k^r \to f_k^s = \begin{cases} 1 & \text{if } r \geq s, \\ f_k^{s-r} & \text{if } r < s. \end{cases}$$

We can also represent f_k by means of an MV-term. Indeed, from Figure 1, it is clear that this term is $t_k = \neg x \vee (k-1)x$.

Now consider the following elements in $\mathbf{Free}_{\mathbb{L}}(2)$: $(1,1)$, $(1, f_k)$, $(1, f_k^2)$, ..., $(1, f_k^k)$. It is immediate that they all belong to $\mathbf{Free}_{\mathbb{L}}(2)$ since they are compatible pairs and they are all greater than or equal to $(x, 0)$. Hence $\mathbf{Free}_{\mathbb{L}}(2)$ has a subalgebra isomorphic to $\mathbf{Ł}_k$. ■

THEOREM 3.9. *Every subquasivariety of* $\mathbb{L}$ *is a variety.*

PROOF. Let Q be a subquasivariety of $\mathbb{L}$ and consider $V = V(Q)$. V is a subvariety of $\mathbb{L}$, so either $V = V(\mathbf{Ł}_k)$ for some $k \in \mathbb{N}$ or $V = \mathbb{L}$.

In the first case, $Q \subseteq V(\mathbf{Ł}_k)$, so Corollary 3.4 implies that Q is a variety. In the second case, we have that $\mathbb{L} = V(Q)$. Then $\mathbf{Free}_{\mathbb{L}}(2) \cong \mathbf{Free}_Q(2)$ so $\mathbf{Free}_{\mathbb{L}}(2) \in Q$. By the last proposition, we conclude that $\mathbf{Ł}_k \in Q$ for every $k \in \mathbb{N}$. Finally, Corollary 3.7 implies that $Q = \mathbb{L}$. ∎

Remark 3.10. Łukasiewicz implication algebras are the $\{\to, 1\}$-subreducts of Wajsberg hoops. So it is natural to try to face the problem of finding the subquasivarieties of the variety $\mathbb{WH}$ of Wajsberg hoops. It turns out that $\mathbb{WH}$ contains subquasivarieties which are not varieties, as is shown in [8]. For example, if $\mathbf{Ł}_n^{\cdot,\to}$ denotes the $\{\cdot, \to, 1\}$-reduct of the $(n+1)$-element MV-chain, it is shown that the class

$$\mathbb{WH} : \mathbf{Ł}_n^{\cdot,\to} = \{\mathbf{A} \in \mathbb{WH} : \mathbf{Ł}_n^{\cdot,\to} \notin IS(\mathbf{A})\}$$

is a proper quasivariety for $n \geq 2$ (see Lemma 2.19 and Example 2.20 in [8]). This shows the great difference between these two subreducts of Wajsberg algebras as well as the poor expression power of $\to$ in the context of Wajsberg algebras when considered alone.

4. Congruence permutable Łukasiewicz implication algebras

In this section we show that congruence permutable Łukasiewicz implication algebras are precisely those Łukasiewicz implication algebras that have an underlying lattice structure, that is, the meet of every pair of elements exists.

We shall need the universal result stated below, which gives a representation of arithmetical algebras as global subdirect products of finitely subdirectly irreducible algebras. The reader can find more information on global subdirect representations in [17], [16] and [12].

Theorem 4.1. *Let* $\mathbf{A}$ *be an arithmetical algebra and suppose that the class* $\mathbb{V}(\mathbf{A})_{FSI} \cup \{$*trivial algebras*$\}$*, which consists of the finitely subdirectly irreducible algebras in the variety generated by* $\mathbf{A}$ *and the trivial algebras, is a universal class. Then the embedding*

$$A \hookrightarrow \prod\{A/\theta : \theta \text{ is meet irreducible or } \theta = A \times A\}$$

is a global subdirect product under the equalizer topology.

Proof. This follows from the fact that every congruence system is solvable in an arithmetical algebra (also known as the Chinese remainder theorem condition) and Theorem 2.1 of [12]. ∎

In our present case, given a congruence permutable Łukasiewicz implication algebra $\mathbf{A}$, the class $\mathbb{V}(\mathbf{A})_{FSI} \cup \{\textit{trivial algebras}\}$ is definable by the universal sentence

$$(\forall x)(\forall y)(x \vee y = 1 \rightarrow (x = 1 \text{ or } y = 1)).$$

Moreover this class of algebras contains only chains. Therefore, the last theorem guarantees that every congruence permutable Łukasiewicz implication algebra is a global subdirect product of chains. We are now in a position to prove the characterization of congruence permutability.

