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Vol. 16, Nos. 3–4, 2011
Distributed worldwide by Springer. *Regular and Chaotic Dynamics* ISSN 1560-3547.

Special Issue: Algebraic Integrability

ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 185–186. © Pleiades Publishing, Ltd., 2011.

Foreword

DOI: 10.1134/S1560354711030014

It is a remarkable fact that for *most* classical integrable systems, the solution with generic initial values admits an analytic continuation which is a meromorphic function, in particular it is single-valued.

In the case of the Euler top (a rigid body which rotates around a fixed point which is its center of gravity), the analytic continuation of the generic solution is expressible in terms of elliptic functions. Thus, after complexification, the circles (ovals) which fill up the phase space and on which occurs the periodic motion of the top, become elliptic curves, likewise called elliptic Riemann surfaces.

At first, the above phenomenon seemed to be a very particular and exceptional one, being restricted to integrable systems in one degree of freedom, or more generally to integrable systems whose solutions are periodic functions of time, as a few other examples of this type were discoverd. However, when Kowalevski integrated the top which bears her name in terms of genus two, rather than elliptic (genus one) theta functions, it became clear that the concepts and tools developed by Jacobi, and further generalized by Riemann, were going to play a decisive role in the study of the integrable problems of classical mechanics: theta functions, Riemann surfaces, Jacobi varieties and Abelian integrals were the new tools, enabling the explicit integration of the integrable problems of mechanics. Nevertheless, when Poincaré showed that the three body problem is not integrable, the further development of the theory of integrable systems slowed down, in particular the remarkable connection with the emerging theory of Riemann surfaces and Abelian varieties was put to rest.

In the 1960's, half a century later, a series of important discoveries were made on the (by now famous) Korteweg–de Vries equation (KdV equation), which describes long waves in a shallow one-dimensional channel. The magical interaction between the solitary waves which appear as solutions, the so-called solitons, led to the discovery that the KdV equation can be viewed as an infinite-dimensional integrable Hamiltonian system. The KdV equation is nowadays still considered as *the* prototype of an integrable PDE: Lax equations, multi-Hamiltonian structures, Virasoro symmetries, matrix integral solutions, Backlund transformations, and so on, have not only be constructed for this equation, the KdV equation is in fact the *first* equation for which each item of this list has been constructed.

The same holds true for what brings us back to Riemann surfaces and Abelian varieties: Its-Matveev show in 1975 that *every* hyperelliptic theta function, with properly scaled space and time coordinates as arguments, is a solution to the Korteweg-de Vries equation (the time and space coordinates being thought of as complex). This observation and its generalization to arbitrary theta functions in connection with the Kadomtsev-Petviashvili equation (KP equation) led to very striking new results. The classical Schottky problem, for example, which asks for a characterization of Jacobi varieties among all so-called principally polarized Abelian varieties gets a very definite answer in terms of the KP equation: the theta function of a polarized Abelian variety comes from a Jacobian if and only if it is a solution of the KP equation (the Novikov conjecture, proven by Shiota in 1986)! These results and several others generated a breakthrough which brought algebraic geometry back in the realm of integrable systems and the classical results on (finite-dimensional) integrable systems were revisited from a new angle, leading often to a better understanding of what was known, and of course to many new results and techniques, both in the finite-dimensional and infinite-dimensional integrable world.

A particular instance of this is the notion of "algebraic integrability", which is the topic of the present special volume. When Adler and van Moerbeke revisited the original works of Kowalevski, they not only adopted the method by which Kowalevski found her top in order to find new cases of integrable systems, they moreover transformed her tool into an efficient instrument which allows one to unveil the beautiful (complex!) geometry that underlies integrable systems, such as the Euler, Lagrange and Kowalevski tops, and basically all classically known integrable systems. While

Kowalevski does an explicit integration of the equations of motion, by a real tour de force, leading to explicit formulas for the motion of the top in terms of genus two theta functions, Adler and van Moerbeke take a much more geometric point of view, considering the complex invariant manifolds, which are the affine algebraic varieties on which the flow takes place and they show that these are Abelian surfaces (compact complex tori of dimension two), minus a pair of genus three curves on which the flow (with complex time) blows up. Thus, the notion of an algebraic completely integrable system (a.c.i. system) is born: it is a complex integrable system for which the invariant manifolds are affine parts of Abelian varieties (commutative algebraic groups, in general) and such that the flow of the integrable vector fields are linear on these tori.

Thirty years have passed by and a lot of progress on algebraic integrability has been made. Not all complex integrable systems are a.c.i., but a vast number of examples is known; they include several types of Toda lattices and similar lattices, potentials such as the Garnier potential, certain geodesic flows, integrable systems on moduli spaces of connections, reductions of equations such as the KdV and KP equation, and many systems which appear naturally in algebraic geometry. Often these systems have deep connections with Lie theory, which is apparent from their Hamiltonian structure or from the definition of their phase space, which account for the integrability, but the direct connection between Lie theory and algebraic integrability remains poorly understood. Similarly, starting from a Lax operator with spectral parameter in some Lie algebra, we have a fair understanding of the connection between the isospectral deformations describing the flow of the corresponding integrable system as a linear flow on the Jacobi variety of the spectral curve; but given an integrable Hamiltonian coming from classical mechanics, with no extra information such as Lax equations, underlying Lie algebras or spectral curves, we have at present only a partial understanding of how the apparently intertwined Lie algebraic and algebraic-geometric properties are encoded (hidden!), in the sparse information which is contained in the Hamiltonian and its constants of motion.

The purpose of the current volume on algebraic integrability is to give an overview of the actual state of the theory, a few of the papers being review papers the other ones being research papers which contain new results. In either case, they are written by researchers which are currently active in the field of algebraic integrability. We hope that these papers will be a significant contribution to the area and will give the reader a overview of the subject as well as of the main problems which remain unsolved.

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 187–209. © Pleiades Publishing, Ltd., 2011.

Algebraic Integrability: the Adler–Van Moerbeke Approach

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Received September 5, 2009; accepted October 19, 2010

Abstract—In this paper, I present an overview of the active area of algebraic completely integrable systems in the sense of Adler and van Moerbeke. These are integrable systems whose trajectories are straight line motions on abelian varieties (complex algebraic tori). We make, via the Kowalewski–Painlevé analysis, a study of the level manifolds of the systems. These manifolds are described explicitly as being affine part of abelian varieties and the flow can be solved by quadrature, that is to say their solutions can be expressed in terms of abelian integrals. The Adler–Van Moerbeke method's which will be used is devoted to illustrate how to decide about the algebraic completely integrable Hamiltonian systems and it is primarily analytical but heavily inspired by algebraic geometrical methods. I will discuss some interesting and well known examples of algebraic completely integrable systems: a five-dimensional system, the Hénon–Heiles system, the Kowalewski rigid body motion and the geodesic flow on the group $SO(n)$ for a left invariant metric.

MSC2010 numbers: 37J35, 70H06, 14H70
DOI: 10.1134/S1560354711030026

Keywords: completely integrable systems, topological structure of phase space, methods of integration

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The problem of finding and integrating Hamiltonian systems, has attracted a considerable amount of attention in recent years. Beside the fact that many integrable Hamiltonian systems have been on the subject of powerful and beautiful theories of mathematics, another motivation for its study is: the concepts of integrability have been applied to an increasing number of physical systems, biological phenomena, population dynamics, chemical rate equations, to mention only a few. However, it seems still hopeless to describe or even to recognize with any facility, those Hamiltonian systems which are integrable, though they are exceptional. The resolution of the well known Korteweg–de Vries equation has generated an enormous number of new ideas in the area of Hamiltonian completely integrable systems. It has led to unexpected connections between

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mechanics, spectral theory, Lie algebra theory, algebraic geometry and even differential geometry. All these connections have generated renewed interest in the questions around complete integrability of finite and infinite dimensional systems, ordinary and partial differential equations. However given a Hamiltonian system, it remains often hard to fit it into any of those general frameworks. But luckily, most of the problems possess the much richer structure of the so called algebraic complete integrability (concept introduced and systematized by Adler and van Moerbeke). A dynamical system is algebraic completely integrable in the sense of Adler–Van Moerbeke [1, 2] if it can be linearized on a complex algebraic torus $\mathbf{C}^n/Lattice$ (=abelian variety). The invariants (often called first integrals or constants) of the motion are polynomials and the phase space coordinates (or some algebraic functions of these) restricted to a complex invariant variety defined by putting these invariants equal to generic constants, are meromorphic functions on an abelian variety. Moreover, in the coordinates of this abelian variety, the flows (run with complex time) generated by the constants of the motion are straight lines. However, besides the fact that many Hamiltonian completely integrable systems possess this structure, another motivation for its study is: algebraic completely integrable systems come up systematically whenever you study the isospectral deformation of some linear operator containing a rational indeterminate. Indeed a theorem by Adler–Kostant–Symes [3] applied to Kac–Moody algebras provides such systems which, by a theorem of van Moerbeke–Mumford [4], are algebraic completely integrable. Also some interesting integrable systems appear as coverings of algebraic completely integrable systems [5, 6]. The invariant varieties are coverings of abelian varieties and these systems are called algebraic completely integrable in the generalized sense. The concept of algebraic complete integrability is quite effective in small dimensions and has the advantage to lead to global results, unlike the existing criteria for real analytic integrability, which, at this stage are perturbation results. The methods used are primarily analytical but heavily inspired by algebraic geometrical methods. Abelian varieties and cyclic coverings of abelian varieties, very heavily studied by algebraic geometers, enjoy certain algebraic properties which can then be translated into differential equations and their Laurent solutions.

In this paper, I present an overview of the concept of algebraic completely integrable systems in the sense of Adler–Van Moerbeke. I will discuss some examples of algebraic completely integrable systems: a five-dimensional system, the Hénon–Heiles system, the Kowalewski rigid body motion and the geodesic flow on the group $SO(n)$ for a left invariant metric. For details the reader is referred to Adler–Van Moerbeke–Vanhaecke excellent books [3, 6], where other interesting problems are being discussed in detail such that the Toda lattice, the odd and the even Mumford systems, the Garnier potential, the Goryachev–Chaplygin top, etc.

1. ALGEBRAIC COMPLETE INTEGRABILITY

We give some results about abelian surfaces which will be used, as well as the basic techniques to study two-dimensional algebraic completely integrable systems (details can be found in [3, 6–8]). Let $M = \mathbb{C}/\Lambda$ be a $n-$dimensional abelian variety where Λ is the lattice generated by the $2n$ columns $\lambda_1, \ldots, \lambda_{2n}$ of the $n \times 2n$ period matrix and let $\mathcal{D} = \sum k_j \mathcal{D}_j,\ k_j \in \mathbb{Z}$, be a divisor on M. Define $\mathcal{L}(\mathcal{D}) = \{f \text{ meromorphic on } M : (f) \geqslant -\mathcal{D}\}$, i.e., a function $f \in \mathcal{L}(\mathcal{D})$ has at worst a k_j-fold pole along $\mathcal{D}_j$. The divisor $\mathcal{D}$ is called ample when a basis $(f_0, \ldots, f_N)$ of $\mathcal{L}(k\mathcal{D})$ embeds M smoothly into $\mathbb{P}^N(\mathbb{C})$ for some k, via the map

$$M \longrightarrow \mathbb{P}^N(\mathbb{C}),\ p \longmapsto [1 : f_1(p) : \ldots : f_N(p)],$$

then $k\mathcal{D}$ is called very ample. It is known that every positive divisor $\mathcal{D}$ on an irreducible abelian variety is ample and thus some multiple of $\mathcal{D}$ embeds M into $\mathbb{P}^N(\mathbb{C})$. By a theorem of Lefschetz, any $k \geqslant 3$ will work. Moreover, there exists a complex basis of $\mathbb{C}^n$ such that the lattice expressed in that basis is generated by the columns of the $n \times 2n$ period matrix $\text{diag}(\delta_1, \ldots, \delta_n|Z)$, with $Z^\top = Z$, $\text{Im}Z > 0$, $\delta_j \in \mathbb{N}^*$ and $\delta_j|\delta_{j+1}$. The integers δ_j which provide the so-called polarization of the abelian variety M are then related to the divisor as follows:

$$\dim \mathcal{L}(\mathcal{D}) = \delta_1 \ldots \delta_n. \tag{1.1}$$

In the case of a $2-$dimensional abelian varieties (surfaces), even more can be stated: the geometric genus g of a positive divisor $\mathcal{D}$ (containing possibly one or several curves) on a surface M is given by the adjunction formula

$$g(\mathcal{D}) = \frac{K_M.\mathcal{D} + \mathcal{D}.\mathcal{D}}{2} + 1, \tag{1.2}$$

where K_M is the canonical divisor on M, i.e., the zero-locus of a holomorphic 2−form, $\mathcal{D}.\mathcal{D}$ denotes the number of intersection points of $\mathcal{D}$ with $a + \mathcal{D}$ (where $a + \mathcal{D}$ is a small translation by a of $\mathcal{D}$ on M), where as the Riemann–Roch theorem for line bundles on a surface tells you that

$$\chi(\mathcal{D}) = p_a(M) + 1 + \frac{1}{2}(\mathcal{D}.\mathcal{D} - \mathcal{D}K_M), \tag{1.3}$$

where $p_a\,(M)$ is the arithmetic genus of M and $\chi(\mathcal{D})$ the Euler characteristic of $\mathcal{D}$. To study abelian surfaces using Riemann surfaces on these surfaces, we recall that

$$\begin{aligned}\chi(\mathcal{D}) &= \dim H^0(M, \mathcal{O}_M(\mathcal{D})) - \dim H^1(M, \mathcal{O}_M(D)),\\ &= \dim \mathcal{L}(\mathcal{D}) - \dim H^1(M, \Omega^2(\mathcal{D}\otimes K_M^*)), \text{(Kodaira–Serre duality)},\\ &= \dim \mathcal{L}(\mathcal{D}), \text{(Kodaira vanishing theorem)},\end{aligned} \tag{1.4}$$

whenever $\mathcal{D}\otimes K_M^*$ defines a positive line bundle. However for abelian surfaces, K_M is trivial and $p_a(M) = -1$; therefore combining relations (1.1), (1.2), (1.3) and (1.4),

$$\chi\,(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}) = \frac{\mathcal{D}.\mathcal{D}}{2} = g\,(\mathcal{D}) - 1 = \delta_1\delta_2.$$

A divisor $\mathcal{D}$ is called projectively normal, when the natural map

$$\mathcal{L}(\mathcal{D})^{\otimes k} \longrightarrow \mathcal{L}(k\mathcal{D}),$$

is surjective, i.e., every function of $\mathcal{L}(k\mathcal{D})$ can be written as a linear combination of k-fold products of functions of $\mathcal{L}(\mathcal{D})$. Not every very ample divisor $\mathcal{D}$ is projectively normal but if $\mathcal{D}$ is linearly equivalent to $k\mathcal{D}_0$ for $k \geqslant 3$ for some divisor $\mathcal{D}_0$, then $\mathcal{D}$ is projectively normal.
Now consider the exact sheaf sequence

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{\pi^*} \mathcal{O}_{\widetilde{C}} \longrightarrow X \longrightarrow 0,$$

where C is a singular connected Riemann surface, $\widetilde{C} = \sum C_j$ the corresponding set of smooth Riemann surfaces after desingularization and $\pi : \widetilde{C} \to C$ the projection. The exactness of the sheaf sequence shows that the Euler characteristic

$$\mathcal{X}(\mathcal{O}) = \dim H^0(\mathcal{O}) - \dim H^1(\mathcal{O}),$$

satisfies

$$\mathcal{X}(\mathcal{O}_C) - \mathcal{X}(\mathcal{O}_{\widetilde{C}}) + \mathcal{X}(X) = 0, \tag{1.5}$$

where $\mathcal{X}(X)$ only accounts for the singular points p of C; $\mathcal{X}(X_p)$ is the dimension of the set of holomorphic functions on the different branches around p taken separately, modulo the holomorphic functions on the Riemann surface C near that singular point. Consider the case of a planar singularity (example a tacnode for which $\mathcal{X}(X) = 2$, as well), i.e., the tangents to the branches lie in a plane. If $f_j(x, y) = 0$ denotes the j^{th} branch of C running through p with local parameter s_j, then

$$\mathcal{X}(X_p) = \dim \Pi_j \mathbb{C}[[s_j]]/\frac{\mathbb{C}[[x, y]]}{\Pi_j f_j(x, y)}.$$

So using (1.4) and Serre duality, we obtain $\mathcal{X}(\mathcal{O}_C) = 1 - g(C)$ and

$$\mathcal{X}(\mathcal{O}_{\widetilde{C}}) = n - \sum_{j=1}^{n} g(C_j).$$

Also, replacing in the formula (1.5), gives

$$g(C)=\sum_{j=1}^{n} g(C_j)+\mathcal{X}(X)+1-n.$$

Finally, recall that a Kähler variety is a variety with a Kähler metric, i.e., a hermitian metric whose associated differential 2-form of type $(1,1)$ is closed. The complex torus $\mathbb{C}^2/lattice$ with the euclidean metric $\sum dz_i\otimes d\overline{z}_i$ is a Kähler variety and any compact complex variety that can be embedded in projective space is also a Kähler variety. A compact complex Kähler variety having as many independent meromorphic functions as its dimension is a projective variety.

Consider now Hamiltonian problems of the form

$$X_H:\dot{x}=J\frac{\partial H}{\partial x}\ \equiv f(x),\ x\in\mathbb{R}^m, \tag{1.6}$$

where H is the Hamiltonian and $J=J(x)$ is a skew-symmetric matrix with polynomial entries in x, for which the corresponding Poisson bracket $\{H_i,H_j\}=\langle\frac{\partial H_i}{\partial x},J\frac{\partial H_j}{\partial x}\rangle$, satisfies the Jacobi identities. The system (1.6) with polynomial right hand side will be called algebraic complete integrable (a.c.i.) in the sense of Adler–Van Moerbeke [1, 2] when:

i) The system possesses $n+k$ independent polynomial invariants $H_1,\dots,H_{n+k}$ of which k lead to zero vector fields $J\frac{\partial H_{n+i}}{\partial x}(x)=0,\ 1\leqslant i\leqslant k$, the n remaining ones are in involution (i.e., $\{H_i,H_j\}=0$) and $m=2n+k$. For most values of $c_i\in\mathbb{R}$, the invariant varieties $\bigcap_{i=1}^{n+k}\{x\in\mathbb{R}^m:H_i=c_i\}$ are assumed compact and connected. Then, according to the Arnold–Liouville theorem [9], there exists a diffeomorphism

$$\bigcap_{i=1}^{n+k}\{x\in\mathbb{R}^m:H_i=c_i\}\longrightarrow\mathbb{R}^n/Lattice,$$

and the solutions of the system (1.6) are straight lines motions on these tori.

ii) The invariant varieties, thought of as affine varieties in $\mathbb{C}^m$ can be completed into complex algebraic tori, i.e.,

$$\bigcap_{i=1}^{n+k}\{x\in\mathbb{C}^m:H_i=c_i\}\cup\mathcal{D}=\mathbb{C}^n/Lattice,$$

where $\mathbb{C}^n/Lattice$ is a complex algebraic torus (i.e., abelian variety) and $\mathcal{D}$ a divisor. Algebraic means that the torus can be defined as an intersection $\bigcap_{i=1}^{l}\{P_i(X_0,\dots,X_N)=0\}$ involving a large number of homogeneous polynomials P_i. In the natural coordinates $(t_1,\dots,t_n)$ of $\mathbb{C}^n/Lattice$ coming from $\mathbb{C}^n$, the functions $x_i=x_i(t_1,\dots,t_n)$ are meromorphic and (1.6) defines straight line motion on $\mathbb{C}^n/Lattice$.

Condition *i*) means, in particular, there is an algebraic map

$$(x_1(t),\dots,x_m(t))\longmapsto(\mu_1(t),\dots,\mu_n(t)),$$

making the following sums linear in t :

$$\sum_{i=1}^{n}\int_{\mu_i(0)}^{\mu_i(t)}\omega_j=d_jt\ ,\ 1\leqslant j\leqslant n,\ d_j\in\mathbb{C},$$

where $\omega_1,\dots,\omega_n$ denote holomorphic differentials on some algebraic curves.

Adler and van Moerbeke [2] have shown that the existence of a coherent set of Laurent solutions:

$$x_i=\sum_{j=0}^{\infty}x_i^{(j)}t^{j-k_i},\quad k_i\in\mathbb{Z},\quad \text{some } k_i>0, \tag{1.7}$$

depending on *dim* (*phase space*) $-1 = m-1$ free parameters is necessary and sufficient for a Hamiltonian system with the right number of constants of motion to be a.c.i. So, if the Hamiltonian flow (1.6) is a.c.i., it means that the variables x_i are meromorphic on the torus $\mathbb{C}^n/Lattice$ and by compactness they must blow up along a codimension one subvariety (a divisor) $\mathcal{D} \subset \mathbb{C}^n/Lattice$. By the a.c.i. definition, the flow (1.6) is a straight line motion in $\mathbb{C}^n/Lattice$ and thus it must hit the divisor $\mathcal{D}$ in at least one place. Moreover through every point of $\mathcal{D}$, there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equations must admit Laurent expansions which depend on the $n-1$ parameters defining $\mathcal{D}$ and the $n+k$ constants c_i defining the torus $\mathbb{C}^n/Lattice$ the total count is therefore $m-1 = dim$ (*phase space*) -1 parameters. Assume now Hamiltonian flows to be (weight)-homogeneous with a weight $\nu_i \in \mathbb{N}$, going with each variable x_i, i.e.,

$$f_i\left(\alpha^{\nu_1}x_1,\ldots,\alpha^{\nu_m}x_m\right) = \alpha^{\nu_i+1}f_i\left(x_1,\ldots,x_m\right),\ \forall\alpha\in\mathbb{C}.$$

Observe that then the constants of the motion H can be chosen to be (weight)-homogeneous:

$$H\left(\alpha^{\nu_1}x_1,\ldots,\alpha^{\nu_m}x_m\right) = \alpha^{k}H\left(x_1,\ldots,x_m\right),\ k\in\mathbb{Z}.$$

If the flow is algebraically completely integrable, the differential equations (6) must admits Laurent series solutions (7) depending on $m-1$ free parameters. We must have $k_i = \nu_i$ and coefficients in the series must satisfy at the 0^{th}step non-linear equations,

$$f_i\left(x_1^{(0)},\ldots,x_m^{(0)}\right) + g_i x_i^{(0)} = 0,\ 1\leqslant i\leqslant m, \tag{1.8}$$

and at the k^{th}step, linear systems of equations:

$$\left(L-kI\right)z^{(k)} = \begin{cases} 0 \ \text{ for } k=1 \\ \text{some polynomial in } x^{(1)},\ldots,x^{(k-1)} \text{ for } k>1, \end{cases} \tag{1.9}$$

where

$$L = \text{ Jacobian map of } (8) = \frac{\partial f}{\partial z} + gI\ |_{z=z^{(0)}}\ .$$

If $m-1$ free parameters are to appear in the Laurent series, they must either come from the non-linear equations (8) or from the eigenvalue problem (9), i.e., L must have at least $m-1$ integer eigenvalues. These are much less conditions than expected, because of the fact that the homogeneity k of the constant H must be an eigenvalue of L. Moreover the formal series solutions are convergent as a consequence of the majorant method [3]. Next we assume that the divisor is very ample and in addition projectively normal. Consider a point $p \in \mathcal{D}$, a chart U_j around p on the torus and a function y_j in $\mathcal{L}(\mathcal{D})$ having a pole of maximal order at p. Then the vector $(1/y_j, y_1/y_j, \ldots, y_N/y_j)$ provides a good system of coordinates in U_j. Then taking the derivative with regard to one of the flows

$$\left(\frac{y_i}{y_j}\right)^{\cdot} = \frac{\dot{y}_i y_j - y_i\dot{y}_j}{y_j^2},\quad 1\leqslant j\leqslant N,$$

are finite on U_j as well. Therefore, since y_j^2 has a double pole along $\mathcal{D}$, the numerator must also have a double pole (at worst), i.e., $\dot{y}_i y_j - y_i\dot{y}_j \in \mathcal{L}(2\mathcal{D})$. Hence, when $\mathcal{D}$ is projectively normal, we have that

$$\left(\frac{y_i}{y_j}\right)^{\cdot} = \sum_{k,l} a_{k,l}\left(\frac{y_k}{y_j}\right)\left(\frac{y_l}{y_j}\right),$$

i.e., the ratios y_i/y_j form a closed system of coordinates under differentiation. At the bad points, the concept of projective normality plays an important role: this enables one to show that y_i/y_j is a bona fide Taylor series starting from every point in a neighborhood of the point in question.

To prove the algebraic complete integrability of a given Hamiltonian system, the main steps of the method are:

– The first step is to show the existence of the Laurent solutions, which requires an argument precisely every time k is an integer eigenvalue of L and therefore $L - kI$ is not invertible.

– One shows the existence of the remaining constants of the motion in involution so as to reach the number $n + k$.

– For given $c_1, \ldots, c_m$, the set

$$\mathcal{D} \equiv \left\{ \begin{array}{l} x_i(t) = t^{-\nu_i}\left(x_i^{(0)} + x_i^{(1)}t + x_i^{(2)}t^2 + \cdots\right), 1 \leqslant i \leqslant m, \\ \text{Laurent solutions such that } : H_j(x_i(t)) = c_j + \text{ Taylor part }, \end{array} \right\}$$

defines one or several $n - 1$ dimensional algebraic varieties (divisor) having the property that

$$\bigcap_{i=1}^{n+k} \{z \in \mathbb{C}^m : H_i = c_i\} \cup \mathcal{D},$$

is a smooth compact, connected variety with n commuting vector fields independent at every point, i.e., a complex algebraic torus $\mathbb{C}^n/Lattice$. The flows $J\frac{\partial H_{k+i}}{\partial z}, \ldots, J\frac{\partial H_{k+n}}{\partial z}$ are straight line motions on this torus.

From the divisor $\mathcal{D}$, a lot of information can be obtained with regard to the periods and the action-angle variables. Some others integrable systems appear as coverings of algebraic completely integrable systems. The manifolds invariant by the complex flows are coverings of abelian varieties and these systems are called algebraic completely integrable in the generalized sense.

2. THE LIOUVILLE–ARNOLD–ADLER–VAN MOERBEKE THEOREM

The idea of the Adler–Van Moerbeke's proof [10] we shall give here is closely related to the geometric spirit of the (real) Arnold–Liouville theorem [9]. Namely, a compact complex n-dimensional variety on which there exist n holomorphic commuting vector fields which are independent at every point is analytically isomorphic to a n-dimensional complex torus $\mathbb{C}^n/Lattice$ and the complex flows generated by the vector fields are straight lines on this complex torus.

Theorem 1. *Let $\overline{M}$ be an irreducible variety defined by an intersection*

$$\overline{M} = \bigcap_i \{Z = (Z_0, Z_1, \ldots, Z_n) \in \mathbb{P}^N(\mathbb{C}) : P_i(Z) = 0\},$$

involving a large number of homogeneous polynomials P_i with smooth and irreducible affine part $M = \overline{M} \cap \{Z_0 \neq 0\}$. Put $\overline{M} \equiv M \cup \mathcal{D}$, i.e., $\mathcal{D} = \overline{M} \cap \{Z_0 = 0\}$ and consider the map

$$f : \overline{M} \longrightarrow \mathbb{P}^N(\mathbb{C}), \quad Z \longmapsto f(Z).$$

Let $\widetilde{M} = f(\overline{M}) = \overline{f(M)}$, $\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_r$, where $\mathcal{D}_i$ are codimension 1 subvarieties and $\mathcal{S} \equiv f(\mathcal{D}) = f(\mathcal{D}_1) \cup \ldots \cup f(\mathcal{D}_r) \equiv \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_r$. Assume that:

(i) f maps M smoothly and 1-1 onto $f(M)$.

(ii) There exist n holomorphic vector fields $X_1, \ldots, X_n$ on M which commute and are independent at every point. One vector field, say $X_k (1 \leqslant k \leqslant n)$, extends holomorphically to a neighborhood of $\mathcal{S}_k$ in $\mathbb{P}^N(\mathbb{C})$.

(iii) For all $p \in \mathcal{S}_k$, the integral curve $f(t) \in \mathbb{P}^N(\mathbb{C})$ of the vector field X_k through $f(0) = p \in \mathcal{S}_k$ has the property that

$$\{f(t) : 0 <\mid t \mid< \varepsilon, t \in \mathbb{C}\} \subset f(M).$$

This condition means that the orbits of X_k through $\mathcal{S}_k$ go immediately into the affine part and in particular, the vector field X_k does not vanish on any point of $\mathcal{S}_k$. Then

a) $\widetilde{M}$ is compact, connected and admits an embedding into $\mathbb{P}^N(\mathbb{C})$.

b) $\widetilde{M}$ is diffeomorphic to a n-dimensional complex torus. The vector fields $X_1, \ldots, X_n$ extend holomorphically and remain independent on $\widetilde{M}$.

c) $\widetilde{M}$ is a Kähler variety.

d) $\widetilde{M}$ a Hodge variety. In particular, M is the affine part of an abelian variety $\widetilde{M}$.

Proof. a) A crucial step is to show that the orbits running through $\mathcal{S}_k$ form a smooth variety Σ_p, $p \in \mathcal{S}_k$ such that $\Sigma_p \backslash \mathcal{S}_k \subseteq M$. Let $p \in \mathcal{S}_k$, $\varepsilon > 0$ small enough, $g^t_{X_k}$ the flow generated by X_k on M and $\{g^t_{X_k} : t \in \mathbb{C}, 0 <\mid t \mid< \varepsilon\}$, the orbit going through the point p. The vector field X_k is holomorphic in the neighborhood of any point $p \in \mathcal{S}_k$ and non-vanishing, by (ii) and (iii). Then the flow $g^t_{X_k}$ can be straightened out after a holomorphic change of coordinates. Let $\mathcal{H} \subset \mathbb{P}^N(\mathbb{C})$ be a hyperplane transversal to the direction of the flow at p and let Σ_p be the surface element formed by the divisor $\mathcal{S}_k$ and the orbits going through p. Consider the segment of $\mathcal{S}' \equiv \mathcal{H} \cap \Sigma_p$ and so locally, we have $\Sigma_p = \mathcal{S}' \times \mathbb{C}$. We shall show that Σ_p is smooth. Note that $\mathcal{S}'$ is smooth. Indeed, suppose that $\mathcal{S}'$ is singular at 0, then Σ_p would be singular along the trajectory (t-axis) which goes immediately into the affine $f(M)$, by condition (iii). Hence, the affine part would be singular which is impossible by condition (i). So, S' is smooth and by the implicit function theorem, Σ_p is smooth too. Consider now the map

$$\overline{M} \subset \mathbb{P}^m(\mathbb{C}) \longrightarrow \mathbb{P}^N(\mathbb{C}), \quad Z \longmapsto f(Z),$$

where $Z = (Z_0, Z_1, \ldots, Z_n) \in \mathbb{P}^m(\mathbb{C})$ and $\widetilde{M} = f(\overline{M}) = \overline{f(M)}$. Recall that the flow exists in a full neighborhood of p in $\mathbb{P}^N(\mathbb{C})$ and it has been straightened out. Therefore, near $p \in \mathcal{S}_k$, we have $\Sigma_p = \widetilde{M}$ and $\Sigma_p \backslash \mathcal{S}_k \subseteq M$. Otherwise, there would exist an element $\Sigma'_p \subset \widetilde{M}$ such that

$$\{g^t_{X_k} : t \in \mathbb{C}, 0 <\mid t \mid< \varepsilon\} = (\Sigma_p \cap \Sigma'_p) \backslash p \subset M,$$

by condition (iii). In other words, $\Sigma_p \cap \Sigma'_p$=t-axis and hence M would be singular along the t-axis which is impossible. Since the variety M is irreducible and since the generic hyperplane section $\mathcal{H}_{gen.}$ of $\widetilde{M}$ is also irreducible, all hyperplane sections are connected and hence $\mathcal{D}$ is also connected. Now consider the graph $G_f \subset \mathbb{P}^m(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$ of the map f, which is irreducible together with $\widetilde{M}$. It follows from the irreducibility of G_f that a generic hyperplane section $G_f \cap (\mathcal{H}_{gen.} \times \mathbb{P}^N(\mathbb{C}))$ is irreducible, hence the special hyperplane section $G_f \cap (\{Z_0 = 0\} \times \mathbb{P}^N(\mathbb{C}))$ is connected and therefore the projection map

$$Proj_{\mathbb{P}^N(\mathbb{C})}[G_f \cap (\{Z_0 = 0\} \times \mathbb{P}^N(\mathbb{C}))] = f(\mathcal{D}) \equiv \mathcal{S},$$

is connected. Hence, the variety

$$\widetilde{M} = M \cup \bigcup_{p \in \mathcal{S}_k} \Sigma_p = M \cup \mathcal{S}_k \subseteq \mathbb{P}^N(\mathbb{C}),$$

is compact, connected and embeds smoothly into $\mathbb{P}^N(\mathbb{C})$ via f.

b) Let g^{t_i} be the flow generated by X_i on M and let $p_1 \in \widetilde{M} \setminus M$. For small $\varepsilon > 0$ and for all $t_1 \in \mathbb{C}$ such that $0 < |t_1| < \varepsilon$, note that $q \equiv g^{t_1}(p_1)$ is well defined and $g^{t_1}(p_1) \in f(M)$, using condition (iii). Let $U(q) \subseteq M$ be a neighborhood of q and let

$$g^{t_2}(p_2) = g^{-t_1} \circ g^{t_2} \circ g^{t_1}(p_2), \quad \forall p_2 \in U(p_1) \equiv g^{-t_1}(U(q)),$$

which is well defined since by commutativity one can see that the right hand side is independent of t_1:

$$\begin{aligned} g^{-(t_1+\varepsilon)} \circ g^{t_2} \circ g^{t_1+\varepsilon}(p_2) &= g^{-(t_1+\varepsilon)} \circ g^{t_2} \circ g^{t_1} \circ g^{\varepsilon}(p_2), \\ &= g^{-(t_1+\varepsilon)} \circ g^{\varepsilon} \circ g^{t_2} \circ g^{t_1}(p_2), \\ &= g^{-t_1} \circ g^{t_2} \circ g^{t_1}(p_2). \end{aligned}$$

Note that $g^{t_2}(p_2)$ is a holomorphic function of p_2 and t_2, because in $U(p_1)$ the function g^{t_1} is holomorphic and its image is away from $\mathcal{S}$, i.e., in the affine, g^{t_2} is holomorphic. The same argument applies to $g^{t_3}(p_3), \ldots, g^{t_n}(p_n)$ where

$$g^{t_n}(p_n) = g^{-t_{n-1}} \circ g^{t_n} \circ g^{t_{n-1}}(p_n), \quad \forall p_n \in U(p_{n-1}) \equiv g^{-t_{n-1}}(U(q)).$$

Thus $X_1, \ldots, X_n$ have been holomorphically extended, remain independent and commuting on $\widetilde{M}$. Therefore, we can show along the same lines as in the Arnold–Liouville theorem [9] that $\widetilde{M}$ is a complex torus $\mathbb{C}^n/lattice$. And that will be done, by considering the local diffeomorphism

$$\mathbb{C}^n \longrightarrow \widetilde{M}, \quad t = (t_1, \ldots, t_n) \longmapsto g^t p = g^{t_1} \circ \ldots \circ g^{t_n}(p),$$

for a fixed origin $p \in f(M)$. The additive subgroup $L = \{t \in \mathbb{C}^n : g^t p = p\}$ is a lattice of $\mathbb{C}^n$ (spanned by $2n$ vectors in $\mathbb{C}^n$, independent over $\mathbb{R}$), hence $\mathbb{C}^n/L \longrightarrow \widetilde{M}$ is a biholomorphic diffeomorphism.
c) Let

$$ds^2 = \sum_{k=1}^{n} dt_k \otimes d\bar{t}_k,$$

be a hermitian metric on the complex variety $\widetilde{M}$ and let ω its fundamental $(1,1)$-form. We have

$$\omega = -\frac{1}{2}\operatorname{Im} ds^2 = \frac{\sqrt{-1}}{2}\sum_{k=1}^{n} dt_k \wedge d\bar{t}_k.$$

So we see that ω is closed and the metric ds^2 is Kähler and consequently $\widetilde{M}$ is a Kähler variety.
d) On the Kähler variety $\widetilde{M}$ are defined periods of ω. If these periods are integers (possibly after multiplication by a number), we obtain a variety of Hodge. More specifically, integrals $\int_{\gamma_k} \omega$ of the form ω (where γ_k are cycles in $H_2(\widetilde{M}, \mathbb{Z})$) determine the periods ω. As they are integers, then $\widetilde{M}$ is a Hodge variety. The variety $\widetilde{M}$ is equipped with n holomorphic vector fields, independent and commuting. From *a*) and *b*) the variety $\widetilde{M}$ is both a projective variety and a complex torus and hence an abelian variety as a consequence of Chow theorem [8]. Another proof is to use the result that we just show since every Hodge torus is abelian, the converse is also true. Note also that by Moishezon's theorem [11], a compact complex Kähler variety having as many independent meromorphic functions as its dimension is an abelian variety. □

3. A FIVE-DIMENSIONAL SYSTEM

Let us consider the following system of five differential equations in the unknowns $z_1, \ldots, z_5$:

$$\begin{aligned} \dot{z}_1 &= 2z_4, & \dot{z}_3 &= z_2(3z_1 + 8z_2^2), \\ \dot{z}_2 &= z_3, & \dot{z}_4 &= z_1^2 + 4z_1 z_2^2 + z_5, \end{aligned} \tag{3.1}$$

$$\dot{z}_5 = 2z_1 z_4 + 4z_2^2 z_4 - 2z_1 z_2 z_3.$$

The system (3.1) possesses three quartic invariants

$$\begin{aligned} F_1 &= \frac{1}{2}z_5 - z_1 z_2^2 + \frac{1}{2}z_3^2 - \frac{1}{4}z_1^2 - 2z_2^4, \\ F_2 &= z_5^2 - z_1^2 z_5 + 4z_1 z_2 z_3 z_4 - z_1^2 z_3^2 + \frac{1}{4}z_1^4 - 4z_2^2 z_4^2, \\ F_3 &= z_1 z_5 + z_1^2 z_2^2 - z_4^2, \end{aligned} \tag{3.2}$$

and is completely integrable in the sense of Liouville. It can be written as a Hamiltonian vector field

$$\dot{z} = J\frac{\partial H}{\partial z}, \quad z = (z_1, z_2, z_3, z_4, z_5)^\top,$$

where $H = F_1$. The Hamiltonian structure is defined by the Poisson bracket

$$\{F, H\} = \left\langle \frac{\partial F}{\partial z}, J\frac{\partial H}{\partial z} \right\rangle = \sum_{k,l=1}^{5} J_{kl}\frac{\partial F}{\partial z_k}\frac{\partial H}{\partial z_l},$$

where $\frac{\partial H}{\partial z}=\left(\frac{\partial H}{\partial z_1},\frac{\partial H}{\partial z_2},\frac{\partial H}{\partial z_3},\frac{\partial H}{\partial z_4},\frac{\partial H}{\partial z_5}\right)^\top$, and

$$J=\begin{bmatrix} 0 & 0 & 0 & 2z_1 & 4z_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2z_1z_2 \\ -2z_1 & 0 & 0 & 0 & 2z_5+4z_1z_2^2 \\ -4z_4 & 0 & -2z_1z_2 & -2z_5-4z_1z_2^2 & 0 \end{bmatrix},$$

is a skew-symmetric matrix for which the corresponding Poisson bracket satisfies the Jacobi identities. The second flow commuting with the first is regulated by the equations $\dot{z}=J\frac{\partial F_2}{\partial z}$, $z=(z_1,z_2,z_3,z_4,z_5)^\top$ and is written explicitly as

$$\begin{aligned}
\dot{z}_1 &= 8z_1^2z_2z_3-16z_1z_2^2z_4+8z_4z_5-4z_1^2z_4,\\
\dot{z}_2 &= 2z_1\left(2z_2z_4-z_1z_3\right),\\
\dot{z}_3 &= -4z_1z_3z_4+8z_2z_4^2+4z_1z_2z_5-2z_1^3z_2,\\
\dot{z}_4 &= 2z_1^2z_5-8z_1z_2z_3z_4+4z_1^2z_3^2-2z_1^4+4z_5^2+8z_1z_2^2z_5-4z_1^3z_2^2,\\
\dot{z}_5 &= 8z_1z_4z_5-16z_2z_3z_4^2+8z_1z_3^2z_4-4z_1^3z_4-8z_1^2z_2^2z_4+4z_1^3z_2z_3\\
&\quad -8z_1z_2z_3z_5+16z_2^2z_4z_5-16z_1^2z_2^3z_3+32z_1z_2^4z_4.
\end{aligned}$$

These vector fields are in involution: $\{F_1,F_2\}=0$, and the remaining one is a Casimir: $J\frac{\partial F_3}{\partial z}=0$. Let $z\in\mathbb{C}^5$, $t\in\mathbb{C}$. By the functional independence of the integrals F_1,F_2,F_3, the map

$$\varphi:(F_1,F_2,F_3):\mathbb{C}^5\longrightarrow\mathbb{C}^3,$$

is submersive, i.e., $dF_1(z)$, $dF_2(z)$ and $dF_3(z)$ are linearly independent on a non empty Zariski open set $\Delta\subset\mathbb{C}^5$. Let

$$\begin{aligned}
\Omega &= \varphi\left(\mathbb{C}^5\backslash\Delta\right),\\
&= \{c\equiv(c_1,c_2,c_3)\in\mathbb{C}^3:\exists z\in\varphi^{-1}(c)\text{ with}\\
&\qquad dF_1(z),dF_2(z),dF_3(z)\text{ linearly dependent}\},
\end{aligned}$$

be the set of critical values of φ. We denote by $\overline{\Omega}$ the Zariski closure of Ω in $\mathbb{C}^3$. The set $\{z\in\mathbb{C}^5:\varphi(z)\in\mathbb{C}^3\backslash\overline{\Omega}\}$ is a non-empty Zariski open set in $\mathbb{C}^5$. Hence this set is everywhere dense in $\mathbb{C}^5$ for the usual topology. Let A be the complex affine variety defined by

$$A=\varphi^{-1}(c)=\bigcap_{k=1}^{2}\{z:F_k(z)=c_k\}\subset\mathbb{C}^5. \tag{3.3}$$

For every $c\equiv(c_1,c_2,c_3)\in\mathbb{C}^3\backslash\overline{\Omega}$, the fibre A is a smooth affine surface. Now, the main problem will be to complete A(3.3) into a non singular compact complex algebraic variety $\widetilde{A}=A\cup\mathcal{D}$ in such a way that the vector fields X_{F_1} and X_{F_2} generated respectively by F_1 and F_2, extend holomorphically along a divisor $\mathcal{D}$ and remain independent there. If this is possible, $\widetilde{A}$ is an algebraic complex torus (an abelian variety) and the coordinates $z_1,\ldots,z_5$ restricted to A are abelian functions. A naive guess would be to take the natural compactification $\overline{A}$ of A by projectivizing the equations: $\overline{A}=\bigcap_{k=1}^{3}\{F_k(Z)=c_kZ_0^4\}\subset\mathbb{P}^5(\mathbb{C})$. Indeed, this can never work for a general reason: an abelian variety $\widetilde{A}$ of dimension bigger or equal than two is never a complete intersection, that is it can never be described in some projective space $\mathbb{P}^n(\mathbb{C})$ by n-dim $\widetilde{A}$ global polynomial homogeneous equations. In other words, if A is to be the affine part of an abelian surface, $\overline{A}$ must have a singularity somewhere along the locus at infinity $\overline{A}\cap\{Z_0=0\}$. In fact, we shall show that the existence of meromorphic solutions to the differential equations (3.1) depending on 4 free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor.

Theorem 2. *The system* (3.1) *possesses Laurent series solutions which depend on 4 free parameters: α, β, γ and θ. These meromorphic solutions restricted to the surface A* (3.3) *are parameterized by two copies $\mathcal{C}_{-1}$ and $\mathcal{C}_1$ of the same Riemann surface* (3.5) *of genus 7.*

Proof. Consider points at infinity which are limit points of trajectories of the flow. To be precise, we search for the set of Laurent solutions which remain confined to the fixed affine invariant surface A (3.3), related to specific values of c_1, c_2 and c_3. The first fact to observe is that if the system is to have Laurent solutions depending on 4 free parameters $\alpha, \beta, \gamma, \theta$, the Laurent decomposition of such asymptotic solutions must have the following form

$$\begin{aligned}
z_1 &= \frac{1}{t}\alpha - \frac{1}{2}\alpha^2 + \beta t - \frac{1}{16}\alpha\left(\alpha^3 + 4\beta\right)t^2 + \gamma t^3 + \cdots,\\
z_2 &= \frac{1}{2t}\varepsilon - \frac{1}{4}\varepsilon\alpha + \frac{1}{8}\varepsilon\alpha^2 t - \frac{1}{32}\varepsilon\left(-\alpha^3 + 12\beta\right)t^2 + \theta t^3 + \cdots,\\
z_3 &= -\frac{1}{2t^2}\varepsilon + \frac{1}{8}\varepsilon\alpha^2 - \frac{1}{16}\varepsilon\left(-\alpha^3 + 12\beta\right)t + 3\theta t^2 + \cdots,\\
z_4 &= -\frac{1}{2t^2}\alpha + \frac{1}{2}\beta - \frac{1}{16}\alpha\left(\alpha^3 + 4\beta\right)t + \frac{3}{2}\gamma t^2 + \cdots,\\
z_5 &= \frac{1}{2t^2}\alpha^2 - \frac{1}{4t}\left(\alpha^3 + 4\beta\right) + \frac{1}{4}\alpha\left(\alpha^3 + 2\beta\right) - \left(\alpha^2\beta - 2\gamma + 4\varepsilon\theta\alpha\right)t + \cdots,
\end{aligned} \tag{3.4}$$

with $\varepsilon = \pm 1$. Using the majorant method [3], we can show that these series are convergent. Substituting the Laurent solutions (3.4) into (3.2): $F_1 = c_1$, $F_2 = c_2$ and $F_3 = c_3$, and equating the t^0-terms yields

$$\begin{aligned}
F_1 &= \frac{7}{64}\alpha^4 - \frac{1}{8}\alpha\beta - \frac{5}{2}\varepsilon\theta = c_1,\\
F_2 &= \frac{1}{16}\left(4\beta - \alpha^3\right)\left(4\alpha^2\beta - \alpha^5 + 64\varepsilon\theta\alpha - 32\gamma\right) = c_2,\\
F_3 &= -\frac{1}{32}\alpha^6 - \beta^2 - \frac{1}{4}\alpha^3\beta - 3\varepsilon\theta\alpha^2 + 4\alpha\gamma = c_3.
\end{aligned}$$

Eliminating γ and θ from these equations, leads to an equation connecting the two remaining parameters α and β:

$$\begin{aligned}
\mathcal{C} : 64\beta^3 - 16\alpha^3\beta^2 - 4\left(\alpha^6 - 32\alpha^2 c_1 - 16c_3\right)\beta\\
+\alpha\left(32c_2 - 32\alpha^4 c_1 + \alpha^8 - 16\alpha^2 c_3\right) = 0.
\end{aligned} \tag{3.5}$$

The Laurent solutions restricted to the surface A(3.3) are thus parametrized by two copies $\mathcal{C}_{-1}$ and $\mathcal{C}_1$ of the same Riemann surface $\mathcal{C}$ (3.5). According to the Riemann–Hurwitz formula, the genus of the Riemann surface $\mathcal{C}$ is 7, which establishes the theorem. □

In order to embed $\mathcal{C}$ into some projective space, one of the key underlying principles used is the Kodaira embedding theorem, which states that a smooth complex manifold can be smoothly embedded into projective space $\mathbb{P}^N(\mathbb{C})$ with the set of functions having a pole of order k along positive divisor on the manifold, provided k is large enough; fortunately, for abelian varieties, k need not be larger than three according to Lefshetz. These functions are easily constructed from the Laurent solutions (3.4) by looking for polynomials in the phase variables which in the expansions have at most a k-fold pole. The nature of the expansions and some algebraic properties of abelian varieties provide a recipe for when to terminate our search for such functions, thus making the procedure implementable. Precisely, we wish to find a set of polynomial functions $\{f_0, \ldots, f_N\}$, of increasing degree in the original variables $z_1, \ldots, z_5$ having the property that the embedding $\mathcal{D}$ of $\mathcal{C}_1 + \mathcal{C}_{-1}$ into $\mathbb{P}^N(\mathbb{C})$ via those functions satisfies the relation: geometric genus of $\mathcal{D} \equiv g(\mathcal{D}) = N + 2$. At this point, it may be not so clear why $\mathcal{D}$ must really live on an abelian surface. Let us say, for the moment, that the equations of the divisor $\mathcal{D}$ (i.e., the place where the solutions blow up), as a Riemann surface traced on the abelian surface $\widetilde{A}$ (to be constructed in theorem 4), must be understood as relations connecting the free parameters as they appear firstly in the expansions (3.4). This means that (3.5) must be understood as relations connecting α and β. Let

$$L^{(r)} = \left\{ \begin{array}{l} \text{polynomials} \quad f = f(z_1, \ldots, z_5) \\ \text{of degree} \leqslant r, \quad \text{such that} \\ f(z(t)) = t^{-1}(z^{(0)} + \cdots), \\ \text{with} \quad z^{(0)} \neq 0 \quad \text{on} \quad \mathcal{D} \\ \text{and with} \quad z(t) \quad \text{as in} \quad (3.4) \end{array} \right\} / [F_k = c_k, k = 1, 2, 3],$$

and let $(f_0, f_1, \ldots, f_{N_r})$ be a basis of $L^{(r)}$. We look for r such that: $g(\mathcal{D}^{(r)}) = N_r + 2$, $\mathcal{D}^{(r)} \subset \mathbb{P}^{N_r}$. We shall show that it is unnecessary to go beyond $r = 4$.

Theorem 3. *a) The spaces $L^{(r)}$, nested according to weighted degree, are generated as follows*

$$\begin{aligned} L^{(1)} &= \{f_0, f_1, f_2\}, \\ L^{(2)} &= L^{(1)} \oplus \{f_3, f_4, f_5, f_6\}, \\ L^{(3)} &= L^{(2)} \oplus \{f_7, f_8, f_9, f_{10}\}, \\ L^{(4)} &= L^{(3)} \oplus \{f_{11}, f_{12}, f_{13}, f_{14}, f_{15}\}, \end{aligned} \tag{3.6}$$

where $f_0 = 1,\ f_1 = z_1,\ f_2 = z_2,\ f_3 = 2z_5 - z_1^2,\ f_4 = z_3 + 2\varepsilon z_2^2,\ f_5 = z_4 + \varepsilon z_1 z_2,\ f_6 = [f_1, f_2],\ f_7 = f_1(f_1 + 2\varepsilon f_4),\ \ f_8 = f_2(f_1 + 2\varepsilon f_4),\ \ f_9 = z_4(f_3 + 2\varepsilon f_6),\ \ f_{10} = z_5(f_3 + 2\varepsilon f_6),\ \ f_{11} = f_5(f_1 + 2\varepsilon f_4),\ f_{12} = f_1 f_2(f_3 + 2\varepsilon f_6),\ \ f_{13} = f_4 f_5 + [f_1, f_4],\ \ f_{14} = [f_1, f_3] + 2\varepsilon\,[f_1, f_6],\ \ f_{15} = f_3 - 2z_5 + 4f_4^2,$ *with* $[s_j, s_k] = \dot{s}_j s_k - s_j \dot{s}_k$, *the wronskian of s_k and s_j.*
b) $L^{(4)}$ provides an embedding of $\mathcal{D}^{(4)}$ into projective space $\mathbb{P}^{15}(\mathbb{C})$ and $\mathcal{D}^{(4)}$ has genus 17.

Proof. a) The proof of a) is straightforward and can be done by inspection of the expansions (3.4).
b) It turns out that neither $L^{(1)}$, nor $L^{(2)}$, nor $L^{(3)}$, yields a Riemann surface of the right genus; in fact $g(\mathcal{D}^{(r)}) \neq \dim L^{(r)} + 1, r = 1, 2, 3$. For instance, the embedding into $\mathbb{P}^2(\mathbb{C})$ via $L^{(1)}$ does not separate the sheets, so we proceed to $L^{(2)}$ and the corresponding embedding into $\mathbb{P}^6(\mathbb{C})$ is unacceptable since $g(\mathcal{D}^{(2)}) - 2 > 6$ and $\mathcal{D}^{(2)} \subset \mathbb{P}^6(\mathbb{C}) \neq \mathbb{P}^{g-2}(\mathbb{C})$, which contradicts the fact that $N_r = g(\mathcal{D}^{(2)}) - 2$. So we proceed to $L^{(3)}$ and we consider the corresponding embedding into $\mathbb{P}^{10}(\mathbb{C})$, according to the functions $(f_0, \ldots, f_{10})$. For finite values of α and β, dividing the vector $(f_0, \ldots, f_{10})$ by f_2 and taking the limit $t \longrightarrow 0$, to yield

$$[0 : 2\varepsilon\alpha : 1 : -\varepsilon(4\beta - \alpha^3) : -\alpha : -\varepsilon\alpha^2 : \frac{1}{2}(4\beta - \alpha^3) : \varepsilon\alpha^3 : \frac{1}{2}\alpha^2 :$$

$$\frac{1}{4}\varepsilon\alpha^3(4\beta - \alpha^3) : -\frac{1}{4}\varepsilon\alpha^4(4\beta - \alpha^3)].$$

The point $\alpha = 0$ requires special attention. Indeed near $\alpha = 0$, the parameter β behaves as follows: $\beta \sim 0, i\sqrt{c_3}, -i\sqrt{c_3}$. Thus near $(\alpha, \beta) = (0, 0)$, the corresponding point is mapped into the point

$$[0 : 0 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0],$$

in $\mathbb{P}^{10}(\mathbb{C})$ which is independent of $\varepsilon = \pm 1$, whereas near the point $(\alpha, \beta) = (0, i\sqrt{c_3})$ (resp. $(\alpha, \beta) = (0, -i\sqrt{c_3})$) leads to two different points:

$$[0 : 0 : 1 : -4\varepsilon i\sqrt{c_3} : 0 : 0 : 2\varepsilon i\sqrt{c_3} : 0 : 0 : 0 : 0],$$

(resp.

$$[0 : 0 : 1 : 4\varepsilon i\sqrt{c_3} : 0 : 0 : -2\varepsilon i\sqrt{c_3} : 0 : 0 : 0 : 0]),$$

according to the sign of ε. The Riemann surface (3.4) has three points covering $\alpha=\infty$, at which β behaves as follows:

$$\beta\sim-\frac{1279}{216}\alpha^3,\ \frac{1}{432}\alpha^3\left(1333-1295i\sqrt{3}\right),\ \frac{1}{432}\alpha^3\left(1333+1295i\sqrt{3}\right).$$

Then by dividing the vector $(f_0,\dots,f_{10})$ by f_{10}, the corresponding point is mapped into the point

$$[0:0:0:0:0:0:0:0:0:0:1],$$

in $\mathbb{P}^{10}(\mathbb{C})$. Thus, $g(\mathcal{D}^{(3)})-2>10$ and $\mathcal{D}^{(2)}\subset\mathbb{P}^{10}\neq\mathbb{P}^{g-2}(\mathbb{C})$, which contradicts the fact that $N_r=g(\mathcal{D}^{(3)})-2$. Consider now the embedding $\mathcal{D}^{(4)}$ into $\mathbb{P}^{15}(\mathbb{C})$ using the 16 functions $f_0,\dots,f_{15}$ of $L^{(4)}$(3.6). It is easily seen that these functions separate all points of the Riemann surface (except perhaps for the points at $\alpha=\infty$ and $\alpha=\beta=0$). The Riemann surfaces $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ are disjoint for finite values of α and β except for $\alpha=\beta=0$; dividing the vector $(f_0,\dots,f_{15})$ by f_2 and taking the limit $t\to0$, to yield

$$[0:2\varepsilon\alpha:1:-\varepsilon(4\beta-\alpha^3):-\alpha:-\varepsilon\alpha^2:\frac{1}{2}(4\beta-\alpha^3):\varepsilon\alpha^3:\frac{1}{2}\alpha^2:\frac{1}{4}\varepsilon\alpha^3(4\beta-\alpha^3):$$

$$-\frac{1}{4}\varepsilon\alpha^4(4\beta-\alpha^3):-\frac{1}{2}\varepsilon\alpha^4:-\frac{1}{4}\alpha^3(4\beta-\alpha^3):\frac{3}{4}\alpha\left(4\beta-\alpha^3\right):\varepsilon\alpha^3\left(4\beta-\alpha^3\right):-2\varepsilon\alpha^3].$$

As before, the point $\alpha=0$ require special attention and the parameter β behaves as follows: $\beta\sim0,i\sqrt{c_3},-i\sqrt{c_3}$. Thus near $(\alpha,\beta)=(0,0)$, the corresponding point is mapped into the point

$$[0:0:1:0:0:0:0:0:0:0:0:0:0:0:0:0],$$

in $\mathbb{P}^{15}(\mathbb{C})$ which is independent of $\varepsilon=\pm1$, whereas near the point $(\alpha,\beta)=(0,i\sqrt{c_3})$ (resp. $(\alpha,\beta)=(0,-i\sqrt{c_3})$) leads to two different points:

$$[0:0:1:-4\varepsilon i\sqrt{c_3}:0:0:2\varepsilon i\sqrt{c_3}:0:0:0:0:0:0:0:0:0],$$

(resp.

$$[0:0:1:4\varepsilon i\sqrt{c_3}:0:0:-2\varepsilon i\sqrt{c_3}:0:0:0:0:0:0:0:0:0]),$$

according to the sign of ε. About the point $\alpha=\infty$, it is appropriate to divide by f_{10}; then the corresponding point is mapped into the point

$$[0:0:0:0:0:0:0:0:0:0:1:0:0:0:0:0],$$

in $\mathbb{P}^{15}(\mathbb{C})$ which is independent of ε. The divisor $\mathcal{D}^{(4)}$ obtained in this way has genus 17 and $\mathcal{D}^{(4)}\subset\mathbb{P}^{15}(\mathbb{C})=\mathbb{P}^{g-2}(\mathbb{C})$, as desired. This ends the proof of the theorem. □

Let $\mathcal{L}=L^{(4)}$ and $\mathcal{D}=\mathcal{D}^{(4)}$. Next we wish to construct a surface strip around $\mathcal{D}$ which will support the commuting vector fields. In fact, $\mathcal{D}$ has a good chance to be very ample divisor on an abelian surface. Following the method (theorem 1), we obtain the following theorem:

Theorem 4. *The variety A (3.3) generically is the affine part of an abelian surface $\widetilde{A}$. The reduced divisor at infinity $\widetilde{A}\backslash A=\mathcal{C}_1+\mathcal{C}_{-1}$, consists of two copies $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ of the same genus 7 Riemann surface $\mathcal{C}$ (3.5). The system of differential equations (3.1) is algebraically completely integrable and the corresponding flows evolve on $\widetilde{A}$.*

Remark 1. a) Note that the reflection σ on the affine variety A amounts to the flip $\sigma:(z_1,z_2,z_3,z_4,z_5)\mapsto(z_1,-z_2,z_3,-z_4,z_5)$, changing the direction of the commuting vector fields. It can be extended to the (-Id)-involution about the origin of $\mathbb{C}^2$ to the time flip $(t_1,t_2)\mapsto(-t_1,-t_2)$ on $\widetilde{A}$, where t_1 and t_2 are the time coordinates of each of the flows X_{F_1} and X_{F_2}. The involution σ acts on the parameters of the Laurent solution (3.4) as follows $\sigma:(t,\alpha,\beta,\gamma,\theta)\longmapsto(-t,-\alpha,-\beta,-\gamma,\theta)$ and the linear space $\mathcal{L}$ can be split into a direct sum of even and odd functions.
b) Consider on $\widetilde{A}$ the holomorphic 1-forms dt_1 and dt_2 defined by $dt_i(X_{F_j})=\delta_{ij}$, where X_{F_1} and

X_{F_2} are the vector fields generated respectively by F_1 and F_2. Taking the differentials of $\zeta = 1/z_1$ and $\xi = z_1/z_2$ viewed as functions of t_1 and t_2, using the vector fields and the Laurent series (3.4) and solving linearly for dt_1 and dt_2, we obtain the holomorphic differentials

$$\omega_1 = dt_1|_{\mathcal{C}_\varepsilon} = \frac{1}{\triangle}(\frac{\partial\xi}{\partial t_2}d\zeta - \frac{\partial\zeta}{\partial t_2}d\xi)|_{\mathcal{C}_\varepsilon} = \frac{8}{\alpha\left(-4\beta+\alpha^3\right)}d\alpha,$$
$$\omega_2 = dt_2|_{\mathcal{C}_\varepsilon} = \frac{1}{\triangle}(\frac{-\partial\xi}{\partial t_1}d\zeta - \frac{\partial\zeta}{\partial t_1}d\xi)|_{\mathcal{C}_\varepsilon} = \frac{2}{\left(-4\beta+\alpha^3\right)^2}d\alpha,$$

with $\Delta \equiv \frac{\partial\zeta}{\partial t_1}\frac{\partial\xi}{\partial t_2} - \frac{\partial\zeta}{\partial t_2}\frac{\partial\xi}{\partial t_1}$. The zeroes of ω_2 provide the points of tangency of the vector field X_{F_1} to $\mathcal{C}_\varepsilon$. We have $\frac{\omega_1}{\omega_2} = \frac{4}{\alpha}\left(-4\beta+\alpha^3\right)$, and X_{F_1} is tangent to $\mathcal{C}_\varepsilon$ at the point covering $\alpha = \infty$.

There are many examples of differential equations which have the weak Painlevé property that all movable singularities of the general solution have only a finite number of branches and some interesting integrable systems appear as coverings of algebraic completely integrable systems. The invariant varieties are coverings of abelian varieties and these systems are called algebraic completely integrable in the generalized sense. These systems are Liouville integrable and by the Arnold–Liouville theorem, the compact connected manifolds invariant by the real flows are tori; the real parts of complex affine coverings of abelian varieties. Most of these systems of differential equations possess solutions which are Laurent series of $t^{1/n}$ (t being complex time) and whose coefficients depend rationally on certain algebraic parameters. It was shown in series of publications of Vanhaecke [12, 13], Abenda and Fedorov [14] and others that Θ-divisor can serve as a carrier of integrability. Let $\mathcal{H}$ be a hyperelliptic curve of genus g and $\mathrm{Jac}(\mathcal{H}) = \mathbb{C}^g/\Lambda$ its jacobian variety where Λ is a lattice of maximal rank in $\mathbb{C}^g$. Let

$$\mathcal{A}_k : \mathrm{Sym}^k(\mathcal{H}) \to \mathrm{Jac}(\mathcal{H}), (P_1,\ldots,P_k) \mapsto \sum_{j=1}^{k}\int_\infty^{P_j}(\omega_1,\ldots,\omega_g)\mathrm{mod}.\Lambda, 0 \leqslant k \leqslant g,$$

be the Abel map where $(\omega_1,\ldots,\omega_g)$ is a canonical basis of the space of differentials of the first kind on $\mathcal{H}$. The theta divisor Θ is a subvariety of $\mathrm{Jac}(\mathcal{H})$ defined as $\Theta \equiv \mathcal{A}\left[\mathrm{Sym}^{g-1}(\mathcal{H})\right]/\Lambda$. By Θ_k we will denote the subvariety (called strata) of $\mathrm{Jac}(\mathcal{H})$ defined by $\Theta_k \equiv \mathcal{A}_k\left[\mathrm{Sym}^k(\mathcal{H})\right]/\Lambda$ and we have the following stratification

$$\{O\} \subset \Theta_0 \subset \Theta_1 \subset \Theta_2 \subset \ldots \subset \Theta_{g-1} \subset \Theta_g = \mathrm{Jac}(\mathcal{H}),$$

where O is the origin of $\mathrm{Jac}(\mathcal{H})$. Vanhaecke [12] showed that these stratifications of the jacobian are connected with stratifications of the Sato grassmannian, via an extension of Krichever's map. He discussed the relation between Laurent solutions for the Master systems and stratifications of the jacobian of a hyperelliptic curve. In [13], Vanhaecke studied Lie-Poisson structure in the jacobian and showed that invariant manifolds associated with Poisson brackets can be identified with these strata. Some problems were considered in [13] and [14], where a connection was established with the flows on these strata. Such varieties or their open subsets often appear as coverings of complex invariants manifolds of finite dimensional integrable systems (Hénon–Heiles and Neumann systems).

Consider the case $F_3 = 0$, and the following change of variables

$$z_1 = q_1^2, \quad z_2 = q_2, \quad z_3 = p_2, \quad z_4 = p_1q_1, \quad z_5 = p_1^2 - q_1^2q_2^2.$$

Substituting this into the constants of motion F_1, F_2, F_3 leads obviously to the relations

$$H_1 = \frac{1}{2}p_1^2 - \frac{3}{2}q_1^2q_2^2 + \frac{1}{2}p_2^2 - \frac{1}{4}q_1^4 - 2q_2^4, \tag{3.7}$$
$$H_2 = p_1^4 - 6q_1^2q_2^2p_1^2 + q_1^4q_2^4 - q_1^4p_1^2 + q_1^6q_2^2 + 4q_1^3q_2p_1p_2 - q_1^4p_2^2 + \frac{1}{4}q_1^8,$$

whereas the last constant leads to an identity. Using the differential equations (3.1) combined with the transformation above leads to the system of differential equations

$$\ddot{q}_1 = q_1\left(q_1^2 + 3q_2^2\right), \tag{3.8}$$
$$\ddot{q}_1 = q_2\left(3q_1^2 + 8q_2^2\right).$$

The last equation (3.1) for z_5 leads to an identity. Thus, we obtain the potential constructed by Ramani, Dorozzi and Grammaticos [15, 16]. Evidently, the functions H_1 and H_2 commute: $\{H_1, H_2\} = 0$. The system (3.8) is weight-homogeneous with q_1, q_2 having weight 1 and p_1, p_2 weight 2, so that H_1 and H_2 have weight 4 and 8 respectively. We show that this system is algebraic completely integrable in the generalized sense. To be more precise, when one examines all possible singularities, one finds that it is possible for the variable q_1 to contain square root terms of the type $t^{1/2}$, which are strictly not allowed by the Painlevé–Kowalewski test (i.e., the general solutions have no movable singularities other than poles). However, these terms are trivially removed by introducing the variables $z_1, \ldots, z_5$ which restores the Painlevé–Kowalewski property to the system. Let B be the affine variety defined by

$$B = \bigcap_{k=1}^{2} \{z \in \mathbb{C}^4 : H_k(z) = b_k\}, \tag{3.9}$$

where $(b_1, b_2) \in \mathbb{C}^2$. We show that the system (3.8) admits 3-dimensional family of Laurent solutions in $t^{1/2}$, depending on three free parameters: u, v and w. There are precisely two such families, labelled by $\varepsilon = \pm i$, and they are explicitly given as follows

$$\begin{aligned}
q_1 &= \frac{1}{\sqrt{t}}\Big(u - \frac{1}{4}u^3 t + vt^2 - \frac{5}{128}u^7 t^3 + \frac{1}{8}u\Big(\frac{3}{4}u^3 v - \frac{7}{256}u^8 + 3\varepsilon w\Big)t^4 + \cdots\Big),\\
q_2 &= \frac{1}{t}\Big(\frac{1}{2}\varepsilon - \frac{1}{4}\varepsilon u^2 t + \frac{1}{8}\varepsilon u^4 t^2 + \frac{1}{4}\varepsilon u\Big(\frac{1}{32}u^5 - 3v\Big)t^3 + wt^4 + \cdots\Big),\\
p_1 &= \frac{1}{2t\sqrt{t}}\Big(-u - \frac{1}{4}u^3 t + 3vt^2 - \frac{25}{128}t^3 u^7\\
&\qquad + \frac{7}{8}u\Big(\frac{3}{4}u^3 v - \frac{7}{256}u^8 + 3\varepsilon w\Big)t^4 + \cdots\Big),\\
p_2 &= \frac{1}{t^2}\Big(-\frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon u^4 t^2 + \frac{1}{2}\varepsilon u\Big(\frac{1}{32}u^5 - 3v\Big)t^3 + 3wt^4 + \cdots\Big).
\end{aligned} \tag{3.10}$$

These formal series solutions are convergent as a consequence of the majorant method. By substituting these series in the constants of the motion $H_1 = b_1$ and $H_2 = b_2$, one eliminates the parameter w linearly, leading to an equation connecting the two remaining parameters u and v:

$$\begin{aligned}
\Gamma: \ &\frac{65}{4}uv^3 + \frac{93}{64}u^6 v^2 + \frac{3}{8192}\left(-9829u^8 + 26112H_1\right)u^3 v\\
&-\frac{10299}{65536}u^{16} - \frac{123}{256}H_1 u^8 + H_2 + \frac{15362\,98731}{52} = 0.
\end{aligned} \tag{3.11}$$

According to Hurwitz's formula, this defines a Riemann surface Γ of genus 16. The Laurent solutions restricted to the surface $B(3.9)$ are parametrized by two copies Γ_{-1} and Γ_1 of the same Riemann surface Γ. Applying the method explained in Piovan [5], we show that the invariant surface $B(3.9)$ can be completed as a cyclic double cover $\overline{B}$ of the abelian surface $\widetilde{A}$, ramified along the divisor $\mathcal{C}_1 + \mathcal{C}_{-1}$. Consequently, the system (3.8) is algebraic complete integrable in the generalized sense. Moreover, $\overline{B}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{C}_1 + \mathcal{C}_{-1}$ and the resolution $\widetilde{B}$ of $\overline{B}$ is a surface of general type with invariants: $\mathcal{X}(\widetilde{B}) = 1$ and $p_g(\widetilde{B}) = 2$. The asymptotic solution (3.10) can be read off from (3.4) and the change of variables: $q_1 = \sqrt{z_1}, q_2 = z_2, p_1 = z_4/q_1, p_2 = z_3$. The function z_1 has a simple pole along the divisor $\mathcal{C}_1 + \mathcal{C}_{-1}$ and a double zero along a Riemann surface of genus 7 defining a double cover of $\widetilde{A}$ ramified along $\mathcal{C}_1 + \mathcal{C}_{-1}$.

4. THE HÉNON–HEILES SYSTEM

The Hénon–Heiles system

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{q}_2 = \frac{\partial H}{\partial p_2}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2},$$

with

$$H \equiv H_1 = \frac{1}{2}\left(p_1^2 + p_2^2 + Aq_1^2 + Bq_2^2\right) + q_1^2 q_2 + 6q_2^3,$$

i.e.,

$$\begin{aligned} \dot{q}_1 &= p_1, & \dot{p}_1 &= -Aq_1 - 2q_1 q_2, \\ \dot{q}_2 &= p_2, & \dot{p}_2 &= -Bq_2 - q_1^2 - 6q_2^2, \end{aligned} \tag{4.1}$$

has another constant of motion

$$H_2 = q_1^4 + 4q_1^2 q_2^2 - 4p_1\left(p_1 q_2 - p_2 q_1\right) + 4Aq_1^2 q_2 + (4A - B)\left(p_1^2 + Aq_1^2\right),$$

where A, B, are constant parameters and q_1, q_2, p_1, p_2 are canonical coordinates and momenta, respectively. First studied as a mathematical model to describe the chaotic motion of a test star in an axisymmetric galactic mean gravitational field this system is widely explored in other branches of physics. It is well-known from applications in stellar dynamics, statistical mechanics and quantum mechanics. It provides a model for the oscillations of atoms in a three-atomic molecule. The system (21) possesses Laurent series solutions depending on 3 free parameters α, β, γ, namely

$$\begin{aligned} q_1 &= \frac{\alpha}{t} + \left(\frac{\alpha^3}{12} + \frac{\alpha A}{2} - \frac{\alpha B}{12}\right) t + \beta t^2 + q_1^{(4)} t^3 + q_1^{(5)} t^4 + q_1^{(6)} t^5 + \cdots, \\ q_2 &= -\frac{1}{t^2} + \frac{\alpha^2}{12} - \frac{B}{12} + \left(\frac{\alpha^4}{48} + \frac{\alpha^2 A}{10} - \frac{\alpha^2 B}{60} - \frac{B^2}{240}\right) t^2 + \frac{\alpha\beta}{3} t^3 + \gamma t^4 + \cdots, \end{aligned}$$

where $p_1 = \dot{q}_1,\ p_2 = \dot{q}_2$ and

$$\begin{aligned} q_1^{(4)} &= \frac{\alpha AB}{24} - \frac{\alpha^5}{72} + \frac{11\alpha^3 B}{720} - \frac{11\alpha^3 A}{120} - \frac{\alpha B^2}{720} - \frac{\alpha A^2}{8}, \\ q_1^{(5)} &= -\frac{\beta\alpha^2}{12} + \frac{\beta B}{60} - \frac{A\beta}{10}, \\ q_1^{(6)} &= -\frac{\alpha\gamma}{9} - \frac{\alpha^7}{15552} - \frac{\alpha^5 A}{2160} + \frac{\alpha^5 B}{12960} + \frac{\alpha^3 B^2}{25920} + \frac{\alpha^3 A^2}{1440} - \frac{\alpha^3 AB}{4320} + \frac{\alpha AB^2}{1440} \\ &\quad - \frac{\alpha B^3}{19440} - \frac{\alpha A^2 B}{288} + \frac{\alpha A^3}{144}. \end{aligned}$$

Let $\mathcal{D}$ be the pole solutions restricted to the surface

$$M = \bigcap_{i=1}^{2} \left\{x \equiv (q_1, q_2, p_1, p_2) \in \mathbb{C}^4, H_i(x) = c_i\right\},$$

to be precise $\mathcal{D}$ is the closure of the continuous components of the set of Laurent series solutions $x(t)$ such that

$$H_i(x(t)) = c_i,\ 1 \leqslant i \leqslant 2,$$

i.e., $\mathcal{D} = t^0 -$ coefficient of M. Thus we find an algebraic curve defined by

$$\mathcal{D} : \beta^2 = P_8(\alpha), \tag{4.2}$$

where

$$\begin{aligned} P_8(\alpha) = &-\frac{7}{15552}\alpha^8 - \frac{1}{432}\left(5A - \frac{13}{18}B\right)\alpha^6 - \frac{1}{36}\left(\frac{671}{15120}B^2 + \frac{17}{7}A^2 - \frac{943}{1260}BA\right)\alpha^4 \\ &-\frac{1}{36}\left(4A^3 - \frac{1}{2520}B^3 - \frac{13}{6}A^2 B + \frac{2}{9}AB^2 - \frac{10}{7}c_1\right)\alpha^2 + \frac{1}{36}c_2. \end{aligned}$$

The curve $\mathcal{D}$ determined by an eight-order equation is smooth, hyperelliptic and its genus is 3. Moreover, the map

$$\sigma : \mathcal{D} \longrightarrow \mathcal{D},\ (\beta, \alpha) \longmapsto (\beta, -\alpha), \tag{4.3}$$

is an involution on $\mathcal{D}$ and the quotient $\mathcal{E} = \mathcal{D}/\sigma$ is an elliptic curve defined by

$$\mathcal{E} : \beta^2 = P_4(\zeta), \tag{4.4}$$

where $P_4(\zeta)$ is the degree 4 polynomial in $\zeta = \alpha^2$ obtained from (22). The hyperelliptic curve $\mathcal{D}$ is thus a 2-sheeted ramified covering of the elliptic curve $\mathcal{E}$ (24),

$$\rho : \mathcal{D} \longrightarrow \mathcal{E},\ (\beta, \alpha) \longmapsto (\beta, \zeta), \tag{4.5}$$

ramified at the four points covering $\zeta = 0$ and ∞. The affine surface M completes into an abelian surface $\widetilde{M}$, by adjoining the divisor $\mathcal{D}$. The latter defines on $\widetilde{M}$ a polarization $(1, 2)$. The divisor $2\mathcal{D}$ is very ample and the functions $1,\ q_1,\ q_1^2,\ q_2,\ p_1,\ p_1^2 + q_1^2 q_2,\ p_2 q_1 - 2p_1 q_2,\ p_1 p_2 + 2Aq_1 q_2 + 2q_1 q_2^2$, embed $\widetilde{M}$ smoothly into $\mathbb{P}^7(\mathbb{C})$ with polarization $(2, 4)$. Then the system (4.1) is algebraically completely integrable and the corresponding flow evolves on an abelian surface $\widetilde{M} = \mathbb{C}^2/Lattice$, where the lattice is generated by the period matrix $\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}$, $\operatorname{Im} \begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$, $(a, b, c \in \mathbb{C})$.

Theorem 5. *The abelian surface $\widetilde{M}$ which completes the affine surface M is the dual Prym variety $Prym^*(\mathcal{D}/\mathcal{E})$ of the genus 3 hyperelliptic curve $\mathcal{D}$(4.2) for the involution σ interchanging the sheets of the double covering ρ(4.5) and the problem linearizes on this variety.*

Proof. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be a canonical homology basis of $\mathcal{D}$ such that $\sigma(a_1) = a_3$, $\sigma(b_1) = b_3$, $\sigma(a_2) = -a_2$, $\sigma(b_2) = -b_2$, for the involution σ(4.3). As a basis of holomorphic differentials $\omega_0, \omega_1, \omega_2$ on the curve $\mathcal{D}$(4.2) we take the differentials $\omega_1 = \frac{\alpha^2 d\alpha}{\beta}$, $\omega_2 = \frac{d\alpha}{\beta}$, $\omega_3 = \frac{\alpha d\alpha}{\beta}$ and obviously $\sigma^*(\omega_1) = -\omega_1$, $\sigma^*(\omega_2) = -\omega_2$, $\sigma^*(\omega_3) = \omega_3$. Recall that the Prym variety $Prym(\mathcal{D}/\mathcal{E})$ is a subabelian variety of the Jacobi variety $Jac(\mathcal{D}) = Pic^0(\mathcal{D}) = H^1(\mathcal{O}_{\mathcal{D}})\ /\ H^1(\mathcal{D}, \mathbb{Z})$ constructed from the double cover ρ, the involution σ on $\mathcal{D}$ interchanging sheets, extends by linearity to a map $\sigma : Jac(\mathcal{D}) \to Jac(\mathcal{D})$ and up to some points of order two, $Jac(\mathcal{D})$ splits into an even part and an odd part: the even part is an elliptic curve (the quotient of $\mathcal{D}$ by σ, i.e., $\mathcal{E}$ (4.4)) and the odd part is a 2−dimensional abelian surface $Prym(\mathcal{D}/\mathcal{E})$. We consider the period matrix Ω of $Jac(D)$

$$\Omega = \begin{pmatrix} \int_{a_1}\omega_1 & \int_{a_2}\omega_1 & \int_{a_3}\omega_1 & \int_{b_1}\omega_1 & \int_{b_2}\omega_1 & \int_{b_3}\omega_1 \\ \int_{a_1}\omega_2 & \int_{a_2}\omega_2 & \int_{a_3}\omega_2 & \int_{b_1}\omega_2 & \int_{b_2}\omega_2 & \int_{b_3}\omega_2 \\ \int_{a_1}\omega_3 & \int_{a_2}\omega_3 & \int_{a_3}\omega_3 & \int_{b_1}\omega_3 & \int_{b_2}\omega_3 & \int_{b_3}\omega_3 \end{pmatrix}.$$

Then,

$$\Omega = \begin{pmatrix} \int_{a_1}\omega_1 & \int_{a_2}\omega_1 & -\int_{a_1}\omega_1 & \int_{b_1}\omega_1 & \int_{b_2}\omega_1 & -\int_{b_1}\omega_1 \\ \int_{a_1}\omega_2 & \int_{a_2}\omega_2 & -\int_{a_1}\omega_2 & \int_{b_1}\omega_2 & \int_{b_2}\omega_2 & -\int_{b_1}\omega_2 \\ \int_{a_1}\omega_3 & 0 & \int_{a_1}\omega_3 & \int_{b_1}\omega_3 & 0 & \int_{b_1}\omega_3 \end{pmatrix},$$

and therefore the period matrices of $Jac(\mathcal{E})$(i.e., $\mathcal{E}$), $Prym(\mathcal{D}/\mathcal{E})$ and $Prym^*(\mathcal{D}/\mathcal{E})$ are respectively $\Delta = \begin{pmatrix} \int_{a_1}\omega_3 & \int_{b_1}\omega_3 \end{pmatrix}$,

$$\Gamma = \begin{pmatrix} 2\int_{a_1}\omega_1 & \int_{a_2}\omega_1 & 2\int_{b_1}\omega_1 & \int_{b_2}\omega_1 \\ 2\int_{a_1}\omega_2 & \int_{a_2}\omega_2 & 2\int_{b_1}\omega_2 & \int_{b_2}\omega_2 \end{pmatrix},$$

and

$$\Gamma^* = \begin{pmatrix} \int_{a_1}\omega_1 & \int_{a_2}\omega_1 & \int_{b_1}\omega_1 & \int_{b_2}\omega_1 \\ \int_{a_1}\omega_2 & \int_{a_2}\omega_2 & \int_{b_1}\omega_2 & \int_{b_2}\omega_2 \end{pmatrix}.$$

Let $L_\Omega = \left\{ \sum_{i=1}^3 m_i \int_{a_i} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + n_i \int_{b_i} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} : m_i, n_i \in \mathbb{Z} \right\}$, be the period lattice associated to Ω.
Let us denote also by L_Δ, the period lattice associated to Δ. We have the following diagram

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & \mathcal{E} & & \mathcal{D} & \\
 & & & \downarrow \rho^* & \swarrow & \downarrow \rho & \\
0 \longrightarrow & \ker N_\rho & \longrightarrow & Prym(\mathcal{D}/\mathcal{E}) \oplus \mathcal{E} = Jac(\mathcal{D}) & \overset{N_\rho}{\longrightarrow} & \mathcal{E} & \longrightarrow 0 \\
 & & \searrow \tau & \downarrow & & & \\
 & & & \widetilde{M} = M \cup 2\mathcal{D} \simeq \mathbb{C}^2/Lattice & & & \\
 & & & \downarrow & & & \\
 & & & 0 & & &
\end{array}
$$

where N_ρ is the norm mapping (surjective). The polarization map

$$\tau : Prym(\mathcal{D}/\mathcal{E}) \longrightarrow \widetilde{M} = Prym^*(\mathcal{D}/\mathcal{E}),$$

has kernel $(\rho^*\mathcal{E}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and the induced polarization on $Prym(\mathcal{D}/\mathcal{E})$ is of type (1,2). Let $\widetilde{M} \to \mathbb{C}^2/L_\Lambda : p \curvearrowright \int_{p_0}^p \binom{dt_1}{dt_2}$, be the uniformizing map where dt_1, dt_2 are two differentials on $\widetilde{M}$ corresponding to the flows generated respectively by H_1, H_2 such that: $dt_1|_{\mathcal{D}} = \omega_1$ and $dt_2|_{\mathcal{D}} = \omega_2$,

$$L_\Lambda = \{\sum_{k=1}^4 n_k \begin{pmatrix} \int_{\nu_k} dt_1 \\ \int_{\nu_k} dt_2 \end{pmatrix} : n_k \in \mathbb{Z}\},$$

is the lattice associated to the period matrix

$$\Lambda = \begin{pmatrix} \int_{\nu_1} dt_1 & \int_{\nu_2} dt_1 & \int_{\nu_3} dt_1 & \int_{\nu_4} dt_1 \\ \int_{\nu_1} dt_2 & \int_{\nu_2} dt_2 & \int_{\nu_3} dt_2 & \int_{\nu_4} dt_2 \end{pmatrix},$$

and $(\nu_1, \nu_2, \nu_3, \nu_4)$ is a basis of $H_1(\widetilde{M}, \mathbb{Z})$. By the Lefschetz theorem on hyperplane section [8], the map $H_1(\mathcal{D}, \mathbb{Z}) \longrightarrow H_1(\widetilde{M}, \mathbb{Z})$ induced by the inclusion $\mathcal{D} \hookrightarrow \widetilde{M}$ is surjective and consequently we can find 4 cycles $\nu_1, \nu_2, \nu_3, \nu_4$ on the curve $\mathcal{D}$ such that

$$\Lambda = \begin{pmatrix} \int_{\nu_1} \omega_1 & \int_{\nu_2} \omega_1 & \int_{\nu_3} \omega_1 & \int_{\nu_4} \omega_1 \\ \int_{\nu_1} \omega_2 & \int_{\nu_2} \omega_2 & \int_{\nu_3} \omega_2 & \int_{\nu_4} \omega_2 \end{pmatrix},$$

and $L_\Lambda = \left\{ \sum_{k=1}^4 n_k \begin{pmatrix} \int_{\nu_k} \omega_1 \\ \int_{\nu_k} \omega_2 \end{pmatrix} : n_k \in \mathbb{Z} \right\}$. The cycles $\nu_1, \nu_2, \nu_3, \nu_4$ in $\mathcal{D}$ which we look for are a_1, a_2, b_1, b_2 and they generate $H_1(\widetilde{M}, \mathbb{Z})$ such that

$$\Lambda = \begin{pmatrix} \int_{a_1} \omega_1 & \int_{a_2} \omega_1 & \int_{b_1} \omega_1 & \int_{b_2} \omega_1 \\ \int_{a_1} \omega_2 & \int_{a_2} \omega_2 & \int_{b_1} \omega_2 & \int_{b_2} \omega_2 \end{pmatrix},$$

is a Riemann matrix. So $\Lambda = \Gamma^*$, i.e., the period matrix of $Prym^*(\mathcal{D}/\mathcal{E})$ dual of $Prym(\mathcal{D}/\mathcal{E})$. Consequently $\widetilde{M}$ and $Prym^*(\mathcal{D}/\mathcal{E})$ are two abelian varieties analytically isomorphic to the same complex torus $\mathbb{C}^2/L_\Lambda$. By Chow's theorem [8], $\widetilde{M}$ and $Prym^*(\mathcal{D}/\mathcal{E})$ are then algebraically isomorphic. □

5. THE KOWALEWSKI RIGID BODY MOTION

The motion for the Kowalewski's top is governed by the equations

$$\dot{m} = m \wedge \lambda m + \gamma \wedge l, \qquad \dot{\gamma} = \gamma \wedge \lambda m, \tag{5.1}$$

where m, γ and l denote respectively the angular momentum, the directional cosine of the z-axis (fixed in space), the center of gravity which after some rescaling and normalization may be taken as $l = (1, 0, 0)$ and $\lambda m = (m_1/2, m_2/2, m_3/2)$. The system (5.1) can be written

$$\begin{aligned} \dot{m}_1 &= m_2\, m_3, & \dot{\gamma}_1 &= 2\, m_3\gamma_2 - m_2\gamma_3, \\ \dot{m}_2 &= -\, m_1\, m_3 + 2\gamma_3, & \dot{\gamma}_2 &= m_1\gamma_3 - 2m_3\gamma_1, \\ \dot{m}_3 &= -2\gamma_2, & \dot{\gamma}_3 &= m_2\gamma_1 - m_1\gamma_2, \end{aligned} \tag{5.2}$$

with constants of motion

$$\begin{aligned} H_1 &= \frac{1}{2}\left(m_1^2 + m_2^2\right) + m_3^2 + 2\gamma_1 = c_1, \\ H_2 &= m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 = c_2, \\ H_3 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = c_3 = 1, \\ H_4 &= \left(\left(\frac{m_1 + im_2}{2}\right)^2 - (\gamma_1 + i\gamma_2)\right)\left(\left(\frac{m_1 - im_2}{2}\right)^2 - (\gamma_1 - i\gamma_2)\right) = c_4. \end{aligned} \tag{5.3}$$

In her famous Acta Mathematica paper [17], Kowalewski integrates the problem in terms of hyperelliptic integrals, using a very beautiful change of variables. Here, we sketch the integration of the problem using the Laurent solutions, as carried out in full detail in [18]. The result is that the invariant surfaces could be completed via the flow into complex algebraic tori (abelian surfaces) $\mathbb{C}^2/Lattice$ were the lattice is spanned by the columns of the period matrix $\begin{pmatrix} 1 & 0 & a & c \\ 0 & 2 & c & b \end{pmatrix}$, $\operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$, i.e., the problem is not expressed in terms of hyperelliptic integrals but rather in terms of abelian integrals associated with the period matrix. As we have seen in the previous section, such abelian surfaces come up naturally as Prym varieties of double covers of elliptic curves ramified over four points. The system (5.2) admits two distinct families of Laurent series solutions:

$$m_1(t) = \begin{cases} \frac{\alpha_1}{t} + i\left(\alpha_1^2 - 2\right)\alpha_2 + \circ(t), \\ \frac{\alpha_1}{t} - i\left(\alpha_1^2 - 2\right)\alpha_2 + \circ(t), \end{cases} \qquad \gamma_1(t) = \begin{cases} \frac{1}{2t^2} + \circ(t), \\ \frac{1}{2t^2} + \circ(t), \end{cases}$$

$$m_2(t) = \begin{cases} \frac{i\alpha_1}{t} - \alpha_1^2\alpha_2 + \circ(t), \\ \frac{-i\alpha_1}{t} - \alpha_1^2\alpha_2 + \circ(t), \end{cases} \qquad \gamma_2(t) = \begin{cases} \frac{i}{2t^2} + \circ(t), \\ \frac{-i}{2t^2} + \circ(t), \end{cases}$$

$$m_3(t) = \begin{cases} \frac{i}{t} + \alpha_1\alpha_2 + \circ(t), \\ \frac{-i}{t} + \alpha_1\alpha_2 + \circ(t), \end{cases} \qquad \gamma_3(t) = \begin{cases} \frac{\alpha_2}{t} + \circ(t), \\ \frac{\alpha_2}{t} + \circ(t), \end{cases}$$

which depend on 5 free parameters $\alpha_1, \ldots, \alpha_5$. By substituting these series in the constants of the motion H_i (5.3), one eliminates three parameters linearly, leading to algebraic relation between the two remaining parameters, which is nothing but the equation of the divisor $\mathcal{D}$ along which the m_i, γ_i blow up. Since the system (5.2) admits two families of Laurent solutions, then $\mathcal{D}$ is a set of two isomorphic curves of genus 3, $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_{-1}$:

$$\mathcal{D}_\varepsilon : \ P(\alpha_1, \alpha_2) = \left(\alpha_1^2 - 1\right)\left(\left(\alpha_1^2 - 1\right)\alpha_2^2 - P(\alpha_2)\right) + c_4 = 0, \tag{5.4}$$

where $P(\alpha_2) = c_1\alpha_2^2 - 2\varepsilon c_2\alpha_2 - 1$ and $\varepsilon = \pm 1$. Each of the curve $\mathcal{D}_\varepsilon$ is a 2-1 ramified cover $(\alpha_1, \alpha_2, \beta)$ of elliptic curves $\mathcal{D}_\varepsilon^0$:

$$\mathcal{D}_\varepsilon^0 : \beta^2 = P^2(\alpha_2) - 4c_4\alpha_2^4, \tag{5.5}$$

ramified at the 4 points $\alpha_1 = 0$ covering the 4 roots of $P(\alpha_2) = 0$. It was shown [18] that each divisor $\mathcal{D}_\varepsilon$ is ample and defines a polarization $(1, 2)$, whereas the divisor $\mathcal{D}$, of geometric genus 9, is very ample and defines a polarization $(2, 4)$. The affine surface $M = \bigcap_{i=1}^4 \{H_i = c_i\} \subset \mathbb{C}^6$, defined by putting the four invariants (5.3) of the Kowalewski flow (5.2) equal to generic constants, is the affine part of an abelian surface $\widetilde{M}$ with

$$\widetilde{M} \setminus M = \mathcal{D} = \begin{array}{l}\text{one genus 9 curve consisting of two genus 3}\\ \text{curves } \mathcal{D}_\varepsilon \text{ (5.4) intersecting in 4 points. Each}\\ \mathcal{D}_\varepsilon \text{ is a double cover of an elliptic curve } \mathcal{D}_\varepsilon^0 \text{ (5.5)}\\ \text{ramified at 4 points.}\end{array}$$