THEOREM 4.2. *Let* $\mathbf{A}$ *be a Łukasiewicz implication algebra. Then* $\mathbf{A}$ *is congruence permutable if and only if for every* $x, y \in A$*, the meet* $x \wedge y$ *exists.*

PROOF. Consider the following sentence

$$\varphi = (\forall x, y)(\exists! z)(z \rightarrow x = 1 \ \& \ z \rightarrow y = 1 \ \& \ (x \rightarrow z) \vee (y \rightarrow z) = 1).$$

It is easy to see that φ is valid in a Łukasiewicz implication algebra $\mathbf{A}$ if and only if the meet $x \wedge y$ exists for every $x, y \in A$. This condition is valid in any chain, hence it is also valid in any congruence permutable Łukasiewicz implication algebra, since validity of this type of sentences is preserved by global subdirect products (see [17]).

Now assume that $\mathbf{A}$ is a Łukasiewicz implication algebra such that every pair of its elements has a meet. Let $f : A^3 \rightarrow A$ be given by

$$f(x, y, z) = ((x \rightarrow y) \rightarrow z) \wedge ((z \rightarrow y) \rightarrow x).$$

Although f is not a term-function in the language of Łukasiewicz implication algebras, by Proposition 2.3, we have that if $x_i \equiv_F y_i$, $1 \leq i \leq 3$, then $f(x_1, x_2, x_3) \equiv_F f(y_1, y_2, y_3)$.

Observe that the function f acts like a Mal'cev term. Indeed, it is immediately verified that $f(x, x, z) = z$ and $f(x, z, z) = x$. Now if $x \equiv_{F_1} z \equiv_{F_2} y$, we have that

$$x = f(x, z, z) \equiv_{F_2} f(x, y, z) \equiv_{F_1} f(y, y, z) = z.$$

This proves that $\mathbf{A}$ is congruence permutable. ■

The global representability of congruence permutable Łukasiewicz implication algebras by means of chains may be derived in a slightly different way using Theorem 3.4 in [16]. To accomplish that, we also need a kind of prime filter theorem for Łukasiewicz implication algebras. We state and prove this theorem because it may be of independent interest.

THEOREM 4.3. *Let $\mathbf{A}$ be a Łukasiewicz implication algebra. Let F be an implicative filter and M a subset of A closed under $\vee$ such that $M \cap F = \emptyset$. Then there exists a prime filter P such that $F \subseteq P$ and $P \cap M = \emptyset$.*

PROOF. Let $\mathcal{F}$ be the family consisting of the filters G of $\mathbf{A}$ such that $F \subseteq G$ and $G \cap M = \emptyset$. Clearly $\mathcal{F}$ is nonempty since $F \in \mathcal{F}$. By a routine application of Zorn's Lemma, there exists a filter P maximal in $\mathcal{F}$. We claim that P is a prime filter.

Indeed, assume $x_1 \vee x_2 \in P$, $x_1, x_2 \notin P$ for some $x_1, x_2 \in A$. Since P is maximal in $\mathcal{F}$, there exists $m_i \in M$, $i = 1, 2$, such that $m_i \in Fg(P, x_i)$. Since M is closed under joins, we may assume that $m_1 = m_2 = m$. Thus $m \in Fg(P, x_1) \cap Fg(P, x_2)$.

Observe now that

$$\begin{aligned} Fg(P, x_1) \cap Fg(P, x_2) &= (P \vee Fg(x_1)) \cap (P \vee Fg(x_2)) \\ &= P \vee (Fg(x_1) \cap Fg(x_2)) \\ &= P \vee Fg(x_1 \vee x_2) \\ &= P. \end{aligned}$$

Hence $m \in P$, a contradiction. ■

The following straightforward lemma gives a partial affirmative answer to the existence of meets in a congruence permutable Łukasiewicz implication algebra.