Moreover, the Hamiltonian flows generated by the vector fields X_{H_1} and X_{H_4} are straight lines on $\widetilde{M}$. The 8 functions 1, $f_1 = m_1$, $f_2 = m_2$, $f_3 = m_3$, $f_4 = \gamma_3$, $f_5 = f_1^2 + f_2^2$, $f_6 = 4f_1f_4 - f_3f_5$, $f_7 = (f_2\gamma_1 - f_1\gamma_2) f_3 + 2f_4\gamma_2$, form a basis of the vector space of meromorphic functions on $\widetilde{M}$ with at worst a simple pole along $\mathcal{D}$ Moreover, the map

$$\widetilde{M} \simeq \mathbb{C}^2/Lattice \to \mathbb{P}^7(\mathbb{C}) \ , \ \ (t_1, t_2) \mapsto [(1, f_1(t_1, t_2), \ldots, f_7(t_1, t_2))] \, ,$$

is an embedding of $\widetilde{M}$ into $\mathbb{P}^7(\mathbb{C})$. Following the method (theorem 5), we obtain the following theorem:

Theorem 6. *The tori $\widetilde{M}$ can be identified as $\widetilde{M} = Prym^*(\mathcal{D}_\varepsilon/\mathcal{D}_\varepsilon^0)$, i.e., dual of $Prym(\mathcal{D}_\varepsilon/\mathcal{D}_\varepsilon^0)$ and the problem linearizes on this Prym variety.*

6. THE GEODESIC FLOW ON $SO(n)$ FOR A LEFT INVARIANT METRIC

Consider the group $SO(n)$ and its Lie algebra $so(n)$ paired with itself, via the customary inner product $\langle X, Y\rangle = -\frac{1}{2}\, tr\ (X.Y)$, where $X, Y \in so(n)$. A left invariant metric on $SO(n)$ is defined by a non-singular symmetric linear map $\Lambda : so(n) \longrightarrow so(n)$, $X \longmapsto \Lambda.X$, and by the following inner product; given two vectors gX and gY in the tangent space $SO(n)$ at the point $g \in SO(n)$, $\langle gX, gY\rangle = \langle X, \Lambda^{-1}.Y\rangle$. The question of classifying the metrics for which geodesic flow on $SO(n)$ is algebraically completely integrable is difficult. As the Euler rigid body motion is always algebraically completely integrable and can be regarded as geodesic flow on $SO(3)$, we consider the case $n = 4$. The problem has been resolved for $SO(4)$ by Adler and van Moerbeke [3, 10]. It more convenient to use the coordinates $u = (x_1, x_2, x_3)$ and $v = (x_4, x_5, x_6)$, they correspond to the decomposition $u \oplus v \in so(4) \simeq so(3) \oplus so(3)$. In these coordinates, the geodesic flow on the group $SO(4)$ can be written as

$$\dot{u} = u \times \frac{\partial H}{\partial u}, \qquad \dot{v} = v \times \frac{\partial H}{\partial v},$$

for the metric defined by the quadratic form

$$H = \frac{1}{2}\sum_{j=1}^{6} \lambda_j x_j^2 + \sum_{j=1}^{3} \mu_j x_j x_{j+3}, \tag{6.1}$$

where $\lambda_1, \ldots, \lambda_6, \mu_1, \mu_2, \mu_3 \in \mathbb{C}$ and $\lambda_{12}\lambda_{23}\lambda_{31}\lambda_{45}\lambda_{56}\lambda_{64}\mu_1\mu_2\mu_3 \neq 0$ with $\lambda_{jk} \equiv \lambda_j - \lambda_k$. The equations have besides the energy $H_1 = H$, two trivial constants of the motion

$$H_2 = x_1^2 + x_2^2 + x_3^2,$$
$$H_3 = x_4^2 + x_5^2 + x_6^2.$$

Adler and van Moerbeke [3, 10] have shown that the geodesic flow on $SO(4)$ for the metric defined by the quadratic form (6.1) is algebraically completely integrable if and only if:
a) The quadratic form H is diagonal with regard to the customary $so(4)$ coordinates (Manakov metric), i.e.,

$$2H = \sum_{\substack{j,k=1 \\ j<k}}^{4} \Lambda_{jk} X_{jk}^2, \quad (X_{jk})_{1\leqslant j,k\leqslant 4}) \in so(4),$$

with

$$\Lambda_{jk} = \frac{\beta_j - \beta_k}{\alpha_j - \alpha_k}, \quad (\alpha_j, \beta_j \in \mathbb{C}, 1 \leqslant j \leqslant 4),$$

all Λ_{jk} distinct. The extra invariant H_4 is quadratic and the flow evolves on abelian surfaces $\mathbb{C}^2/lattice \subseteq \mathbb{P}^7(\mathbb{C})$, having period matrix $\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}$, $\operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$, $(a, b, c \in \mathbb{C})$. According to Haine [19] and Mumford [1],

$$\bigcap_{j=1}^{4} \left\{ x \in \mathbb{C}^6 : H_j(x) = c_j \right\} = \operatorname{Prym}(\mathcal{C}/\mathcal{C}_0) \backslash \mathcal{D},$$

where $\mathcal{D}$ is a curve of genus 9, $\mathcal{C}_0$ is an elliptic curve defined as

$$\mathcal{C}_0 = \left\{ (t_1, t_2, t_3, t_4) \in \mathbb{P}^3(\mathbb{C}) \text{ such that } \sum t_j H_j \text{ has rank } 3 \right\},$$

and $\mathcal{C}$ is a double cover of $\mathcal{C}_0$ ramified at the 4 points $\mathcal{C}_0 \cap \{\sum t_j c_j = 0\}$. The periods of this Prym variety $\operatorname{Prym}(\mathcal{C}/\mathcal{C}_0)$ provide the exact periods of the motion in terms of abelian integrals. The problem of the solid body in a fluid in the case of Clebsch is a particular case of this metric.
b) The quadratic form H satisfies the conditions

$$\left(\mu_1^2, \mu_2^2, \mu_3^2\right) = \frac{\lambda_{12}\lambda_{23}\lambda_{31}\lambda_{45}\lambda_{56}\lambda_{64}}{\left(\lambda_{46}\lambda_{32} - \lambda_{65}\lambda_{13}\right)^2} \left(\frac{(\lambda_{23} - \lambda_{56})^2}{\lambda_{23}\lambda_{56}}, \frac{(\lambda_{31} - \lambda_{64})^2}{\lambda_{31}\lambda_{64}}, \frac{(\lambda_{12} - \lambda_{45})^2}{\lambda_{12}\lambda_{45}} \right),$$

with the product $\mu_1\mu_2\mu_3$ being rational in $\lambda_1, \ldots, \lambda_6$ and with the following sign specification

$$\mu_1\mu_2\mu_3 = \frac{\lambda_{12}\lambda_{23}\lambda_{31}\lambda_{45}\lambda_{56}\lambda_{64}}{\left(\lambda_{46}\lambda_{32} - \lambda_{65}\lambda_{13}\right)^3} (\lambda_{12} - \lambda_{45})(\lambda_{23} - \lambda_{56})(\lambda_{31} - \lambda_{64}).$$

The extra invariant H_4 is quadratic and the flow linearizes on 2-dimensional hyperelliptic jacobians. More precisely

$$\bigcap_{j=1}^{4} \left\{ x \in \mathbb{C}^6 : H_j(x) = c_j \right\} = Jac \text{ (hyperelliptic curve } \mathcal{C} \text{ of genus 2)} \backslash \mathcal{D},$$

where $\mathcal{D}$ is a divisor of genus 17, which contains 4 translates of the Θ-divisor in $Jac(\mathcal{C})$, each of which is isomorphic to $\mathcal{C}$. The hyperelliptic curve $\mathcal{C}$ is a double cover of the curve $\mathcal{C}_0$ (see a)) of rank 4 quadrics which in this case is isomorphic to $\mathbb{P}^1(\mathbb{C})$. The periods of the motion are given by the periods of the hyperelliptic curve $\mathcal{C}$. The problem of the solid body in a fluid in the case of Lyapunov–Steklov is a particular case of this metric.
c) The form H satisfies

$$\left(\mu_1^4, \mu_2^4, \mu_3^4\right) = \lambda_{13}\lambda_{46}\lambda_{21}\lambda_{54}\lambda_{32}\lambda_{65} \left(\frac{1}{\lambda_{32}\lambda_{65}}, \frac{1}{\lambda_{13}\lambda_{46}}, \frac{1}{\lambda_{21}\lambda_{54}} \right).$$

The quantities ζ, ξ and η defined by

$$\zeta^2 \equiv \frac{\lambda_{46}}{\lambda_{13}}, \quad \xi^2 \equiv \frac{\lambda_{54}}{\lambda_{21}}, \quad \eta^2 \equiv \frac{\lambda_{65}}{\lambda_{32}},$$

satisfy the quadratic relations

$$\zeta\xi + \xi\eta + \eta\zeta + 1 = 0, \quad 3\xi\eta + \eta - \xi + 1 = 0.$$

The geodesic flow has a quartic invariant H_4, evolves on abelian surfaces $\widetilde{A} \subseteq \mathbb{P}^{23}(\mathbb{C})$ having period matrix $\begin{pmatrix} 2 & 0 & a & c \\ 0 & 12 & c & b \end{pmatrix}$, $\operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$, $(a, b, c \in \mathbb{C})$ and it will be expressed in terms of abelian integrals. More precisely

$$\bigcap_{j=1}^{4} \left\{x \in \mathbb{C}^6 : H_j(x) = c_j\right\} = \widetilde{A} \backslash \mathcal{D},$$

where $\mathcal{D}$ is a curve of genus 25 with 8 singular points. Put in a more geometrical language, the tori $\widetilde{A}$ contain a very ample and projectively normal curve $\mathcal{D}$ of geometric genus 25, with 8 normal crossings whose smooth version $\mathcal{C}$ is a 4-1 unramified cover of a curve $\mathcal{C}_0$ of genus 5. The curve $\mathcal{C}_0$ itself is a double cover ramified over 4 points of a genus 2 hyperelliptic curve $\mathcal{H}$. Moreover, the linearization takes place on a 2-dimensional subtorus of the 3-dimensional Prym variety $\operatorname{Prym}(\mathcal{C}_0/\mathcal{H})$ with

$$\operatorname{Prym}(\mathcal{C}_0/\mathcal{H}) = \widetilde{A} \oplus \mathcal{E},$$

where $\mathcal{E}$ is an elliptic curve. This situation provides a full description of the moduli for the abelian surfaces of polarization $(1, 6)$.

We have seen throughout this work that if a system is algebraically completely integrable, then it has a family of meromorphic Laurent series depending on "dim (phase space) − 1" free parameters. Now, trying to generalize the result to the geodesic flow on $SO(n)$ for $n \geqslant 5$ using the same method leads to insurmountable calculations. As was shown by Haine [20], for $n \geqslant 5$ Manakov's metrics are the only left invariant diagonal metrics on $SO(n)$ for which the geodesic flow is algebraically completely integrable. Note that it turns out that the geodesic flow on $SO(n)$ admits a lot of invariant manifolds on which they reduce to geodesic flow on $SO(3)$ and the solutions of the differential equation with initial conditions on these manifolds are elliptic functions and this without any condition on the metric. Haine [20] has shown that looking at solutions near these special a priori known solutions and imposing these solutions to be single-valued functions of $t \in \mathbb{C}$, suffices to single out the left invariant diagonal metrics for which the geodesic flow is algebraically completely integrable.

I wish to express my thanks to an anonymous referee for his valuable helpful comments and suggestions.

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 210–222. © Pleiades Publishing, Ltd., 2011.

On Rosenhain–Göpel Configurations and Integrable Systems

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Received April 28, 2010; accepted August 21, 2010

Abstract—We give a birational morphism between two types of genus 2 Jacobians in $\mathbb{P}^{15}$. One of them is related to an Algebraic Completely Integrable System: the Geodesic Flow on SO(4), metric II (so termed after Adler and van Moerbeke). The other Jacobian is related to a linear system in $|4\Theta|$ with 12 base points coming from a Göpel tetrad of 4 translates of the Θ divisor. A correspondence is given on the base spaces so that the Poisson structure of the SO(4) system can be pulled back to the family of Göpel Jacobians.

MSC2010 numbers: 58F07, 14K25

DOI: 10.1134/S1560354711030038

Keywords: integrable systems

1. INTRODUCTION

It was Jacobi who realized that new Integrable Systems were very difficult to find. In the best cases, if one has a good change of coordinates that relates a given System to a known Integrable System, one could expect to integrate the given one via this change of coordinates. We asked ourselves whether there would be an Algebraic Completely Integrable System related with genus two Jacobians in $\mathbb{P}^{15}$ and associated to a divisor $\mathcal{D}_0$ in $|4\Theta|$ formed by four translates of the Θ divisor by $\frac{1}{2}$-periods intersecting into 12 nodes $e_1, \ldots, e_{12}$ as shown in Fig. 1. Such a divisor is an even section of $|4\Theta|$ cut out by a hyperplane section at infinity. The affine variables expressed in terms of the complex time flow have to blow up at $\mathcal{D}_0$ in order for this to be associated with an Algebraic Completely Integrable System in the sense of Adler and van Moerbeke.

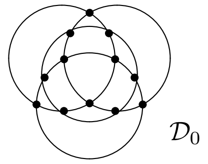

Fig. 1

This is the classical configuration that comes up when in the related Kummer surfaces in $\mathbb{P}^3$ we pick a Göpel tetrad of theta functions as coordinates of the ambient 3-space. This subject is extensively described in [1–3].

The Geodesic Flow on SO(4), metric II is an Algebraic Completely Integrable System that was studied by Adler and van Moerbeke [4, 5], (See also [6–8]). In [8] an alternative way of obtaining this system via algebro-geometric considerations is given. This Integrable System linearizes on genus two Jacobians and has at infinity divisors $\mathcal{D}_1$ which consists of four translates by $\frac{1}{2}$-periods of the Θ divisor that intersect into four triple points (Fig. 2).

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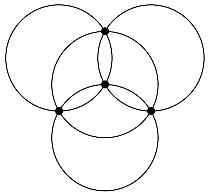

Fig. 2

The configuration divisor $\mathcal{D}_1$ comes up when in the related Kummer surface in $\mathbb{P}^3$ we choose a Rosenhain tetrad of theta functions as coordinates of the ambient 3-space [1–3]. In terms of the (-1)-involution applied to the theta functions defining $\mathcal{D}_1$, this divisor becomes odd, as opposed to $\mathcal{D}_0$.

We realized there should be a map between the variables associated with the functions in $\mathcal{L}(\mathcal{D}_0)$ and those in $\mathcal{L}(\mathcal{D}_1)$ in the following way. Write $\mathcal{D}_0 = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3$ where the Θ_i's are the four translates by $\frac{1}{2}$-periods of the Θ divisor and therefore homologous to Θ. Analogously, $\mathcal{D}_1 = \Theta_0{}' + \Theta_1{}' + \Theta_2{}' + \Theta_3{}'$, where the $\Theta_i{}'$'s are homologous to Θ. Since the Θ_i's are related to a Göpel tetrad and the $\Theta_i{}'$'s to a Rosenhain tetrad, they must be linearly equivalent to 4Θ. In particular, we have the linear equivalences $4\Theta_i \sim \mathcal{D}_0 \sim 4\Theta \sim \mathcal{D}_1 \sim 4\Theta_i{}'$ that imply a linear map between $\mathcal{L}(\mathcal{D}_0)$ and $\mathcal{L}(\mathcal{D}_1)$ on each Jacobian. Now, the divisors $2\Theta_i$ and $2\Theta_i{}'$ are algebraically equivalent and because they are totally symmetric we have the linear equivalence $2\Theta_i \sim 2\Theta_i{}'$ (see [7]). The functions of $\mathcal{L}(2\Theta_i)$ and $\mathcal{L}(2\Theta_i{}')$ embed the respective Kummer surfaces in $\mathbb{P}^3$ and they are therefore related by a linear transformation in $\mathbb{P}^3$. This map gives an isomorphism between the Kummer surfaces of the Rosenhain and Göpel configurations and can be extended to a linear map in $\mathbb{P}^{15}$ by a procedure that was described in [7]. Thus we get a biregular map at the level of the Jacobians.

The main result of this paper is Theorem 1, the linear map in $\mathbb{P}^3$ that relates the Göpel and Rosenhain Kummer surfaces biregularly. As a bypass we can translate the Poisson structure of the SO(4) system to the Göpel family of Jacobians.

2. GENUS TWO THETA FUNCTIONS

Let τ be the 2×2 Riemann matrix of a (generic and principally polarized) genus 2 Jacobian. A pair of real vectors (m, m^*) is associated univocally with the point $m^* + m\tau$ of $\mathbb{C}^2$.

For the pair of row vectors (m, m^*) (called characteristics) we define the classical theta functions [3, §8.5] as (1) below, where $e(z) = exp(2\pi i z)$, $z \in \mathbb{C}$. They have the properties (2), (3), (3′), (4).

(1) $\vartheta_{m,m^*}(\tau,\zeta) = \sum_{\psi\in\mathbb{Z}^n} e(\frac{1}{2}(\psi+m)\tau^t(\psi+m) + (\psi+m)^t(\zeta+m^*))$

(2) $\vartheta_{m,m^*}(\tau,-\zeta) = \vartheta_{-m,-m^*}(\tau,\zeta)$

(3) $\vartheta_{m+\psi,m^*+\psi^*}(\tau,\zeta) = e(m^t\psi^*)\vartheta_{m,m^*}(\tau,\zeta), \quad \text{for} \psi,\psi^* \in \mathbb{Z}^n$

(3′) $\vartheta_{m,m^*}(\tau,\zeta+u\tau+u^*) = e(-\frac{1}{2}u\tau^t u - u^t(\zeta+u^*))e(-u^t m^*)\vartheta_{m+u,m^*+u^*}(\tau,\zeta).$

We also use the customary notation $\vartheta_{m,m^*}(\tau,\zeta) = \vartheta\begin{bmatrix} m \\ m^* \end{bmatrix}(\tau,\zeta)$, and agree to represent the point $m^* + m\tau$ either by $\begin{bmatrix} m \\ m^* \end{bmatrix}$ or $\begin{Bmatrix} m \\ m^* \end{Bmatrix}$, when τ is fixed.

If $\begin{Bmatrix} m \\ m^* \end{Bmatrix} \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ is a half period, then we have the formula (Prop. 3.14 — Ch. II. p. 167 [9]).

(4) $\vartheta_{m,m^*}(\tau,-\zeta) = e(2m^t m^*)\vartheta_{m,m^*}(\tau,\zeta) = e_*(m^*+m\tau)\vartheta_{m,m^*}(\tau,\zeta)$

There are 2^{2g} half periods on an abelian variety of dimension g. We say that a half period of characteristic $\begin{Bmatrix} m \\ m^* \end{Bmatrix}$ is odd (even) if the factor $e_*\left(\begin{bmatrix} m \\ m^* \end{bmatrix}_\tau\right)$ is negative (positive).

For a genus 2 Jacobian the even half period characteristics are given by

$$e_{35} = \begin{Bmatrix} 0\ 0 \\ 0\ 0 \end{Bmatrix}, \quad e_{23} = \begin{Bmatrix} 0 & 0 \\ 1/2 & 0 \end{Bmatrix}, \quad e_{45} = \begin{Bmatrix} 0\ 0 \\ 0\ 1/2 \end{Bmatrix}, \quad e_{13} = \begin{Bmatrix} 1/2 & 0 \\ 0 & 0 \end{Bmatrix}$$

$$e_{12} = \begin{Bmatrix} 0\ 1/2 \\ 0\ 0 \end{Bmatrix}, \quad e_{25} = \begin{Bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{Bmatrix}, \quad e_{14} = \begin{Bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{Bmatrix}, e_{15} = \begin{Bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{Bmatrix}$$

$$e_{24} = \begin{Bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{Bmatrix}, \quad e_{34} = \begin{Bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{Bmatrix}.$$

While the odd characteristics are the following

$$e_0 = \begin{Bmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{Bmatrix}, \quad e_1 = \begin{Bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{Bmatrix}, \quad e_2 = \begin{Bmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{Bmatrix}, \quad e_3 = \begin{Bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{Bmatrix}$$

$$e_4 = \begin{Bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{Bmatrix}, \quad e_5 = \begin{Bmatrix} 0\ 1/2 \\ 0\ 1/2 \end{Bmatrix}$$

It follows that the theta functions $\vartheta[e_i]$ are odd functions with respect to the involution, while the $\vartheta[e_{ij}]$'s are even. One has the relations $\overline{e_i} + \overline{e_j} = \overline{e_{ij}} + \overline{e_0}, \ 0 < i < j \leqslant 5$, and $\sum_{i=0}^5 \overline{e_i} = 0$ on the Jacobian.

The odd half periods are the Weierstrass points of the theta divisor $\overline{\{\zeta : \vartheta_{0,0}(\tau, \zeta) = 0\}} = \Theta$ and Θ is also the genus 2 curve into its Jacobian.

Any Rosenhain divisor $\mathcal{D}_1$ (i.e. related to the SO(4) system) and any Göpel divisor $\mathcal{D}_0$ can be constructed from one of the 16 symmetric curves $\{\Theta_{0i} = \Theta + \overline{e_0} + \overline{e_i}, \ 0 \leqslant i \leqslant 5; \ \Theta_{ij} = \Theta + \overline{e_i} + \overline{e_j}, \ 0 < i < j \leqslant 5\}$ by acting on it with a particular group of translates $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, t_1, t_2, t_3 = t_1 + t_2\}$ and by selecting the origin to be one of the $\overline{e_i}$'s. As well known, there are 80 Rosenhain tetrads and 60 Göpel tetrads that can be chosen among these 16 curves possibly after choosing the right origin [1], [2]. For any curve $\mathcal{C}$, define $\mathcal{C}_\ell := \mathcal{C} + \overline{e_\ell}, \ \ell = 0, \dots 5$.

The 60 Göpel divisors have the form $\mathcal{D}_0 = \Theta_\ell + (\Theta_{ij})_\ell + (\Theta_{kl})_\ell + (\Theta_{mn})_\ell$, where $i, j, \dots, n$ are all different and $0 \leqslant \ell \leqslant 5$. The 80 Rosenhain divisors are $\mathcal{D}_1 = \Theta_\ell + (\Theta_{0i})_\ell + (\Theta_{0j})_\ell + (\Theta_{ij})_\ell$ with $0 < i < j \leqslant 5$, and $\mathcal{D}_1' = \Theta_\ell + (\Theta_{ij})_\ell + (\Theta_{jk})_\ell + (\Theta_{ki})_\ell$, with $0 < i < j < k \leqslant 5$; $0 \leqslant \ell \leqslant 5$.

These divisors are all linearly equivalent in $|4\Theta|$. Also, any divisor $2(\Theta_{ij})_\ell$ is linearly equivalent to 2Θ. However, there may be issues once an origin is chosen.

Let us call $G_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, t_1, t_2, t_3 = t_1 + t_2\}$ a Göpel group if acting on Θ_ℓ by translations it defines a Göpel divisor. $G_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, t_1', t_2', t_3' = t_1' + t_2'\}$ is a Rosenhain group if acting on Θ_ℓ by translations it defines a Rosenhain divisor.

Assume that on the Jacobian A we have chosen an origin and fixed a Göpel or Rosenhain divisor $\mathcal{D}$ given as the zeroes of theta function $s_0 s_1 s_2 s_3$, where the s_i's are translates of the Riemann theta function by $\frac{1}{2}$-periods. We have the following:

Proposition 1 ([7]). *Let $\mathcal{D} = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3$ be a Göpel or Rosenhain divisor defined by the theta sections s_0, s_1, s_2, s_3 Then, the sections of 2Θ: $\{s_0^2, s_1^2, s_2^2, s_3^2\}$ are linearly independent and therefore give a basis for $H^0(A, 2\Theta)$. If X represents a generic global vector field, then a basis for the even sections of $H^0(A, 4\Theta)$ is given by the sections $\{s_0^4, s_0^2s_1^2, s_0^2s_2^2, s_0^2s_3^2, s_1^4, s_1^2s_2^2, s_1^2s_3^2, s_2^4, s_2^2s_3^2, s_3^4\}$, and a basis for the odd sections of $H^0(A, 4\Theta)$ is $\{W_X(s_0^2, s_1^2), W_X(s_0^2, s_2^2), W_X(s_0^2, s_3^2), W_X(s_1^2, s_2^2), W_X(s_1^2, s_3^2), W_X(s_2^2, s_3^2)\}$, where s_0, s_1, s_2, s_3 are theta functions vanishing on $\Theta_0, \Theta_1, \Theta_2, \Theta_3$ respectively.*

3. KUMMER EQUATIONS

Here we will give the equations of the Kummer surfaces in the Rosenhain and Göpel basis of $H^0(A, 2\Theta)$. We will work with a particular choice of theta functions and the action of the Rosenhain (Göpel) group on the respectively chosen Rosenhain (Göpel) basis is one of the possible many actions that essentially differ by a symplectic automorphism.

Pick three points $\overline{e_0}, \overline{e_1}, \overline{e_2}$ in $\Theta = \overline{\{\zeta : \vartheta(\tau, \zeta) = 0\}}$, $\overline{e_0}$ as origin and consider the group G_1 generated by $\overline{e_1} - \overline{e_0}, \overline{e_2} - \overline{e_0}$. This has an extra element $\overline{e_{12}} - \overline{e_0}$. We write

$$t_1 = e_1 - e_0 = \left\{ \begin{matrix} 0 & 0 \\ -1/2 & 1/2 \end{matrix} \right\} = e_{24} + \left\{ \begin{matrix} 0 & 0 \\ -1 & 0 \end{matrix} \right\}$$

$$t_2 = e_2 - e_0 = \left\{ \begin{matrix} -1/2 & 0 \\ 0 & 1/2 \end{matrix} \right\} = e_{14} + \left\{ \begin{matrix} -1 & 0 \\ 0 & 0 \end{matrix} \right\}$$

$$t_3 = e_{12} - e_0 = \left\{ \begin{matrix} -1/2 & 0 \\ -1/2 & 0 \end{matrix} \right\} = e_4 + \left\{ \begin{matrix} -1 & 0 \\ -1 & 0 \end{matrix} \right\}$$

and consider the translates of the ϑ-divisor Θ by these elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. These translates are given by the sections

$$s_o = \vartheta[e_{35}](\tau, \zeta) = \vartheta(\tau, \zeta), s_1 = \vartheta[e_{24}](\tau, \zeta), s_2 = \vartheta[e_{14}](\tau, \zeta), \text{ and } s_3 = \vartheta[e_4](\tau, \zeta).$$

Thus, the zero locus of $Z_1(\tau, \zeta) = \vartheta[e_{35}]\vartheta[e_{24}]\vartheta[e_{14}]\vartheta[e_4]$ gives a typical SO(4) divisor. As Z_1 is the product of 3 even sections and one odd section, Z_1 is odd.

We will make a table with the action of t_i defined by $t_{-x}\vartheta(\zeta) = \vartheta(\zeta + x)$. This action is associated with the Schrodinger representation of the Theta group (see [3] and [7] for those matters).

	$s_0 = \vartheta[e_{35}](\tau, \zeta)$	$s_1 = \vartheta[e_{24}]$	$s_2 = \vartheta[e_{14}]$	$s_3 = \vartheta[e_4]$
t_1	$\vartheta[e_{35}](\tau, \zeta + e_1 - e_0) = \vartheta[e_{24}]$	$\vartheta[e_{35}]$	$-\vartheta[e_4]$	$\vartheta[e_{14}]$
t_2	$\vartheta[e_{35}](\tau, \zeta + e_2 - e_0) = f(\zeta)\vartheta[e_{14}]$	$f(\zeta)i\vartheta[e_4]$	$f(\zeta)\vartheta[e_{35}]$	$f(\zeta)i\vartheta[e_{24}]$
t_3	$\vartheta[e_{35}](\tau, \zeta + e_{12} - e_0) = -g(\zeta)\vartheta[e_4]$	$g(\zeta)i\vartheta[e_{14}]$	$g(\zeta)\vartheta[e_{24}]$	$g(\zeta)i\vartheta[e_{35}]$

(3.1)

This induces the following action on the Rosenhain basis of $H^0(A, 2\Theta)$

Table I

	$x_1 = s_0^2$	$x_2 = s_1^2$	$x_3 = s_2^2$	$x_4 = s_3^2$
t_1	x_2	x_1	x_4	x_3
t_2	f^2x_3	$-f^2x_4$	f^2x_1	$-f^2x_2$
t_3	g^2x_4	$-g^2x_3$	g^2x_2	$-g^2x_1$

where f, g are factors corresponding to trivial ϑ-functions.

Using Table I we deduce the equation of the Kummer surface in terms of a Rosenhain basis. This is the following (compare with [1, p. 83] for a slightly different equation due to a different action on the basis)

$$\begin{aligned} 2sx_1x_2x_3x_4 + u^2\left(x_1^2x_4^2 + x_2^2x_3^2\right) + 2uv(x_1x_4 - x_2x_3)(x_1x_3 - x_2x_4) \\ +2uw(x_1x_4 + x_2x_3)(x_1x_2 + x_3x_4) + v^2\left(x_1^2x_3^2 + x_2^2x_4^2\right) \\ -2vw(x_1x_3 + x_2x_4)(x_1x_2 - x_3x_4) + w^2\left(x_1^2x_2^2 + x_3^2x_4^2\right) = 0 \end{aligned} \tag{3.2}$$

The intersection of this surface with the coordinate planes $x_i = 0$ give double conics. Also, the equation can be written as $x_1^2p_2(x_2, x_3, x_4) + x_1p_3(x_2, x_3, x_4) + (q_2(x_2, x_3, x_4))^2 = 0$, where p_i, q_i are homogeneous polynomials of degree i. Having the equation written down in this form in a Rosenhain basis allows us to quickly find the branch locus of the projection from $[1 : 0 : 0 : 0]$ onto the plane $\{x_1 = 0\}$. This is given by decomposing $\Delta = p_3^2 - 4p_2q_2^2$ into its 6 linear forms (possibly using roots of u, v, w), each one of these forms defines a plane that intersects the surface into a double conic. The branch locus of the projection is precisely the union of these 6 double conics. This branch locus is important to find if we want to match two Kummer surfaces. As it was shown in [7, Prop. II, Ch. V], two Kummer surfaces with chosen double points are biregularly equivalent $(K, p) \cong (K', p')$ if and only if there is a linear map in $\mathbb{P}^3$ preserving the branch loci of the projections from p (respectively p') onto a plane. So essentially in order to match a Rosenhain Kummer surface and a Göpel one we have to carry the Göpel equation into Rosenhain form by a linear change which will probably involve a choice of an origin in the Jacobian.

Notice that four of the double points of the surface are the corners of the tetrahedron formed by the planes $x_i = 0$; namely the points $\{[1 : 0 : 0 : 0],\ [0 : 1 : 0 : 0],\ [0 : 0 : 1 : 0],\ [0 : 0 : 0 : 1]\}$ of $\mathbb{P}^3$. These double points correspond in the Jacobian to the triple points of a Rosenhain divisor $\mathcal{D}_1$.

Now, let us write down for later use the action of G_1 on the sections $v_{ij} = s_i^2s_j^2,\ i < j$ of 4Θ. These sections are closely related to one of the incarnations of the SO(4) system that was worked out in [7] and [8].

	v_{01}	v_{02}	v_{03}	v_{12}	v_{13}	v_{23}	Z_1
t_1	v_{01}	v_{13}	v_{12}	v_{03}	v_{02}	v_{23}	$-Z_1$
t_2	$-f^4v_{23}$	f^4v_{02}	$-f^4v_{12}$	$-f^4v_{03}$	f^4v_{13}	$-f^4v_{01}$	$-f^4Z_1$
t_3	$-g^4v_{23}$	g^4v_{13}	$-g^4v_{03}$	$-g^4v_{12}$	g^4v_{02}	$-g^4v_{01}$	g^4Z_1

(3.3)

In the Göpel situation we choose the $\frac{1}{2}$-periods e_0, e_{15}, e_4, e_{23}. Fix the origin $\overline{e_0}$ and let G_0 be the group generated by $\{e_{15} - e_0, e_4 - e_0, e_{23} - e_0\}$. Write

$$t_1' = e_{15} - e_0 = \left\{\begin{matrix} -1/2 & 0 \\ 0 & 0 \end{matrix}\right\} = e_{13} + \left\{\begin{matrix} -1 & 0 \\ 0 & 0 \end{matrix}\right\}$$

$$t_2' = e_4 - e_0 = \left\{\begin{matrix} 0 & -1/2 \\ 0 & 0 \end{matrix}\right\} = e_{12} + \left\{\begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix}\right\}$$

$$t_3' = e_{23} - e_0 = \left\{\begin{matrix} -1/2 & -1/2 \\ 0 & 0 \end{matrix}\right\} = e_{25} + \left\{\begin{matrix} -1 & -1 \\ 0 & 0 \end{matrix}\right\}.$$

Therefore, consider the translates of the ϑ-divisor Θ by these elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. These translates are given by the sections

$$s'_o = \vartheta[e_{35}](\tau,\zeta) = \vartheta(\tau,\zeta), s'_1 = \vartheta[e_{13}](\tau,\zeta), s'_2 = \vartheta[e_{12}](\tau,\zeta), \text{ and } s'_3 = \vartheta[e_{25}](\tau,\zeta).$$

Thus, the zero locus of $Z_0(\tau,\zeta) = \vartheta[e_{35}]\vartheta[e_{13}]\vartheta[e_{12}]\vartheta[e_{25}]$ gives a Göpel divisor. As Z_0 is the product of 4 even sections, Z_0 is even.

We will make a table with the action of t'_i on the theta functions as in the Rosenhain case.

	$s'_o = \vartheta[e_{35}](\tau,\zeta)$	$s'_1 = \vartheta[e_{13}]$	$s'_2 = \vartheta[e_{12}]$	$s'_3 = \vartheta[e_{25}]$
t'_1	$\vartheta[e_{35}](\tau,\zeta+e_{15}-e_0) = g(\zeta)\vartheta[e_{13}]$	$g(\zeta)\vartheta[e_{35}]$	$g(\zeta)\vartheta[e_{25}]$	$g(\zeta)\vartheta[e_{12}]$
t'_2	$\vartheta[e_{35}](\tau,\zeta+e_4-e_0) = f(\zeta)\vartheta[e_{12}]$	$f(\zeta)\vartheta[e_{25}]$	$f(\zeta)\vartheta[e_{35}]$	$f(\zeta)\vartheta[e_{13}]$
t'_3	$\vartheta[e_{35}](\tau,\zeta+e_{23}-e_0) = f(\zeta)g(\zeta)\vartheta[e_{25}]$	$f(\zeta)g(\zeta)\vartheta[e_{12}]$	$f(\zeta)g(\zeta)\vartheta[e_{13}]$	$f(\zeta)g(\zeta)\vartheta[e_{35}]$

(3.4)

This leads to the following action on the Göpel basis of $H^0(A, 2\Theta)$

Table II

	$x = (s'_0)^2$	$y = (s'_1)^2$	$z = (s'_2)^2$	$t = (s'_3)^2$
t'_1	g^2y	g^2x	g^2t	g^2z
t'_2	f^2z	f^2t	f^2x	f^2y
t'_3	f^2g^2t	f^2g^2z	f^2g^2y	f^2g^2x

Again, f and g are factors corresponding to trivial ϑ-functions.

The equation in these variables is already given in [1, p. 85]:

$$\begin{aligned}\left(2p(tx+yz)+2q(ty+xz)+2r(tz+xy)+t^2+x^2+y^2+z^2\right)^2 \\ -16txyz\left(p^2-2pqr+q^2+r^2-1\right)=0.\end{aligned} \tag{3.5}$$

Using Table II and the invariance of the Kummer surface under this action, we deduce the same equation as Hudson for which we get the relation $Z_0^2 = xyzt$ and moreover, our section x (in Table II) and x_1 (in Rosenhain case) is the same theta function $\vartheta[e_{35}]^2$.

The 12 double points corresponding to the 12 half periods in a Göpel divisor $\mathcal{D}_0$ are met at the intersection of each pair of coordinate planes (i.e. at an edge of the fundamental Göpel tetrahedron whose faces are the planes $\{x=0\},\{y=0\},\{z=0\},\{t=0\}$). There are two double points per edge. The remaining 4 double points are outside any of the double conics obtained by intersecting the surface with a face of the tetrahedron. They come from the $\frac{1}{2}$-periods in the affine part of the Jacobian in $\mathbb{P}^{15}$ whose hyperplane section at infinity is precisely Z_0. One checks that if $P_0 = [1 : y_0 : z_0 : t_0]$ (with all entries $\neq 0$) is one of these 4 double points, then $p, q,$ and r are obtained in the following way:

$$\begin{cases} p = -\dfrac{t_0^2-y_0^2-z_0^2+1}{2(t_0-y_0z_0)}, \\ q = -\dfrac{t_0^2+y_0^2-z_0^2-1}{2(t_0y_0-z_0)}, \\ r = \dfrac{t_0^2-y_0^2+z_0^2-1}{2(y_0-t_0z_0)}, \\ \phi_0 = 2p(t_0+y_0z_0)+2q(t_0y_0+z_0)+2r(t_0z_0+y_0)+t_0^2+1+y_0^2+z_0^2 = \\ \quad = -\dfrac{2t_0y_0z_0(t_0-y_0-z_0+1)(t_0+y_0-z_0-1)(t_0-y_0+z_0-1)(t_0+y_0+z_0+1)}{(t_0y_0-z_0)(t_0z_0-y_0)(t_0-y_0z_0)}. \end{cases} \tag{3.6}$$

These relations are deduced from the condition that P_0 be a singular double point in the Kummer surface, and as soon as this happens, the orbit $G_0.P_0$ gives the 4 double points corresponding to the $\frac{1}{2}$-periods in the affine part. It is by using (3.6) into the general quartic equation invariant under G_0 that one obtains (after several calculations) equation (3.5). It matters here that the intersection of a coordinate plane and the surface has to be a double conic.

The four points $G_0.P_0$ are the vertices of a Rosenhain tetrahedron. We would like to map the vertices of the Rosenhain tetrahedron coming from the SO(4) system to the tetrahedron with vertices in $G_0.P_0$ (although this is not unique). In order to do that, we write Göpel's equation (3.5) around the point P_0. We need to do a Taylor expansion of (3.5) about P_0 keeping in mind that P_0 is a double point. The change of coordinates involved is the following: $\{x = \boldsymbol{x}, y = \boldsymbol{y} + y_0\boldsymbol{x}, z = \boldsymbol{z} + z_0\boldsymbol{x}, t = \boldsymbol{t} + t_0\boldsymbol{x}\}$ and in these new variables the equation can be written down as follows:

$$\begin{cases} \boldsymbol{x}^2 P_2(\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}) + \boldsymbol{x}P_3(\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}) + (Q_2(\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}))^2 = 0, \qquad \text{where} \\ P_2(\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}) = \dfrac{64s^2(\boldsymbol{t}y_0z_0 + t_0\boldsymbol{y}z_0 + t_0y_0\boldsymbol{z})^2}{\phi_0^2} \\ \qquad + 2\phi_0\left(2(p\boldsymbol{yz} + q\boldsymbol{ty} + r\boldsymbol{tz}) + \boldsymbol{t}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2\right) - 16s(\boldsymbol{ty}z_0 + \boldsymbol{t}y_0\boldsymbol{z} + t_0\boldsymbol{yz}), \\ P_3(\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}) = \dfrac{16s\left(2(p\boldsymbol{yz} + q\boldsymbol{ty} + r\boldsymbol{tz}) + \boldsymbol{t}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2\right)(\boldsymbol{t}y_0z_0 + t_0\boldsymbol{y}z_0 + t_0y_0\boldsymbol{z})}{\phi_0} - 16s\boldsymbol{tyz}, \\ Q_2(\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}) = 2(p\boldsymbol{yz} + q\boldsymbol{ty} + r\boldsymbol{tz}) + \boldsymbol{t}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2, \qquad \text{and} \\ s = p^2 + q^2 + r^2 - 2pqr - 1. \end{cases} \tag{3.7}$$

Although (3.7) looks like a Rosenhain form, we will need to go to the roots of the moduli numbers $\{x_0, y_0, z_0\}$ in order to fully factorize these expressions. In fact, the double conics that should appear by intersecting equation (3.7) with the coordinate planes $\{\boldsymbol{y} = 0\}, \{\boldsymbol{z} = 0\}$, and $\{\boldsymbol{t} = 0\}$, only appear after taking the root $\left\{\alpha = \sqrt{(t_0-z_0-y_0+1)(t_0+y_0-z_0-1)(t_0-y_0+z_0-1)(t_0+y_0+z_0+1)}\right\}$. Also, notice that the groups G_0 and G_1 do not correspond well under this translation.

4. THE SO(4) SYSTEM

The SO(4) system for the metric II is the system of differential equations [5], [6], [7].

$$\begin{aligned} \dot{\tau_1} &= \tau_2\tau_6 & \dot{\tau_4} &= \tau_3\tau_5 \\ \dot{\tau_2} &= \frac{1}{2}\tau_3(\tau_1 + \tau_4) & \dot{\tau_5} &= \tau_3\tau_4 \\ \dot{\tau_3} &= \frac{1}{2}\tau_3(\tau_1 + \tau_4) & \dot{\tau_6} &= \tau_1\tau_2. \end{aligned}$$

One can pick a Poisson matrix for this system

$$J_{SO(4)} = \begin{bmatrix} 0 & \tau_3 & \tau_2 & 0 & 0 & (2\tau_2 - \tau_5) \\ -\tau_5 & 0 & 0 & 0 & 0 & 0 \\ -\tau_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_5 \\ 0 & 0 & 0 & 0 & 0 & \tau_4 \\ -(\tau_4 - \tau_5) & 0 & 0 & -\tau_5 & -\tau_4 & 0 \end{bmatrix}$$

with Poisson bracket $\{f, g\} = \left\langle \frac{\partial f}{\partial \tau}, J_{SO(4)} \cdot \frac{\partial g}{\partial \tau} \right\rangle$.

By making the change of variables

$$v_1 = \tau_1 + \tau_6,\ v_2 = \tau_6 - \tau_1,\ v_3 = \tau_2 + \tau_3,\ v_4 = \tau_2 - \tau_3,\ v_5 = \tau_4 + \tau_5,\ v_6 = \tau_5 - \tau_4$$

the Poisson matrix takes the form

$$J_{SO(4)} = \begin{bmatrix} 0 & 2(v_3+v_4)-(v_5+v_6) & v_3 & -v_4 & -v_5 & v_6 \\ -2(v_3+v_4)+(v_5+v_6) & 0 & -v_3 & v_4 & -v_5 & v_6 \\ -v_3 & v_3 & 0 & 0 & 0 & 0 \\ v_4 & -v_4 & 0 & 0 & 0 & 0 \\ v_5 & v_5 & 0 & 0 & 0 & 0 \\ v_6 & -v_6 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the invariants are $Q_2 = v_1v_2 = a_2$, $Q_3 = v_3v_4 = \frac{a_3}{4}$, $Q_1 = v_5v_6 = a_1$; $Q_4 = \frac{1}{2}(v_4+v_3-v_5-v_6)^2 + \frac{1}{2}(v_3-v_4-v_1-v_2)^2 - \frac{1}{4}(v_1-v_2-v_5+v_6)^2 = a_4$ with Q_1 and Q_3 as the Casimirs. The SO(4) system possess a group $\mathbb{Z}_2 \times \mathbb{Z}_2 =< \sigma, \tau >$ of translations by $\frac{1}{2}$-periods leaving invariant the equations and the vector fields whose action on the affine variables v_i's is given in the following

Table III

	v_1	v_2	v_3	v_4	v_5	v_6
σ	$-v_2$	$-v_1$	$-v_3$	$-v_4$	$-v_6$	$-v_5$
τ	$-v_1$	$-v_2$	v_4	v_3	v_6	v_5

This Algebraic Completely Integrable System linearizes on Jacobians of genus 2 curves and has an odd divisor $\mathcal{D}_1 = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3$ at infinity given by 4 translates of the theta divisor (Rosenhain divisor). The functions $\{1, k_1 = v_2v_6, k_2 = v_2v_3, k_3 = v_3v_6\}$ yield a basis of $H^0(A, 2\Theta)$ (see [7]). With the four invariants at hand one easily deduces the equation for the Kummer surface in $\mathbb{P}^3$ in terms of the homogeneous variables $\{k_0, k_1, k_2, k_3\}$

$$\begin{aligned} -(-a_1k_0k_2 + a_2k_0k_3 - k_1k_2 + k_1k_3)^2 + 2\left(-a_1k_0k_2 + \frac{a_3k_0k_1}{4} - k_1k_3 + k_2k_3\right)^2 \\ +2\left(a_2k_0k_3 + \frac{a_3k_0k_1}{4} + k_1k_2 - k_2k_3\right)^2 - 16a_4k_0k_1k_2k_3 = 0 \end{aligned} \tag{4.1}$$

We need to exhibit an expression for the branch locus of this Kummer surface in order to relate equation (4.1) to equation (3.7). First, we need to connect the Kummer equation (4.1) with the basis $\{s_0^2, s_1^2, s_2^2, s_3^2\}$ of $H^0(A, 2\Theta)$. The action of G_1 on the affine variables $\{V_1 = \frac{v_{02}}{Z_1}, V_2 = \frac{v_{13}}{Z_1}, V_3 = \frac{v_{23}}{Z_1}, V_4 = \frac{v_{01}}{Z_1}, V_5 = \frac{v_{03}}{Z_1}, V_6 = \frac{v_{12}}{Z_1}\}$ is the same as that given in Table III by substituting the v_i's with the V_i's (see [8] and [7]). Moreover $V_1V_2 = 1, V_3V_4 = 1, V_5V_6 = 1$, and also $V_2V_6 = \frac{s_1^2}{s_0^2}, V_2V_3 = \frac{s_3^2}{s_0^2}, V_3V_6 = \frac{s_2^2}{s_0^2}$. As shown in [7], the divisors of the functions $k_i, i = 1,2,3$ have precisely the form $(k_i) = 2\Theta_i - 2\Theta_0$. Thus, the v_i's are just a rescaling of the V_i's. We can pick k_0 to be cs_0^2 for some constant c. This is done by conveniently fixing an origin for the Kummer surface (3.2).

For the SO(4) system the four triple points of $\mathcal{D}_1$ go down to the points $\{p_0 = [1:0:0:0], p_1 = [0:1:0:0], p_2 = [0:0:1:0], p_3 = [0:0:0:1]\}$ in the Kummer surface (4.1). We will project from the point p_0 to the plane $\{k_0 = 0\}$ and obtain

Proposition 2. *The branch locus of the projection from $p_0 = [1:0:0:0]$ to the plane $\{k_0 = 0\}$ in the Kummer surface* (4.1) *of the SO(4) system is given by the zeroes of the linear forms in the sextic form*

$$\Delta = k_1k_2k_3 \prod_{i=1}^{3} \left(a_3\lambda_ik_1 + \frac{2a_1\lambda_i}{\lambda_i - 1}k_2 + 2a_2k_3\right) \tag{4.2}$$

where the roots λ_i*'s are obtained from the relations*

$$\begin{cases} a_1 = -a_3(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1), \\ a_2 = -a_3\lambda_1\lambda_2\lambda_3, \\ a_4 = -a_3(\lambda_1 - \frac{1}{2})(\lambda_2 - \frac{1}{2})(\lambda_3 - \frac{1}{2}). \end{cases} \tag{4.3}$$

Proof. Write equation (4.1) as $P_2k_0^2 + P_3k_0 + Q_2^2 = 0$, so that the branch locus will be a multiple of $P_3^2 - 4P_2Q_2^2 = \prod_{i=1}^{6}(\mu_{i1}k_1 + \mu_{i2}k_2 + \mu_{i3}k_3)$. We compute

$$\begin{aligned} P_3^2 - 4P_2Q_2^2 = 4k_1k_2k_3 \big(&-8a_1^2k_1k_2^2 + 8a_1^2k_1k_2k_3 + 8a_1^2k_2^3 - 8a_1^2k_2^2k_3 - 8a_1a_2k_1k_2^2 + 16a_1a_2k_1k_2k_3 \\ &- 8a_1a_2k_1k_3^2 + 6a_1a_3k_1^2k_2 - 2a_1a_3k_1^2k_3 - 6a_1a_3k_1k_2^2 - 2a_1a_3k_1k_2k_3 + 4a_1a_3k_2^2k_3 \\ &+ 16a_1a_4k_1k_2^2 - 48a_1a_4k_1k_2k_3 + 32a_1a_4k_2^2k_3 + 8a_2^2k_1k_2k_3 - 8a_2^2k_1k_3^2 - 8a_2^2k_2k_3^2 \\ &+ 8a_2^2k_3^3 + 2a_2a_3k_1^2k_2 - 6a_2a_3k_1^2k_3 + 2a_2a_3k_1k_2k_3 + 6a_2a_3k_1k_3^2 - 4a_2a_3k_2k_3^2 \\ &- 48a_2a_4k_1k_2k_3 + 16a_2a_4k_1k_3^2 + 32a_2a_4k_2k_3^2 - a_3^2k_1^3 + a_3^2k_1^2k_2 + a_3^2k_1^2k_3 - a_3^2k_1k_2k_3 \\ &-8a_3a_4k_1^2k_2 + 8a_3a_4k_1^2k_3 + 64a_4^2k_1k_2k_3\big) . \end{aligned} \tag{4.4}$$

Look at the quadratic polynomial in equation (4.1) when we put $k_0 = 1$ (i.e. when we do a Taylor expansion around the point p_0 in affine coordinates). This is the polynomial

$$\frac{1}{4}\left(4a_1^2k_2^2 + 8a_1a_2k_2k_3 - 4a_1a_3k_1k_2 + 4a_2^2k_3^2 + 4a_2a_3k_1k_3 + a_3^2k_1^2\right), \tag{4.5}$$

which already gives a double point at p_0; but when intersecting with the form $(\mu_{i1}k_1 + \mu_{i2}k_2 + \mu_{i3}k_3) = 0$, there has to be a double tangent because the intersection of one of these planes with the surface is a double conic. Assume $\mu_{i3} \neq 0$, then $k_3 = \mu_1k_1 + \mu_2k_2$ and by substituting this into (4.5) the condition of double tangency is met if $\mu_2 = -\frac{2a_1\mu_1}{a_3+2a_2\mu_1}$. If $\mu_1 = 0$ we get $k_3 = 0$. If $\mu_1 = \infty$ we get $k_1 = 0$, and $\mu_1 = -\frac{a_3}{2a_2}$ implies $k_2 = 0$. The remaining linear forms are proportional to $\left(-\mu_ik_1 + \frac{2a_1\mu_i}{a_3+2a_2\mu_i}k_2 + k_3\right)$. Therefore, after rescaling $\mu_i = -\frac{a_3}{2a_2}\lambda_i$, we get (4.2) up to a constant. Now, by comparing (4.4) with (4.2) we get the relations (4.3).

5. BRANCH LOCI

Here we will write the linear forms that appear in the factorization of the branch locus about the point P_0 for the Göpel equation (3.7) in its Rosenhain form. As a first approximation we try to factorize the polynomial $\boldsymbol{\Delta} = P_3(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})^2 - 4P_2(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})Q_2(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})^2$. Using the relations given in (3.6) we get (up to a constant c' depending on y_0, z_0, t_0) that $\boldsymbol{\Delta} = c'R_1R_2R_3$, where the R_i's are the quadrics

$$R_i(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}) = \boldsymbol{y}^2 + \boldsymbol{z}^2 + \boldsymbol{t}^2 + \varphi_{i1}\boldsymbol{yz} + \varphi_{i2}\boldsymbol{yt} + \varphi_{i3}\boldsymbol{zt}, \qquad i = 1, 2, 3,$$

and the coefficients φ_{ij} are given in the following list

$$\begin{cases} \varphi_{11} = \dfrac{t_0^4 - t_0^2y_0^2 - t_0^2z_0^2 - 2t_0^2 + 4t_0y_0z_0 - y_0^2 - z_0^2 + 1}{(t_0y_0 - z_0)(t_0z_0 - y_0)}, & \varphi_{12} = \varphi_{22} = -\dfrac{t_0^2 + y_0^2 - z_0^2 - 1}{t_0y_0 - z_0}, \\ \varphi_{23} = -\dfrac{-t_0^2y_0^2 - t_0^2 + 4t_0y_0z_0 + y_0^4 - y_0^2z_0^2 - 2y_0^2 - z_0^2 + 1}{(t_0y_0 - z_0)(t_0 - y_0z_0)}, & \varphi_{31} = \varphi_{21} = -\dfrac{t_0^2 - y_0^2 - z_0^2 + 1}{t_0 - y_0z_0}, \\ \varphi_{32} = -\dfrac{-t_0^2z_0^2 - t_0^2 + 4t_0y_0z_0 - y_0^2z_0^2 - y_0^2 + z_0^4 - 2z_0^2 + 1}{(t_0z_0 - y_0)(t_0 - y_0z_0)}, & \varphi_{13} = \varphi_{33} = -\dfrac{t_0^2 - y_0^2 + z_0^2 - 1}{t_0z_0 - y_0}. \end{cases} \tag{5.1}$$

These coefficients are related to the original p, q, r in the following way:

$$\begin{cases} \varphi_{11} = \varphi_{21} - \dfrac{\phi_0}{2y_0z_0}, & \varphi_{12} = \varphi_{22} = 2q, \\ \varphi_{23} = \varphi_{33} - \dfrac{\phi_0}{2z_0t_0}, & \varphi_{31} = \varphi_{21} = 2p, \\ \varphi_{32} = \varphi_{22} - \dfrac{\phi_0}{2y_0t_0}, & \varphi_{13} = \varphi_{33} = 2r. \end{cases}$$

In order to factorize $R_i(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}) = (\boldsymbol{y} + \beta_i \boldsymbol{z} + \gamma_i \boldsymbol{t})(\boldsymbol{y} + \beta_i' \boldsymbol{z} + \gamma_i' \boldsymbol{t})$, we get that β_i, β_i' must be the roots of the equation $\zeta^2 - \varphi_{i1}\zeta + 1 = 0$. Analogously, γ_i, γ_i' are the roots of $\zeta^2 - \varphi_{i2}\zeta + 1 = 0$, but the signs of these roots must be carefully chosen so that the relation $\beta_i\gamma_i' + \gamma_i\beta_i' = \varphi_{i3}$ holds. Call $r_{ij}^{\pm}$ the roots of the equation $\zeta^2 - \varphi_{ij}\zeta + 1 = 0$. We get the following

$$\begin{cases} r_{11}^{\pm} = \dfrac{1}{2}\left(\varphi_{11} \pm \dfrac{(t_0^2 - 1)\alpha}{(t_0y_0 - z_0)(t_0z_0 - y_0)}\right), & r_{22}^{\pm} = \dfrac{1}{2}\left(\varphi_{22} \pm \dfrac{\alpha}{t_0y_0 - z_0}\right), \\ r_{32}^{\pm} = \dfrac{1}{2}\left(\varphi_{32} \pm \dfrac{(z_0^2 - 1)\alpha}{(y_0 - t_0z_0)(y_0z_0 - t_0)}\right), & r_{21}^{\pm} = \dfrac{1}{2}\left(\varphi_{21} \pm \dfrac{\alpha}{t_0 - y_0z_0}\right), \end{cases} \tag{5.2}$$

where $\{\alpha^2 = (t_0 - z_0 - y_0 + 1)(t_0 + y_0 - z_0 - 1)(t_0 - y_0 + z_0 - 1)(t_0 + y_0 + z_0 + 1)\}$.

We state now the following

Proposition 3. *Let $P_0 = [1 : y_0 : z_0 : t_0]$ be a double point of the Kummer surface coming from one of the four $\frac{1}{2}$-period not sitting on a Göpel divisor $\mathcal{D}_0$. After translating to P_0 Göpel equation* (3.5)*, the projection from P_0 to the plane $\{\boldsymbol{x} = 0\}$ has a branch locus determined by the zeroes of the following sextic form*

$$\begin{aligned} \boldsymbol{\Delta} = &(\boldsymbol{y} + r_{11}^{+}\boldsymbol{z} + r_{22}^{-}\boldsymbol{t})(\boldsymbol{y} + r_{11}^{-}\boldsymbol{z} + r_{22}^{+}\boldsymbol{t})(\boldsymbol{y} + r_{21}^{+}\boldsymbol{z} + r_{22}^{+}\boldsymbol{t}) \\ &\times (\boldsymbol{y} + r_{21}^{-}\boldsymbol{z} + r_{22}^{-}\boldsymbol{t})(\boldsymbol{y} + r_{21}^{-}\boldsymbol{z} + r_{32}^{+}\boldsymbol{t})(\boldsymbol{y} + r_{21}^{+}\boldsymbol{z} + r_{32}^{-}\boldsymbol{t}), \end{aligned} \tag{5.3}$$

where the $r_{ij}^{\pm}$'s are in (5.2)

Proof. This follows from the considerations above. Since $r_{ij}^{+} + r_{ij}^{-} = \varphi_{ij}$, and $r_{ij}^{+}r_{ij}^{-} = 1$, in order to obtain a decomposition into linear factors for the quadratic polynomial $R_i(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$, we only need to check that the roots $\beta_i, \beta_i', \gamma_i, \gamma_i'$ satisfy the relation $\frac{\beta_i'}{\gamma_i'} + \frac{\beta_i}{\gamma_i} = \varphi_{i3}$. This tells how to pick the right roots for each linear factor. We check that $\frac{r_{11}^{+}}{r_{12}^{-}} + \frac{r_{11}^{-}}{r_{12}^{+}} = \varphi_{13}$; $\frac{r_{21}^{+}}{r_{22}^{+}} + \frac{r_{21}^{-}}{r_{22}^{-}} = \varphi_{23}$; and $\frac{r_{31}^{+}}{r_{32}^{+}} + \frac{r_{31}^{-}}{r_{32}^{-}} = \varphi_{33}$ by using (5.1) and (5.2). Thus, having decomposed into linear factors each R_i according to this prescription and using the equalities in (5.1) we obtain the stated decomposition for $\boldsymbol{\Delta}$.

6. MAPPING THE GÖPEL AND ROSENHAIN KUMMER SURFACES

The plan now is to match the branch loci of the Göpel and Rosenhain Kummer surfaces. As said before, the linear map in $\mathbb{P}^3$ has to take $P_0 \to p_0$ and $\boldsymbol{\Delta} \to \Delta$ so as to give the required isomorphism. We discussed in section 4 that $k_0 = cs_0^2 = cx = c\boldsymbol{x}$ (after choosing a different origin for the SO(4) system if necessary). Therefore one of the possible linear maps that take $P_0 \to p_0$ and $\boldsymbol{\Delta} \to \Delta$ is (in the affine chart $\{\boldsymbol{x} \neq 0\}$) the following

$$\begin{cases} k_1 = c_1(\boldsymbol{y} + r_{11}^{+}\boldsymbol{z} + r_{22}^{-}\boldsymbol{t}), \\ k_2 = c_2(\boldsymbol{y} + r_{21}^{-}\boldsymbol{z} + r_{22}^{-}\boldsymbol{t}), \\ k_3 = c_3(\boldsymbol{y} + r_{21}^{-}\boldsymbol{z} + r_{32}^{+}\boldsymbol{t}). \end{cases} \tag{6.1}$$

After a relabeling of the roots λ_i's if necessary, we need to check that the following relations hold

$$\begin{cases} \left(a_3\lambda_1 k_1 + \dfrac{2a_1\lambda_1}{\lambda_1 - 1}k_2 + 2a_2k_3\right) = d_1\left(\boldsymbol{y} + r_{11}^-\boldsymbol{z} + r_{22}^+\boldsymbol{t}\right), \\ \left(a_3\lambda_2 k_1 + \dfrac{2a_1\lambda_2}{\lambda_2 - 1}k_2 + 2a_2k_3\right) = d_2\left(\boldsymbol{y} + r_{21}^+\boldsymbol{z} + r_{22}^+\boldsymbol{t}\right), \\ \left(a_3\lambda_3 k_1 + \dfrac{2a_1\lambda_3}{\lambda_3 - 1}k_2 + 2a_2k_3\right) = d_3\left(\boldsymbol{y} + r_{21}^+\boldsymbol{z} + r_{32}^-\boldsymbol{t}\right). \end{cases} \tag{6.2}$$

This implies the d_i's are determined once the constants c_i's are given. Just substitute (6.1) into (6.2) and compare the coefficients of $\boldsymbol{y}$ on each side of the three equations (6.2). We obtain

$$d_i = a_3\lambda_i c_1 + \frac{2a_1\lambda_i}{\lambda_i - 1}c_2 + 2a_2c_3, \qquad \text{for } \ i = 1, 2, 3. \tag{6.3}$$

The remaining equations that appear by comparing the coefficients of $\boldsymbol{z}$ and $\boldsymbol{t}$ in (6.2) allow us to determine c_2, c_3 and the λ_i's in terms of y_0, z_0, t_0 and α. So the correspondence is written as follows

$$\begin{cases} c_3 = -\dfrac{c_2(\lambda_1 - 1)}{\lambda_1}, \\ c_1 = \dfrac{2c_2(\lambda_1 - 1)(r_{21}^+ - r_{21}^-)}{r_{11}^+ - r_{21}^+}, \\ \lambda_3 = -\dfrac{(r_{11}^+ - r_{21}^-)(r_{22}^+ - r_{22}^-)}{(r_{11}^+ - r_{21}^+)(r_{22}^- - r_{32}^+)}, \\ \lambda_2 = \dfrac{(r_{11}^+ - r_{21}^-)(r_{22}^- - r_{32}^-)}{(r_{11}^+ - r_{21}^+)(r_{22}^- - r_{32}^+)}, \\ \lambda_1 = \dfrac{(r_{21}^- - r_{11}^+)(r_{32}^+ - r_{22}^+)}{(r_{21}^+ - r_{21}^-)(r_{22}^- - r_{32}^+)}. \end{cases} \tag{6.4}$$

Here, we substituted (4.3) and (6.3) into (6.2). Then we factored and solved the equations. Finally we used (5.2) and the expression for α^2. Notice that there is one degree of freedom in the coefficients c_i's since we can pick any c_2. This is due to homogeneity of the branch locus. We have

Theorem 1. *The Göpel Kummer surface and the SO(4) Kummer surface are biregularly equivalent. A linear mapping in $\mathbb{P}^3$ taking one into the other in such a way that the point P_0 goes to p_0 is given by combining* (6.1) *and* (6.4) *with the translation that takes $P_0 \to p_0$. The constants of motion of the SO(4) system are obtained by substituting* (6.4) *into* (4.3)*. Thus, they depend on y_0, z_0, t_0 and the root α. By inverting this map and using the extension procedure of Proposition 1 we induce on the Göpel family a Poisson structure. Moreover, The Jacobians of the SO(4) and Göpel configuration are isomorphic.*

Proof. The proofs of the procedure have been described in [7] and the procedure is carried out as described above. Notice that the constant c such that $k_0 = \boldsymbol{cx} = cx$ is determined in terms of c_2, a_3 and other constants y_0, z_0, t_0, α. We will not make this relation precise here. If we are given a set a_i's of SO(4) constants of motion, we find the λ_i's by solving a cubic equation because by using (4.3) we get the symmetric functions in λ_i's. Obtaining the quantities y_0, z_0, t_0 by inverting (6.4) is much more difficult and has to be done locally. Once we get y_0, z_0, t_0 as functions of the a_i's and we have y, z, t (affine variables) as functions of $\{1, k_1 = v_2v_6, k_2 = v_2v_3, k_3 = v_3v_6\}$ (also affine variables of $\mathbb{P}^3$) then we use the SO(4) Poisson structure to translate the SO(4) vector fields to the Göpel family. In doing that we have to keep in mind that the constants a_i's have to be treated as functions of the entire phase space. Namely, computing $\{f, a_3\}$ we get 0 since Q_3 is a Casimir, while $\{f, a_2\}$ is differentiation under the nontrivial vector field generated by Q_2. In other words, we replace all the a_i's by the functions Q_i's of the SO(4) affine space. Let $'$ denote differentiation with regard to the SO(4) vector field. Now, with the given functions of $H^0(A, 2\Theta)$: $\{1, y, z, t\}$ we construct the embedding of the Jacobian in $\mathbb{P}^{15}$ via the functions of $H^0(A, 4\Theta) = \{1, y, z, t, y^2, yz, yt, z^2, zt, t^2\}^+ \oplus \{y', z', t', yz' - zy', yt' - ty', zt' - tz'\}^-$ which are split into even and odd functions. The isomorphism obtained from the SO(4) Kummer surface to the Göpel Kummer surface is then extended to a linear map in $\mathbb{P}^{15}$ [7] that yields an isomorphism of the respective Jacobians. Let Z_0 be the Göpel function of $H^0(A, 4\Theta)$. That is, if $s_0s_1s_2s_3$ is the Göpel section then $Z_0 = \frac{s_1s_2s_3}{s_0^3}$. We have $Z_0^2 = yzt$, then we will be able to compute the SO(4) Poisson bracket of any two of the functions in $H^0(A, Z_0) = \{\frac{1}{Z_0}, \frac{y}{Z_0}, \frac{z}{Z_0}, \frac{t}{Z_0}, \frac{y^2}{Z_0}, \frac{yz}{Z_0}, \frac{yt}{Z_0}, \frac{z^2}{Z_0}, \frac{zt}{Z_0}, \frac{t^2}{Z_0}\}^+ \oplus \{\frac{y'}{Z_0}, \frac{z'}{Z_0}, \frac{t'}{Z_0}, \frac{yz'-zy'}{Z_0}, \frac{yt'-ty'}{Z_0}, \frac{zt'-tz'}{Z_0}\}^-$. This leads to expressions that close up in terms of the affine variables v_i, $i = 1, \ldots, 6$ of the SO(4) system. However, using the isomorphism $H^0(A, Z_1) \cong H^0(A, Z_0)$ between the SO(4) and Göpel variables in $\mathbb{P}^{15}$ one gets the closure of the Göpel variables in $\mathbb{P}^{15}$.

7. CONCLUDING REMARKS

A more direct way would be desirable to get a Poisson structure and integrals for functions related to a Göpel configuration. In some cases this has been carried out successfully [8]. For instance one starts with the sections s_0', s_1', s_2', s_3' of a Göpel configuration so that $Z_0 = s_0's_1's_2's_3'$ gives the section at infinity. Then from Table II one gets the action of a Göpel group G_0 and this can be extended to the even functions $u_{ij} = \frac{(s_i')^2(s_j')^2}{Z_0}$, $0 \leqslant i \leqslant j \leqslant 3$; and the odd functions $w_{ij} = \frac{(s_i')^2X((s_j')^2)-(s_j')^2X((s_i')^2)}{Z_0}$, $0 \leqslant i < j \leqslant 3$ (X a generic global vector field) of $H^0(A, Z_0)$ just by tensoring the action. Then we ask whether there would be a subset of these functions that entitle as the phase space variables of an Integrable System with integrals and Poisson structure invariant under G_0 and $\mathcal{D}_0$ as divisor at infinity. One way is to look at the linear system $|\mathcal{D}_0 - \sum_{i=1}^{12} e_i|$ of functions vanishing at the 12 half periods of $\mathcal{D}_0$ or $|\mathcal{D}_0 - \sum_{i=1}^{12} e_i + \sum_{i=13}^{16} e_i|$; then exhibit a basis of functions for these systems, compute dimensions of the spaces of odd and even functions for linear, quadratic, cubic expressions (etc.) and find the invariants. Unfortunately, the first linear system is not symmetric and the second does not have fixed components. So they do not apply as candidates in the theory developed by Bauer [10] and Szemberg [11]. A new algebro-geometric theory is needed. Moreover, functions that are related to these linear systems are the Wronskians w_{ij}. However, relations for the Wronskians of genus 2 theta functions are very difficult to work out. Some of them are explained in the old books by Baker [12] and M. Krause [13].

The other functions on which we may try to define a phase space for the Göpel family are the 6 even functions u_{ij}, $0 \leqslant i < j \leqslant 3$. The advantage here is that with the relations $u_{ii} = u_{ij}u_{ik}u_{il}$, i, j, k, l all different, and Kummer equation (3.5) we already have four invariants: $u_{01}u_{23} = 1$, $u_{02}u_{13} = 1$, $u_{03}u_{12} = 1$, and $(u_{01}u_{02}u_{03} + u_{01}u_{12}u_{13} + u_{02}u_{12}u_{23} + u_{03}u_{13}u_{23} + 2p(u_{03} + u_{12}) + 2q(u_{02} + u_{13}) + 2r(u_{01} + u_{23}))^2 = 16s = 16(p^2 + q^2 + r^2 - 2pqr - 1)$. In a way this is how one obtains the SO(4) system from Rosenhain Kummer equation (3.2) by rescaling the theta functions and using a basis v_{ij}'s (3.3), which are the cousins of the u_{ij}'s because they are constructed similarly but for the Rosenhain sections s_0, s_1, s_2, s_3. The problem here is that Z_0 is also an even section and therefore the quadratic vector fields (in the variables of $\mathbb{P}^{15}$) have to be sums of products of an odd and an even function. In the variables of $H^0(A, 4\Theta)$ for genus 2 Jacobians the square of an odd function can be written as a quadratic polynomial of even functions, and because of the relations held by the u_{ii}'s, as a sextic polynomial in the u_{ij}, $i \neq j$. Thus, and this seems awkward, square roots of the u_{ij} would have to be introduced. These are odd sections and the (-1)-involution changes the sign of the roots. A Poisson matrix $J_{\text{Göpel}}$ that is equivariant with respect to G_0 and changes sign with regard to the (-1)-involution will have to contain some of these square roots in its entries, otherwise if only polynomial in the even variables u_{ij}'s we obtain the null matrix. The other approach is to extend the phase space with some odd functions and again it will be difficult to find Wronskian relations and new invariants, now with the addition of even functions. Both solutions seem workable, although kind of difficult may give an invariant way of expressing what was shown here and a new Algebraic Completely Integrable System whose Jacobians are isomorphic to those of the SO(4) metric II system. Although Rosenhain and Göpel divisors are linearly equivalent (and therefore follows the setting explained in this paper) we can state that the two families are in a way different. This is explained because the actions of the groups G_0 and G_1 are not preserved under the obtained isomorphism. They are related to two different choices of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group inside the Heisemberg group. In any case if the matrix $J_{\text{Göpel}}$ is found it may lead to a system quite different from the SO(4) metric II.

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 223–244. © Pleiades Publishing, Ltd., 2011.

Poisson Pencils, Algebraic Integrability, and Separation of Variables

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Received May 11, 2010; accepted October 7, 2010

Abstract—In this paper we review a recently introduced method for solving the Hamilton–Jacobi equations by the method of Separation of Variables. This method is based on the notion of pencil of Poisson brackets and on the bihamiltonian approach to integrable systems. We discuss how separability conditions can be intrinsically characterized within such a geometrical set-up, the definition of the separation coordinates being encompassed in the bihamiltonian structure itself. We finally discuss these constructions studying in details a particular example, based on a generalization of the classical Toda Lattice.

MSC2010 numbers: 14H70, 37J35, 37K10, 70H20

DOI: 10.1134/S156035471103004X

Keywords: Hamilton–Jacobi equations, bihamiltonian manifolds, separation of variables, generalized Toda lattices

Contents

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1. INTRODUCTION

The study of the separability of the Hamilton–Jacobi (HJ) equations associated with a given Hamiltonian function H is a very classical issue in Mechanics, dating back to the foundational works of Jacobi, Stäckel, Levi-Civita, and others. It has recently received a strong renewed interest thanks to its applications to the theory of integrable PDEs of KdV type (namely, the theory of finite gap integration) and to the theory of quantum integrable systems (see, e.g., [1, 2]).

As it is well known, the problem can be formulated as follows. Let (M, ω) be a $2n$ dimensional symplectic manifold, and let $(p_1, \dots, p_n, q_1, \dots, q_n) \equiv (\mathbf{p}, \mathbf{q})$ be canonical coordinates in $U \subset M$, i.e., $\omega_{|U} = \sum_{i=1}^{n} dp_i \wedge dq_i$. The (stationary) HJ equation reads

$$H(q_1, \dots, q_n; \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}) = E \,. \tag{1.1}$$

Definition 1. *A complete integral* $S(\mathbf{q}; \alpha_1, \dots, \alpha_n)$ *of the HJ equation is a solution of* (1.1)*, depending on* n *parameters* $(\alpha_1, \dots, \alpha_n)$ *such that* $Det \frac{\partial^2 S}{\partial q_i \partial \alpha_j} \neq 0$*. The Hamiltonian* H *is said to be* separable *in the coordinates* $(\mathbf{p}, \mathbf{q})$ *if the HJ equation admits an additively separated complete integral, that is, a complete integral of the form*

$$S(\mathbf{q}; \alpha_1, \dots, \alpha_n) = \sum_{i=1}^{n} S_i(q_i; \alpha_1, \dots, \alpha_n). \tag{1.2}$$

In this paper we will focus on an equivalent definition of separability, originally due to Jacobi and recently widely used by Sklyanin and his collaborators. Let us consider an *integrable* Hamiltonian H, that is, let us suppose that, along with $H = H_1$ we have further $(n-1)$ mutually commuting integrals of the motion $H_2, \dots H_n$, with $dH_1 \wedge \dots \wedge dH_n \neq 0$.

Definition 2. *An integrable system* $(H_1, \dots, H_n)$ *is separable in the coordinates* $(\mathbf{p}, \mathbf{q})$ *if there exist* n *non-trivial relations*

$$\Phi_i(q_i, p_i; H_1, \dots, H_n) = 0, \qquad i = 1, \dots n \,, \tag{1.3}$$

connecting single pairs (q_i, p_i) *of canonical coordinates with the* n *Hamiltonians* H_i*.*

This alternative definition is indeed a *constructive* approach to separability, since the knowledge of the separation relations (1.3) allows one to reduce the problem of finding a separated solution of HJ to quadratures. In fact, let us suppose that the relations $\Phi_i(q_i, p_i; H_1, \dots, H_n) = 0$, for $i = 1, \dots, n$, can be solved in terms of the p_i to get $p_i = p_i(q_i; H_1, \dots, H_n)$. Then one can define:

$$S(\mathbf{q}; \alpha_1, \dots, \alpha_n) = \sum_{i=1}^{n} \int^{q_i} p_i(q_i'; H_1, \dots, H_n)\big|_{H_i = \alpha_i} dq_i' \,. \tag{1.4}$$

This is by construction a separated solution of HJ; the fact that it is a complete integral is equivalent to the (already assumed) fact that the integrals of the motion depend non trivially on the momenta.

In intrinsic terms, one notices that the equations $H_i = \alpha_i$, for $i = 1, \dots, n$, define a foliation $\mathcal{F}$ of M. The leaves of $\mathcal{F}$ are nothing but the (generalized) tori of the Arnol'd–Liouville theorem. The foliation is *Lagrangian*, that is, the restriction of the two-form ω to $\mathcal{F}$ vanishes. Hence the restriction to $\mathcal{F}$ of the Liouville form $\theta = \sum_{i=1}^{n} p_i dq_i$ is (locally) exact. Indeed, the function S defined by (1.4) is a (local) potential for such restriction. What is non intrinsic, and singles out the separation coordinates $(\mathbf{p}, \mathbf{q})$, is that the separation relations (1.3), which are another set of defining equations for the foliation $\mathcal{F}$, have the very special property of containing a single pair of canonical coordinates at a time. The problem to find such a system of coordinates and relations is the core of the theory of SoV. In particular, a natural question arises:

Is it possible to formulate intrinsic condition(s) on the Hamiltonians $(H_1, \dots, H_n)$ *to* a priori *ensure separability in a (given) set of canonical coordinates?*

Actually, this is the main issue studied by both the "classical" Eisenhart–Benenti theory [3] of separability of natural systems defined on cotangent bundles to Riemannian manifolds (M, g), as well as the "modern" theory, mainly due to the St. Petersburg [2] and Montreal [4, 5] schools, of SoV for systems admitting a Lax representation. Even though a detailed survey of the huge mass of results in these fields is clearly outside the aim of this paper, we notice that both such general approaches require, generally speaking, the presence of *an additional structure* to solve the problem. Indeed, the Eisenhart–Benenti theory requires the existence of a conformal Killing tensor for the metric g (but we mention that, under additional hypotheses, such tensor can be computed starting from a given Hamiltonian function, see [6, 7] and references cited therein), while the Lax theory requires – in addition to the knowledge of a Lax representation with spectral parameter for the Hamiltonian system under study – the existence of an r-matrix structure for such a Lax representation. To be more precise, the existence of an r-matrix is needed to prove the involutivity of the integrals of motion given by the Lax representation, and, as shown in [8], the converse statement is also true. In this case too, we have to point out that, in some special circumstances, a multi-Hamiltonian structure can be provided by the Lax matrix (see, e.g., [9–11]).

The method we review in this paper has recently been exposed in the literature (see, e.g., [12–24]), and can be seen as a kind of bridge between the classical and the modern points of view, putting an emphasis on the geometrical aspects of the Hamiltonian theory. Its "additional" structure is simply the requirement of the existence, on the symplectic manifold (M, ω), of a *second* Hamiltonian structure, compatible with the one defined by ω. Namely, the bihamiltonian structure on M will allow us:

1. To encompass the definition of a special set of coordinates, to be called *Darboux–Nijenhuis (DN)* coordinates, within a well defined geometrical object.

2. To formulate intrinsic (i.e., tensorial) conditions for the separability of a Hamiltonian integrable system, in the DN coordinates associated with the bihamiltonian structure.

3. To give recipes to characterize, find and handle sets of DN coordinates.

A very important issue that is close to the separability problem is the notion of algebraically completely integrable Hamiltonian systems (see, e.g., [25]). In general, a (Hamiltonian) system is said to be algebraically integrable whenever its flow(s) linearize on the Jacobian variety of an algebraic curve (the spectral curve). The latter is usually recovered as the characteristic polynomial of the Lax matrix of the system (provided the latter is known/given), and the integration of the equation of motion reconducted to a Jacobi inversion problem.

It is fair to say that in the bihamiltonian setting we are herewith discussing we are, so far, not able to provide general criteria for the algebraic integrability of our systems. However, as we shall see below, we are in a position to make contact with the problem of algebraic integrability, at least in the (slightly different and weaker) setting of Veselov and Novikov [26], that can be summarized as follows.

Given a Hamiltonian systems one assumes that the phase space M fulfills the following properties:

a) M has the fibered structure

$$M \xrightarrow{S^k\Gamma} B, \tag{1.5}$$

where the base B is an n-dimensional manifold whose points b determine an algebraic curve $\Gamma(b)$, and the fiber is the k–th symmetric product of that curve. In more details, one requires that $\Gamma(b)$ be given as an m–sheeted covering $\Gamma(b) \xrightarrow{m} \mathbb{C}$ of the complex λ-plane, and that points of M can be parameterized via the curve $\Gamma(b)$, and a set of k points on it, that is, the coordinates $\lambda_1, \dots, \lambda_k$ of the projection on the λ-plane of a set of points on it, as well as discrete parameters ϵ_i that specify on which sheet of the covering the points live.

b) An Abelian differential $Q(\Gamma)$ on Γ (or possibly on a covering of Γ), smoothly depending on the point $b \in B$, is defined. It is furthermore required that, if $Q(\Gamma)$ is given by

$$Q(\Gamma) = Q(b;\lambda)d\lambda \tag{1.6}$$

according to the representation of Γ as a covering of the λ-plane, the closed two-form

$$\omega_Q = \sum_{i=1}^{k} dQ(b;\lambda_i) \wedge d\lambda_i \tag{1.7}$$

gives rise to a Poisson bracket, conveniently called *algebro-geometric* Poisson bracket, with λ_i and $\mu_i = Q(b;\lambda_i)$ playing the role of Darboux coordinates on the symplectic leaves of this bracket.

In such a case, it was proven in [26] that functions that depend only on the curve Γ — i.e., on the points of B — are in involution with respect to the Poisson bracket defined by (1.7), and these geometric data explicitly define action-angle variables for the corresponding Hamiltonian flows. The Veselov–Novikov picture was further generalized in more recent works of Krichever and Phong [27, 28], who showed (besides generalizing the construction to include also $1+1$ PDEs) how to construct out of algebro-geometric data some "universal" symplectic form on suitable moduli spaces, whose Darboux coordinates are, by construction, algebro-geometrical coordinates. Also, Sklyanin's interpretation [2] of the method of the poles of the Baker–Akhiezer function can be seen as a particularly efficient scheme of implementing the Veselov–Novikov axiomatic picture also for the case of quantum integrable systems, especially quantum spin chains.

In [29] we studied some relations between the bihamiltonian approach to SoV and the Veselov–Novikov description of algebraic integrability, especially within the example of the Volterra lattice. It might be useful to remark the following change of "perspective" that (*si parva licet*) qualifies the approach described in the above-mentioned paper of ours, as well as the present paper, with respect to the Novikov–Veselov (and Krichever–Phong) approach. We *start* from a bi-Hamiltonian structure to produce separation coordinates that eventually turn out to be associated with an algebro-geometric structure; on the other hand, it is fair to say that in the approach of the "Moscow school" the building block is the algebraic geometry of Riemann surfaces and moduli thereof, and Poisson structure(s) an output.

As far as the organization of this paper is concerned, in Section 2 we briefly introduce the notions of bihamiltonian geometry relevant for the subsequent sections. In particular, we discuss the notion of DN coordinates, as well as methods to find them. In Section 3 we present the main theorems of the bihamiltonian set-up for SoV, namely, the tensorial conditions ensuring separability of the HJ equations in DN coordinates. Section 4 is devoted to separable systems coming from bihamiltonian systems by means of a reduction along a suitable transversal distribution. Then we discuss our constructions in a specific example, whose separability, to the best of our knowledge, has not been considered in the literature yet. It is a generalization of the periodic Toda lattice with four sites. In Section 5 we recall its definition, and show how the "bihamiltonian recipe" for SoV can be applied to it. Although our constructions can be generalized to the generic N-site system, for the sake of concreteness and brevity we choose to consider the four-site system only, and sometimes rely on direct computations to prove some of its properties. In the last subsection we apply our geometrical scheme to study a specific reduction of this generalized Toda system, and to find integrals of the motion which are not encompassed in the Lax representation. This result can possibly be a suitable step towards an alternative approach to the so–called chopping method [30] for the full (non-periodic) Toda Lattice.

2. SOME ISSUES IN THE GEOMETRY OF BIHAMILTONIAN MANIFOLDS

We start this section recalling some well known facts in the theory of Poisson manifolds (see, e.g., [31]).

Definition 3. *A Poisson manifold $(M, \{\cdot,\cdot\})$ is a manifold endowed with a Poisson bracket, that is, a bilinear antisymmetric composition laws defined on the space $C^\infty(M)$ satisfying:*

1. *The Leibniz rule: $\{fg, h\} = f\{g, h\} + g\{f, h\}$;*
2. *The Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.*

A Poisson bracket (or Poisson structure) can be equivalently described with the corresponding Poisson tensor, i.e., with the application $P : T^*M \to TM$, smoothly varying with $m \in M$, defined by

$$\{f, g\} = \langle df, Pdg\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical pairing between T^*M and TM. In a given coordinate system $(x^1, \dots, x^n)$ on M, the Poisson tensor P associated with the Poisson bracket $\{\cdot,\cdot\}$ is represented as

$$P = \sum_{i,j=1}^{n} P^{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}, \qquad \text{with } P^{ij} = \{x^i, x^j\}.$$

The Jacobi identity is translated into a quadratic differential condition on the matrix $\left(P^{ij}\right)$, known as the vanishing of the Schouten bracket, which in local coordinates reads

$$\sum_{s=1}^{n}\left(P^{is}\frac{\partial P^{jk}}{\partial x^s} + P^{js}\frac{\partial P^{ki}}{\partial x^s} + P^{ks}\frac{\partial P^{ij}}{\partial x^s}\right) = 0, \qquad \forall\, i > j > k. \tag{2.1}$$

A function in $C^\infty(M)$ is said to be a Casimir function if its Poisson bracket with any other function on M vanishes, or, equivalently, if its differential lies in the kernel of P.

The local structure of a Poisson manifold is described in details in [1, 31, 32]. For our purposes, we just need to recall that (in the open subset of M where the rank $r = 2n$ of the Poisson tensor is maximal) M is foliated in symplectic leaves, that (locally) are the common level sets of k Casimir functions $C_1, \dots, C_k$ of P. The dimension of M is related with the integers n and k by $\dim M = k + 2n$.

Let us now come to the definition of bihamiltonian manifold.

Definition 4. *A manifold M is called a bihamiltonian manifold if it is endowed with* two *Poisson brackets $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}'$ such that, for any $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$ if M is complex), the linear combination*

$$\{f, g\}' - \lambda\{f, g\} \equiv \langle df, (P' - \lambda P)dg\rangle \tag{2.2}$$

defines a Poisson bracket. This property is known as the compatibility condition *between the two brackets.*

The expression (2.2) will be referred to as *pencil* of Poisson brackets, and the sum $P_\lambda = P' - \lambda P$ as pencil of Poisson tensors. The most "popular" property of bihamiltonian manifolds is contained in the following

Proposition 1. *Let f and f' be two functions on a bihamiltonian manifold M, which satisfy the characteristic condition $Pdf = P'df'$. Then the Poisson brackets $\{f, f'\}$ and $\{f, f'\}'$ vanish.*

Proof. It consists of a one-line computation. Let us consider, e.g., $\{f, f'\}$:

$$\{f, f'\} = \langle df, Pdf'\rangle = -\langle df', Pdf\rangle = -\langle df', P'df'\rangle = 0.$$

The vanishing of the other Poisson bracket is even easier. □

Definition 5. *A vector field X that can be written as $X = Pdf = P'df'$ is called a* bihamiltonian vector field.

Corollary 1. *Let f_i, with $i \in \mathbb{Z}$, be a sequence of functions satisfying*

$$Pdf_i = P'df_{i-1}. \tag{2.3}$$

Then $\{f_i, f_k\} = \{f_i, f_k\}' = 0$ for all $i, k \in \mathbb{Z}$.

Proof. Using twice equation (2.3) and the antisymmetry of the Poisson brackets we have

$$\{f_i, f_k\} = \langle df_i, Pdf_k\rangle = \langle df_i, P'df_{k-1}\rangle = -\langle df_{k-1}, P'df_i\rangle$$

$$= -\langle df_{k-1}, Pdf_{i+1}\rangle = \langle df_{i+1}, Pdf_{k-1}\rangle = \{f_{i+1}, f_{k-1}\}.$$

Supposing $k > i$ and iterating this procedure $(k - i)$ times, we get $\{f_i, f_k\} = \{f_k, f_i\}$, so that $\{f_i, f_k\} = 0$. The vanishing of $\{f_i, f_k\}'$ is an easy consequence. □

Using the same technique, we can prove

Amplification 1. *Let* $\{f_n\}_{n\geqslant 0}$ *and* $\{g_n\}_{n\geqslant 0}$ *be two sequences of functions satisfying*

$$Pdf_i = P'df_{i-1},\ Pdf_0 = 0; \qquad Pdg_i = P'dg_{i-1},\ Pdg_0 = 0. \tag{2.4}$$

Then, along with $\{f_n, f_m\} = \{f_n, f_m\}' = \{g_n, g_m\} = \{g_n, g_m\}' = 0$*, it holds*

$$\{f_n, g_m\} = \{f_n, g_m\}' = 0 \quad \forall\, n, m \geqslant 0.$$

The family of vector fields associated with a sequence of functions satisfying the recursion relations (2.3) are customarily said to form a Lenard–Magri sequence. Those sequences starting from the null vector field, as in Amplification 1, are pictorially called *anchored* Lenard–Magri sequences. Notice that anchored Lenard sequences can occur in bihamiltonian manifold where at least one of the Poisson brackets is non-symplectic (indeed, e.g., df_0 is a non-trivial element of the kernel of P). We can compactly express equations (2.4) relative, say, to the sequence f_i by considering the formal Laurent series $f(\lambda) = \sum_{i=0}^{\infty} f_i/\lambda^i$ and writing the equation

$$(P' - \lambda P)df(\lambda) = 0. \tag{2.5}$$

If, as it often happens in the applications, inside the family f_i we have an element f_n satisfying $P'df_n = 0$, we can form a *polynomial* Casimir of the pencil as

$$F(\lambda) = \lambda^n f_0 + \lambda^{n-1} f_1 + \cdots + f_n. \tag{2.6}$$

In analogy with the definition of Casimir of a Poisson bracket, Laurent series satisfying (2.5) are called Casimirs of the Poisson *pencil*. The reader should, however, bear in mind that while Casimir functions for a single Poisson bracket are, in a sense, uninteresting functions, Casimirs of a *pencil* of Poisson bracket compactly encode non-trivial dynamics and constants of the motion. More precisely, anchored Lenard sequences may give rise to families of integrable systems. Let us see how this happens in the case of a $(2n+1)$-dimensional manifold endowed with a rank-$2n$ pencil of Poisson tensors. Let us suppose that we have found a polynomial Casimir of the form (2.6), such that the $(n+1)$ functions $f_0, \ldots, f_n$ are independent. Let $\mathcal{S}_c$ be a generic symplectic leaf of P, corresponding to $f_0 = c$. The vector fields X_{f_i}, with $i = 1, \ldots, n$, are tangent to $\mathcal{S}_c$, are Hamiltonian on $\mathcal{S}_c$ (with respect to the symplectic form given by the restriction of P), and the restrictions of the functions $f_1, \ldots, f_n$ provide n commuting integrals for each of them. In general, it holds [33, 34]:

Proposition 2. *Let* (M, P, P') *be a bihamiltonian manifold of dimension* $d = 2n + k$*, and let* $dim(Ker(P' - \lambda P)) = k$ *for generic values of* λ*. Let us suppose that* $H^{(1)}(\lambda), \ldots, H^{(k)}(\lambda)$ *are* k *polynomial Casimirs of the pencil* P_λ *of the form*

$$H^{(a)}(\lambda) = \lambda^{n_a} H_0^{(a)} + \lambda^{n_a - 1} H_1^{(a)} + \cdots + H_{n_a}^{(a)},$$

such that the collection of differentials $\{dH_j^{(a)}\}_{j=0,\ldots n_a}^{a=1,\ldots,k}$ *defines a* $(k+n)$*-dimensional distribution in* T^*M*. Then the vector fields defined by the anchored sequences associated with the* $H^{(a)}$ *are integrable (in the Arnol'd–Liouville sense) on the generic symplectic leaves of* P*.*

2.1. Geometry of Regular Bihamiltonian Manifolds and Darboux–Nijenhuis Coordinates

An important class of bihamiltonian manifold occurs when an element of the Poisson pencil (which without loss of generality we will assume to be P) is everywhere *invertible*, i.e., the Poisson bracket $\{\cdot, \cdot\}$ associated with P is symplectic. The possibility of defining the inverse to one of the Poisson tensors leads us to introduce a fundamental object in the bihamiltonian theory of SoV: the Nijenhuis (or Hereditary, or Recursion) operator

$$N = P'\, P^{-1}, \tag{2.7}$$

together with its transpose $N^* = P^{-1}\, P'$. By definition, N (resp., N^*) is an endomorphism of the tangent bundle to M (resp., of the cotangent bundle). As a remarkable consequence of the compatibility between P and P', the Nijenhuis torsion of N, defined by its action on a pair of vector fields X, Y as

$$T(N)(X,Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]), \tag{2.8}$$

identically vanishes [35]. So, from the classical Frölicher-Nijenhuis theory, we know that its eigenspaces are *integrable* distributions. Such distributions will be the building blocks of the bihamiltonian set-up for SoV.

To explain this point, we have to make some remarks and a genericity assumption. It can be shown that, owing to the antisymmetry of the Poisson tensors defining N, the eigenspaces of N are pointwise even dimensional. Throughout this paper, we will assume that, for generic points $m \in M$, the operator N_m has the maximal number $n = \frac{1}{2}\dim M$ of different eigenvalues $\lambda_1, \ldots, \lambda_n$, so that the dimension of the eigenspace relative to any eigenvalue is 2. Otherwise stated, the characteristic polynomial of N is the square of its degree-n minimal polynomial $\Delta_N(\lambda)$, whose roots are pairwise distinct. We will call *regular* a bihamiltonian manifolds endowed with a Poisson pencil with at least one of the elements of the Poisson pencil invertible, and such that the eigenvalues of the associated Nijenhuis tensor are maximally distinct.