LEMMA 4.4. *Let $\mathbf{A}$ be a congruence permutable Łukasiewicz implication algebra. For every $x, y \in A$ such that $x \vee y = 1$, the meet $x \wedge y$ exists.*

PROOF. Let $F_1 = Fg(x)$ and $F_2 = Fg(y)$ and let θ_1, θ_2 be the corresponding congruences. Clearly $(x, 1) \in \theta_1$ and $(1, y) \in \theta_2$, that is, $(x, y) \in \theta_1 \circ \theta_2$. Since θ_1 and θ_2 permute, there exists $z \in A$ such that $(x, z) \in \theta_2$ and $(z, y) \in \theta_1$. Thus $z \to x \in F_2$ and $z \to y \in F_1$. Moreover, we get $z \to x, z \to y \in F_1 \cap F_2$. Now observe that $F_1 \cap F_2 = Fg(x \vee y) = \{1\}$. Hence, $z \leq x$ and $z \leq y$, so $x \wedge y$ exists. ■

We are now in a position to show the global representation of congruence permutable Łukasiewicz implication algebras by means of chains. In fact, by Theorem 3.4 in [16], we only need to show that the set $MI(\mathbf{A})$ of meet irreducible congruences on $\mathbf{A}$ is compact in the following sense: given $S \subseteq A \times A$ such that for every $\theta \in MI(\mathbf{A})$, there exists $(a, b) \in S$ such that $(a, b) \in \theta$, then there exists some finite subset $S_0 \subseteq S$ such that for every $\theta \in MI(\mathbf{A})$, there is $(a, b) \in S_0$ with $(a, b) \in \theta$.

Recall that for any $a, b \in A$ we have that $(a \to b) \vee (b \to a) = 1$ and since **A** is congruence permutable, Lemma 4.4 implies that the meet $(a \to b) \wedge (b \to a)$ exists. Let $X = \{\bigvee_{i=1}^{n}(a_i \to b_i) \wedge (b_i \to a_i) : (a_i, b_i) \in S\}$ and assume $1 \notin X$. By Theorem 4.3 there exists a prime filter P in **A** such that $P \cap X = \emptyset$. On the other hand, since P is a prime filter, $\mathbf{A}/P$ is a chain, so $\theta_P \in MI(\mathbf{A})$. Hence, there exists $(a, b) \in S$ such that $(a, b) \in \theta_P$. It follows that $(a \to b) \wedge (b \to a) \in P \cap X$, a contradiction.

This shows that $1 \in X$. Hence, $1 = \bigvee_{i=1}^{n}(a_i \to b_i) \wedge (b_i \to a_i)$ for some $(a_i, b_i) \in S$. Now, given any $\theta \in MI(\mathbf{A})$, the associated filter P_θ is a prime filter, and since $1 \in P$, there exists $j \in \{1, \ldots, n\}$ such that $(a_j \to b_j) \wedge (b_j \to a_j) \in P_\theta$, or equivalently, $(a_j, b_j) \in \theta$. This concludes the proof that $MI(\mathbf{A})$ is compact.

5. Another characterization of permutability

We say that an algebra **A** is *minimally non-permutable* if it is not congruence permutable, yet each one of its non trivial quotients is congruence permutable. The following result is immediate.

LEMMA 5.1. *Given a finite algebra* **A**, **A** *is not congruence permutable if and only if one of its quotients is minimally non-permutable.*

We now give a simple characterization of minimally non-permutable Łukasiewicz implication algebras.

LEMMA 5.2. *A finite Łukasiewicz implication algebra is minimally non-permutable if and only if it is isomorphic to an upwardly-closed proper implicative subalgebra of a product of finite chains,* **B**, *which contains the atoms of the Boolean skeleton of* **B**.

PROOF. Let **A** be a minimally non-permutable finite Łukasiewicz implication algebra. We may consider **A** as an upwardly-closed implicative subalgebra of a product of finite chains **B** such that A contains the coatoms of the Boolean skeleton of **B** (see Theorem 2.1). Since **A** is not congruence-permutable, **A** is a proper implicative subalgebra of **B**.