Theorem 1. *On a regular bihamiltonian manifold there exists a class of coordinates (y_i, x_i), to be called* Darboux–Nijenhuis (DN) *coordinates, satisfying the two properties:*

(Darboux) *They are canonical, that is, $\{x_i, y_j\} = \delta_{ij}$, $\{x_i, x_j\} = \{y_i, y_j\} = 0$.*

(Nijenhuis) *They diagonalize N, that is, $N = \sum_i \lambda_i \big(\dfrac{\partial}{\partial y_i} \otimes dy_i + \dfrac{\partial}{\partial x_i} \otimes dx_i\big)$.*

The proof of this theorem can be found in [33, 36]. Here we will sketch it and discuss its meaning. In words, the assertion states that DN coordinates are defined by the spectral properties of N, as follows. For all m in the open set U where the eigenvalues λ_i of N (which are the same as the eigenvalues of N^*) satisfy $\lambda_i \neq \lambda_j$ for $i \neq j$, the cotangent space $T^*_m M$ admits the decomposition

$$T^*_m M = \oplus_{i=1}^n \mathcal{D}_{m,\lambda_i}, \;\; \dim\, \mathcal{D}_{m,\lambda_i} = 2, \tag{2.9}$$

into eigenspaces of N^*. Thanks to the vanishing of the torsion of N, each eigenspace $\mathcal{D}_{m,\lambda_i}$ is locally generated by differentials of pairs of independent functions (f_i, g_i). This means that the pointwise decomposition (2.9) holds (in $U' \subset U$) as

$$T^*M_{|_{U'}} = \oplus_{i=1}^n \mathcal{D}_{\lambda_i},$$

where $\mathcal{D}_{\lambda_i}$ is spanned by df_i and dg_i, with $N^* df_i = \lambda_i df_i$ and $N^* dg_i = \lambda_i dg_i$.

Functions whose differential belong to *different* summands $\mathcal{D}_{\lambda_i}$ are in involution with respect to the Poisson brackets defined both by P and P'. Indeed, suppose that f_1 and f_2 satisfy $N^* df_1 = \lambda_1 df_1$ and $N^* df_2 = \lambda_2 df_2$, with $\lambda_1 \neq \lambda_2$. The relation $N^* = P^{-1}P'$ implies that $P' df_1 = \lambda_1 P df_1$ and $P' df_2 = \lambda_2 P df_2$. So,

$$\{f_1, f_2\}' = \begin{cases} \langle df_1, P' df_2 \rangle = \lambda_2 \langle df_1, P df_2 \rangle = \lambda_2 \{f_1, f_2\} \\ -\langle df_2, P' df_1 \rangle = -\lambda_1 \langle df_2, P df_1 \rangle = \lambda_1 \{f_1, f_2\} \end{cases}$$

whence the assertion. It is equally straightforward to realize that the only non vanishing Poisson brackets have the form

$$\{f_i, g_i\} = F_i(f_i, g_i), \quad \{f_i, g_i\}' = F'_i(f_i, g_i), \quad i = 1, \ldots, n.$$

This means that from the n pairs of functions (f_i, g_i) we can construct by quadratures a set of canonical coordinates satisfying the Nijenhuis property of Theorem 1. Thus the class of coordinates where to frame the bihamiltonian set-up for SoV admits a clearcut and simple geometrical description. Admittedly, in the general case the computation of DN coordinates requires the integration of the two-dimensional distributions $\mathcal{D}_{\lambda_i}$ associated with the eigenvalues λ_i of N^*. Fortunately enough, there are instances (that frequently occur in the applications) in which DN coordinates can be found in an easier way.

For an analysis of Darboux–Nijenhuis coordinates within the theory of multi-hamiltonian structure on loop algebras, see [15, 37, 38].

2.2. On Darboux–Nijenhuis Coordinates

In this subsection we will briefly discuss conditions and "recipes" to algebraically find and/or characterize sets of Darboux–Nijenhuis coordinates on regular bihamiltonian manifolds. A very simplifying instance occurs whenever the eigenvalues λ_i of N (that are, in general, functions of the point $m \in M$) are *functionally independent.* It holds (see, e.g., [39]):

Proposition 3. *Let us define $I_k = \frac{1}{2k} TrN^k$ for $k = 1, \dots, n$. In the open set U where $dI_1 \wedge \cdots \wedge dI_n \neq 0$ the eigenvalues λ_i , $i = 1, \dots, n$, are functionally independent, satisfy $N^* d\lambda_i = \lambda_i d\lambda_i$, and so may be used to construct a set of Darboux–Nijenhuis coordinates.*

Proof. We express the normalized traces I_k of the Nijenhuis tensor N in terms of its eigenvalues as $kI_k = \sum_{i=1}^n \lambda_i^k$. Hence $dI_k = \sum_{i=1}^n \lambda_i^{k-1} d\lambda_i$, that is, in matrix terms:

$$\begin{bmatrix} dI_1 \\ dI_2 \\ \vdots \\ dI_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} d\lambda_1 \\ d\lambda_2 \\ \vdots \\ d\lambda_n \end{bmatrix} \tag{2.10}$$

So we have

$$dI_1 \wedge \cdots \wedge dI_n = \left(\prod_{i \neq j} (\lambda_i - \lambda_j) \right) d\lambda_1 \wedge \cdots \wedge d\lambda_n,$$

i.e., on the open set where the traces of the powers of the Nijenhuis tensor are functionally independent, we have that the eigenvalues λ_i are different and functionally independent.

To proceed further we need to recall [35] that the normalized traces I_k of the powers of Nijenhuis operator satisfy the recursion relation

$$N^* dI_k = dI_{k+1}. \tag{2.11}$$

This can be proved as follows. At first one notices that (2.11) is equivalent to the relation

$$L_{NX}(I_k) = L_X(I_{k+1}) \qquad \text{for all vector field } X,$$

as it can be easily seen evaluating the equality (2.11) on a generic vector field X. Thanks to the Leibniz property of the Lie derivative and the cyclicity of the trace, we see that

$$L_{NX}(I_k) = \mathrm{Tr}(L_{NX}(N) \cdot N^{k-1}) \quad \text{and } L_X(I_{k+1}) = \mathrm{Tr}(L_X(N) \cdot N^k). \tag{2.12}$$

Since the vanishing of the Nijenhuis torsion of N implies that $L_{NX}(N) = N \cdot L_X(N)$, the validity of (2.11) is proved.

We now express the relations (2.11) in terms of the eigenvalues λ_i as

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} N^* d\lambda_1 - \lambda_1 d\lambda_1 \\ N^* d\lambda_2 - \lambda_2 d\lambda_2 \\ \vdots \\ N^* d\lambda_n - \lambda_n d\lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.13}$$

Since the Vandermond matrix in the left-hand side of this equation is, by assumption, invertible, we conclude that $N^* d\lambda_i = \lambda_i d\lambda_i$ for all $i = 1, \dots, n$. □

This proposition can be rephrased saying that "half of" the DN coordinates are algebraically provided by the Nijenhuis tensor itself. The remaining "half" $\mu_1, \dots, \mu_n$ can always be found by quadratures. Actually, there is a condition leading to the algebraic solution of this problem too. To elucidate this, the following two considerations are crucial.

The first argument goes as follows. Let us consider the distinguished functions I_k introduced in Proposition 3, and trade them for the coefficients p_i of the minimal polynomial

$$\Delta_N(\lambda) = \lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \cdots - p_n$$

of N. The functions p_k and I_k are related by the triangular Newton formulas

$$\begin{aligned} &I_1 = p_1; \quad I_2 = p_2 + \frac{1}{2}p_1^2; \quad I_3 = p_3 + p_2p_1 + \frac{1}{3}p_1^3; \\ &I_4 = p_4 + p_1p_3 + p_1^2p_2 + \frac{1}{2}p_2^2 + \frac{1}{4}p_1^4; \quad I_5 = p_5 + \ldots. \end{aligned} \tag{2.14}$$

As a consequence of the recursion relations (2.11), it can be easily shown that the p_i's satisfy the "Frobenius" recursion relations

$$N^*dp_i = dp_{i+1} + p_idp_1, \text{ with } p_{n+1} \equiv 0. \tag{2.15}$$

We can compactly write these relations as a single relation for the polynomial $\Delta_N(\lambda)$; indeed, a straightforward computation shows that they are equivalent to

$$N^*d\Delta_N(\lambda) = \lambda d\Delta_N(\lambda) + \Delta_N(\lambda)dp_1\,. \tag{2.16}$$

Actually, relations of this kind are very important for our purposes. Indeed, in [21] we proved the following

Proposition 4. *Let $\Phi(\lambda)$ be a smooth function defined on the manifold M, depending on an additional parameter λ. Suppose that there exists a one-form α_Φ such that*

$$N^*d\Phi(\lambda) = \lambda d\Phi(\lambda) + \Delta_N(\lambda)\alpha_\Phi\,. \tag{2.17}$$

Then, the n functions Φ_i obtained evaluating the "generating" function $\Phi(\lambda)$ for $\lambda = \lambda_i$, with $i = 1, \ldots, n$, are Nijenhuis functions, that is, they satisfy $N^*d\Phi_i = \lambda_i d\Phi_i$.

Definition 6. *We will call a generating function $\Phi(\lambda)$ satisfying equation* (2.17) *a* Nijenhuis functions generator.

Secondly, one remarks [21] that the $n(n-1)/2$ equations $\{\lambda_i, \mu_j\} = \delta_{ij}$ can be replaced with the requirement $N^*d\mu_j = \lambda_j d\mu_j$ and the n equations

$$\{\lambda_1 + \cdots + \lambda_n, \mu_j\} = 1, \qquad j = 1, \ldots, n,$$

that do not involve the individual coordinates λ_i but only their sum $\sum_{i=1}^n \lambda_i = I_1$. In terms of the Hamiltonian vector field $Y = -PdI_1 = \sum_i \frac{\partial}{\partial\mu_i}$, the condition we are looking for is

$$L_Y(\mu_i) = 1. \tag{2.18}$$

The relevance of Definition 6 in the search for DN coordinates stems from the fact that Nijenhuis functions generators form an algebra $\mathcal{N}(M)$, *which is closed* under the action of the vector field $Y = -PdI_1$. In this way, knowing a set of Nijenhuis functions generators, we can obtain further elements of the algebra $\mathcal{N}(M)$ by repeated applications of the vector field Y. Clearly, in such an extended algebra, the characteristic equation

$$L_Y(\Psi(\lambda)) = 1 + \Delta_N(\lambda)f_\Psi,$$

corresponding to (2.18), may be easier to solve, thus yielding the missing Darboux–Nijenhuis coordinates μ_i as $\mu_i = \Psi(\lambda_i)$. The following remark is very important in view of the relations with algebraic integrability.

Remark 1. Suppose $(H_1, \ldots, H_n)$ to be a separable system in the DN coordinates constructed above. Then the separation relations (1.3) do not depend on the pair (λ_i, μ_i), i.e., they collapse to the single relation

$$\Phi(\lambda, \mu; H_1, \ldots, H_n) = 0\,. \tag{2.19}$$

Indeed, $\mu_i = \Psi(\lambda_i)$, where the λ_i are the eigenvalues of N. Since the I_k are invariant with respect to the exchange $\lambda_i \leftrightarrow \lambda_j$, every function globally defined on M is invariant with respect to the exchange $(\lambda_i, \mu_i) \leftrightarrow (\lambda_j, \mu_j)$. This is in particular true for the Hamiltonians H_k, and the assertion about (2.19) follows.

In many cases, equation (2.19) defines an algebraic curve, possibly coinciding with the spectral curve associated with a Lax matrix for the Hamiltonian system at hand. We will see an instance of this situation in the example of Section 5. The application of this scheme to (a particular class of) Gaudin models have been spelled out in [23, 24].

3. SEPARABILITY CONDITIONS IN THE BIHAMILTONIAN SETTING

As we have briefly recalled in Section 2, on a bihamiltonian manifold one is usually led to consider bihamiltonian vector fields, that is, vector fields X admitting the twofold Hamiltonian representation $X = Pdf = P'dg$. Let us now suppose that (M, P, P') be a regular bihamiltonian manifold of dimension $2n$, and that we were able to construct, by means of the Lenard–Magri iteration procedure, a sequence of functions $H_1, H_2, \ldots$ satisfying $P'dH_i = PdH_{i+1}$. Let us also suppose that the first n of them be functionally independent. Then one easily shows that all the further Hamiltonians $H_{n+1}, \ldots$ are functionally dependent from the first n. (This follows from the fact that a regular Poisson manifold of dimension $2n$ cannot have more than n mutually commuting independent functions). This means that, if we consider the Hamiltonian H_{n+1}, there must be a relation of the form

$$\psi(H_1, \ldots, H_n; H_{n+1}) = 0, \quad \text{with } \psi_{H_{n+1}} \equiv \frac{\partial \psi}{\partial H_{n+1}} \neq 0, \tag{3.1}$$

relating it with the independent Hamiltonians H_i, with $i = 1, \ldots, n$.

Actually, the case of $H_i = I_i \equiv \frac{1}{2i}\mathrm{Tr} N^i$ is an instance of this situation. In fact, since by the Cayley–Hamilton theorem N annihilates its minimal polynomial, we have $N^n - \sum_{i=1}^{n} p_i N^{n-i} = 0$, yielding the relation

$$2(n+1)I_{n+1} - \sum_{i=1}^{n} 2(n-i+1)p_i I_{n-i+1} = 0.$$

Differentiating equation (3.1) we see that, along with $P'dH_i = PdH_{i+1}$, for $i = 1, \ldots, n-1$, it holds:

$$P'dH_n = PdH_{n+1} = -\frac{1}{\psi_{H_{n+1}}} \sum_{i=1}^{n} \frac{\partial \psi}{\partial H_i} PdH_i, \tag{3.2}$$

that is, the vector field $X_{n+1} = PdH_{n+1} = P'dH_n$ is a linear combination of the vector fields $X_1 = PdH_1, \ldots, X_n = PdH_n$.

This innocent looking observation is the clue for the bihamiltonian theory of SoV. Indeed, let $\{H_1, H_2, \ldots H_n\}$ be any integrable system on M, that is, suppose that the H_i are mutually commuting (with respect to P) independent functions. We can construct an n-dimensional distribution, namely the distribution $\mathcal{D}_H$ spanned by the n mutually commuting vector fields $X_i = PdH_i$. This is nothing but the very classical tangent distribution to the invariant (generalized) tori of the Liouville Arnol'd theory of integrable systems. Since M comes equipped with a second Poisson tensor P', we can as well consider the distribution $\mathcal{D}'_H$ generated by the Hamiltonians H_i under the action of P', that is, generated by the vector fields $X'_i = P'dH_i$. It holds:

Theorem 2. *Let $\{H_1, \ldots, H_n\}$ define, as explained above, an integrable system on a regular bihamiltonian manifold (M, P, P'). The Hamilton–Jacobi equations associated with any of the Hamiltonians H_i are separable in the DN coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ defined by $N = P'P^{-1}$ if and only if the distribution $\mathcal{D}'_H$ is contained in $\mathcal{D}_H$, or, equivalently, if and only if the distribution $\mathcal{D}_H$ is invariant along N.*

Proof. We will first prove the equivalence between the invariance of $\mathcal{D}_H$ under N and the inclusion $\mathcal{D}'_H \subset \mathcal{D}_H$. To say that $\mathcal{D}'_H$ is contained in $\mathcal{D}_H$ is tantamount to saying that there exists a matrix F_{ij}, whose entries are, in general, functions defined on M, such that

$$X'_i \equiv P'dH_i = \sum_j F_{ij} P dH_j = \sum_j F_{ij} X_j \qquad \text{for } i = 1, \dots, n. \tag{3.3}$$

Writing $P' = NP$, we can translate these equalities into $NX_i = \sum_j F_{ij} X_j$ for all $i = 1, \dots, n$.

The full proof of the fact that the invariance of $\mathcal{D}_H$ insures separability in DN coordinates can be found in [21]. It goes as follows.

At first we notice that the translation in terms of the codistribution $\mathcal{D}^*_H$ generated by the differentials of the Hamiltonians H_i of the invariance condition for $\mathcal{D}_H$ is the invariance condition $N^*\mathcal{D}^*_H \subset \mathcal{D}^*_H$. This can be easily seen applying to (3.3) the operator P^{-1}, to get $N^* dH_i = \sum_j F_{ij} dH_j$.

Since all the Poisson brackets $\{H_i, H_j\}$ vanish and M is a regular bihamiltonian manifold, the matrix F defined by (3.3) can be shown to have simple eigenvalues, that coincide with the eigenvalues λ_i of N. So there exists a matrix S satisfying $SF = \Lambda S$, where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ If we introduce the n one-forms $\theta_i = \sum_j S_{ij} dH_j$, we get

$$N^*\theta_i = \sum_j S_{ij} N^* dH_j = \sum_{j,k} S_{ij} F_{jk} dH_k = \sum_{j,k} \lambda_i \delta_{ij} S_{jk} dH_k = \lambda_i \theta_i, \tag{3.4}$$

meaning that θ_i is an eigenvector of N^* relative to λ_i. Hence there must exist functions F_i, G_i such that

$$\sum_j S_{ij} dH_j = F_i dx_i + G_i dy_i, \tag{3.5}$$

whence the existence of a separation relation $\Phi_i(x_i, y_i; H_1, \dots, H_n)$ for all $i = 1, \dots, n$. The converse statement can be trivially proved. □

We would like to stress that the separability condition of Theorem 2 is a *tensorial one*. That is, given a regular bihamiltonian manifold (M, P, P') this separability criterion can be checked in *any system of coordinates*, without the *a priori* calculation of the DN coordinates themselves. Notice, also, that the validity of the statement does not (as it should be!) depend on the choice of mutually commuting integrals $\{H_1, \dots, H_n\}$. That is, if we consider a "change of coordinates in the space of the actions", that is, we trade the H_i's for another complete set of integrals of the motion $K_i = K_i(H_1, \dots, H_n)$, then the separability of the new Hamiltonians K_i will hold if and only if the separability of the original ones holds. Indeed, the dual distributions generated by the H_i's and the K_i's coincide.

A second remark is important and deserves to be explicitly spelled out. Although we have started our discussion considering the case of a family of bihamiltonian vector fields, that is, the case of Lenard–Magri sequences, the hypotheses of Theorem 2 concern only the relations of the distributions generated respectively under the action of P and P' by the Hamiltonians H_i, *without any mention* of the fact that the generators of the distribution be bihamiltonian vector fields. Thus, although it might seem a somewhat odd statement, *the vector fields that are separable by means of the bihamiltonian approach are not necessarily bihamiltonian vector fields!* It is also important to notice that it is not only a matter of choice of generators. Indeed, in [40] it has been shown that the only bihamiltonian vector fields on a regular bihamiltonian manifold turn out to be associated with *separated functions of the eigenvalues of* N, i.e., functions of the form $H = \sum_{i=1}^n f_i(\lambda_i)$. This means that, in such a case, the distribution $\mathcal{D}_H$ coincides with that generated by the distinguished functions I_i. However, this is by far a very special example, that is, the range of applicability of the method is much wider than that, as it has already been shown in the literature.

The separation condition of Theorem 2 is based on the analysis of the behaviour of the characteristic distribution associated with an integrable system under the Nijenhuis tensor N. An equivalent criterion, based on the analysis of the Poisson brackets associated with the tensor P', can be formulated as follows.

Theorem 3. *Let $\{H_1, H_2, \dots H_n\}$ be an integrable system defined on a regular bihamiltonian manifold (M, P, P'). The Hamiltonians H_i are separable in the DN coordinates defined by $N = P'P^{-1}$ if and only if, along with the commutation relations $\{H_i, H_j\} = 0$, there also hold*

$$\{H_i, H_j\}' \equiv \langle dH_i, P'dH_j\rangle = 0, \quad \text{for } i, j = 1, \dots, n. \tag{3.6}$$

Proof. The key formula is the relation between P, P' and N^*. Indeed, suppose that $\mathcal{D}_H^*$ be invariant along N^*. Then:

$$\{H_i, H_j\}' = \langle dH_i, P'dH_j\rangle = \langle dH_i, NPdH_j\rangle = \langle N^*dH_i, PdH_j\rangle$$

$$= \sum_k F_{ik}\langle dH_k, PdH_j\rangle = \sum_k F_{ik}\{H_k, H_i\} = 0,$$

which, in view of Theorem 2, proves the statement in one direction. Now, let us suppose that (3.6) holds. Then, for every $i, j = 1, \dots, n$, we have:

$$0 = \{H_i, H_j\}' = \langle dH_i, P'dH_j\rangle = \langle dH_i, NPdH_j\rangle = \langle N^*dH_i, PdH_j\rangle,$$

meaning that, for all $i = 1, \dots, n$, the one-form N^*dH_i belongs to the annihilator (with respect to P) of the distribution $\mathcal{D}_H$. Since such an annihilator coincides with $\mathcal{D}_H^*$, this means that $N^*dH_i \in \mathcal{D}_H^*$ for all $i = 1, \dots, n$. □

This results lead to the following, (somewhat daring), comparison. The Liouville–Arnol'd theorem on finite dimensional integrable Hamiltonian systems says that the geometrical structure underlying integrability of a Hamiltonian vector field defined on a symplectic manifold (M, ω) is a *Lagrangian* foliation of M. We can rephrase the content of Theorem 3 saying that the geometrical structure underlying the separability of a system defined on a regular bihamiltonian manifold (M, P, P') is a *bilagrangian* foliation of M.

We end our presentation of the bihamiltonian set-up for SoV with the following remark. Theorem 2 concerns only the existence of the separation relations. In principle, one could try to find these relations in concrete examples by actually diagonalizing the matrix F, and explicitly finding and integrating the relations (3.5). However, there is a very simple tensorial criterion which can be used to determine the *functional form* of the separation relations $\Phi_i(x_i, y_i; H_1, \dots, H_n)$, whose proof can be found in [21].

Proposition 5. *Let $\{H_1, \dots, H_n\}$ be an integrable system defined on a regular bihamiltonian manifold, which is separable in the Darboux–Nijenhuis coordinates associated with $N = P^{-1}P'$. Consider the matrix F_{ij} fulfilling the relations* (3.3)*. Then the separation relations are* affine *in the Hamiltonians H_i, that is, of the form*

$$\Phi_i(x_i, y_i; H_1, \dots, H_n) = \sum_j S_{ij}(x_i, y_i)H_j + U_i(x_i, y_i), \tag{3.7}$$

if and only if the matrix F satisfies the relation $N^*dF_{ij} = \sum_k F_{ik}dF_{kj}$.

The matrix S on (3.7) can be shown to be a suitably normalized matrix of eigenvectors of the matrix F. Its characteristic property is that, as expressed in the equation, the entries S_{ij} of the i–th row depend only on the pair (x_i, y_i) of Darboux–Nijenhuis coordinates. For this reason it can be called a Stäckel matrix.

4. TRANSVERSAL DISTRIBUTIONS AND SEPARATION RELATIONS

A very natural source of integrable systems fulfilling the separability conditions given in Theorems 2 and 3 is described in [21]. In this short section we recall this construction, and we comment on the resulting separation relations, with a particular emphasis on the relations with algebraic integrability.

Suppose that the hypotheses of Proposition 2 hold, and that there exists a k-dimensional foliation $\mathcal{Z}$ on M, spanned by the vector fields $Z_1, \dots, Z_k$, with the following properties:

1. It is transversal to the symplectic foliation of P; more precisely, the vector fields Z_a are normalized in such a way that $Z_a(H_0^{(b)}) = \delta_a^b$;

2. The Z_a are symmetries of P:

$$L_{Z_a}(P) = 0 \; ;$$

3. There exist vector fields Y_a^b such that

$$L_{Z_a}P' = \sum_b Y_a^b \wedge Z_b.$$

It turns out that the Z_a commute, and that $Y_a^b = P\, d(Z_a(H_1^{(b)}))$. But the important point is that any symplectic leaf $\mathcal{S}$ of P can be seen as a quotient space and inherits a (quotient) bi-Hamiltonian structure from M. Moreover, the reduction of P coincides with the symplectic form of $\mathcal{S}$, and therefore $\mathcal{S}$ is a regular bihamiltonian manifold (if the eigenvalues of the associated Nijenhuis tensor are maximally distinct).

Now, it can be shown that the integrable system described in Proposition 2 is separable in the DN coordinates on $\mathcal{S}$. As far as the Stäckel separability is concerned, a necessary and sufficient condition is that $Z_b(Z_c(H_j^{(a)})) = 0$ on $\mathcal{S}$, for all $a, b, c = 1, \ldots, k$ and for all $j = 1, \ldots, n_a$.

The search for DN coordinates is made easier by the fact that the determinant of the matrix

$$\mathcal{G}(\lambda) = \begin{pmatrix} Z_1(H^{(1)}(\lambda)) & \cdots & Z_k(H^{(1)}(\lambda)) \\ \vdots & & \vdots \\ Z_1(H^{(k)}(\lambda)) & \cdots & Z_k(H^{(k)}(\lambda)) \end{pmatrix} \tag{4.1}$$

coincides on $\mathcal{S}$ with the minimal polynomial of the recursion operator N. Thus the coordinates λ_i are the solutions of $\mathcal{G}(\lambda) = 0$. To find the μ_i, one can use the results of Subsection 2.2 and the following proposition, whose proof is given in [29].

Proposition 6. *In the above setting, let us consider a generating function $\Gamma(\lambda, \mu)$ of the Casimirs $H^{(a)}(\lambda)$ of the Poisson pencil, and let us suppose that $\Gamma(\lambda, \mu) = 0$ defines a smooth algebraic curve. Suppose that f is a $\mathcal{Z}$-invariant root of the minimal polynomial of N, i.e.,*

$$N^* df = f df, \quad \text{and } Z_i(f) = 0, \; i = 1, \ldots, k, \tag{4.2}$$

and suppose that $\Gamma(\mu, f) = 0$ defines generic point(s) of the affine curve $\Gamma(\lambda, \mu) = 0$. Then, any solution g of the equation $\Gamma(g, f) = 0$ which is invariant as well under $\mathcal{Z}$ satisfies $N^* dg = f dg$.

5. EXAMPLE: A GENERALIZED TODA LATTICE

In this final section we will apply the general scheme outlined in the previous sections to a specific model, with the aim of showing how the recipes discussed so far from a theoretical standpoint can be concretely applied. We will study a generalization of the four site Toda lattice, to be termed *$Toda_3^4$ model*. This system is a member of a family introduced in [41] as reductions of the discrete KP hierarchy, whose continuous limits are further discussed in [42]. It can be described as follows.

We consider on $M = \mathbb{C}^{12}$, endowed with global coordinates $\{b_i, a_i, c_i\}_{i=1,2,3,4}$, the Hamiltonian

$$H_{GT} = \frac{1}{2}(b_1^2 + b_2^2 + b_3^2 + b_4^2) - (a_1 + a_2 + a_3 + a_4), \tag{5.1}$$

and the linear Poisson tensor given by the matrix

$$P = \begin{bmatrix} \mathbf{0} & \mathbf{A_1} & \mathbf{C_1} \\ -\mathbf{A_1^T} & \mathbf{C_2} & \mathbf{0} \\ -\mathbf{C_1^T} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{where } \mathbf{A_1} = \begin{bmatrix} -a_1 & 0 & 0 & a_4 \\ a_1 & -a_2 & 0 & 0 \\ 0 & a_2 & -a_3 & 0 \\ 0 & 0 & a_3 & -a_4 \end{bmatrix}, \tag{5.2}$$

$$\mathbf{C_1} = \begin{bmatrix} -c_1 & 0 & c_3 & 0 \\ 0 & -c_2 & 0 & c_4 \\ c_1 & 0 & -c_3 & 0 \\ 0 & c_2 & 0 & -c_4 \end{bmatrix}, \quad \mathbf{C_2} = \begin{bmatrix} 0 & -c_1 & 0 & c_4 \\ c_1 & 0 & -c_2 & 0 \\ 0 & c_2 & 0 & -c_3 \\ -c_4 & 0 & c_3 & 0 \end{bmatrix},$$

and we denoted by $\mathbf{0}$ the 4×4 matrix with vanishing entries. Using (here and in the sequel) the cyclic identifications $a_{i+4} = a_i$, $b_{i+4} = b_i$, and $c_{i+4} = c_i$, the Hamiltonian vector field $X_{H_{GT}} = PdH_{GT}$ can be written as

$$\begin{bmatrix} \dot{b}_i \\ \dot{a}_i \\ \dot{c}_i \end{bmatrix} = \begin{bmatrix} a_{i-1} - a_i \\ a_i(b_{i+1} - b_i) + c_{i-1} - c_i \\ c_i(b_{i-2} - b_i) \end{bmatrix}, \quad i = 1, \dots, 4. \tag{5.3}$$

The expert reader surely noticed that H_{GT} coincides with the Hamiltonian of the periodic four-site Toda lattice, written in the Flaschka coordinates $b_i = p_i$, $a_i = \exp(q_i - q_{i+1})$. Indeed, on the hyperplane $M_T \simeq \mathbb{C}^8$ defined by $c_i = 0$ for $i = 1, \dots, 4$, the vector field $X_{H_{GT}}$ defines the periodic Toda flow.

Proposition 7. *The Hamiltonian vector field $X_{H_{GT}}$ admits the Lax representation $\dot{L}(\nu) = [L(\nu), \Phi]$, where*

$$L(\nu) = \begin{bmatrix} b_1 & -\nu & \frac{c_3}{\nu^2} & \frac{a_4}{\nu} \\ \frac{a_1}{\nu} & b_2 & -\nu & \frac{c_4}{\nu^2} \\ \frac{c_1}{\nu^2} & \frac{a_2}{\nu} & b_3 & -\nu \\ -\nu & \frac{c_2}{\nu^2} & \frac{a_3}{\nu} & b_4 \end{bmatrix}, \qquad \Phi = \begin{bmatrix} 0 & 0 & \frac{c_3}{\nu^2} & \frac{a_4}{\nu} \\ \frac{a_1}{\nu} & 0 & 0 & \frac{c_4}{\nu^2} \\ \frac{c_1}{\nu^2} & \frac{a_2}{\nu} & 0 & 0 \\ 0 & \frac{c_2}{\nu^2} & \frac{a_3}{\nu} & 0 \end{bmatrix}. \tag{5.4}$$

The bihamiltonian aspects of this system have been discussed in [43] (see also [44]). In particular, it has been noticed that on M there exists a second Hamiltonian structure for the vector field $X_{H_{GT}}$. Namely, one considers the bivector P' having the following form:

$$P' = \begin{bmatrix} \mathbf{A_2} & \mathbf{B_1} & \mathbf{C_3} \\ -\mathbf{B_1^T} & \mathbf{A_3} & \mathbf{C_4} \\ -\mathbf{C_3^T} & -\mathbf{C_4^T} & \mathbf{A_4} \end{bmatrix}, \text{ where } \mathbf{C_3} = \begin{bmatrix} -b_1c_1 & 0 & b_1c_3 & 0 \\ 0 & -b_2c_2 & 0 & b_2c_4 \\ c_1b_3 & 0 & -b_3c_3 & 0 \\ 0 & c_2b_4 & 0 & -b_4c_4 \end{bmatrix}, \tag{5.5}$$

$$\mathbf{A_2} = \begin{bmatrix} 0 & a_1 & 0 & -a_4 \\ -a_1 & 0 & a_2 & 0 \\ 0 & -a_2 & 0 & a_3 \\ a_4 & 0 & -a_3 & 0 \end{bmatrix}, \quad \mathbf{B_1} = \begin{bmatrix} -b_1a_1 & c_1 & -c_3 & b_1a_4 \\ b_2a_1 & -b_2a_2 & c_2 & -c_4 \\ -c_1 & b_3a_2 & -b_3a_3 & c_3 \\ c_4 & -c_2 & b_4a_3 & -b_4a_4 \end{bmatrix},$$

$$\mathbf{A_3} = \begin{bmatrix} 0 & -b_2c_1 - a_1a_2 & 0 & b_1c_4 + a_1a_4 \\ b_2c_1 + a_1a_2 & 0 & -b_3c_2 - a_2a_3 & 0 \\ 0 & b_3c_2 + a_2a_3 & 0 & -b_4c_3 - a_3a_4 \\ -b_1c_4 - a_1a_4 & 0 & b_4c_3 + a_3a_4 & 0 \end{bmatrix},$$

$$\mathbf{C_4} = \begin{bmatrix} -a_1c_1 & -a_1c_2 & a_1c_3 & a_1c_4 \\ c_1a_2 & -a_2c_2 & -a_2c_3 & a_2c_4 \\ c_1a_3 & c_2a_3 & -a_3c_3 & -a_3c_4 \\ -c_1a_4 & c_2a_4 & c_3a_4 & -a_4c_4 \end{bmatrix}, \quad \mathbf{A_4} = \begin{bmatrix} 0 & -c_1c_2 & 0 & c_1c_4 \\ c_1c_2 & 0 & -c_2c_3 & 0 \\ 0 & c_2c_3 & 0 & -c_3c_4 \\ -c_1c_4 & 0 & c_3c_4 & 0 \end{bmatrix}.$$

It can be easily checked that $X_{H_{GT}} = P'd(-\sum_{i=1}^4 b_i)$. More in general, we have the following

Proposition 8. *The pencil $P' - \lambda P$ is a pencil of Poisson brackets. The rank of the generic element of the pencil is eight. The characteristic polynomial $Det\big(\lambda\mathbf{1} - L(\nu)\big)$ can be expanded as:*

$$Det\big(\lambda\mathbf{1} - L(\nu)\big) = \lambda^4 - \nu^4 + H(\lambda) + (K(\lambda) - \lambda^2 J_1)/\nu^4 + J_2/\nu^8. \tag{5.6}$$

The functions J_1 and J_2 are common Casimirs of P and P'. The polynomials $H(\lambda)$ and $K(\lambda)$ are Casimirs of the pencil $P_\lambda = P' - \lambda P$. They have the form

$$H(\lambda) = \lambda^3 H_0 - \lambda^2 H_1 + \lambda H_2 - H_3, \qquad K(\lambda) = K_0\lambda + K_1. \tag{5.7}$$

Explicitly, $J_1 = c_1c_3 + c_2c_4$ and $J_2 = c_1c_2c_3c_4$, while the coefficients of $H(\lambda)$ and $K(\lambda)$ are given by:

$$H_0 = \sum_{i=1}^4 b_i,\ H_1 = \sum_{i>j=1}^4 b_ib_j + \sum_{i=1}^4 a_i,\ H_2 = \sum_{i=1}^4 \big(c_i + b_i(a_{i+1} + a_{1+2}) + b_ib_{i+1}b_{i+2}\big)$$

$$H_3 = \sum_{i=1}^4 b_ic_{i+1} + a_1a_3 + a_2a_4 + \text{cubic and quartic terms;}$$

$$K_0 = \sum_{i=1}^4 (b_ic_{i-1}c_{i+1} - c_ia_{i-1}a_{i+2}),\ K_1 = \sum_{i=1}^4 a_ic_{i+1}c_{i+2} + \text{quartic terms}$$

One can show via a direct computation that the eight functions $H_0, H_1, H_2, H_3, K_0, K_1, J_1, J_2$ are functionally independent and, thanks to the fact that they fill in Lenard sequences, are mutually in involution. The kernel of P is generated (at generic points $m \in M$) by the differentials of the four functions H_0, K_0, J_1, J_2. Hence, on the 8-dimensional manifold $\mathcal{S}_{\boldsymbol{\kappa}}$ defined by the equations $H_0 = \kappa_1, K_0 = \kappa_2, J_1 = \kappa_3, J_2 = \kappa_4$, that is, the generic symplectic leaf of P, the vector field $X_{H_{GT}}$ is completely integrable. To realize this we simply have to notice that H_{GT} can be expressed as $\frac{1}{2}H_0^2 - H_1$, and apply the properties of anchored Lenard–Magri sequences collected in Proposition 2.

5.1. Separation of Variables

We will now show how to apply the ideas and recipes of the bihamiltonian set-up for SoV to the Toda_3^4 model introduced above. The first problem to deal with is that the Poisson tensor P' does not restrict to $\mathcal{S}_{\boldsymbol{\kappa}}$, but must be projected according to the procedure outlined in Section 4. This can be rephrased as follows, by means of a kind of Dirac reduction process (see [21, 37, 45, 46] for details and the geometric background).

We consider the vector fields $Z_1 = -\dfrac{\partial}{\partial b_4}$ and $Z_2 = \dfrac{\partial}{\partial a_4}$, and we notice that the matrix

$$G = \begin{pmatrix} L_{Z_1}(H_0) & L_{Z_2}(H_0) \\ L_{Z_1}(K_0) & L_{Z_2}(K_0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c_1c_3 & -c_1a_3 - a_1c_2 \end{pmatrix}$$

is invertible. Then we form the bivector

$$R = \sum_{i,j=1}^{2} \left(G^{-1}\right)_{ij} Z_i \wedge X_1^j, \text{ where } X_1^1 = P'dH_0 \text{ and } X_1^2 = P'dK_0. \tag{5.8}$$

Lemma 1. *The modified bivector $Q = P' - R$ defines a Poisson bracket, compatible with P; moreover, Q restricts to $\mathcal{S}_{\boldsymbol{\kappa}}$.*

Proof. The proof of the fact that $Q_\lambda = Q - \lambda P$ is a Poisson pencil follows (see, e.g., [45]) from the equalities

$$\begin{aligned} L_{Z_1}P = 0, \quad & L_{Z_1}P' = Y_1^1 \wedge Z_1 - c_3\frac{\partial}{\partial a_3} \wedge Z_2 \\ L_{Z_2}P = (\frac{\partial}{\partial b_1} - \frac{\partial}{\partial a_1}) \wedge Z_2, \quad & L_{Z_2}P' = (b_4\frac{\partial}{\partial a_4} + \frac{\partial}{\partial b_1}) \wedge Z_1 + Y_2^2 \wedge Z_2, \end{aligned} \tag{5.9}$$

where

$$Y_1^1 = a_3\frac{\partial}{\partial a_3} - a_4\frac{\partial}{\partial a_4} + c_2\frac{\partial}{\partial c_2} - c_4\frac{\partial}{\partial c_4}$$

$$Y_2^2 = b_4\frac{\partial}{\partial b_4} - b_1\frac{\partial}{\partial b_1} - a_3\frac{\partial}{\partial a_3} - a_4\frac{\partial}{\partial a_4} - c_1\frac{\partial}{\partial c_1} + c_2\frac{\partial}{\partial c_2} + c_3\frac{\partial}{\partial c_3} - c_4\frac{\partial}{\partial c_4},$$

as well as from the fact that

$$QdH_0 = QdK_0 = QdJ_1 = QdJ_2 = 0. \tag{5.10}$$

To show that (5.9) holds true is simply a matter of an explicit computation, while (5.10) follows from the definition of Q. In fact, the last two equations hold since J_1 and J_2 are Casimirs of P' invariant under Z_1 and Z_2. For, e.g., H_0 one computes

$$\begin{aligned} QdH_0 = P'dH_0 - RdH_0 &= X_1^1 - \sum_{i,j=1}^{2} \left(G^{-1}\right)_{ij} L_{Z_j}(H_0) \cdot X_1^i \\ &= X_1^1 - \sum_{i,j=1}^{2} \left(G^{-1}\right)_{ij} G_{j1} \cdot X_1^i = X_1^1 - \sum_i \delta_{i1} \cdot X_1^i = 0, \end{aligned}$$

where the second equality follows from the fact that all the functions H_i, K_α, J_α are in involution with respect to P. □

Thanks to the above lemma, the generic symplectic leaf $\mathcal{S}_{\boldsymbol{\kappa}}$ is endowed with the structure of a regular bihamiltonian manifold. We know from Section 4 that the non trivial Hamiltonians H_1, H_2, H_3, K_1 (more precisely, the restriction to $\mathcal{S}_{\boldsymbol{\kappa}}$ of these Hamiltonians) satisfy the hypothesis of Theorem 2 with respect to the (restriction to $\mathcal{S}_{\boldsymbol{\kappa}}$) of the pencil $Q - \lambda P$. This fact can be directly shown as follows:

$$QdH_i = P'dH_i - \sum_{i,j=1}^{2} \left(G^{-1}\right)_{ij} (Z_i \wedge X_1^j)(dH_i) = PdH_{i+1} - \sum_{i,j=1}^{2} \left(G^{-1}\right)_{ij} L_{Z_i}(H_i) X_1^j$$

(where we understand $H_4 = 0$) and

$$QdK_2 = P'dK_1 - \sum_{i,j=1}^{2} \left(G^{-1}\right)_{ij} (Z_i \wedge X_1^j)(dK_2) = \sum_{i,j=1}^{2} \left(G^{-1}\right)_{ij} L_{Z_i}(dK_1) X_1^j$$

So we proved that, for generic values κ_i, with $i = 1, \ldots, 4$, of the Casimirs, the system obtained by restriction of the Toda_3^4 flows on $\mathcal{S}_{\boldsymbol{\kappa}}$ is separable in the DN coordinates associated with the restriction to $\mathcal{S}_{\boldsymbol{\kappa}}$ of the pencil $Q - \lambda P$. To finish our job we finally have to:
a) explicitly compute the DN coordinates;
b) find the separation relations.

To solve the first problem, we will use the tools briefly described in Subsection 2.2. We rely on a result of [21], as well as on explicit computations, to state the following proposition, whose first part has been already discussed in Section 4.

Proposition 9. *Let us consider the matrix*

$$\mathcal{G}(\lambda) = \begin{pmatrix} L_{Z_1} H(\lambda) & L_{Z_2} H(\lambda) \\ L_{Z_1} K(\lambda) & L_{Z_2} K(\lambda) \end{pmatrix}. \tag{5.11}$$

The roots of the degree-4 polynomial $Det(\mathcal{G}(\lambda))$ *are the roots of the minimal polynomial* $\Delta(\lambda) = \lambda^4 - \sum_{i=1}^4 p_i \lambda^{4-i}$ *of the Nijenhuis tensor* $N = P^{-1}Q$ *associated with the regular Poisson pencil* Q_λ. *The coefficients* p_i *are functionally independent on the generic symplectic leaf* $\mathcal{S}_{\boldsymbol{\kappa}}$. *Furthermore, the ratios*

$$\rho(\lambda) = -\mathcal{G}_{22}/\mathcal{G}_{12}, \qquad \sigma(\lambda) = -\mathcal{G}_{21}/\mathcal{G}_{11}$$

are Nijenhuis function generators.

Thus, one half of the Darboux–Nijenhuis coordinates will be given by the roots of $\mathrm{Det}(\mathcal{G}(\lambda))$. To find the remaining half we consider the vector field $Y = -Pdp_1$, whose role has been discussed in Subsection 2.2. Since an explicit computations shows that $L_Y \log(\rho(\lambda)) = 1$, we can state the following

Proposition 10. *A set of Darboux–Nijenhuis coordinates for the restriction to the generic symplectic leaf* $\mathcal{S}_{\boldsymbol{\kappa}}$ *of the* Toda_3^4 *flows are given by the four roots* λ_i *of* $Det(\mathcal{G}(\lambda))$ *and by the values* μ_i *of the function* $\log(\rho(\lambda))$ *for* $\lambda = \lambda_i$, *where*

$$\rho(\lambda) = \frac{(-c_1 a_3 - a_1 c_2)\,\lambda + c_2 a_1 b_3 - a_1 a_2 a_3 + c_1 b_2 a_3 + c_1 c_2}{c_1 c_3 \lambda + a_1 a_2 c_3 - c_1 b_2 c_3}.$$

(We assume that $c_3 \neq 0$ *and* $c_1 \lambda_i + a_1 a_2 - b_2 c_1 \neq 0$ *for all* $i = 1, \ldots, 4$.*)*

To find the separation relations, we notice that the pairs $(\lambda_i, \rho(\lambda_i))$ are common solutions to the system

$$\begin{cases} \rho \mathcal{G}_{11} + \mathcal{G}_{21} = 0 \\ \rho \mathcal{G}_{12} + \mathcal{G}_{22} = 0 \end{cases} \tag{5.12}$$

since the rank of $\mathcal{G}(\lambda_i)$ is equal to 1. Then we reconsider the Lax matrix (5.4), and we compute the Lie derivatives of the matrix $\mathcal{L}(\lambda,\nu)=\lambda\mathbf{1}-L(\nu)$ along the vector fields Z_i:

$$L_{Z_1}(\mathcal{L}(\lambda,\nu)) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad L_{Z_2}(\mathcal{L}(\lambda,\nu)) = \begin{bmatrix} 0 & 0 & 0 & -\nu^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since

$$L_{Z_a}\left[\mathrm{Det}(\mathcal{L}(\lambda,\nu))\right] = \mathrm{Tr}\left[L_{Z_a}(\mathcal{L}(\lambda,\nu))\mathcal{L}(\lambda,\nu)^{\vee}\right], \qquad a=1,2,$$

$\mathcal{L}(\lambda,\nu)^{\vee}$ being the classical adjoint to $\mathcal{L}(\lambda,\nu)=\lambda\mathbf{1}-L(\nu)$, it follows from Proposition 8 and the definition (5.11) of $\mathcal{G}(\lambda)$ that the solutions of the system (5.12) are related to those of the system

$$\begin{cases} [\mathcal{L}(\lambda,\nu)^{\vee}]_{44} = 0 \\ [\mathcal{L}(\lambda,\nu)^{\vee}]_{41} = 0 \end{cases}$$

via $\rho=\nu^4$. Now we can state

Proposition 11. *The separation relations connecting pairs of Darboux–Nijenhuis coordinates (λ_i,μ_i), the Hamiltonians H_1,H_2,H_3,K_2, and the Casimirs H_0,K_1,K_2,J_0 are, on the generic symplectic leaf $\mathcal{S}_{\boldsymbol{\kappa}}$, given by the evaluation of the characteristic polynomial Det$(\mathcal{L}(\lambda,\nu))$ in $\lambda=\lambda_i$ and $\nu=\nu_i=\exp(\frac{\mu_i}{4})$.*

Proof. We know that the pairs (λ_i,ν_i) solve system (5.1), and we have to show that they satisfy $\mathrm{Det}(\mathcal{L}(\lambda_i,\nu_i))=0$. This can be done with the following adaptation of the algebro-geometrical technique of finding the poles of the (normalized) Baker–Akhiezer function. Let us consider the 5×4 matrix $\mathcal{M}_i$ obtained by putting the vector $(0,0,0,1)$ on the top of $\mathcal{L}(\lambda_i,\nu_i)$. As we have assumed at the end of Proposition 10, the 3×3 matrix extracted from $\mathcal{M}_i$ by removing the 3rd column and the 2nd and the 4th rows, is invertible. Since the system (5.1) is satisfied, the rank of $\mathcal{M}_i$ is 3, and therefore $\mathrm{Det}(\mathcal{L}(\lambda_i,\nu_i))$ vanishes. □

We notice that, *a posteriori,* the separation coordinates for the Toda$^4_3$ system fall in the class described in, e.g., [2, 4, 47–50]. Namely, the DN coordinates that separate the Toda$^4_3$ system are algebro-geometrical Darboux coordinates associated with the spectral curve (5.6), and fulfill the so-called Sklyanin's "magic recipe". This is a quite general fact, as discussed in Section 4 (see also [51] and Remark 1).

As a final remark, in connection with the discussion on the relation between the bihamiltonian property of an integrable vector field and the separability of the associated HJ equations of Section 3, we notice that the Hamiltonians H_1,H_2,H_3,K_2 are functionally independent from the coefficients of the minimal polynomial of the Nijenhuis tensor obtained from $Q-\lambda P$. So, this is a further instance of a system which is not bihamiltonian on a regular manifold, but turns out to be separable via the bihamiltonian method of SoV.

5.2. A Remarkable Subsystem: the Open Toda$^4_3$ System

In this last subsection we will discuss a remarkable reduction of the periodic Toda$^4_3$ system (even though less interesting from the point of view of algebraic integrability), leading to the corresponding generalization of the open (or non-periodic) one. In the manifold $M\simeq\mathbb{C}^{12}$ we consider the nine-dimensional submanifold M_0 defined by the equations

$$a_4 = c_3 = c_4 = 0. \tag{5.13}$$

One can easily verify that the restriction $X^0_{H_{GT}}$ to M_0 of the vector field $X_{H_{GT}}$ is tangent to M_0. Also, the tensor P can be restricted to M_0; indeed, the expression of its restriction P_0 with respect to the natural coordinates $\{b_1, \ldots, b_4, a_1, \ldots, a_3, c_1, c_2\}$ of M_0 is obtained from (5.2) simply by removing the 9th, 11th, 12th rows and columns. Moreover, one can check that $X^0_{H_{GT}} = P_0 dH_{GT,0}$, with

$$H_{GT,0} = \frac{1}{2}(b_1^2 + b_2^2 + b_3^2 + b_4^2) - (a_1 + a_2 + a_3), \tag{5.14}$$

and recognize that this function is the Hamiltonian of the open Toda lattice. Also, a Lax pair for $X^0_{H_{GT}}$ is

$$L_0 = \begin{bmatrix} b_1 & -\nu & 0 & 0 \\ \frac{a_1}{\nu} & b_2 & -\nu & 0 \\ \frac{c_1}{\nu^2} & \frac{a_2}{\nu} & b_3 & -\nu \\ -\nu & \frac{c_2}{\nu^2} & \frac{a_3}{\nu} & b_4 \end{bmatrix}, \qquad \Phi_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{a_1}{\nu} & 0 & 0 & 0 \\ \frac{c_1}{\nu^2} & \frac{a_2}{\nu} & 0 & 0 \\ 0 & \frac{c_2}{\nu^2} & \frac{a_3}{\nu} & 0 \end{bmatrix}. \tag{5.15}$$

It should be clear from the form of the Lax pair that the system associated to the vector field $X^0_{H_{GT}}$ is an extension of the standard open Toda lattice towards the so-called full open Toda lattice, which is a system describing a flow on the lower Borel subgroup of $sl(N)$. The integrability of the full (open) Toda lattice was established in [30] (see, also, [52]). The idea was to complement the integrals of the motion coming from the Lax representation with additional integrals obtained by means of the "chopping method", within the group–theoretical point of view.

The need to supply the standard results of the Lax theory with further methods should be clear from the following considerations. The only Casimir function of P_0 is $h_0 = \sum_{i=1}^4 b_i$. Hence, its symplectic leaves $\mathcal{S}_\xi \subset M_0$ are the eight-dimensional manifolds defined by $h_0 = \xi$, and $X^0_{H_{GT}}$ can be seen as a Hamiltonian system with *four* degrees of freedom. The characteristic polynomial of the matrix L_0 is

$$\mathrm{Det}(\lambda \mathbf{1} - L_0(\nu)) = -\nu^4 + \lambda^4 - h_0\lambda^3 + h_1\lambda^2 - h_2\lambda + h_3, \tag{5.16}$$

that is, it provides us with only *three* non trivial Hamiltonians,

$$\begin{aligned} &h_1 = \sum_{i>j=1}^{4} b_i b_j + \sum_{i=1}^{3} a_i, \quad h_2 = \sum_{i>j>k=1}^{4} b_i b_j b_k + \sum_{i=1}^{3} a_i(b_{i+2} + b_{i+3}) + c_1 + c_2 \\ &h_3 = b_1 a_2 b_4 + a_1 b_3 b_4 + b_1 b_2 a_3 + b_1 c_2 + a_1 a_3 + c_1 b_4 + b_1 b_2 b_3 b_4. \end{aligned} \tag{5.17}$$

We will now show how the tools we previously introduced can be used to geometrically prove the complete integrability of such a system and, moreover, yield the existence of an additional integral of the motion. The main property is that, along with P, the tensor Q restricts to M_0. This can be proven as follows: one checks by direct inspection that this is true for P'; then the assertion follows from the fact that the vector field $X_1^2 = P' dK_0$ vanishes on M_0, while Z_1 and X_1^1, which coincides with X_{GT}, are tangent to M_0.

Furthermore, we add two observations. The first one concerns the restriction $\mathcal{G}_0$ to M_0 of the matrix $\mathcal{G}$. It has the form

$$\mathcal{G}_0 = \begin{bmatrix} \mathcal{G}^0_{11} & \lambda^2 - (b_2 + b_3)\,\lambda + b_2 b_3 + a_2 \\ 0 & -(c_1 a_3 + a_1 c_2)\,\lambda - a_1 a_2 a_3 + c_1 c_2 + c_1 b_2 a_3 + a_1 c_2 b_3 \end{bmatrix} \tag{5.18}$$

and therefore its determinant (that is, the minimal polynomial of the Nijenhuis tensor N_0 induced by the pencil $Q_0 - \lambda P_0$ on $\mathcal{S}_\xi$) factors as $\mathcal{G}^0_{11}\mathcal{G}^0_{22}$, where $\mathcal{G}^0_{11} = \lambda^3 - \pi_1\lambda^2 - \pi_2\lambda - \pi_3$ is a degree-three polynomial. The second observation consists in the fact that the three surviving Hamiltonians h_1, h_2, h_3 given by (5.17) satisfy the conditions:

$$Q_0 dh_i = \sum_{j=1}^{3} F_{ij}^0 P_0 dh_i, \text{ with } F_{ij}^0 = \begin{bmatrix} \pi_1 & 1 & 0 \\ \pi_2 & 0 & 1 \\ \pi_3 & 0 & 0 \end{bmatrix}. \tag{5.19}$$

We notice that the functions π_1, π_2, π_3, and the root

$$\lambda_4 = \frac{-a_1 a_2 a_3 + c_1 c_2 + a_1 c_2 b_3 + c_1 b_2 a_3}{c_1 a_3 + a_1 c_2} \tag{5.20}$$

of $\mathcal{G}_{22}^0$ are still functionally independent and hence (generically) different on $\mathcal{S}_\xi$.

Lemma 2. *Let σ be any function satisfying $Q_0 d\sigma = \lambda_4 P_0 d\sigma$. Under the above hypotheses, the brackets $\{\sigma, h_i\}_{P_0}$ and $\{\sigma, h_i\}_{Q_0}$ vanish.*

Proof. Evaluating both sides of $Q_0 d\sigma = \lambda_4 P_0 d\sigma$ on the differentials (dh_1, dh_2, dh_3), and switching the action of the Poisson tensors on the dh_i's, we get

$$\langle d\sigma, Q_0 dh_i \rangle = \lambda_4 \langle d\sigma, P_0 dh_i \rangle, \qquad i = 1, 2, 3.$$

Inserting (5.19) we get the equation $\sum_{j=1}^{3} \left(F_{ij}^0 - \lambda_4 \delta_{ij}\right) \langle d\sigma, P_0 dh_j \rangle = 0$. Since λ_4 is *not* an eigenvalue of F_{ij}^0, the lemma is proved. □

So a fourth integral of the motion, that commutes with the Hamiltonian H_{GT}^0 of the open Toda_3^4 lattice, is given indeed by the distinguished root λ_4 of equation (5.20); this constructively proves the integrability of the system.

Finally, we notice that this method proves the existence of a *fifth* integral of the motion. Indeed, we know that, along with λ_4, there must exist another independent function μ_4, satisfying the hypotheses of Lemma 2 and functionally independent of λ_4 and of the h_i's. In such a comparatively low dimensional case, such a function can be explicitly found to be

$$\mu_4 = \frac{c_2 \left(a_1 b_2 c_1 - a_2 {a_1}^2 - {c_1}^2 - c_1 b_3 a_1\right)}{c_1 \left(c_1 a_3 + a_1 c_2\right) \left({\lambda_4}^3 - \pi_1 {\lambda_4}^2 - \pi_2 \lambda_4 - \pi_3\right)}.$$

ACKNOWLEDGMENTS

We wish to thank Franco Magri for stimulating our interest in the theory of bihamiltonian manifolds and, in particular, for sharing with us his insights about the problem of SoV. The results presented in Sections 2 and 3 have been obtained in a long-standing joint-work with him. We are also grateful to Boris Dubrovin and Giorgio Tondo for a number of useful discussions. Some computations were performed with Maple $V^{®}$. We also thank the anonymous referee for some useful remarks that helped to improve the presentation of the paper.

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 245–289. © Pleiades Publishing, Ltd., 2011.

On Integrability of Hirota–Kimura Type Discretizations

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Received August 3, 2010; accepted October 24, 2010

Abstract—We give an overview of the integrability of the Hirota–Kimura discretization method applied to algebraically completely integrable (a.c.i.) systems with quadratic vector fields. Along with the description of the basic mechanism of integrability (Hirota–Kimura bases), we provide the reader with a fairly complete list of the currently available results for concrete a.c.i. systems.

MSC2010 numbers: 37K10, 14E05, 37J35, 37M15, 70E40

DOI: 10.1134/S1560354711030051

Keywords: algebraic integrability, integrable systems, integrable discretizations, birational dynamics

Contents

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1. INTRODUCTION

The discretization method studied in this paper seems to have been introduced in the geometric integration literature by W. Kahan in the unpublished notes [1]. It is applicable to any system of ordinary differential equations for $x : \mathbb{R} \to \mathbb{R}^n$ with a quadratic vector field:

$$\dot{x} = Q(x) + Bx + c,$$

where each component of $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a quadratic form, while $B \in \mathrm{Mat}_{n\times n}$ and $c \in \mathbb{R}^n$. Kahan's discretization reads as

$$\frac{\widetilde{x} - x}{\epsilon} = Q(x, \widetilde{x}) + \frac{1}{2}B(x + \widetilde{x}) + c, \tag{1.1}$$

where

$$Q(x, \widetilde{x}) = \frac{1}{2}\left(Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x})\right)$$

is the symmetric bilinear form corresponding to the quadratic form Q. Here and below we use the following notational convention which will allow us to omit a lot of indices: for a sequence $x : \mathbb{Z} \to \mathbb{R}$ we write x for x_k and $\widetilde{x}$ for x_{k+1}. Eq. (1.1) is *linear* with respect to $\widetilde{x}$ and therefore defines a *rational* map $\widetilde{x} = f(x, \epsilon)$. Clearly, this map approximates the time-ϵ-shift along the solutions of the original differential system, so that $x_k \approx x(k\epsilon)$. (Sometimes it will be more convenient to use 2ϵ for the time step, in order to avoid appearance of various powers of 2 in numerous formulas.) Since Eq. (1.1) remains invariant under the interchange $x \leftrightarrow \widetilde{x}$ with the simultaneous sign inversion $\epsilon \mapsto -\epsilon$, one has the *reversibility* property

$$f^{-1}(x, \epsilon) = f(x, -\epsilon).$$

In particular, the map f is *birational*.

W. Kahan applied this discretization scheme to the famous Lotka–Volterra system and showed that in this case it possesses a very remarkable non-spiralling property. This example is briefly discussed in [2]. Some further applications of this discretization have been explored in [3].

The next, even more intriguing appearance of this discretization was in the two papers by R. Hirota and K. Kimura who (being apparently unaware of the work by Kahan) applied it to two famous *integrable* systems of classical mechanics, the Euler top and the Lagrange top [4, 5]. For the purposes of the present text, integrability of a dynamical system is synonymous with the existence of a sufficient number of functionally independent conserved quantities, or integrals of motion, that is, functions constant along the orbits. We leave aside other aspects of the multi-faceted notion of integrability, such as Hamiltonian ones or explicit solutions. Surprisingly, the Kahan–Hirota–Kimura discretization scheme produced in both the Euler and the Lagrange cases of the rigid body motion *integrable* maps. Even more surprisingly, the mechanism which assures integrability in these two cases seems to be rather different from the majority of examples known in the area of integrable discretizations, and, more generally, integrable maps, cf. [6]. The case of the discrete time Euler top is relatively simple, and the proof of its integrability given in [4] is rather straightforward and easy to verify by hands. As it often happens, no explanation was given in [4] about how this result has been *discovered*. The "derivation" of integrals of motion for the discrete time Lagrange top in [5] is rather cryptic and almost uncomprehensible.

We use the term "Hirota–Kimura (HK) type discretization" for Kahan discretization in the context of integrable systems. At the Oberwolfach meeting on Geometric Integration in 2006,

T. Ratiu proposed to apply the Hirota–Kimura discretization to the Clebsch case of the rigid body motion in an ideal fluid and to the Kovalevsky top. The claim on the integrability of the HK discretization of the Clebsch case was proven only several years later in [2], while the integrability of the HK discretization of the Kovalevsky top remains an open problem (although there are some indications in favor of its non-integrability). Anyway, the general question of integrability of the HK type discretizations turns out to be very intriguing and rather difficult. In the present overview, we will present a rather long list of examples of integrable HK discretizations. Actually, this list is so impressive that in [2] we conjectured that HK discretizations of algebraically integrable systems always remain integrable. At present, we have some indications that this conjecture is wrong (although a rigorous proof of non-integrability for any of the apparently non-integrable cases remains elusive). Nevertheless, the sheer length of our list of examples clearly shows that there exist some general mechanisms that ensure integrability at least under certain additional assumptions. We think that to uncover general structures behind the integrability of HK discretizations is a big and important challenge for the modern theory of (algebraically) integrable systems.

The structure of our overview is as follows. In Section 2 we demonstrate some general sufficient conditions for a HK discretization to be measure preserving. These conditions are not related to integrability and do not cover all special cases considered in the main text. All our examples turn out to be measure preserving but for the majority of them we only can prove this property individually and do not know any general mechanisms. In Section 3 we present a formalization of the HK mechanism from [5], which will hopefully unveil its main idea and contribute towards demystifying at least some of its aspects. We introduce a notion of a "Hirota–Kimura basis" (HK basis) for a given map f. Such a basis Φ is a set of simple (often monomial) functions, $\Phi = (\varphi_l, \ldots, \varphi_l)$, such that for every orbit $\{f^i(x)\}$ of the map f there is a certain linear combination $c_1\varphi_1 + \ldots + c_l\varphi_l$ of functions from Φ vanishing on this orbit. As explained in Section 3, this is a *new* mathematical notion, not reducible to that of integrals of motion, although closely related to the latter. We lay a theoretical fundament for the search for HK bases for a given discrete time system, and discuss some practical recipes and tricks for doing this. Sections 4–15 contain our list of examples of algebraically integrable systems with quadratic vector fields, for which the HK discretization preserves integrability. For all the examples we provide the reader with a rather complete set of currently available results. The proofs are omitted almost everywhere. One of the reasons is that our investigations are based mainly on computer experiments, which are used both for *discovery* of new results and for their rigorous *proof*. For a given system, a search for HK bases can be done with the help of *numerical experiments*. If the search has been successful and a certain set of functions Φ has been identified as a HK basis for a given map f, then numerical experiments can provide a very convincing evidence in favor of such a statement. A rigorous proof of such a statement turns out to be much more demanding. At present, we are not in possession of any theoretical proof strategies and are forced to verify the corresponding statements by means of *symbolic computations* which in some cases turn out to be hardly feasible due to complexity issues. All these issues are intentionally avoided in our presentation here; an interested reader may consult [2] for a detailed discussion of one concrete example (Clebsch system). Our main goal here is to document the available results on integrable HK discretizations and to attract attention of specialists in integrable systems and in algebraic geometry to these beautiful and mysterious objects which are definitely worth further investigation.

2. SOME GENERAL RESULTS ON INVARIANT MEASURES

In this section, we prove the existence of invariant measures for Kahan discretizations of two classes of dynamical systems with quadratic vector fields (not necessarily integrable).

The first class reads:

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij}x_j^2 + c_i, \qquad 1 \leqslant i \leqslant N,$$

with a skew-symmetric matrix $A = (a_{ij})_{i,j=1}^N = -A^{\mathrm{T}}$. The Kahan's discretization reads:

$$\widetilde{x}_i - x_i = \epsilon \sum_{j=1}^{N} a_{ij}x_j\widetilde{x}_j + \epsilon c_i, \qquad 1 \leqslant i \leqslant N. \tag{2.1}$$

Proposition 1. *The map $\widetilde{x} = f(x,\epsilon)$ defined by Eqs. (2.1) has an invariant volume form:*

$$\det\frac{\partial\widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x},\epsilon)}{\phi(x,\epsilon)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1\wedge\ldots\wedge dx_N}{\phi(x,\epsilon)},$$

where $\phi(x,\epsilon) = \det(\mathbb{1} - \epsilon AX)$, with $X = \mathrm{diag}(x_1,\ldots,x_N)$, is an even polynomial in ϵ.

Proof. Eqs. (2.1) can be put as

$$\widetilde{x} = A^{-1}(x,\epsilon)(x+\epsilon c), \qquad A(x,\epsilon) = \mathbb{1} - \epsilon AX.$$

As for any Kahan type discretization, there holds the formula

$$\det\frac{\partial\widetilde{x}}{\partial x} = \frac{\det A(\widetilde{x},-\epsilon)}{\det A(x,\epsilon)}. \tag{2.2}$$

Indeed, differentiation of eqs. (2.1) with respect to x_j will give the j-th column of the matrix equation

$$A(x,\epsilon)\frac{\partial\widetilde{x}}{\partial x} = A(\widetilde{x},-\epsilon).$$

It remains to notice that $\det A(x,\epsilon) = \det A(x,-\epsilon)$. Indeed, due to the skew-symmetry of A, we have: $\det(\mathbb{1}-\epsilon AX) = \det(\mathbb{1}-\epsilon X^{\mathrm{T}}A^{\mathrm{T}}) = \det(\mathbb{1}+\epsilon XA) = \det(\mathbb{1}+\epsilon AX)$.

The second class consists of equations of the Lotka–Volterra type:

$$\dot{x}_i = x_i\left(b_i + \sum_{j=1}^N a_{ij}x_j\right), \qquad 1\leqslant i\leqslant N,$$

with a skew-symmetric matrix $A = (a_{ij})_{i,j=1}^N = -A^{\mathrm{T}}$. The Kahan's discretization (with the stepsize 2ϵ) reads:

$$\widetilde{x}_i - x_i = \epsilon b_i(x_i+\widetilde{x}_i) + \epsilon\sum_{j=1}^N a_{ij}(x_i\widetilde{x}_j + \widetilde{x}_i x_j), \qquad 1\leqslant i\leqslant N. \tag{2.3}$$

Proposition 2. *The map $\widetilde{x} = f(x,\epsilon)$ defined by eqs. (2.3) has an invariant volume form:*

$$\det\frac{\partial\widetilde{x}}{\partial x} = \frac{\widetilde{x}_1\widetilde{x}_2\cdots\widetilde{x}_N}{x_1x_2\cdots x_N} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1\wedge\ldots\wedge dx_N}{x_1x_2\cdots x_N}.$$

Proof. Eqs. (2.3) are equivalent to

$$\frac{\widetilde{x}_i}{1+\epsilon b_i + \epsilon\sum_{j=1}^N a_{ij}\widetilde{x}_j} = \frac{x_i}{1-\epsilon b_i - \epsilon\sum_{j=1}^N a_{ij}x_j} =: y_i. \tag{2.4}$$

We denote $d_i(x,\epsilon) = 1 - \epsilon b_i - \epsilon\sum_{j=1}^N a_{ij}x_j$. In the matrix form equation (2.3) can be put as

$$\widetilde{x} = A^{-1}(x,\epsilon)(\mathbb{1}+\epsilon B)x,$$

where the i-th diagonal entry of $A(x,\epsilon)$ equals $d_i(x,\epsilon)$, while the (i,j)-th off-diagonal entry equals $-\epsilon x_i a_{ij}$. In other words, $A(x,\epsilon) = D(\mathbb{1}-\epsilon YA)$, where $D = D(x,\epsilon) = \mathrm{diag}(d_1,\ldots,d_N)$ and $Y = \mathrm{diag}(y_1,\ldots y_N)$. Formula (2.2) holds true also in the present case, and it implies:

$$\det\frac{\partial\widetilde{x}}{\partial x} = \frac{\det D(\widetilde{x},-\epsilon)}{\det D(x,\epsilon)}\,\frac{\det(\mathbb{1}+\epsilon YA)}{\det(\mathbb{1}-\epsilon YA)}.$$

The second factor equals 1 due to the skew-symmetry of A, while the first factor equals

$$\frac{d_1(\widetilde{x},-\epsilon)\cdots d_N(\widetilde{x},-\epsilon)}{d_1(x,\epsilon)\cdots d_N(x,\epsilon)} = \frac{\widetilde{x}_1\cdots\widetilde{x}_N}{x_1\cdots x_N},$$

by virtue of (2.4).

The statement of proposition 2 for the Kahan discretization of the classical Lotka–Volterra model with two species, $\dot{x} = x(a - by)$, $\dot{y} = y(cx - d)$, was found in [7] and used to explain the non-spiralling behavior of the numerical orbits in this case.

3. HIROTA–KIMURA BASES AND INTEGRALS

In this section a general formulation of a remarkable mechanism will be given, which seems to be responsible for the integrability of the Hirota–Kimura type (or Kahan type) discretizations of algebraically completely integrable systems. This mechanism is so far not well understood, in fact at the moment we do not know what mathematical structures make it actually work.

Throughout this section $f : \mathbb{R}^n \to \mathbb{R}^n$ is a birational map, while $h_i, \varphi_i : \mathbb{R}^n \to \mathbb{R}$ stand for rational, usually polynomial functions on the phase space. We start with recalling a well known definition.

Definition 1. *A function $h : \mathbb{R}^n \to \mathbb{R}$ is called an* **integral***, or a* **conserved quantity***, of the map f, if for every $x \in \mathbb{R}^n$ there holds*

$$h(f(x)) = h(x),$$

so that $h(f^i(x)) = h(x)$ for all $i \in \mathbb{Z}$.

Thus, each orbit of the map f lies on a certain level set of its integral h. As a consequence, if one knows d functionally independent integrals $h_1, \ldots, h_d$ of f, one can claim that each orbit of f is confined to an $(n - d)$-dimensional invariant set, which is a common level set of the functions $h_1, \ldots, h_d$.

Definition 2. *A set of functions $\Phi = (\varphi_1, \ldots, \varphi_l)$, linearly independent over $\mathbb{R}$, is called a* **Hirota–Kimura basis (HK basis)***, if for every $x \in \mathbb{R}^n$ there exists a vector $c = (c_1, \ldots, c_l) \neq 0$ such that*

$$c_1\varphi_1(f^i(x)) + \ldots + c_l\varphi_l(f^i(x)) = 0 \qquad (3.1)$$

holds true for all $i \in \mathbb{Z}$. For a given $x \in \mathbb{R}^n$, the vector space consisting of all $c \in \mathbb{R}^l$ with this property will be denoted by $K_\Phi(x)$ and called the null-space of the basis Φ (at the point x).

Thus, for a HK basis Φ and for $c \in K_\Phi(x)$ the function $h = c_1\varphi_1 + \ldots + c_l\varphi_l$ vanishes along the f-orbit of x. Let us stress that we cannot claim that $h = c_1\varphi_1 + \ldots.. + c_l\varphi_l$ is an integral of motion, since vectors $c \in K_\Phi(x)$ do not have to belong to $K_\Phi(y)$ for initial points y not lying on the orbit of x. However, for any x the orbit $\{f^i(x)\}$ is confined to the common zero level set of d functions

$$h_j = c_1^{(j)}\varphi_1 + \ldots + c_l^{(j)}\varphi_l = 0, \quad j = 1, \ldots, d,$$

where the vectors $c^{(j)} = \left(c_1^{(j)}, \ldots, c_l^{(j)}\right) \in \mathbb{R}^l$ form a basis of $K_\Phi(x)$. We will say that the HK basis Φ is **regular**, if the differentials $dh_1, \ldots, dh_d$ are lineraly independent along the common zero level set of the functions $h_1, \ldots, h_d$. Thus, knowledge of a regular HK basis with a d-dimensional null-space leads to a similar conclusion as knowledge of d independent integrals of f, namely to the conclusion that the orbits lie on $(n - d)$-dimensional invariant sets. Note, however, that a HK basis gives no immediate information on how these invariant sets foliate the phase space $\mathbb{R}^n$, since the vectors $c^{(j)}$, and therefore the functions h_j, change from one initial point x to another.

Although the notions of integrals and of HK bases cannot be immediately translated into one another, they turn out to be closely related.

The simplest situation for a HK basis corresponds to $l = 2$, $\dim K_\Phi(x) = d = 1$. In this case we immediately see that $h = \varphi_1/\varphi_2$ is an integral of motion of the map f. Conversely, for any rational integral of motion $h = \varphi_1/\varphi_2$ its numerator and denominator φ_1, φ_2 satisfy

$$c_1\varphi_1(f^i(x)) + c_2\varphi_2(f^i(x)) = 0, \quad i \in \mathbb{Z},$$

with $c_1 = 1$, $c_2 = -h(x)$, and thus build a HK basis with $l = 2$. Thus, the notion of a HK basis *generalizes* (for $l \geqslant 3$) the notion of integrals of motion.

On the other hand, knowing a HK basis Φ with $\dim K_\Phi(x) = d \geqslant 1$ allows one to find integrals of motion for the map f. Indeed, from Definition 2 there follows immediately:

Proposition 3. *If Φ is a HK basis for a map f, then*

$$K_\Phi(f(x)) = K_\Phi(x).$$

Thus, the d-dimensional null-space $K_\Phi(x) \in Gr(d,l)$, regarded as a function of the initial point $x \in \mathbb{R}^n$, is constant along trajectories of the map f, i.e., it is a $Gr(d,l)$-valued integral. Its Plücker coordinates are then scalar integrals:

Corollary 1. *Let Φ be a HK basis for f with $\dim K_\Phi(x) = d$ for all $x \in \mathbb{R}^n$. Take a basis of $K_\Phi(x)$ consisting of d vectors $c^{(i)} \in \mathbb{R}^l$ and put them into the columns of a $l \times d$ matrix $C(x)$. For any d-index $\alpha = (\alpha_1, \ldots, \alpha_d) \subset \{1, 2, \ldots, n\}$ let $C_\alpha = C_{\alpha_1 \ldots \alpha_d}$ denote the $d \times d$ minor of the matrix C built from the rows $\alpha_1, \ldots, \alpha_d$. Then for any two d-indices α, β the function C_α / C_β is an integral of f.*

Especially simple is the situation where the null-space of a HK basis has dimension $d = 1$.

Corollary 2. *Let Φ be a HK basis for f with $\dim K_\Phi(x) = 1$ for all $x \in \mathbb{R}^n$. Let $K_\Phi(x) = [c_1(x) : \ldots : c_l(x)] \in \mathbb{RP}^{l-1}$. Then the functions c_j / c_k are integrals of motion for f.*

An interesting (and difficult) question is about the number of functionally independent integrals obtained from a given HK basis according to Corollaries 1 and 2. It is possible for a HK basis with a one-dimensional null-space to produce more than one independent integral. The first examples of this mechanism (with $d = 1$) were found in [5] and (somewhat implicitly) in [4].

We note, however, that HK bases appeared in a disguised form in the continuous time theory long ago. We mention here two relevant examples.

- Classically, integration of a given system of ODEs in terms of elliptic functions started with the derivation of an equation of the type $\dot{y}^2 = P_4(y)$, where y is one of the components of the solution, and $P_4(y)$ is a polynomial of degree 4 with constant coefficients (depending on parameters of the system and on its integrals of motion), see examples in Sections 12, 13. This can be interpreted as the claim about $\Phi = (\dot{y}^2, y^4, y^3, y^2, y, 1)$ being a HK basis with a one-dimensional null-space.

- According to [8, Sect. 7.6.6], for any algebraically integrable system, one can choose projective coordinates $y_0, y_1, \ldots, y_n$ so that *quadratic Wronskian equations* are satisfied:

$$\dot{y}_i y_j - y_i \dot{y}_j = \sum_{k,l=0}^{n} \alpha_{ij}^{kl} y_k y_l,$$

  with coefficients α_{ij}^{kl} depending on integrals of motion of the original system. Again, this admits an immediate interpretation in terms of HK bases consisting of the Wronskians and the quadratic monomials of the coordinate functions: $\Phi_{ij} = \big(\dot{y}_i y_j - y_i \dot{y}_j,\ \{y_k y_l\}_{k,l=0}^n\big)$.

Thus, these HK bases consist not only of simple monomials, but include also more complicated functions composed of the vector field of the system at hand. We will encounter discrete counterparts of these HK bases, as well.

At present, we cannot give any theoretical sufficient conditions for existence of a HK basis Φ for a given map f, and the only way to find such a basis remains the experimental one. Definition 2 requires to verify condition (3.1) for all $i \in \mathbb{Z}$, which is, of course, impractical. However, it is enough to check this condition for a finite number of iterates f^i.

Typically (that is, for general maps f and general monomial sets Φ), the dimension of the vector space of solutions of the homogeneous system of s linear equations (3.1) with $i = i_0, i_0 + 1, \ldots, i_0 + s - 1$ decays with the growing s, from $l - 1$ for $s = 1$ down to 0 for $s = l$. If, however, Φ is a HK basis for f with a d-dimensional null-space, then this dimension fails to drop starting with $s = l - d + 1$. Thus, the dimension of the solution space of the system (3.1) with $i = i_0, i_0 + 1, \ldots, i_0 + s - 1$ is equal to $l - s$ for all $1 \leqslant s \leqslant l - d$, and remains equal to d for $s = l - d + 1$. It is easy to see that this situation can be also characterized as follows: the d-dimensional spaces of solutions of the system (3.1) with $i = i_0, i_0 + 1, \ldots, i_0 + l - d - 1$ and of the system (3.1) with $i = i_0 + 1, i_0 + 2, \ldots, i_0 + l - d$ coincide with each other and with $K_\Phi(x)$. The most important particular case of this characterization corresponds to $d = 1$ and reads as follows:

Proposition 4. *A set $\Phi = (\varphi_1, \ldots, \varphi_l)$ is a HK basis for a map f with $\dim K_\Phi(x) = 1$ if and only if the unique solution $[c_1 : \ldots : c_l] \in \mathbb{RP}^{l-1}$ of the system (3.1) with $i = i_0, i_0+1, \ldots, i_0+l-2$ coincides with the unique solution of the analogous system with $i = i_0+1, i_0+2, \ldots, i_0+l-1$, in other words, if this unique solution $[c_1 : \ldots : c_l]$ is an integral of motion:*

$$[c_1(x) : \ldots : c_l(x)] = [c_1(f(x)) : \ldots : c_l(f(x))].$$

At this point it should be mentioned that a numerical testing of the above criterion usually represents no problems, but the corresponding symbolic computations might be extremely complicated, due to the complexity of the iterates $f^i(x)$. While the expression for $f(x)$ is typically of a moderate size, already the second iterate $f^2(x)$ becomes typically prohibitively big. See the detailed discussion of the complexity issue for the case of the Clebsch system in [2]. In such a situation it becomes crucial to reduce the number of iterates involved in (3.1) as far as possible. Several tricks which can be used for this aim are also described in [2]. Here is one of them.

Proposition 5. *Consider the non-homogeneous system of $l-1$ equations*

$$c_1\varphi_1(f^i(x)) + \ldots + c_{l-1}\varphi_{l-1}(f^i(x)) = \varphi_l(f^i(x)), \quad i = i_0, i_0+1, \ldots, i_0+l-2. \tag{3.2}$$

Suppose that the index range $i \in [i_0, i_0+l-2]$ in Eq. (3.2) contains 0 but is non-symmetric. If the solution of this system $\big(c_1(x,\epsilon), \ldots, c_{l-1}(x,\epsilon)\big)$ is unique and is even with respect to ϵ, then all $c_k(x,\epsilon)$ are conserved quantities of the map f, and $\Phi = (\varphi_1, \ldots, \varphi_l)$ is a HK basis for f with $\dim K_\Phi(x) = 1$.

Proof. Considering the non-homogeneous system (3.2) instead of the homogeneous one (3.1) corresponds just to fixing an affine representative of the projective solution $[c_1 : \ldots : c_l]$ by $c_l = -1$. The reversibility of the map $f^{-1}(x,\epsilon) = f(x,-\epsilon)$ yields that equations of the system (3.2) are satisfied not only for $i \in [i_0, i_0+l-2]$ but for $i \in [-(i_0+l-2), -i_0]$, as well. Since, by condition, the intervals $[i_0, i_0+l-2]$ and $[-(i_0+l-2), -i_0]$ overlap but do not coincide, their union is an interval containing more than l integers.

Of course, it would be highly desirable to find some structures, like Lax representation, bi-Hamiltonian structure, etc., which would allow one to check the conservation of integrals in a more clever way, but up to now no such structures have been found for any of the HK type discretizations.

4. WEIERSTRASS DIFFERENTIAL EQUATION

Consider the second-order differential equation

$$\ddot{x} = 6x^2 - \alpha. \tag{4.1}$$

Its general solution is given by the Weierstrass elliptic function $\wp(t) = \wp(t, g_2, g_3)$ with the invariants $g_2 = 2\alpha$, g_3 arbitrary, and by its time shifts. Actually, the parameter g_3 can be interpreted as the value of an integral of motion (conserved quantity) of system (4.1):

$$\dot{x}^2 - 4x^3 + 2\alpha x = -g_3.$$

Being re-written as a system of first-order equations with a quadratic vector field,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 6x^2 - \alpha, \end{cases} \tag{4.2}$$

equation (4.1) becomes suitable for an application of the Kahan–Hirota–Kimura discretization:

$$\begin{cases} \widetilde{x} - x = \dfrac{\epsilon}{2}\left(\widetilde{y} + y\right), \\ \widetilde{y} - y = \epsilon\left(6x\widetilde{x} - \alpha\right). \end{cases} \tag{4.3}$$

Eqs. (4.3), put as a linear system for $(\widetilde{x}, \widetilde{y})$, read:

$$\begin{pmatrix} 1 & -\epsilon/2 \\ -6\epsilon x & 1 \end{pmatrix} \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} x + \epsilon y/2 \\ y - \epsilon\alpha \end{pmatrix}.$$

This can be immediately solved, thus yielding an explicit birational map $(\widetilde{x}, \widetilde{y}) = f(x, y, \epsilon)$:

$$\begin{cases} \widetilde{x} = \dfrac{x + \epsilon y - \epsilon^2\alpha/2}{1 - 3\epsilon^2 x}, \\ \widetilde{y} = \dfrac{y + \epsilon(6x^2 - \alpha) + 3\epsilon^2 xy}{1 - 3\epsilon^2 x}. \end{cases} \tag{4.4}$$

This map turns out to be integrable: it possesses an invariant two-form

$$\omega = \frac{dx \wedge dy}{1 - 3\epsilon^2 x}, \tag{4.5}$$

and an integral of motion (conserved quantity):

$$I(x, y, \epsilon) = \frac{y^2 - 4x^3 + 2\alpha x + \epsilon^2 x(y^2 - 2\alpha x) - \epsilon^4\alpha^2 x}{1 - 3\epsilon^2 x}. \tag{4.6}$$

Both these objects are $O(\epsilon^2)$-perturbations of the corresponding objects for the continuous time system (4.2). The statement about the invariant two-form (4.5) is not difficult to prove. The following argument exemplifies considerations which hold for an arbitrary Kahan discretization (1.1). Differentiating Eqs. (4.3) with respect to x and to y, we obtain the columns of the matrix equation

$$\begin{pmatrix} 1 & -\epsilon/2 \\ -6\epsilon x & 1 \end{pmatrix} \frac{\partial(\widetilde{x}, \widetilde{y})}{\partial(x, y)} = \begin{pmatrix} 1 & \epsilon/2 \\ 6\epsilon\widetilde{x} & 1 \end{pmatrix},$$

whence

$$\det \frac{\partial(\widetilde{x}, \widetilde{y})}{\partial(x, y)} = \frac{1 - 3\epsilon^2\widetilde{x}}{1 - 3\epsilon^2 x}.$$

This is equivalent to the preservation of (4.5). The statement about the conserved quantity is most simply verified with any computer system for symbolic manipulations.

System (4.3) is known in the literature on integrable maps, although in a somewhat different form. Indeed, it is equivalent to the second order difference equation

$$\widetilde{x} - 2x + \underset{\sim}{x} = \epsilon^2\big(3x(\widetilde{x} + \underset{\sim}{x}) - \alpha\big) \quad \Leftrightarrow \quad \widetilde{x} - 2x + \underset{\sim}{x} = \frac{\epsilon^2(6x^2 - \alpha)}{1 - 3\epsilon^2 x}.$$

This equation belongs to class of integrable *QRT systems* [9, 10]; in order to see this, one should re-write it as

$$\widetilde{x} - 2x + \underset{\sim}{x} = \frac{\epsilon^2(6x^2 - \alpha)(1 + \epsilon^2 x)}{1 - 2\epsilon^2 x - 3\epsilon^4 x^2}.$$

This difference equation generates a map $(x, \underset{\sim}{x}) \mapsto (\widetilde{x}, x)$ which is symplectic, that is, preserves the two-form $\omega = dx \wedge d\widetilde{x}$, and possesses a biquadratic integral of motion

$$I(x, \widetilde{x}, \epsilon) = (\widetilde{x} - x)^2 - 2\epsilon^2 x\widetilde{x}(x + \widetilde{x}) + \epsilon^2\alpha(x + \widetilde{x}) - \epsilon^4(3x^2\widetilde{x}^2 - \alpha x\widetilde{x}).$$

Under the change of variables $(x, \widetilde{x}) \mapsto (x, y)$ given by the first equation in (4.4), these integrability attributes turn into the two-form (4.5) and the conserved quantity (4.6) (up to an additive constant).

We note that a more usual QRT discretization of the Weierstrass second order equation (4.1) would be

$$\widetilde{x} - 2x + \underset{\sim}{x} = \frac{\epsilon^2(6x^2 - \alpha)}{1 - 2\epsilon^2 x}, \tag{4.7}$$

with a simpler conserved quantity

$$J(x,\widetilde{x},\epsilon) = (\widetilde{x}-x)^2 - 2\epsilon^2 x\widetilde{x}(x+\widetilde{x}) + \epsilon^2\alpha(x+\widetilde{x}).$$

Eq. (4.7) is equivalent to

$$\widetilde{x} - 2x + \underset{\sim}{x} = \epsilon^2\big(2x(\widetilde{x}+\underset{\sim}{x}) + 2x^2 - \alpha\big),$$

which is not obtained by the Kahan–Hirota–Kimura method.

5. SOME TWO-DIMENSIONAL INTEGRABLE SYSTEMS

5.1. The Three-dimensional Suslov System

The three-dimensional nonholonomic Suslov problem [11] is defined by the following system of differential equations

$$\dot{m} = m \times \omega + \lambda a, \qquad \langle a,\omega\rangle = 0, \tag{5.1}$$

where $\omega = (\omega_1,\omega_2,\omega_3)^{\mathrm{T}}$ is the angular velocity, $m = (m_1,m_2,m_3)^{\mathrm{T}} = I\omega$ is the angular momentum, I is the inertia operator, a is a unit vector fixed in the body, and λ is the Lagrange multiplier. In a basis where $a = (0,0,1)^{\mathrm{T}}$ and

$$I = \begin{pmatrix} I_1 & 0 & I_{13} \\ 0 & I_2 & I_{23} \\ I_{13} & I_{23} & I_3 \end{pmatrix},$$

the constraint $\langle a,\omega\rangle = 0$ reduces to $\omega_3 = 0$, and equations of motion (5.1) read

$$\begin{cases} I_1\dot{\omega}_1 = -(I_{13}\omega_1 + I_{23}\omega_2)\omega_2, \\ I_2\dot{\omega}_2 = (I_{13}\omega_1 + I_{23}\omega_2)\omega_1, \\ I_{13}\dot{\omega}_1 + I_{23}\dot{\omega}_2 = (I_1 - I_2)\omega_1\omega_2 + \lambda. \end{cases} \tag{5.2}$$

The first two equations in (5.2) form a closed system for ω_1 and ω_2. It possesses a conserved quantity $H = I_1\omega_1^2 + I_2\omega_2^2$. After the solution of this system is found (Suslov gave it in terms of trigonometric and exponential functions), one finds the Lagrange multiplier λ from the third equation in (5.2).

To put the first two equations in (5.2) into a more convenient form, one can introduce the coordinates $x = I_{13}\omega_1 + I_{23}\omega_2$, $y = I_{23}I_1\omega_1 - I_{13}I_2\omega_2$, and arrives at

$$\begin{cases} \dot{x} = \alpha xy, \\ \dot{y} = -x^2, \end{cases} \tag{5.3}$$

where $\alpha = 1/I_1I_2$. This system admits a conserved quantity

$$H = x^2 + \alpha y^2.$$

Proposition 6 ([12]). *HK discretization of system (5.3),*

$$\begin{cases} \widetilde{x} - x = \epsilon\alpha(\widetilde{x}y + x\widetilde{y}), \\ \widetilde{y} - y = -2\epsilon\widetilde{x}x, \end{cases}$$

possesses an invariant two-form and an integral of motion, given by

$$\omega = \frac{dx\wedge dy}{x(x^2+\alpha y^2)}, \qquad H(\epsilon) = \frac{x^2+\alpha y^2}{1+\epsilon^2\alpha x^2}.$$

Actually, the invariant two-form was not given in [12]; rather, this paper contains an explicit solution of the discrete time Suslov system and its qualitative analysis.

5.2. Reduced Nahm Equations

In [13] Nahm equations associated with symmetric monopoles are considered. Assuming rotational symmetry groups of regular polytops leads to solutions of Nahm equations in terms of elliptic functions. Reduced equations corresponding to tetrahedrally symmetric monopoles of charge 3, to octahedrally symmetric monopoles of charge 4, and to icosahedrally symmetric monopoles of charge 6 are two-dimensional algebraically integrable systems with quadratic vector fields. More concretely, the reductions of Nahm equations derived in [13] read:

(i) tetrahedral symmetry:

$$\begin{cases} \dot{x} = x^2 - y^2, \\ \dot{y} = -2xy, \end{cases} \tag{5.4}$$

with an integral of motion $H = y(3x^2 - y^2)$;

(ii) octahedral symmetry:

$$\begin{cases} \dot{x} = 2x^2 - 12y^2, \\ \dot{y} = -6xy - 4y^2, \end{cases} \tag{5.5}$$

with an integral of motion $H = y(2x + 3y)(x - y)^2$;

(iii) icosahedral symmetry:

$$\begin{cases} \dot{x} = 2x^2 - y^2, \\ \dot{y} = -10xy + y^2, \end{cases} \tag{5.6}$$

with an integral of motion $H = y(3x - y)^2(4x + y)^3$.

HK discretizations of systems (5.4)–(5.6) turn out to be algebraically integrable.

Proposition 7.