We may write $\mathbf{B} = \prod_{i=1}^{n} \mathbf{B}_i$, where each $\mathbf{B}_i$ is a finite MV-chain. For each $j \in \{1, \ldots, n\}$, let $F_j = \{x \in A : x(i) = 1 \text{ for } i \neq j\}$. It is clear that F_j is a non-trivial implicative filter of **A**. Thus $\mathbf{A}/F_j$ is congruence permutable, and by Theorem 4.2, every pair of elements in $\mathbf{A}/F_j$ has a meet. Since $\mathbf{A}/F_j \cong \{x \in A, x(j) = 1\}$, it follows that $(0, \ldots, 0, 1, 0, \ldots, 0) \in A$, where 1 occupies the j-th position.

This shows that A contains all the atoms of the Boolean skeleton of $\mathbf{B}$.

Conversely, assume A contains all the atoms of the Boolean skeleton. It is clear then that every non-trivial quotient has a least element and hence is congruence permutable. ■

COROLLARY 5.3. *A finite Łuksaiewicz implication algebra is congruence permutable if and only if none of its quotients is isomorphic to a proper implicative subalgebra of a product of finite chains containing the atoms of the Boolean skeleton.*

Acknowledgements. The authors are partially supported by CONICET.

References

[1] ABBOTT, J. C., 'Implicational algebras', *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, 11 (59):3–23 (1968), 1967.

[2] ABBOTT, J. C., 'Semi-boolean algebras', *Mat. Vesnik*, 4 (19):177–198, 1967.

[3] BERMAN, J., and W. J. BLOK, 'Free Łukasiewicz and hoop residuation algebras', *Studia Logica*, 77(2):153–180, 2004.

[4] BURRIS, S., and H. P. SANKAPPANAVAR, *A course in universal algebra*, volume 78 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1981.

[5] CHANG, C. C., 'Algebraic analysis of many valued logics', *Trans. Amer. Math. Soc.*, 88:467–490, 1958.

[6] DÍAZ VARELA, J. P., 'Free Łukasiewicz implication algebras', *Arch. Math. Logic*, 47(1):25–33, 2008.

[7] DÍAZ VARELA, J. P., and A. TORRENS, 'Decomposability of free Tarski algebras', *Algebra Universalis*, 50(1):1–5, 2003.

[8] FERREIRIM, I. M. A., *On varieties and quasivarieties of hoops and their reducts*, ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.), University of Illinois at Chicago, 1992.

[9] FONT, J. M., A. J. RODRÍGUEZ, and A. TORRENS, 'Wajsberg algebras', *Stochastica*, 8(1):5–31, 1984.

[10] GISPERT, J., and D. MUNDICI, 'MV-algebras: a variety for magnitudes with Archimedean units', *Algebra Universalis*, 53(1):7–43, 2005.

[11] GISPERT, J., and A. TORRENS, 'Locally finite quasivarieties of MV-algebras', preprint.

[12] GRAMAGLIA, H., and D. VAGGIONE. 'Birkhoff-like sheaf representation for varieties of lattice expansions', *Studia Logica*, special issue on Priestley duality, 56(1-2):111–131, 1996.

[13] KOMORI, Y., 'The separation theorem of the $\aleph_0$-valued Łukasiewicz propositional logic', *Rep. Fac. Sci. Shizuoka Univ.*, 12:1–5, 1978.

[14] KOMORI, Y., 'Super-Łukasiewicz implicational logics', *Nagoya Math. J.*, 72:127–133, 1978.

[15] MUNDICI, D., 'MV-algebras are categorically equivalent to bounded commutative BCK-algebras', *Math. Japon.*, 31(6):889–894, 1986.

[16] Vaggione, D., 'Sheaf representation and Chinese remainder theorems', *Algebra Universalis*, 29(2):232–272, 1992.

[17] Volger, H., 'Preservation theorems for limits of structures and global sections of sheaves of structures', *Math. Z.*, 166(1):27–54, 1979.

Miguel Campercholi
Facultad de Matemática, Astronomía y Física
Universidad Nacional de Córdoba
Medina Allende s/n, Ciudad Universitaria
Córdoba (5000), Argentina
camper@mate.uncor.edu

Diego Castaño
Departamento de Matemática
Universidad Nacional del Sur
Alem 1253
Bahía Blanca (8000), Argentina
diego.castano@uns.edu.ar

José Patricio Díaz Varela
Departamento de Matemática
Universidad Nacional del Sur
Alem 1253
Bahía Blanca (8000), Argentina
usdiavar@criba.edu.ar