(i) *HK discretization of system* (5.4),

$$\begin{cases} \widetilde{x} - x = \epsilon(\widetilde{x}x - \widetilde{y}y), \\ \widetilde{y} - y = -\epsilon(\widetilde{x}y + x\widetilde{y}), \end{cases}$$

possesses an invariant two-form and an integral of motion, given by

$$\omega = \frac{dx \wedge dy}{y(3x^2 - y^2)}, \quad H(\epsilon) = \frac{y(3x^2 - y^2)}{1 - \epsilon^2(x^2 + y^2)};$$

(ii) *HK discretization of system* (5.5),

$$\begin{cases} \widetilde{x} - x = \epsilon\left(2\widetilde{x}x - 12\widetilde{y}y\right), \\ \widetilde{y} - y = -\epsilon(3\widetilde{x}y + 3x\widetilde{y} + 4\widetilde{y}y), \end{cases}$$

possesses an invariant two-form and an integral of motion given by

$$\omega = \frac{dx \wedge dy}{y(2x + 3y)(x - y)},$$

$$H(\epsilon) = \frac{y(2x + 3y)(x - y)^2}{1 - 10\epsilon^2(x^2 + 4y^2) + \epsilon^4(9x^4 + 272x^3y - 352xy^3 + 696y^4)};$$

(iii) *HK discretization of system* (5.6),

$$\begin{cases} \widetilde{x}-x=\epsilon\left(2\widetilde{x}x-\widetilde{y}y\right), \\ \widetilde{y}-y=\epsilon(-5\widetilde{x}y-5x\widetilde{y}+\widetilde{y}y), \end{cases}$$

possesses an invariant two-form and an integral of motion given by

$$\omega=\frac{dx\wedge dy}{y(3x-y)(4x+y)},\quad H(\epsilon)=\frac{y(3x-y)^2(4x+y)^3}{1+\epsilon^2c_2+\epsilon^4c_4+\epsilon^6c_6},$$

with

$$\begin{aligned} &c_2=-7(5x^2-y^2),\\ &c_4=7(37x^4+22x^2y^2-2xy^3+2y^4),\\ &c_6=-225x^6+3840x^5y+80xy^5-514x^3y^3-19x^4y^2-206x^2y^4. \end{aligned}$$

6. EULER TOP

The differential equations of motion of the Euler top read

$$\begin{cases} \dot{x}_1=\alpha_1x_2x_3,\\ \dot{x}_2=\alpha_2x_3x_1,\\ \dot{x}_3=\alpha_3x_1x_2, \end{cases} \tag{6.1}$$

with real parameters α_i. This is one of the most famous integrable systems of the classical mechanics, with a big literature devoted to it. It can be explicitly integrated in terms of elliptic functions, and admits two functionally independent integrals of motion. Actually, a quadratic function $H(x)=\gamma_1x_1^2+\gamma_2x_2^2+\gamma_3x_3^2$ is an integral for Eqs. (6.1) as soon as $\gamma_1\alpha_1+\gamma_2\alpha_2+\gamma_2\alpha_2=0$. In particular, the following three functions are integrals of motion:

$$H_1=\alpha_2x_3^2-\alpha_3x_2^2,\qquad H_2=\alpha_3x_1^2-\alpha_1x_3^2,\qquad H_3=\alpha_1x_2^2-\alpha_2x_1^2.$$

Clearly, only two of them are functionally independent because of $\alpha_1H_1+\alpha_2H_2+\alpha_3H_3=0$. These integrals appear also on the right-hand sides of the quadratic (in this case even linear) expressions for the Wronskians of the coordinates x_j:

$$\begin{cases} \dot{x}_2x_3-x_2\dot{x}_3=H_1x_1,\\ \dot{x}_3x_1-x_3\dot{x}_1=H_2x_2,\\ \dot{x}_1x_2-x_1\dot{x}_2=H_3x_3. \end{cases} \tag{6.2}$$

Moreover, one easily sees that the coordinates x_j satisfy the following differential equations with the coefficients depending on the integrals of motion:

$$\begin{cases} \dot{x}_1^2=(H_3+\alpha_2x_1^2)(\alpha_3x_1^2-H_2),\\ \dot{x}_2^2=(H_1+\alpha_3x_2^2)(\alpha_1x_2^2-H_3),\\ \dot{x}_3^2=(H_2+\alpha_1x_3^2)(\alpha_2x_3^2-H_1). \end{cases}$$

The fact that the polynomials on the right-hand sides of these equations are of degree four implies that the solutions are given by elliptic functions.

The HK discretization of the Euler top [4] is:

$$\begin{cases} \widetilde{x}_1 - x_1 = \epsilon\alpha_1(\widetilde{x}_2 x_3 + x_2\widetilde{x}_3), \\ \widetilde{x}_2 - x_2 = \epsilon\alpha_2(\widetilde{x}_3 x_1 + x_3\widetilde{x}_1), \\ \widetilde{x}_3 - x_3 = \epsilon\alpha_3(\widetilde{x}_1 x_2 + x_1\widetilde{x}_2). \end{cases} \tag{6.3}$$

(In this form it corresponds to the stepsize 2ϵ rather than ϵ.) The map $f : x \mapsto \widetilde{x}$ obtained by solving (6.3) for $\widetilde{x}$ is given by:

$$\widetilde{x} = f(x,\epsilon) = A^{-1}(x,\epsilon)x, \qquad A(x,\epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix}. \tag{6.4}$$

It might be instructive to have a look at the explicit formulas for this map:

$$\begin{cases} \widetilde{x}_1 = \dfrac{x_1 + 2\epsilon\alpha_1 x_2 x_3 + \epsilon^2 x_1(-\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\[2ex] \widetilde{x}_2 = \dfrac{x_2 + 2\epsilon\alpha_2 x_3 x_1 + \epsilon^2 x_2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\[2ex] \widetilde{x}_3 = \dfrac{x_3 + 2\epsilon\alpha_3 x_1 x_2 + \epsilon^2 x_3(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \end{cases} \tag{6.5}$$

where

$$\Delta(x,\epsilon) = \det A(x,\epsilon) = 1 - \epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2) - 2\epsilon^3\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3. \tag{6.6}$$

We will use the abbreviation dET for this map. As is always the case for a HK discretization, dET is birational, with the reversibility property expressed as $f^{-1}(x,\epsilon) = f(x,-\epsilon)$.

Proposition 8. ([4, 14]) *The quantities*

$$F_1 = \frac{1 - \epsilon^2\alpha_3\alpha_1 x_2^2}{1 - \epsilon^2\alpha_1\alpha_2 x_3^2}, \qquad F_2 = \frac{1 - \epsilon^2\alpha_1\alpha_2 x_3^2}{1 - \epsilon^2\alpha_2\alpha_3 x_1^2}, \qquad F_3 = \frac{1 - \epsilon^2\alpha_2\alpha_3 x_1^2}{1 - \epsilon^2\alpha_3\alpha_1 x_2^2},$$

are conserved quantities of dET. Of course, there are only two independent integrals since $F_1F_2F_3 = 1$.

The relation between F_i and the integrals H_i of the continuous time Euler top is straightforward: $F_i = 1 + \epsilon^2\alpha_i H_i + O(\epsilon^4)$. As a corollary of Proposition 8, we find that, for any conserved quantity H of the Euler top which is a linear combination of the integrals H_1, H_2, H_3, the three functions $H/(1 - \epsilon^2\alpha_j\alpha_k x_i^2)$ are conserved quantities of dET. Hereafter (i,j,k) are cyclic permutations of $(1,2,3)$. In particular, the functions

$$H_i(\epsilon) = \frac{\alpha_j x_k^2 - \alpha_k x_j^2}{1 - \epsilon^2\alpha_j\alpha_k x_i^2} \tag{6.7}$$

are conserved quantities of dET. Again, only two of them are independent, since

$$\alpha_1 H_1(\epsilon) + \alpha_2 H_2(\epsilon) + \alpha_3 H_3(\epsilon) + \epsilon^4\alpha_1\alpha_2\alpha_3 H_1(\epsilon)H_2(\epsilon)H_3(\epsilon) = 0.$$

Proposition 9. ([14]) *The map dET possesses an invariant volume form:*

$$\det\frac{\partial\widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)},$$

where $\phi(x)$ is any of the functions

$$\phi(x) = (1-\epsilon^2\alpha_i\alpha_j x_k^2)(1-\epsilon^2\alpha_j\alpha_k x_i^2) \quad \text{or} \quad (1-\epsilon^2\alpha_i\alpha_j x_k^2)^2.$$

(The ratio of any two functions $\phi(x)$ is an integral of motion, due to Proposition 8).

The proof is based on formula (2.2) with the matrix $A(x,\epsilon)$ given in (6.4). Its determinant is given in (6.6).

A proper discretization of the Wronskian differential equations (6.2) is given by the following statement.

Proposition 10. *The following relations hold true for dET:*

$$\begin{cases} \widetilde{x}_2 x_3 - x_2\widetilde{x}_3 = \epsilon H_1(\epsilon)(\widetilde{x}_1 + x_1), \\ \widetilde{x}_3 x_1 - x_3\widetilde{x}_1 = \epsilon H_2(\epsilon)(\widetilde{x}_3 + x_3), \\ \widetilde{x}_1 x_2 - x_1\widetilde{x}_2 = \epsilon H_3(\epsilon)(\widetilde{x}_3 + x_3), \end{cases} \tag{6.8}$$

with the functions $H_i(\epsilon)$ from (6.7).

The proof is based on relations

$$\widetilde{x}_i + x_i = \frac{2(1-\epsilon^2\alpha_j\alpha_k x_i^2)(x_i + \epsilon\alpha_i x_j x_k)}{\Delta(x,\epsilon)}, \tag{6.9}$$

$$\widetilde{x}_j x_k - x_j\widetilde{x}_k = \frac{2\epsilon(\alpha_j x_k^2 - \alpha_k x_j^2)(x_i + \epsilon\alpha_i x_j x_k)}{\Delta(x,\epsilon)}, \tag{6.10}$$

which follow easily from the explicit formulas (6.5). They should be compared with

$$\widetilde{x}_i - x_i = \epsilon\alpha_i(\widetilde{x}_j x_k + x_j\widetilde{x}_k) = \frac{2\epsilon\alpha_i(x_j + \epsilon\alpha_j x_k x_i)(x_k + \epsilon\alpha_k x_i x_j)}{\Delta(x,\epsilon)}. \tag{6.11}$$

As pointed out in [2], a probable way to the discovery of the conserved quantities of dET in [4] was through finding the HK bases for this map. In this respect, one has the following results.

Proposition 11. ([2])

(a) *The set $\Phi = (x_1^2, x_2^2, x_3^2, 1)$ is a HK basis for dET with* $\dim K_\Phi(x) = 2$. *Therefore, any orbit of dET lies on the intersection of two quadrics in $\mathbb{R}^3$.*

(b) *The set $\Phi_0 = (x_1^2, x_2^2, x_3^2)$ is a HK basis for dET with* $\dim K_{\Phi_0}(x) = 1$. *At each point $x \in \mathbb{R}^3$ we have:*

$$K_{\Phi_0}(x) = [c_1 : c_2 : c_3] = [\,\alpha_2 x_3^2 - \alpha_3 x_2^2 : \alpha_3 x_1^2 - \alpha_1 x_3^2 : \alpha_1 x_2^2 - \alpha_2 x_1^2\,].$$

Setting $c_3 = -1$, the following functions are integrals of motion of dET:

$$c_1(x) = \frac{\alpha_3 x_2^2 - \alpha_2 x_3^2}{\alpha_1 x_2^2 - \alpha_2 x_1^2}, \qquad c_2(x) = \frac{\alpha_1 x_3^2 - \alpha_3 x_1^2}{\alpha_1 x_2^2 - \alpha_2 x_1^2}. \tag{6.12}$$

(c) *The set $\Phi_{12} = (x_1^2, x_2^2, 1)$ is a further HK basis for dET with* $\dim K_{\Phi_{12}}(x) = 1$. *At each point $x \in \mathbb{R}^3$ we have: $K_{\Phi_{12}}(x) = [d_1 : d_2 : -1]$, where*

$$d_1(x) = -\frac{\alpha_2(1-\epsilon^2\alpha_3\alpha_1 x_2^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2}, \qquad d_2(x) = \frac{\alpha_1(1-\epsilon^2\alpha_2\alpha_3 x_1^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2}. \tag{6.13}$$

These functions are integrals of motion of dET independent on the integrals (6.12). *We have: $K_\Phi(x) = K_{\Phi_0} \oplus K_{\Phi_{12}}$.*

Proof. To prove statement (b), we solve the system

$$\begin{cases} c_1x_1^2 + c_2x_2^2 = x_3^2, \\ c_1\widetilde{x}_1^2 + c_2\widetilde{x}_2^2 = \widetilde{x}_3^2. \end{cases}$$

The solution is given, according to the Cramer's rule, by ratios of determinants of the type

$$\begin{vmatrix} x_i^2 & x_j^2 \\ \widetilde{x}_i^2 & \widetilde{x}_j^2 \end{vmatrix} = \frac{4\epsilon(\alpha_j x_i^2 - \alpha_i x_j^2)(x_1 + \epsilon\alpha_1 x_2 x_3)(x_2 + \epsilon\alpha_2 x_3 x_1)(x_3 + \epsilon\alpha_3 x_1 x_2)}{\Delta^2(x,\epsilon)}. \tag{6.14}$$

(Here we used (6.10), (6.11)). In the ratios of such determinants everything cancels out, except for the factors $\alpha_j x_i^2 - \alpha_i x_j^2$, so we end up with (6.12). The cancelation of the denominators $\Delta^2(x,\epsilon)$ is, of course, no wonder, but the cancelation of all the non-even factors in the numerators is rather *remarkable and miraculous* and is not granted by any well-understood mechanism. Since the components of the solution do not depend on ϵ, we conclude that functions (6.12) are integrals of motion of dET.

To prove statement (c), we solve the system

$$\begin{cases} d_1x_1^2 + d_2x_2^2 = 1, \\ d_1\widetilde{x}_1^2 + d_2\widetilde{x}_2^2 = 1. \end{cases}$$

The solution is given by Eq. (6.13), due to Eq. (6.14) and the similar formula

$$\begin{vmatrix} 1 & x_i^2 \\ 1 & \widetilde{x}_i^2 \end{vmatrix} = \frac{4\epsilon\alpha_i(1 - \epsilon^2\alpha_j\alpha_k x_i^2)(x_1 + \epsilon\alpha_1 x_2 x_3)(x_2 + \epsilon\alpha_2 x_3 x_1)(x_3 + \epsilon\alpha_3 x_1 x_2)}{\Delta^2(x,\epsilon)},$$

which, in turn, follows from (6.9) and (6.10). This time the solution does depend on ϵ, but consists of manifestly even functions of ϵ. Everything non-even luckily cancels, again. Therefore, functions (6.13) are integrals of motion of dET.

Although each one of the HK bases Φ_0, Φ_1 delivers apparently two integrals of motion (6.12), each pair turns out to be *functionally dependent*, as

$$\alpha_1 c_1(x) + \alpha_2 c_2(x) = \alpha_3, \quad \alpha_1 d_1(x) + \alpha_2 d_2(x) = \epsilon^2\alpha_1\alpha_2\alpha_3.$$

However, functions c_1, c_2 are independent on d_1, d_2, since the former depend on x_3, while the latter do not.

Of course, permutational symmetry yields that each of the sets of monomials $\Phi_{23} = (x_2^2,\, x_3^2,\, 1)$ and $\Phi_{13} = (x_1^2,\, x_3^2,\, 1)$ is a HK basis, as well, with $\dim K_{\Phi_{23}}(x) = \dim K_{\Phi_{13}}(x) = 1$. But we do not obtain additional linearly independent null-spaces, as any two of the four found one-dimensional null-spaces span the full null-space $K_\Phi(x)$.

Summarizing, we have found a HK basis with a two-dimensional null-space, as well as two functionally independent conserved quantities for the HK discretization of the Euler top. Both results yield integrability of this discretization, in the sense that its orbits are confined to closed curves in $\mathbb{R}^3$. Moreover, each such curve is an intersection of two quadrics, which in the general position case is an elliptic curve.

Proposition 12. *Each component x_i of any solution of dET satisfies a relation of the type $P_i(x_i, \widetilde{x}_i) = 0$, where P_i is a biquadratic polynomial whose coefficients are integrals of motion of dET:*

$$P_i(x_i, \widetilde{x}_i) = p_i^{(3)} x_i^2 \widetilde{x}_i^2 + p_i^{(2)}(x_i^2 + \widetilde{x}_i^2) + p_i^{(1)} x_i \widetilde{x}_i + p_i^{(0)} = 0,$$

with

$$p_i^{(3)} = -4\epsilon^2\alpha_j\alpha_k, \quad p_i^{(2)} = \big(1 + \epsilon^2\alpha_j H_j(\epsilon)\big)\big(1 - \epsilon^2\alpha_k H_k(\epsilon)\big),$$
$$p_i^{(1)} = -2\big(1 - \epsilon^2\alpha_j H_j(\epsilon)\big)\big(1 + \epsilon^2\alpha_k H_k(\epsilon)\big), \quad p_i^{(0)} = 4\epsilon^2 H_j(\epsilon) H_k(\epsilon).$$

Proof. From Eqs. (6.3) and (6.8) there follows:

$$\frac{(\widetilde{x}_i - x_i)^2}{(\epsilon\alpha_i)^2} + \epsilon^2 H_i^2(\epsilon)(\widetilde{x}_i + x_i)^2 = 2(\widetilde{x}_j^2 x_k^2 + x_j^2 \widetilde{x}_k^2).$$

It remains to express x_j^2 and x_k^2 through x_i^2 and integrals $H_j(\epsilon)$, $H_k(\epsilon)$ given in Eq. (6.7).

It follows from Proposition 12 that solutions $x_i(t)$ as functions of the discrete time $t \in 2\epsilon\mathbb{Z}$ are given by elliptic functions of order 2 (the order of an elliptic function is the number of the zeroes or poles it possesses in a period parallelogram).

We would like to point out that Propositions 10 and 12 can be interpreted as existence of further HK bases. For instance, according to Proposition 10, each pair $(\widetilde{x}_j x_k - x_j \widetilde{x}_k, \widetilde{x}_i + x_i)$ is a HK basis with a 1-dimensional null-space. Similarly, Proposition 12 says that for each $i = 1, 2, 3$, the set $x_i^p \widetilde{x}_i^q$ $(0 \leqslant p, q \leqslant 2)$ is a HK basis with a 1-dimensional null-space. Of course, due to the dependence on the shifted variables $\widetilde{x}$, these HK bases consist of complicated functions of x rather than of monomials. A further instance of HK bases of this sort is given in the following statement. Compared with Proposition 11, it says that for dET, *for each HK basis consisting of monomials quadratic in x, the corresponding set of monomials bilinear in $x, \widetilde{x}$ is a HK basis, as well.* This seems to be a quite general phenomenon, further issues of which will appear later several times.

Proposition 13.

(a) *The set* $\Psi = (\widetilde{x}_1 x_1,\ \widetilde{x}_2 x_2,\ \widetilde{x}_3 x_3,\ 1)$ *is a HK basis for dET with* $\dim K_\Psi(x) = 2$.

(b) *The set* $\Psi_0 = (\widetilde{x}_1 x_1,\ \widetilde{x}_2 x_2,\ \widetilde{x}_3 x_3)$ *is a HK basis for dET with* $\dim K_{\Psi_0}(x) = 1$. *At each point* $x \in \mathbb{R}^3$, *the homogeneous coordinates* $\bar{c}_i$ *of the null-space* $K_{\Psi_0}(x) = [\bar{c}_1 : \bar{c}_2 : \bar{c}_3]$ *are given by*

$$\bar{c}_i = (\alpha_j x_k^2 - \alpha_k x_j^2)\big(1 - \epsilon^2(\alpha_i\alpha_j x_k^2 + \alpha_k\alpha_i x_j^2 - \alpha_j\alpha_k x_i^2)\big).$$

The quotients $\bar{c}_i/\bar{c}_j$ *are integrals of motion of dET.*

(c) *The set* $\Psi_{12} = (\widetilde{x}_1 x_1,\ \widetilde{x}_2 x_2,\ 1)$ *is a further HK basis for dET with* $\dim K_{\Psi_{12}}(x) = 1$. *At each point* $x \in \mathbb{R}^3$, *there holds:* $K_{\Psi_{12}}(x) = [\bar{d}_1 : \bar{d}_2 : -1]$, *where*

$$\bar{d}_1(x) = -\frac{\alpha_2(1 - \epsilon^2\alpha_3\alpha_1 x_2^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2}\,\frac{1 - \epsilon^2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{1 - \epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2)},$$

$$\bar{d}_2(x) = \frac{\alpha_1(1 - \epsilon^2\alpha_2\alpha_3 x_1^2)}{\alpha_1 x_2^2 - \alpha_2 x_1^2}\,\frac{1 - \epsilon^2(\alpha_3\alpha_1 x_2^2 - \alpha_2\alpha_3 x_1^2 + \alpha_1\alpha_2 x_3^2)}{1 - \epsilon^2(\alpha_3\alpha_1 x_2^2 + \alpha_2\alpha_3 x_1^2 - \alpha_1\alpha_2 x_3^2)},$$

are integrals of dET. We have: $K_\Psi(x) = K_{\Psi_0}(x) \oplus K_{\Psi_{12}}(x)$.

7. ZHUKOVSKI–VOLTERRA SYSTEM

The gyroscopic Zhukovski–Volterra (ZV) system is a generalization of the Euler top. It describes the free motion of a rigid body carrying an asymmetric rotor (gyrostat) [15]. Equations of motion of the ZV system read

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3 + \beta_3 x_2 - \beta_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1 + \beta_1 x_3 - \beta_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2 + \beta_2 x_1 - \beta_1 x_2, \end{cases} \tag{7.1}$$

with α_i, β_i being real parameters of the system. For $(\beta_1, \beta_2, \beta_3) = (0, 0, 0)$, the flow (7.1) reduces to the Euler top (6.1). The ZV system is (Liouville and algebraically) integrable under the condition

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{7.2}$$

It can be explicitly integrated in terms of elliptic functions, see [15] and also [16] for a more recent exposition. The following quantities are integrals of motion of the ZV system:

$$H_1 = \alpha_2 x_3^2 - \alpha_3 x_2^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3),$$

$$H_2 = \alpha_3 x_1^2 - \alpha_1 x_3^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3), \qquad (7.3)$$
$$H_3 = \alpha_1 x_2^2 - \alpha_2 x_1^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3).$$

Clearly, only two of them are functionally independent because of $\alpha_1 H_1 + \alpha_2 H_2 + \alpha_3 H_3 = 0$. Note that

$$H_2 - H_1 = \alpha_3 C, \quad H_3 - H_2 = \alpha_1 C, \quad H_1 - H_3 = \alpha_2 C,$$

with $C = x_1^2 + x_2^2 + x_3^2$.

As in the Euler case, the Wronskians of the coordinates x_j admit quadratic expressions with coefficients dependent on the integrals of motion:

$$\begin{cases} \dot{x}_2 x_3 - x_2 \dot{x}_3 = H_1 x_1 + x_1(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \beta_1 C, \\ \dot{x}_3 x_1 - x_3 \dot{x}_1 = H_2 x_2 + x_2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \beta_2 C, \\ \dot{x}_1 x_2 - x_1 \dot{x}_2 = H_3 x_3 + x_3(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \beta_3 C. \end{cases} \qquad (7.4)$$

The HK discretization of the ZV system is:

$$\begin{cases} \widetilde{x}_1 - x_1 = \epsilon\alpha_1(\widetilde{x}_2 x_3 + x_2 \widetilde{x}_3) + \epsilon\beta_3(\widetilde{x}_2 + x_2) - \epsilon\beta_2(\widetilde{x}_3 + x_3), \\ \widetilde{x}_2 - x_2 = \epsilon\alpha_2(\widetilde{x}_3 x_1 + x_3 \widetilde{x}_1) + \epsilon\beta_1(\widetilde{x}_3 + x_3) - \epsilon\beta_3(\widetilde{x}_1 + x_1), \\ \widetilde{x}_3 - x_3 = \epsilon\alpha_3(\widetilde{x}_1 x_2 + x_1 \widetilde{x}_2) + \epsilon\beta_2(\widetilde{x}_1 + x_1) - \epsilon\beta_1(\widetilde{x}_2 + x_2). \end{cases} \qquad (7.5)$$

The map $f : x \mapsto \widetilde{x}$ obtained by solving (7.5) for $\widetilde{x}$ is given by:

$$\widetilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)(\mathbb{1} + \epsilon B)x,$$

with

$$A(x, \epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix} - \epsilon B, \quad B = \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix}.$$

We will call this map dZV. Formula (2.2) holds true for dZV, as for any HK discretization.

7.1. ZV System with Two Vanishing β_k's

In the case where two out of three β_k's vanish, say $\beta_2 = \beta_3 = 0$, the condition (7.2) is not necessary for integrability of the ZV system. The functions H_2 and H_3 as given in (7.3) (with $\beta_2 = \beta_3 = 0$) are in this case conserved quantities without any condition on α_k's, while their linear combinations H_1 and C are given by

$$H_1 = -\frac{1}{\alpha_1}(\alpha_2 H_2 + \alpha_3 H_3) = \alpha_2 x_3^2 - \alpha_3 x_2^2 + 2\beta_1 \frac{\alpha_2 + \alpha_3}{\alpha_1} x_1,$$
$$C = \frac{1}{\alpha_1}(H_3 - H_2) = x_2^2 + x_3^2 - \frac{\alpha_2 + \alpha_3}{\alpha_1} x_1^2.$$

Wronskian relations (7.4) are replaced by

$$\begin{cases} \dot{x}_2 x_3 - x_2 \dot{x}_3 = H_1 x_1 - \beta_1 \dfrac{\alpha_2 + \alpha_3}{\alpha_1} x_1^2 + \beta_1 C, \\ \dot{x}_3 x_1 - x_3 \dot{x}_1 = H_2 x_2 + \beta_1 x_1 x_2, \\ \dot{x}_1 x_2 - x_1 \dot{x}_2 = H_3 x_3 + \beta_1 x_1 x_3. \end{cases} \qquad (7.6)$$

The HK discretization of the ZV system with $\beta_2 = \beta_3 = 0$ turns out to possess two conserved quantities (without imposing condition (7.2)) and an invariant measure.

Proposition 14. *The functions*

$$H_2(\epsilon) = \frac{\alpha_3 x_1^2 - \alpha_1 x_3^2 - 2\beta_1 x_1 + \epsilon^2\beta_1^2\alpha_1 x_2^2}{1 - \epsilon^2\alpha_3\alpha_1 x_2^2},$$
$$H_3(\epsilon) = \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2 - 2\beta_1 x_1 - \epsilon^2\beta_1^2\alpha_1 x_3^2}{1 - \epsilon^2\alpha_1\alpha_2 x_3^2},$$

are conserved quantities of dZV with $\beta_2 = \beta_3 = 0$.

Proposition 15. *The map dZV with* $\beta_2 = \beta_3 = 0$ *possesses an invariant volume form:*

$$\det\frac{\partial\widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)},$$

with $\phi(x) = (1 - \epsilon^2\alpha_3\alpha_1 x_2^2)(1 - \epsilon^2\alpha_1\alpha_2 x_3^2)$.

The conserved quantities of Proposition 14 appear on the right-hand sides of the following relations which are the discrete versions of the Wronskian relations (7.6):

Proposition 16. *The following relations hold true for dZV with* $\beta_2 = \beta_3 = 0$*:*

$$\begin{cases} \widetilde{x}_2 x_3 - x_2\widetilde{x}_3 = \epsilon c_1(\widetilde{x}_1 + x_1) + 2\epsilon c_2\widetilde{x}_1 x_1 + 2\epsilon c_3, \\ \widetilde{x}_3 x_1 - x_3\widetilde{x}_1 = \epsilon H_2(\epsilon)(\widetilde{x}_2 + x_2) + \epsilon\beta_1(\widetilde{x}_1 x_2 + x_1\widetilde{x}_2), \\ \widetilde{x}_1 x_2 - x_1\widetilde{x}_2 = \epsilon H_3(\epsilon)(\widetilde{x}_3 + x_3) + \epsilon\beta_1(\widetilde{x}_1 x_3 + x_1\widetilde{x}_3), \end{cases}$$

with

$$c_1 = -\frac{\alpha_2 H_2(\epsilon) + \alpha_3 H_3(\epsilon)}{\alpha_1\Delta}, \quad c_2 = -\frac{\beta_1(\alpha_2 + \alpha_3)}{\alpha_1\Delta}, \quad c_3 = \frac{\beta_1\big(H_3(\epsilon) - H_2(\epsilon)\big)}{\alpha_1\Delta},$$

$$\Delta = 1 + \epsilon^4\big(\alpha_2 H_3(\epsilon) - \beta_1^2\big)\big(\alpha_3 H_2(\epsilon) + \beta_1^2\big).$$

Next, we describe the HK bases found in this case.

Proposition 17.

(a) *The set* $\Phi = (x_1^2, x_2^2, x_3^2, x_1, 1)$ *is a HK basis for dZV with* $\beta_2 = \beta_3 = 0$, *with* $\dim K_\Phi(x) = 2$. *Any orbit of dZV with* $\beta_2 = \beta_3 = 0$ *is thus confined to the intersection of two quadrics in* $\mathbb{R}^3$.

(b) *The set* $\Phi_0 = (x_1^2, x_2^2, x_3^2, 1)$ *is a HK basis for dZV with* $\beta_2 = \beta_3 = 0$, *with* $\dim K_{\Phi_0}(x) = 1$. *At each point* $x \in \mathbb{R}^3$ *we have:* $K_{\Phi_0}(x) = [-1 : d_2 : d_3 : d_4]$, *where*

$$d_2 = \frac{\alpha_1}{\alpha_2 + \alpha_3}\big(1 - \epsilon^2\beta_1^2 - \epsilon^2\alpha_3 H_2(\epsilon)\big), \quad d_3 = \frac{\alpha_1}{\alpha_2 + \alpha_3}\big(1 - \epsilon^2\beta_1^2 + \epsilon^2\alpha_2 H_3(\epsilon)\big),$$

$$d_4 = \frac{1}{\alpha_2 + \alpha_3}\big(H_2(\epsilon) - H_3(\epsilon)\big).$$

(c) *The set* $\Phi_{23} = (x_2^2, x_3^2, x_1, 1)$ *is a HK basis for dZV with* $\beta_2 = \beta_3 = 0$, *with* $\dim K_{\Phi_{23}}(x) = 1$. *At each point* $x \in \mathbb{R}^3$ *we have:* $K_{\Phi_{23}}(x) = [c_1 : c_2 : c_3 : c_4]$, *where*

$$c_1 = \alpha_1\big(\alpha_3 + \epsilon^2\beta_1^2\alpha_2 + \epsilon^2\alpha_2\alpha_3 H_2(\epsilon)\big), \quad c_2 = -\alpha_1\big(\alpha_2 + \epsilon^2\beta_1^2\alpha_3 - \epsilon^2\alpha_2\alpha_3 H_3(\epsilon)\big),$$

$$c_3 = -2\beta_1(\alpha_2 + \alpha_3), \quad c_4 = -\big(\alpha_2 H_2(\epsilon) + \alpha_3 H_3(\epsilon)\big).$$

Unlike the case of dET, we see that here a HK basis with a one dimensional null-space already provides more than one independent integral of motion.

"Bilinear" versions of the above HK bases also exist:

Proposition 18. *The set* $\Psi = (x_1\widetilde{x}_1,\, x_2\widetilde{x}_2,\, x_3\widetilde{x}_3,\, x_1+\widetilde{x}_1,\, 1)$ *is a HK basis for dZV with* $\beta_2 = \beta_3 = 0$*, with* $\dim K_\Psi(x) = 2$*. The sets*

$$\Psi_0 = (x_1\widetilde{x}_1,\, x_2\widetilde{x}_2,\, x_3\widetilde{x}_3,\, 1) \quad \text{and} \quad \Psi_{23} = (x_2\widetilde{x}_2,\, x_3\widetilde{x}_3,\, x_1+\widetilde{x}_1,\, 1)$$

are HK bases with one-dimensional null-spaces.

The following statement is a starting point towards an explicit integration of the map dZV with $\beta_2 = \beta_3 = 0$ in terms of elliptic functions.

Proposition 19. *The component* x_1 *of the solution of the difference equations (7.5) satisfies a relation of the type*

$$P(x_1, \widetilde{x}_1) = p_0 x_1^2\widetilde{x}_1^2 + p_1 x_1\widetilde{x}_1(x_1+\widetilde{x}_1) + p_2(x_1^2+\widetilde{x}_1^2) + p_3 x_1\widetilde{x}_1 + p_4(x_1+\widetilde{x}_1) + p_5 = 0,$$

coefficients of the biquadratic polynomial P *being conserved quantities of dZV with* $\beta_2 = \beta_3 = 0$*.*

Proof is parallel to that of Proposition 12.

7.2. ZV System with One Vanishing β_k

In the case $\beta_3 = 0$ (say) and generic values of other parameters, the ZV system has only one integral H_3 and is therefore non-integrable. One of the Wronskian relations holds true in this general situation:

$$\dot{x}_1 x_2 - x_1\dot{x}_2 = H_3 x_3 + \beta_1 x_1 x_3 + \beta_2 x_2 x_3. \tag{7.7}$$

Under condition (7.2), the ZV system becomes integrable, with all the results formulated in the general case.

Similarly, the map dZV with $\beta_3 = 0$ and generic values of other parameters possesses one conserved quantity:

$$H_3(\epsilon) = \frac{\alpha_1 x_2^2 - \alpha_2 x_1^2 - 2(\beta_1 x_1 + \beta_2 x_2) - \epsilon^2(\beta_1^2\alpha_1 + \beta_2^2\alpha_2)x_3^2}{1 - \epsilon^2\alpha_1\alpha_2 x_3^2}.$$

Clearly, this fact can be re-formulated as the existence of a HK basis $\Phi = (x_1^2, x_2^2, x_3^2, x_1, x_2, 1)$ with $\dim K_\Phi = 1$. The Wronskian relation (7.7) possesses a decent discretization:

$$\widetilde{x}_1 x_2 - x_1\widetilde{x}_2 = \epsilon H_3(\epsilon)(x_3+\widetilde{x}_3) + \epsilon\beta_1(\widetilde{x}_1 x_3 + x_1\widetilde{x}_3) + \epsilon\beta_2(\widetilde{x}_2 x_3 + x_2\widetilde{x}_3).$$

However, it seems that the map dZV with $\beta_3 = 0$ does not acquire an additional integral of motion under condition (7.2). It might be conjectured that in order to assure the integrability of the dZV map with $\beta_3 = 0$, its other parameters have to satisfy some relation which is an $O(\epsilon)$-deformation of (7.2).

7.3. ZV System with All β_k*'s Non-vanishing*

Numerical experiments indicate non-integrability for the map (7.5) with non-vanishing β_k's, even under condition (7.2). Nevertheless, some other relation between the parameters might yield integrability. In this connection we notice that the map dZV with $(\alpha_1, \alpha_2, \alpha_3) = (\alpha, -\alpha, 0)$ admits a polynomial conserved quantity

$$H = -\alpha x_3^2 - 2(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) + \epsilon^2\alpha(\beta_2 x_1 - \beta_1 x_2)^2.$$

8. VOLTERRA CHAIN

8.1. Periodic Volterra Chain with $N = 3$ Particles

Equations of motion of the periodic Volterra chain with three particles (VC_3, for short):

$$\begin{cases} \dot{x}_1 = x_1(x_2 - x_3), \\ \dot{x}_2 = x_2(x_3 - x_1), \\ \dot{x}_3 = x_3(x_1 - x_2). \end{cases} \tag{8.1}$$

This system is Liouville and algebraically integrable, with the following two independent integrals of motion:

$$H_1 = x_1 + x_2 + x_3, \qquad H_2 = x_1x_2x_3.$$

There hold the following Wronskian relations:

$$\dot{x}_i x_j - x_i \dot{x}_j = H_1 x_i x_j - 3H_2. \tag{8.2}$$

Eliminating x_j, x_k from equation of motion for x_i with the help of integrals of motion, one arrives at

$$\dot{x}_i^2 = x_i^2(x_i - H_1)^2 - 4H_2 x_i,$$

which yields a solution in terms of elliptic functions.

The HK discretization of system (8.1) (with the time step 2ϵ) is:

$$\begin{cases} \widetilde{x}_1 - x_1 = \epsilon x_1(\widetilde{x}_2 - \widetilde{x}_3) + \epsilon \widetilde{x}_1(x_2 - x_3), \\ \widetilde{x}_2 - x_2 = \epsilon x_2(\widetilde{x}_3 - \widetilde{x}_1) + \epsilon \widetilde{x}_2(x_3 - x_1), \\ \widetilde{x}_3 - x_3 = \epsilon x_3(\widetilde{x}_1 - \widetilde{x}_2) + \epsilon \widetilde{x}_3(x_1 - x_2). \end{cases} \tag{8.3}$$

The map $f : x \mapsto \widetilde{x}$ obtained by solving (8.3) for $\widetilde{x}$ is given by:

$$\widetilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)x,$$

with

$$A(x, \epsilon) = \begin{pmatrix} 1 + \epsilon(x_3 - x_2) & -\epsilon x_1 & \epsilon x_1 \\ \epsilon x_2 & 1 + \epsilon(x_1 - x_3) & -\epsilon x_2 \\ -\epsilon x_3 & \epsilon x_3 & 1 + \epsilon(x_2 - x_1) \end{pmatrix}.$$

Explicitly:

$$\widetilde{x}_i = x_i \, \frac{1 + 2\epsilon(x_j - x_k) + \epsilon^2\big((x_j + x_k)^2 - x_i^2\big)}{1 - \epsilon^2(x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1)}. \tag{8.4}$$

This map will be called dVC_3. From Proposition 2 there follows immediately:

Proposition 20. *The map dVC_3 possesses an invariant volume form:*

$$\det \frac{\partial \widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)},$$

with $\phi(x) = x_1x_2x_3$.

Concerning integrability of dVC_3, we note first of all that H_1 is an obvious conserved quantity. The second one is most easily obtained from the following discretization of Wronskian relations (8.2).

Proposition 21. *For the map dVC_3, the following relations hold:*

$$\widetilde{x}_i x_j - x_i \widetilde{x}_j = \epsilon H_1(\widetilde{x}_i x_j + x_i \widetilde{x}_j) - 6\epsilon H_2(\epsilon)\left(1 - \tfrac{1}{3}\epsilon^2 H_1^2\right), \tag{8.5}$$

where $H_2(\epsilon)$ is a conserved quantity, given by

$$H_2(\epsilon) = \frac{x_1 x_2 x_3}{1 - \epsilon^2(x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1)}. \tag{8.6}$$

Proof. Define $H_2(\epsilon)$ by Eq. (8.5). It is easily computed with explicit formulas (8.4). The result given by (8.6) is a manifestly even function of ϵ and therefore an integral of motion.

Proposition 22.

(a) *The set $\Phi_{ij} = (x_i x_j(x_i + x_j),\ x_i^2 + x_j^2,\ x_i x_j,\ x_i + x_j,\ 1)$ is a HK basis for the map dVC_3 with $\dim K_{\Phi_{ij}}(x) = 1$. In other words, the pairs (x_i, x_j) lie on a cubic curve*

$$P(x_i, x_j) = p_0 x_i x_j (x_i + x_j) + p_1(x_i^2 + x_j^2) + p_2 x_i x_j + p_3(x_i + x_j) + p_4 = 0,$$

whose coefficients p_m are constant (expressed through integrals of motion).

(b) *The set $\Psi_i = (x_i^2 \widetilde{x}_i^2,\ x_i \widetilde{x}_i(x_i + \widetilde{x}_i),\ x_i^2 + \widetilde{x}_i^2,\ x_i \widetilde{x}_i,\ x_i + \widetilde{x}_i,\ 1)$ is a HK basis for the map dVC_3 with $\dim K_{\Psi_i}(x) = 1$. In other words, the pairs $(x_i, \widetilde{x}_i)$ lie on a symmetric biquadratic curve with constant coefficients (which can be expressed through integrals of motion).*

Proof. Statement (a) follows by eliminating x_k from (8.6) via $x_k = H_1 - x_i - x_j$. Statement (b) is obtained with the help of MAPLE; it implies that x_i as functions of t are elliptic functions of degree 2 (i.e., with two poles within one parallelogram of periods).

8.2. Periodic Volterra Chain with $N = 4$ Particles

Equations of motion of VC_4 are:

$$\begin{cases} \dot{x}_1 = x_1(x_2 - x_4), \\ \dot{x}_2 = x_2(x_3 - x_1), \\ \dot{x}_3 = x_3(x_4 - x_2), \\ \dot{x}_4 = x_4(x_1 - x_3). \end{cases}$$

This system possesses three obvious integrals of motion: $H_1 = x_1 + x_2 + x_3 + x_4$, $H_2 = x_1 x_3$, and $H_3 = x_2 x_4$. One easily finds that x_1, x_3 satisfy the differential equation

$$\dot{x}_1^2 = (x_1^2 - H_1 x_1 + H_2)^2 - 4H_3 x_1^2,$$

while x_2, x_4 satisfy a similar equation with $H_2 \leftrightarrow H_3$. This immeadiately leads to solution in terms of elliptic functions. There are two types of Wronskian relations:

$$\dot{x}_1 x_3 - x_1 \dot{x}_3 = 2H_2(x_2 - x_4), \qquad \dot{x}_2 x_4 - x_2 \dot{x}_4 = 2H_3(x_3 - x_1), \tag{8.7}$$

and

$$\dot{x}_1 x_2 - x_1 \dot{x}_2 = H_1 x_1 x_2 - 2H_2 x_2 - 2H_3 x_1.$$

HK discretization (denoted by dVC_4):

$$\begin{cases} \widetilde{x}_1 - x_1 = \epsilon x_1(\widetilde{x}_2 - \widetilde{x}_4) + \epsilon \widetilde{x}_1(x_2 - x_4), \\ \widetilde{x}_2 - x_2 = \epsilon x_2(\widetilde{x}_3 - \widetilde{x}_1) + \epsilon \widetilde{x}_2(x_3 - x_1), \\ \widetilde{x}_3 - x_3 = \epsilon x_3(\widetilde{x}_4 - \widetilde{x}_2) + \epsilon \widetilde{x}_3(x_4 - x_2), \\ \widetilde{x}_4 - x_4 = \epsilon x_4(\widetilde{x}_1 - \widetilde{x}_3) + \epsilon \widetilde{x}_4(x_1 - x_4). \end{cases}$$

It possesses an obvious integral $H_1 = x_1 + x_2 + x_3 + x_4$.

Proposition 23. *For the map dVC_4, the following natural discretization of Eqs.* (8.7) *holds:*

$$\widetilde{x}_1 x_3 - x_1 \widetilde{x}_3 = 2\epsilon H_2(\epsilon)(x_2 + \widetilde{x}_2 - x_4 - \widetilde{x}_4),$$
$$\widetilde{x}_2 x_4 - x_2 \widetilde{x}_4 = 2\epsilon H_3(\epsilon)(x_1 + \widetilde{x}_1 - x_3 - \widetilde{x}_3),$$

with the conserved quantities

$$H_2(\epsilon) = \frac{x_1 x_3}{1 - \epsilon^2 (x_2 - x_4)^2}, \qquad H_3(\epsilon) = \frac{x_2 x_4}{1 - \epsilon^2 (x_1 - x_3)^2}.$$

Proof. This can be shown directly; the fact that $H_2(\epsilon)$, $H_3(\epsilon)$ are even functions of ϵ, assures that they are conserved quantities. One can also show this immediately from equations of motion: for instance, multiplying the equations

$$\frac{\widetilde{x}_1}{1 + \epsilon(\widetilde{x}_2 - \widetilde{x}_4)} = \frac{x_1}{1 - \epsilon(x_2 - x_4)}, \qquad \frac{\widetilde{x}_3}{1 - \epsilon(\widetilde{x}_2 - \widetilde{x}_4)} = \frac{x_3}{1 + \epsilon(x_2 - x_4)},$$

shows that $H_2(\epsilon)$ is a conserved quantity.

Proposition 24.

(a) *For the iterates of map dVC_4, the pairs (x_1, x_2) lie on a quartic curve whose coefficients are constant (expressed through integrals of motion).*

(b) *The pairs $(x_i, \widetilde{x}_i)$ lie on a biquartic curve of genus 1 with constant coefficients (which can be expressed through integrals of motion).*

Proof. Statement (a) follows by eliminating x_3, x_4 from integrals $H_1, H_2(\epsilon), H_3(\epsilon)$. Statement (b) is obtained with the help of MAPLE; it implies that x_i as functions of t are elliptic functions of degree 4 (i.e., with four poles within one parallelogram of periods).

Note that the reduction $x_4 = 0$ of the periodic Volterra chain with $N = 4$ leads to the open-end Volterra chain with $N = 3$ particles.

9. DRESSING CHAIN ($N = 3$)

The three-dimensional dressing chain (DC_3, for short) is described by the following system of quadratic ordinary equations [17]:

$$\begin{cases} \dot{x}_1 = x_3^2 - x_2^2 + \alpha_3 - \alpha_2, \\ \dot{x}_2 = x_1^2 - x_3^2 + \alpha_1 - \alpha_3, \\ \dot{x}_3 = x_2^2 - x_1^2 + \alpha_2 - \alpha_1, \end{cases} \tag{9.1}$$

with real parameters α_i. The system (9.1) is (Liouville and algebraically) integrable. The following quantities are integrals of motion:

$$I_1 = x_1 + x_2 + x_3,$$
$$I_2 = (x_1 + x_2)(x_2 + x_3)(x_3 + x_1) - \alpha_1 x_1 - \alpha_2 x_2 - \alpha_3 x_3.$$

Sometimes it is more convenient to use the following integral instead of I_2:

$$H_2 = x_1^3 + x_2^3 + x_3^3 + 3\alpha_1 x_1 + 3\alpha_2 x_2 + 3\alpha_3 x_3 = I_1^3 - 3I_2.$$

There hold the following Wronskian relations:

$$\dot{x}_i x_j - x_i \dot{x}_j = I_1 x_k^2 + 2(\alpha_i - \alpha_k)x_i + 2(\alpha_j - \alpha_k)x_j + 3\alpha_k I_1 - H_2. \tag{9.2}$$

Excluding x_j, x_k from equations of motion for x_i with the help of integrals of motion, one arrives at

$$\dot{x}_i^2 = x_i^4 + 6a_2 x_i^2 + 4a_3 x_i + a_4, \tag{9.3}$$

with

$$a_2 = -\tfrac{1}{3}(I_1^2 + \alpha_j + \alpha_k - 2\alpha_i), \qquad a_3 = (\alpha_j + \alpha_k - \alpha_i)I_1 + I_2,$$
$$a_4 = I_1^4 - 2(\alpha_j + \alpha_k)I_1^2 + (\alpha_j - \alpha_k)^2 - 4I_1I_2.$$

All three elliptic curves corresponding to (9.3) with $i = 1, 2, 3$, have equal Weierstrass invariants (expressed through the parameters α_i and the integrals of motion):

$$g_2 = a_4 + 3a_2^2, \qquad g_3 = a_2a_4 - a_2^3 - a_3^2, \quad \text{so that} \quad a_3^2 = -4a_2^3 + g_2a_2 - g_3.$$

The coefficients in (9.3) can be thus parametrized in terms of the Weierstrass elliptic function with the invariants g_2, g_3 as follows: $a_2 = -\wp(A_i)$, $a_3 = \wp'(A_i)$, so that $a_4 = g_2 - 3\wp^2(A_i)$. One can show that $A_1 + A_2 + A_3 = 0$ (modulo the period lattice), so that one can introduce B_i, defined up to a common additive shift, through $A_i = B_i - B_{i+1}$. The solution of the dressing chain DC_3 is then given as

$$x_i(t) = \zeta(t - B_{i+1}) - \zeta(t - B_i) - \zeta(B_i - B_{i+1}).$$

The HK discretization of system (9.1) is:

$$\begin{cases} \widetilde{x}_1 - x_1 = \epsilon(\widetilde{x}_3x_3 - \widetilde{x}_2x_2 + \alpha_3 - \alpha_2), \\ \widetilde{x}_2 - x_2 = \epsilon(\widetilde{x}_1x_1 - \widetilde{x}_3x_3 + \alpha_1 - \alpha_3), \\ \widetilde{x}_3 - x_3 = \epsilon(\widetilde{x}_2x_2 - \widetilde{x}_1x_1 + \alpha_2 - \alpha_1). \end{cases} \tag{9.4}$$

The map $f : x \mapsto \widetilde{x}$ obtained by solving (9.4) for $\widetilde{x}$ is given by:

$$\widetilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)(x + \epsilon c),$$

with

$$A(x, \epsilon) = \begin{pmatrix} 1 & \epsilon x_2 & -\epsilon x_3 \\ -\epsilon x_1 & 1 & \epsilon x_3 \\ \epsilon x_1 & -\epsilon x_2 & 1 \end{pmatrix}, \qquad c = (\alpha_3 - \alpha_2, \alpha_1 - \alpha_3, \alpha_2 - \alpha_1)^{\mathrm{T}}.$$

Explicitly:

$$\widetilde{x}_i = \frac{x_i + \epsilon(x_k^2 - x_j^2 + \alpha_k - \alpha_j) + \epsilon^2\big(I_1x_jx_k + (\alpha_k - \alpha_i)x_j + (\alpha_j - \alpha_i)x_k\big)}{1 + \epsilon^2(x_1x_2 + x_2x_3 + x_3x_1)}. \tag{9.5}$$

This map will be called dDC_3. From Proposition 1 there follows immediately:

Proposition 25. *The map dDC_3 possesses an invariant volume form:*

$$\det\frac{\partial\widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\phi(x)},$$

with $\phi(x) = 1 + \epsilon^2(x_1x_2 + x_2x_3 + x_3x_1)$.

Concerning integrability of dDC_3, we note first of all that I_1 is an obvious conserved quantity. The second one is most easily obtained from the following discretization of Wronskian relations (9.2).

Proposition 26. *For the map dDC_3, the following relations hold:*

$$\frac{\widetilde{x}_ix_j - x_i\widetilde{x}_j}{\epsilon} = I_1x_k\widetilde{x}_k + (\alpha_i - \alpha_k)(x_i + \widetilde{x}_i) + (\alpha_j - \alpha_k)(x_j + \widetilde{x}_j) + 3\alpha_kI_1 - H_2(\epsilon), \tag{9.6}$$

where $H_2(\epsilon)$ is a conserved quantity, given by

$$H_2(\epsilon) = \frac{H_2 + \epsilon^2G_2}{1 + \epsilon^2(x_1x_2 + x_2x_3 + x_3x_1)}, \tag{9.7}$$

where

$$\begin{aligned} G_2 = \; & I_1^2 x_1x_2x_3 + (\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3)(x_1x_2 + x_2x_3 + x_3x_1) \\ & +2I_1(\alpha_1x_2x_3 + \alpha_2x_3x_1 + \alpha_3x_1x_1) \\ & -(\alpha_2-\alpha_3)^2x_1 - (\alpha_3-\alpha_1)^2x_2 - (\alpha_1-\alpha_2)^2x_3. \end{aligned}$$

Proof. Define $H_2(\epsilon)$ by Eq. (9.6). It is easily computed with explicit formulas (9.5). The result given by (9.7) is a manifestly even function of ϵ and therefore an integral of motion.

Proposition 27.

(a) *The set* $\Phi_{ij} = (x_i^3,\, x_j^3,\, x_ix_j(x_i+x_j),\, x_i^2,\, x_ix_j,\, x_j^2,\, x_i,\, x_j,\, 1)$ *is a HK basis for the map dDC*$_3$ *with* $\dim K_{\Phi_{ij}}(x) = 1$. *In other words, the pairs* x_i, x_j *satisfy equations of degree 3,*

$$P_{ij}(x_i,x_j) = p_0x_i^3 + p_1x_j^3 + p_2x_ix_j(x_i+x_j) + p_3x_i^2 + p_4x_ix_j + p_5x_j^2 + p_6x_i + p_7x_j + p_8 = 0,$$

whose coefficients $p_m = p_m^{(ij)}$ *are constant (expressed through parameters* α_k *and integrals of motion).*

(b) *The set* $\Psi_i = (x_i^m\widetilde{x}_i^n)_{m,n=0}^3$ *is a HK basis for the map dDC*$_3$ *with* $\dim K_{\Psi_i}(x) = 1$. *In other words, the pairs* $x_i, \widetilde{x}_i$ *lie on bicubic curves of genus 1:*

$$Q_i(x_i,\widetilde{x}_i) = \sum_{m,n=0}^{3} q_{mn}x_i^m\widetilde{x}_i^n = 0,$$

whose coefficients $q_{mn} = q_{mn}^{(i)}$ *are constant (expressed through parameters* α_k *and integrals of motion). Moreover,* $q_{13} = q_{31}$.

Proof. Statement (a) follows by eliminating x_k from (9.7) via $x_k = I_1 - x_i - x_j$. Statement (b) is obtained with the help of MAPLE. One can also show that these bicubic curves are of genus 1, so that x_i as functions of t are elliptic functions of degree 3 (i.e., with three poles within one parallelogram of periods).

10. COUPLED EULER TOPS

In [18] a remarkable mechanical system was introduced, which can be interpreted as a chain of coupled three-dimensional Euler tops. The differential equations governing the system are given by:

$$\begin{cases} \dot{x}_1 = \alpha_1x_2x_3, \\ \dot{x}_{2j} = \alpha_{3j-1}x_{2j-1}x_{2j+1}, \\ \dot{x}_{2j+1} = \alpha_{3j}x_{2j}x_{2j-1} + \alpha_{3j+1}x_{2j+2}x_{2j+3}, \\ \dot{x}_{2j+2} = \alpha_{3j+2}x_{2j+1}x_{2j+3}, \\ \dot{x}_{2N+1} = \alpha_{3N}x_{2N}x_{2N-1}, \end{cases} \tag{10.1}$$

with real parameters α_i. Each triple of variables $(x_{2j-1}, x_{2j}, x_{2j+1})$ can be considered as a 3D Euler top, coupled with the neighboring triple $(x_{2j+1}, x_{2j+2}, x_{2j+3})$ via the variable x_{2j+1}. We will denote system (10.1) by CET$_N$. It has $N+1$ independent conserved quantities:

$$\begin{aligned} & H_1 = \alpha_2x_1^2 - \alpha_1x_2^2, \\ & H_j = \alpha_{3j-3}\alpha_{3j-1}x_{2j-2}^2 - \alpha_{3j-4}\alpha_{3j-1}x_{2j-1}^2 + \alpha_{3j-4}\alpha_{3j-2}x_{2j}^2, \qquad 2 \leqslant j \leqslant N, \\ & H_{N+1} = \alpha_{3N}x_{2N}^2 - \alpha_{3N-1}x_{2N+1}^2. \end{aligned}$$

Nothing is known about the possible Hamiltonian formulation of this system, and therefore about its integrability in the Liouville–Arnold sense.

For $N = 1$ system (10.1) reduces to the usual Euler top (6.1). We will consider in detail the HK discretization of the system CET_2 given by

$$\begin{cases} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2 + \alpha_4 x_4 x_5, \\ \dot{x}_4 = \alpha_5 x_5 x_3, \\ \dot{x}_5 = \alpha_6 x_3 x_4. \end{cases} \tag{10.2}$$

It can be interpreted as two Euler tops, described by the two sets of variables (x_1, x_2, x_3) and (x_3, x_4, x_5), respectively, coupled via the variable x_3. It has three independent integrals of motion:

$$H_1 = \alpha_2 x_1^2 - \alpha_1 x_2^2, \qquad H_3 = \alpha_6 x_4^2 - \alpha_5 x_5^2,$$
$$H_2 = \alpha_3 \alpha_5 x_2^2 - \alpha_2 \alpha_5 x_3^2 + \alpha_2 \alpha_4 x_4^2,$$

and it can be solved in terms of elliptic functions. We will be mainly interested in its particular case which is superintegrable.

Proposition 28. *If the coefficients α_i satisfy the following condition,*

$$\alpha_1 \alpha_2 = \alpha_5 \alpha_6, \tag{10.3}$$

then the system CET_2 is superintegrable: it has two additional integrals,

$$H_4 = \alpha_5 x_2 x_5 - \alpha_2 x_1 x_4, \qquad H_5 = \alpha_5 x_1 x_5 - \alpha_1 x_2 x_4,$$

and among the functions $H_1, \ldots, H_5$ there are four independent ones.

In this case, the variable x_3 satisfies the following differential equation:

$$\dot{x}_3^2 = \left(x_3^2 + \frac{H_2}{\alpha_2 \alpha_5} \right) \left(\alpha_1 \alpha_2 x_3^2 + \frac{\alpha_1}{\alpha_5} H_2 + \alpha_3 H_1 - \alpha_4 H_3 \right) - \frac{\alpha_3 \alpha_4}{\alpha_2 \alpha_5} H_4^2, \tag{10.4}$$

so that its time evolution is described by an elliptic function of degree 2.

The HK discretization of the system CET_2 reads:

$$\begin{cases} \widetilde{x}_1 - x_1 = \epsilon \alpha_1 (\widetilde{x}_2 x_3 + x_2 \widetilde{x}_3), \\ \widetilde{x}_2 - x_2 = \epsilon \alpha_2 (\widetilde{x}_3 x_1 + x_3 \widetilde{x}_1), \\ \widetilde{x}_3 - x_3 = \epsilon \alpha_3 (\widetilde{x}_1 x_2 + x_1 \widetilde{x}_2) + \epsilon \alpha_4 (\widetilde{x}_4 x_5 + x_4 \widetilde{x}_5), \\ \widetilde{x}_4 - x_4 = \epsilon \alpha_5 (\widetilde{x}_5 x_3 + x_5 \widetilde{x}_3), \\ \widetilde{x}_5 - x_5 = \epsilon \alpha_6 (\widetilde{x}_3 x_4 + x_3 \widetilde{x}_4). \end{cases} \tag{10.5}$$

The map $f : x \mapsto \widetilde{x}$ obtained by solving (10.5) for $\widetilde{x}$ is given by:

$$\widetilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon) x,$$

with

$$A(x,\epsilon)=\begin{pmatrix} 1 & \epsilon\alpha_1x_3 & \epsilon\alpha_1x_2 & 0 & 0\\ \epsilon\alpha_2x_3 & 1 & \epsilon\alpha_2x_1 & 0 & 0\\ \epsilon\alpha_3x_2 & \epsilon\alpha_3x_1 & 1 & \epsilon\alpha_4x_5 & \epsilon\alpha_4x_4\\ 0 & 0 & \epsilon\alpha_5x_5 & 1 & \epsilon\alpha_5x_3\\ 0 & 0 & \epsilon\alpha_6x_4 & \epsilon\alpha_6x_3 & 1 \end{pmatrix}.$$

This map will be called dCET_2 in the sequel.

Proposition 29. *The functions*

$$H_1(\epsilon)=\frac{\alpha_2x_1^2-\alpha_1x_2^2}{1-\epsilon^2\alpha_1\alpha_2x_3^2},\qquad H_3(\epsilon)=\frac{\alpha_6x_4^2-\alpha_5x_5^2}{1-\epsilon^2\alpha_5\alpha_6x_3^2},$$

are conserved quantities of the map $dCET_2$.

Proposition 29 gives only two independent integrals of motion for the map dCET_2. Numerical experiments indicate that the third integral does not exist in general. The situation is different under condition (10.3). Note that in this case the denominators of the integrals $H_1(\epsilon)$ and $H_3(\epsilon)$ coincide.

Proposition 30. *If condition (10.3) holds, then the map* $dCET_2$ *has in addition to* $H_1(\epsilon)$ *and* $H_3(\epsilon)$ *also the following conserved quantities*

$$H_2(\epsilon)=\frac{\alpha_3\alpha_5x_2^2-\alpha_2\alpha_5x_3^2+\alpha_2\alpha_4x_4^2}{1-\epsilon^2\alpha_1\alpha_2x_3^2},$$

$$H_4(\epsilon)=\frac{\alpha_5x_2x_5-\alpha_2x_1x_4}{1-\epsilon^2\alpha_1\alpha_2x_3^2},\qquad H_5(\epsilon)=\frac{\alpha_5x_1x_5-\alpha_1x_2x_4}{1-\epsilon^2\alpha_1\alpha_2x_3^2}.$$

There are four independent functions among $H_1(\epsilon),\ldots,H_5(\epsilon)$.

We now present HK bases for the map dCET_2.

Proposition 31. *Under condition* (10.3)*, the map* $dCET_2$ *has the following HK bases.*

(a) *The set* $\Phi=(x_1^2,\, x_2^2,\, x_3^2,\, x_4^2,\, x_5^2,\, 1)$ *is a HK basis with* $\dim K_\Phi(x)=3$.

(b) *The sets* $\Phi_1=(x_1^2,\, x_2^2,\, x_3^2,\, 1)$ *and* $\Phi_2=(x_3^2,\, x_4^2,\, x_5^2,\, 1)$ *are HK bases with one-dimensional null-spaces. At each point* $x\in\mathbb{R}^5$ *we have:* $K_{\Phi_1}(x)=[e_1:e_2:\epsilon^2\alpha_1\alpha_2:-1]$ *and* $K_{\Phi_2}(x)=[\epsilon^2\alpha_1\alpha_2:f_4:f_5:-1]$. *The functions* e_i *and* f_i *are conserved quantities given by*

$$e_1=\frac{\alpha_2(1-\epsilon^2\alpha_1\alpha_2x_3^2)}{\alpha_2x_1^2-\alpha_1x_2^2},\qquad e_2=-\frac{\alpha_1(1-\epsilon^2\alpha_1\alpha_2x_3^2)}{\alpha_2x_1^2-\alpha_1x_2^2},$$

and

$$f_4=\frac{\alpha_5(1-\epsilon^2\alpha_1\alpha_2x_3^2)}{\alpha_5x_4^2-\alpha_4x_5^2},\qquad f_5=-\frac{\alpha_4(1-\epsilon^2\alpha_1\alpha_2x_3^2)}{\alpha_5x_4^2-\alpha_4x_5^2}.$$

The set $\Phi_3=(x_1^2,\, x_2^2,\, x_3^2,\, x_4^2)$ *is a HK basis with a one-dimensional null-space. At each point* $x\in\mathbb{R}^5$ *we have:* $K_{\Phi_3}(x)=[g_1:g_2:\alpha_5:-\alpha_4]$. *The functions* g_i *are conserved quantities given by*

$$g_1=\frac{\alpha_3\alpha_5x_2^2-\alpha_2\alpha_5x_3^2+\alpha_2\alpha_4x_4^2}{\alpha_2x_1^2-\alpha_1x_2^2},\qquad g_2=-\frac{\alpha_3\alpha_5x_1^2-\alpha_1\alpha_5x_3^2+\alpha_1\alpha_4x_4^2}{\alpha_2x_1^2-\alpha_1x_2^2}.$$

Similar claim hold for the sets $(x_1^2,\, x_2^2,\, x_3^2,\, x_5^2)$, $(x_1^2,\, x_3^2,\, x_4^2,\, x_5^2)$, *and* $(x_2^2,\, x_3^2,\, x_4^2,\, x_5^2)$.

(c) *The set* $\Psi = (x_1, x_2, x_3, x_4, x_5)$ *is a HK basis with* $\dim K_\Psi(x) = 2$.

(d) *The sets* $\Psi_1 = (x_1, x_2, x_4)$, $\Psi_2 = (x_1, x_2, x_5)$ *are HK bases with one-dimensional null-spaces. At each point* $x \in \mathbb{R}^5$ *we have:* $K_{\Psi_1}(x) = [c_1 : c_2 : -1]$, $K_{\Psi_2}(x) = [d_1 : d_2 : -1]$. *The functions* c_1, c_2 *and* d_1, d_2 *are conserved quantities given by*

$$c_1 = \frac{\alpha_2 x_1 x_4 - \alpha_5 x_2 x_5}{\alpha_2 x_1^2 - \alpha_1 x_2^2}, \qquad c_2 = \frac{\alpha_5 x_1 x_5 - \alpha_1 x_2 x_4}{\alpha_2 x_1^2 - \alpha_1 x_2^2},$$

while

$$d_1 = \frac{\alpha_2}{\alpha_5} c_2, \qquad d_2 = \frac{\alpha_1}{\alpha_5} c_1.$$

A similar claim holds for the sets (x_2, x_4, x_5) *and* (x_1, x_4, x_5).

We see that the map dCET_2 under condition (10.3) possesses four functionally independent conserved quantities. It might look paradoxical that $\Phi \cup \Psi$ is a HK basis with a 5-dimensional null-space, thus imposing seemingly 5 restrictions on any orbit of the map, which would yield 0-dimensional invariant sets (instead of invariant curves in the continuous time case). The resolution of this paradox is that the five restrictions are functionally dependent on their common set (i.e., along any orbit). In other words, the HK basis $\Phi \cup \Psi$ is not regular. This is the first and the only instance of a non-regular HK basis in this paper.

The map dCET_2 possesses, in its (super)-integrable regime, an invariant volume form:

Proposition 32. *Under condition (10.3), the map* dCET_2 *preserves the following volume form:*

$$\det \frac{\partial \widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5}{\phi(x)},$$

with $\phi(x) = (1 - \epsilon^2 \alpha_1 \alpha_2 x_3^2)^3$.

In this regime, the solutions can be found in terms of elliptic functions, as the following statement shows.

Proposition 33. *Under condition (10.3), the component* x_3 *of any orbit of the map* dCET_2 *satisfies a relation of the type*

$$Q(x_3, \widetilde{x}_3) = q_0 x_3^2 \widetilde{x}_3^2 + q_1 x_3 \widetilde{x}_3 (x_3 + \widetilde{x}_3) + q_2 (x_3^2 + \widetilde{x}_3^2) + q_3 x_3 \widetilde{x}_3 + q_4 (x_3 + \widetilde{x}_3) + q_5 = 0,$$

coefficients of the biquadratic polynomial Q *being conserved quantities of* dCET_2.

This statement is a proper discretization of Eq. (10.4).

11. THREE WAVE SYSTEM

The three wave interaction system of ordinary differential equations is [19]:

$$\begin{cases} \dot{z}_1 = \mathrm{i}\alpha_1 \bar{z}_2 \bar{z}_3, \\ \dot{z}_2 = \mathrm{i}\alpha_2 \bar{z}_3 \bar{z}_1, \\ \dot{z}_3 = \mathrm{i}\alpha_3 \bar{z}_1 \bar{z}_2. \end{cases} \tag{11.1}$$

Here $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, while the parameters α_i of the system are supposed to be real numbers. If (i, j, k) stands for any cyclic permutation of (123), then we can write system (11.1) in the abbreviated form

$$\dot{z}_i = \mathrm{i}\alpha_i \bar{z}_j \bar{z}_k, \tag{11.2}$$

Writing $z_i = x_i + \mathrm{i}y_i$, $i = 1, 2, 3$, we put system (11.2) into the form

$$\begin{cases} \dot{x}_i = \alpha_i(x_j y_k + y_j x_k), \\ \dot{y}_i = \alpha_i(x_j x_k - y_j y_k). \end{cases} \tag{11.3}$$

System (11.3) is completely integrable and can be integrated in terms of elliptic functions. It has three independent integrals of motion: quadratic ones,

$$H_i = \alpha_j|z_k|^2 - \alpha_k|z_j|^2,$$

among which there are only two independent ones because of $\alpha_1 H_1 + \alpha_2 H_2 + \alpha_3 H_3 = 0$, and a cubic one,

$$K = \frac{1}{2}\left(z_1 z_2 z_3 + \bar{z}_1 \bar{z}_2 \bar{z}_3\right) = \mathfrak{Re}(z_1 z_2 z_3).$$

The HK discretization of system (11.2) reads

$$\tilde{z}_i - z_i = \mathrm{i}\epsilon\alpha_i(\bar{z}_j \tilde{\bar{z}}_k + \tilde{\bar{z}}_j \bar{z}_k),$$

or, in the real variables (x_i, y_i),

$$\begin{cases} \widetilde{x}_i - x_i = \epsilon\alpha_i(x_j \widetilde{y}_k + \widetilde{x}_j y_k + y_j \widetilde{x}_k + \widetilde{y}_j x_k), \\ \widetilde{y}_i - y_i = \epsilon\alpha_i(x_j \widetilde{x}_k + \widetilde{x}_j x_k - y_j \widetilde{y}_k - \widetilde{y}_j y_k). \end{cases}$$

In the matrix form, this can be put as

$$A(x, y, \epsilon)\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = f(x, y, \epsilon) = A^{-1}(x, y, \epsilon)\begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$A(x, y, \epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 y_3 & -\epsilon\alpha_1 y_2 & 0 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 y_3 & 1 & -\epsilon\alpha_2 y_1 & -\epsilon\alpha_2 x_3 & 0 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 y_2 & -\epsilon\alpha_3 y_1 & 1 & -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 0 \\ 0 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 & 1 & \epsilon\alpha_1 y_3 & \epsilon\alpha_1 y_2 \\ -\epsilon\alpha_2 x_3 & 0 & -\epsilon\alpha_2 x_1 & \epsilon\alpha_2 y_3 & 1 & \epsilon\alpha_2 y_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 0 & \epsilon\alpha_3 y_2 & \epsilon\alpha_3 y_1 & 1 \end{pmatrix}.$$

The birational map $f : \mathbb{R}^6 \to \mathbb{R}^6$ will be called d3W hereafter.

Proposition 34. *The map d3W has three independent conserved quantities, namely, any two of*

$$H_i(\epsilon) = \frac{\alpha_j|z_k|^2 - \alpha_k|z_j|^2}{1 - \epsilon^2\alpha_j\alpha_k|z_i|^2},$$

supplied with any one of

$$K_i(\epsilon) = \frac{\mathfrak{Re}(z_1 z_2 z_3)(1 - \epsilon^2\alpha_k\alpha_i|z_j|^2)(1 - \epsilon^2\alpha_i\alpha_j|z_k|^2)}{\Delta(z, \bar{z}, \epsilon)},$$

where

$$\begin{aligned} \Delta(z, \bar{z}, \epsilon) = \det A(x, y, \epsilon) \ &= \ 1 - 2\epsilon^2(\alpha_2\alpha_3|z_1|^2 + \alpha_3\alpha_1|z_2|^2 + \alpha_1\alpha_2|z_3|^2) \\ &+ \epsilon^4(\alpha_2\alpha_3|z_1|^2 + \alpha_3\alpha_1|z_2|^2 + \alpha_1\alpha_2|z_3|^2)^2 - 4\epsilon^6\alpha_1^2\alpha_2^2\alpha_3^2|z_1|^2|z_2|^2|z_3|^2. \end{aligned}$$

Proposition 35. *The map d3W possesses an invariant volume form:*

$$\det \frac{\partial \tilde{z}}{\partial z} = \frac{\phi(\tilde{z})}{\phi(z)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_1 \wedge dy_2 \wedge dy_3}{\phi(z)},$$

where $\phi(z) = \Delta(z, \bar{z}, \epsilon)$.

Next, we give the results on the HK bases for the map d3W, which yield a complete set of integrals of motion.

Proposition 36.

(a) *The sets* $\Phi_i = (|z_j|^2,\ |z_k|^2,\ 1)$, $i = 1, 2, 3$, *are HK bases for the map d3W with* $\dim K_{\Phi_i}(z) = 1$. *At each point* $z \in \mathbb{C}^3$ *there holds:* $K_{\Phi_i}(z) = [d_1 : d_2 : -1]$, *where the coefficients*

$$d_1(z) = \frac{\alpha_k(1 - \epsilon^2\alpha_i\alpha_j|z_k|^2)}{\alpha_k|z_j|^2 - \alpha_j|z_k|^2}\,, \qquad d_2(z) = -\frac{\alpha_j(1 - \epsilon^2\alpha_k\alpha_i|z_j|^2)}{\alpha_k|z_j|^2 - \alpha_j|z_k|^2}\,,$$

are integrals of motion of the map d3W. They are functionally dependent because of $\alpha_j d_1(z) + \alpha_k d_2(z) = \epsilon^2\alpha_1\alpha_2\alpha_3$.

(b) *The sets* $\Psi_i = (\mathfrak{Re}(z_1z_2z_3),\ |z_i|^2,\ 1)$, $i = 1, 2, 3$, *are HK bases for the map d3W with* $\dim K_{\Psi_i}(z) = 1$. *At each point* $z \in \mathbb{C}^3$ *there holds:* $K_{\Psi_i}(z) = [e_1 : e_2 : -1]$, *where the coefficients*

$$e_1(z) = -\frac{\Delta(z, \bar{z}, \epsilon)}{\mathfrak{Re}(z_1z_2z_3)\big(1 - \epsilon^2\left(-\alpha_j\alpha_k|z_i|^2 + \alpha_k\alpha_i|z_j|^2 + \alpha_i\alpha_j|z_k|^2\right)\big)^2}\,,$$

$$e_2(z) = \frac{4\alpha_j\alpha_k\epsilon^2(1 - \epsilon^2\alpha_k\alpha_i|z_j|^2)(1 - \epsilon^2\alpha_i\alpha_j|z_k|^2)}{\big(1 - \epsilon^2\left(-\alpha_j\alpha_k|z_i|^2 + \alpha_k\alpha_i|z_j|^2 + \alpha_i\alpha_j|z_k|^2\right)\big)^2},$$

are independent integrals of motion of the map d3W.

12. LAGRANGE TOP

Lagrange top was the second integrable system, after Euler top, to which the HK discretization was successfully applied [5]. We reproduce and re-derive here the results of that paper, and add some new results.

Equations of motion of the Lagrange top are of the general Kirchhoff type:

$$\begin{cases} \dot{m} = m \times \nabla_m H + p \times \nabla_p H, \\ \dot{p} = p \times \nabla_m H, \end{cases} \tag{12.1}$$

where $m = (m_1, m_2, m_3)^{\mathrm{T}}$ and $p = (p_1, p_2, p_3)^{\mathrm{T}}$. Any Kirchhoff type system is Hamiltonian with the Hamilton function $H = H(m, p)$ with respect to the Lie–Poisson bracket on $\mathfrak{e}(3)^*$,

$$\{m_i, m_j\} = \epsilon_{ijk}m_k, \qquad \{m_i, p_j\} = \epsilon_{ijk}p_k, \qquad \{p_i, p_j\} = 0,$$

and admits the Hamilton function H and the Casimir functions

$$C_1 = p_1^2 + p_2^2 + p_3^2, \qquad C_2 = m_1p_1 + m_2p_2 + m_3p_3, \tag{12.2}$$

as integrals of motion. For the complete integrability of a Kirchhoff type system, it should admit a fourth independent integral of motion.

The Hamilton function of the Lagrange top (LT) is $H = H_1/2$, where

$$H_1 = m_1^2 + m_2^2 + \alpha m_3^2 + 2\gamma p_3.$$

Thus, equations of motion of LT read

$$\begin{cases} \dot{m}_1 = (\alpha-1)m_2m_3 + \gamma p_2, \\ \dot{m}_2 = (1-\alpha)m_1m_3 - \gamma p_1, \\ \dot{m}_3 = 0, \\ \dot{p}_1 = \alpha p_2 m_3 - p_3 m_2, \\ \dot{p}_2 = p_3 m_1 - \alpha p_1 m_3, \\ \dot{p}_3 = p_1 m_2 - p_2 m_1. \end{cases} \tag{12.3}$$

It follows immediately that the fourth integral of motion is simply $H_2 = m_3$. Traditionally, the explicit integration of the LT in terms of elliptic functions starts with the following observation: the component p_3 of the solution satisfies the differential equation

$$\dot{p}_3^2 = P_3(p_3), \tag{12.4}$$

with a cubic polynomial P_3 whose coefficients are expressed through integrals of motion:

$$P_3(p_3) = (H_1 - \alpha m_3^2 - 2\gamma p_3)(C_1 - p_3^2) - (C_2 - m_3 p_3)^2.$$

We mention also the following Wronskian relation which follows easily from equations of motion:

$$(\dot{m}_1 p_1 - m_1 \dot{p}_1) + (\dot{m}_2 p_2 - m_2 \dot{p}_2) + (2\alpha - 1)(\dot{m}_3 p_3 - m_3 \dot{p}_3) = 0. \tag{12.5}$$

Applying the HK discretization scheme to Eqs. (12.3), we obtain the following discrete system:

$$\begin{cases} \widetilde{m}_1 - m_1 = \epsilon(\alpha-1)(\widetilde{m}_2 m_3 + m_2\widetilde{m}_3) + \epsilon\gamma(p_2 + \widetilde{p}_2), \\ \widetilde{m}_2 - m_2 = \epsilon(1-\alpha)(\widetilde{m}_1 m_3 + m_1\widetilde{m}_3) - \epsilon\gamma(p_1 + \widetilde{p}_1), \\ \widetilde{m}_3 - m_3 = 0, \\ \widetilde{p}_1 - p_1 = \epsilon\alpha(p_2\widetilde{m}_3 + \widetilde{p}_2 m_3) - \epsilon(p_3\widetilde{m}_2 + \widetilde{p}_3 m_2), \\ \widetilde{p}_2 - p_2 = \epsilon(p_3\widetilde{m}_1 + \widetilde{p}_3 m_1) - \epsilon\alpha(p_1\widetilde{m}_3 + \widetilde{p}_1 m_3), \\ \widetilde{p}_3 - p_3 = \epsilon(p_1\widetilde{m}_2 + \widetilde{p}_1 m_2 - p_2\widetilde{m}_1 - \widetilde{p}_2 m_1). \end{cases}$$

As usual, this can be solved for $(\widetilde{m}, \widetilde{p})$, thus yielding the reversible and birational map $x \mapsto \widetilde{x} = f(x,\epsilon) = A^{-1}(x,\epsilon)(\mathbb{1} + \epsilon B)x$, where $x = (m_1, m_2, m_3, p_1, p_2, p_3)^{\mathrm{T}}$, and

$$A(x,\epsilon) = \begin{pmatrix} 1 & \epsilon(1-\alpha)m_3 & \epsilon(1-\alpha)m_2 & 0 & 0 & 0 \\ -\epsilon(1-\alpha)m_3 & 1 & -\epsilon(1-\alpha)m_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \epsilon p_3 & -\epsilon\alpha p_2 & 1 & -\epsilon\alpha m_3 & \epsilon m_2 \\ -\epsilon p_3 & 0 & \epsilon\alpha p_1 & \epsilon\alpha m_3 & 1 & -\epsilon m_1 \\ \epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1 \end{pmatrix} - \epsilon B,$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This map will be called dLT in the sequel. Obviously, m_3 serves as a conserved quantity for dLT. The remaining three conserved quantities can be found with the help of the HK bases approach. A simple conserved quantity can be found from the following statement which serves as a natural discretization of the Wronskian relation (12.5).

Proposition 37. *The set* $\Gamma = (\widetilde{m}_1 p_1 - m_1 \widetilde{p}_1,\, \widetilde{m}_2 p_2 - m_2 \widetilde{p}_2,\, \widetilde{m}_3 p_3 - m_3 \widetilde{p}_3)$ *is a HK basis for the map dLT with* $\dim K_\Gamma(x) = 1$. *At each point* $x \in \mathbb{R}^6$ *we have:* $K_\Gamma(x) = [1 : 1 : b_3]$, *where* b_3 *is a conserved quantity of dLT given by*

$$b_3 = \frac{(2\alpha - 1)m_3 + \epsilon^2(\alpha - 1)m_3(m_1^2 + m_2^2) + \epsilon^2\gamma(m_1 p_1 + m_2 p_2)}{m_3 \Delta_1}, \tag{12.6}$$

where

$$\Delta_1 = 1 + \epsilon^2 \alpha(1 - \alpha)m_3^2 - \epsilon^2 \gamma p_3. \tag{12.7}$$

Proof. A straightforward computation with MAPLE of the quantity

$$b_3 = -\frac{(\widetilde{m}_1 p_1 - m_1 \widetilde{p}_1) + (\widetilde{m}_2 p_2 - m_2 \widetilde{p}_2)}{(\widetilde{m}_3 p_3 - m_3 \widetilde{p}_3)}$$

leads to the value (12.6). It is an even function of ϵ and therefore a conserved quantity.

Further integrals of motion were found by Hirota and Kimura. We reproduce here their results with new simplified proofs.

Proposition 38. ([5])

(a) *The set* $\Phi = (m_1^2 + m_2^2,\, p_1 m_1 + p_2 m_2,\, p_1^2 + p_2^2,\, p_3^2,\, p_3,\, 1)$ *is a HK basis for the map dLT with* $\dim K_\Phi(x) = 3$.

(b) *The set* $\Phi_1 = (1,\, p_3,\, p_3^2,\, m_1^2 + m_2^2)$ *is a HK basis for the map dLT with a one-dimensional null-space. At each point* $x \in \mathbb{R}^6$ *we have:* $K_{\Phi_1}(x) = [c_0 : c_1 : c_2 : -1]$. *The functions* c_0, c_1, c_2 *are conserved quantities of the map dLT, given by*

$$c_0 = \frac{m_1^2 + m_2^2 + 2\gamma p_3 + \epsilon^2 c_0^{(4)} + \epsilon^4 c_0^{(6)} + \epsilon^6 c_0^{(8)} + \epsilon^8 c_0^{(10)}}{\Delta_1 \Delta_2},$$

$$c_1 = -\frac{2\gamma\big(1 - \epsilon^2\alpha(1 - \alpha)m_3^2\big)\big(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)}\big)}{\Delta_1 \Delta_2},$$

$$c_2 = -\frac{\epsilon^2\gamma^2\big(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)}\big)}{\Delta_1 \Delta_2}.$$

Here Δ_1 *is given in (12.7), and* $\Delta_2 = 1 + \epsilon^2 \Delta_2^{(2)} + \epsilon^4 \Delta_2^{(4)} + \epsilon^6 \Delta_2^{(6)}$; *coefficients* $\Delta^{(q)}$ *and* $c_k^{(q)}$ *are polynomials of degree* q *in the phase variables. In particular:*

$$c_2^{(2)} = m_1^2 + m_2^2 + (1 - 2\alpha + 2\alpha^2)m_3^2 - 2\gamma p_3,$$

$$\Delta_2^{(2)} = m_1^2 + m_2^2 + (1 - 3\alpha + 3\alpha^2)m_3^2 - \gamma p_3.$$

(c) *The set* $\Phi_2 = (1,\, p_3,\, p_3^2,\, m_1p_1 + m_2p_2)$ *is a HK basis for the map dLT with a one-dimensional null-space. At each point* $x \in \mathbb{R}^6$ *we have:* $K_{\Phi_2}(x) = [d_0 : d_1 : d_2 : -1]$. *The functions* d_0, d_1, d_2 *are conserved quantities of the map dLT, given by*

$$d_0 = \frac{m_1p_1 + m_2p_2 + m_3p_3 + \epsilon^2 d_0^{(4)} + \epsilon^4 d_0^{(6)} + \epsilon^6 d_0^{(8)} + \epsilon^8 d_0^{(10)}}{\Delta_1\Delta_2},$$

$$d_1 = -\frac{m_3 + \epsilon^2 d_1^{(3)} + \epsilon^4 d_1^{(5)} + \epsilon^6 d_1^{(7)} + \epsilon^8 d_1^{(9)}}{\Delta_1\Delta_2},$$

$$d_2 = -\frac{\epsilon^2\gamma(1-\alpha)m_3\big(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)}\big)}{\Delta_1\Delta_2},$$

where $d_k^{(q)}$ *are polynomials of degree q in the phase variables. In particular,*

$$d_1^{(3)} = \gamma(m_1p_1 + m_2p_2) - \gamma(3-2\alpha)m_3p_3 + \alpha m_3(m_1^2 + m_2^2) + (1 - 3\alpha + 3\alpha^2)m_3^3.$$

(d) *The set* $\Phi_3 = (1,\, p_3,\, p_3^2,\, p_1^2 + p_2^2)$ *is a HK basis for the map dLT with a one-dimensional null-space. At each point* $x \in \mathbb{R}^6$ *we have:* $K_{\Phi_3}(x) = [e_0 : e_1 : e_2 : -1]$. *The functions* e_0, e_1, e_2 *are conserved quantities of the map dLT, given by*

$$e_0 = \frac{p_1^2 + p_2^2 + p_3^2 + \epsilon^2 e_0^{(4)} + \epsilon^4 e_0^{(6)} + \epsilon^6 e_0^{(8)} + \epsilon^8 e_0^{(10)}}{\Delta_1\Delta_2},$$

$$e_1 = -\frac{2\epsilon^2\big(e_1^{(3)} + \epsilon^2 e_1^{(5)} + \epsilon^4 e_1^{(7)} + \epsilon^6 e_1^{(9)}\big)}{\Delta_1\Delta_2},$$

$$e_2 = -\frac{\big(1 + \epsilon^2(1-\alpha)^2 m_3^2\big)\big(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)}\big)}{\Delta_1\Delta_2},$$

where $e_k^{(q)}$ *are polynomials of degree q in the phase variables. In particular,*

$$e_1^{(3)} = \gamma(p_1^2 + p_2^2 + p_3^2) - (1-\alpha)m_3(m_1p_1 + m_2p_2 + m_3p_3).$$

Proof. (b) We consider a linear system of equations

$$(c_0 + c_1p_3 + c_2p_3^2) \circ f^i(m,p,\epsilon) = (m_1^2 + m_2^2) \circ f^i(m,p,\epsilon), \tag{12.8}$$

for all $i \in \mathbb{Z}$. Numerically one sees that it admits a unique solution, and one can identify the linear relation

$$\frac{1}{2}\gamma\epsilon^2 c_1 = \big(1 - \epsilon^2\alpha(1-\alpha)m_3^2\big)c_2. \tag{12.9}$$

The system of three equations for three unknowns c_0, c_1, c_2 consisting of (12.8) with $i = 0, 1$ and (12.9) can easily be solved with MAPLE. Its solutions are even functions of ϵ, which proves that they are integrals of motion.

(c) This time we consider the linear system of equations

$$(d_0 + d_1p_3 + d_2p_3^2) \circ f^i(m,p,\epsilon) = (m_1p_1 + m_2p_2) \circ f^i(m,p,\epsilon), \tag{12.10}$$

for all $i \in \mathbb{Z}$. Numerically we see that it admits a unique solution, and we can identify the linear relation

$$\gamma d_2 = (1-\alpha)m_3c_2. \tag{12.11}$$

The system of three equations for the three unknowns d_0, d_1, d_2 consisting of (12.10) for $i = 0, 1$ and of (12.11) with c_2 already found in part b) can easily be solved with MAPLE. Its solutions are even functions of ϵ and therefore are integrals.

(d) Completely analogous to the last two proofs: we solve the linear system of three equations for the three unknowns e_0, e_1, e_2, consisting of the equations

$$(e_0 + e_1 p_3 + e_2 p_3^2) \circ f^i(m, p, \epsilon) = (p_1^2 + p_2^2) \circ f^i(m, p, \epsilon),$$

for $i = 0, 1$, and of the linear relation

$$\epsilon^2 \gamma^2 e_2 = \big(1 + \epsilon^2 (1-\alpha)^2 m_3^2\big) c_2,$$

and verify that they are even functions of ϵ.

We note that for $\alpha = 1$ the integrals d_0, d_1, d_2 simplify to

$$d_0 = \frac{m_1 p_1 + m_2 p_2 + m_3 p_3}{1 - \epsilon^2 \gamma p_3}, \quad d_1 = -\frac{m_3 + \epsilon^2 \gamma (m_1 p_1 + m_2 p_2)}{1 - \epsilon^2 \gamma p_3}, \quad d_2 = 0.$$

It is possible to find a further simple, in fact polynomial, integral for the map dLT.

Proposition 39. ([5]) *The function*

$$F = m_1^2 + m_2^2 + 2\gamma p_3 - \epsilon^2 \big((1-\alpha) m_3 m_1 + \gamma p_1\big)^2 - \epsilon^2 \big((1-\alpha) m_3 m_2 + \gamma p_2\big)^2,$$

is a conserved quantity for the map dLT.

Proof. Setting $C = 1 - \epsilon^2 (1-\alpha)^2 m_3^2$, $D = -2\epsilon^2 \gamma (1-\alpha) m_3$, $E = -\epsilon^2 \gamma^2$, one can check that $Cc_1 + Dd_1 + Ee_1 = 0$ and $Cc_2 + Dd_2 + Ee_2 = -2\gamma$. This yields for the conserved quantity $F = Cc_0 + Dd_0 + Ee_0$ the expression given in the Proposition.

Considering the leading terms of the power expansions in ϵ, one sees immediately that the integrals c_0, d_0, e_0, and m_3 are functionally independent. Using exact evaluation of gradients we can also verify independence of other sets of integrals. It turns out that for $\alpha \neq 1$ each one of the quadruples $\{d_0, d_1, d_2, m_3\}$ and $\{e_0, e_1, e_2, m_3\}$ consists of independent integrals.

A direct "bilinearization" of the HK bases of Proposition 38 provides us with an alternative source of integrals of motion:

Proposition 40. *The set*

$$\Psi = (m_1 \widetilde{m}_1 + m_2 \widetilde{m}_2,\ p_1 \widetilde{m}_1 + \widetilde{p}_1 m_1 + p_2 \widetilde{m}_2 + \widetilde{p}_2 m_2,\ p_1 \widetilde{p}_1 + p_2 \widetilde{p}_2,\ p_3 \widetilde{p}_3,\ p_3 + \widetilde{p}_3,\ 1)$$

is a HK basis for the map dLT with $\dim K_\Psi(x) = 3$. *Each of the following subsets of* Ψ*,*

$$\Psi_1 = (1,\ p_3 + \widetilde{p}_3,\ p_3 \widetilde{p}_3,\ m_1 \widetilde{m}_1 + m_2 \widetilde{m}_2),$$
$$\Psi_2 = (1,\ p_3 + \widetilde{p}_3,\ p_3 \widetilde{p}_3,\ m_1 \widetilde{p}_1 + \widetilde{m}_1 p_1 + m_2 \widetilde{p}_2 + \widetilde{m}_2 p_2),$$
$$\Psi_3 = (1,\ p_3 + \widetilde{p}_3,\ p_3 \widetilde{p}_3,\ p_1 \widetilde{p}_1 + p_2 \widetilde{p}_2),$$

is a HK basis with a one-dimensional null-space.

Concerning solutions of dLT as functions of the (discrete) time t, the crucial result is given in the following statement which should be considered as the proper discretization of Eq. (12.4).

Proposition 41. ([5]) *The component* p_3 *of the solution of difference equations (12.6) satisfies a relation of the type*

$$Q(p_3, \widetilde{p}_3) = q_0 p_3^2 \widetilde{p}_3^2 + q_1 p_3 \widetilde{p}_3 (p_3 + \widetilde{p}_3) + q_2 (p_3^2 + \widetilde{p}_3^2) + q_3 p_3 \widetilde{p}_3 + q_4 (p_3 + \widetilde{p}_3) + q_5 = 0,$$

coefficients of the biquadratic polynomial Q *being conserved quantities of dLT. Hence,* $p_3(t)$ *is an elliptic function of degree 2.*

Although it remains unknown whether the map dLT admits an invariant Poisson structure, we have the following statement.

Proposition 42. *The map dLT possesses an invariant volume form:*

$$\det \frac{\partial \widetilde{x}}{\partial x} = \frac{\phi(\widetilde{x})}{\phi(x)} \quad \Leftrightarrow \quad f^* \omega = \omega, \quad \omega = \frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\phi(x)},$$

with $\phi(x) = \Delta_2(x, \epsilon)$.

13. KIRCHHOFF CASE OF THE RIGID BODY MOTION IN AN IDEAL FLUID

The motion of a rigid body in an ideal fluid is described by Kirchhoff equations (12.1) with H being a quadratic form in $m = (m_1, m_2, m_3)^{\mathrm{T}} \in \mathbb{R}^3$ and $p = (p_1, p_2, p_3)^{\mathrm{T}} \in \mathbb{R}^3$. The physical meaning of m is the total angular momentum, whereas p represents the total linear momentum of the system. A detailed introduction to the general context of rigid body dynamics and its mathematical foundations can be found in [20].

The integrable case of this system found in the original paper by Kirchhoff [21] and carrying his name is characterized by the Hamilton function $H = H_1/2$, where

$$H_1 = a_1(m_1^2 + m_2^2) + a_3m_3^2 + b_1(p_1^2 + p_2^2) + b_3p_3^2.$$

The differential equations of the Kirchhoff case are:

$$\begin{cases} \dot{m}_1 = (a_3 - a_1)m_2m_3 + (b_3 - b_1)p_2p_3, \\ \dot{m}_2 = (a_1 - a_3)m_1m_3 + (b_1 - b_3)p_1p_3, \\ \dot{m}_3 = 0, \\ \dot{p}_1 = a_3p_2m_3 - a_1p_3m_2, \\ \dot{p}_2 = a_1p_3m_1 - a_3p_1m_3, \\ \dot{p}_3 = a_1(p_1m_2 - p_2m_1). \end{cases} \tag{13.1}$$

Along with the Hamilton function H and the Casimir functions (12.2), it possesses the obvious fourth integral, due to the rotational symmetry of the system: $H_2 = m_3$. Traditionally, the explicit integration of the Kirchhoff case in terms of elliptic functions starts with the following observation: the component p_3 of the solution satisfies the differential equation

$$\dot{p}_3^2 = P_4(p_3),$$

with a quartic polynomial P_4 whose coefficients are expressed through integrals of motion:

$$P_4(p_3) = a_1\big(H_1 - a_3m_3^2 - b_1(C_1 - p_3^2) - b_3p_3^2\big)(C_1 - p_3^2) - a_1^2(C_2 - m_3p_3)^2.$$

We mention also the following Wronskian relation which follows easily from equations of motion:

$$a_1(\dot{m}_1p_1 - m_1\dot{p}_1) + a_1(\dot{m}_2p_2 - m_2\dot{p}_2) + (2a_3 - a_1)(\dot{m}_3p_3 - m_3\dot{p}_3) = 0. \tag{13.2}$$

Applying the HK approach to (13.1), we obtain the following system of equations:

$$\begin{cases} \widetilde{m}_1 - m_1 = \epsilon(a_3 - a_1)(\widetilde{m}_2m_3 + m_2\widetilde{m}_3) + \epsilon(b_3 - b_1)(\widetilde{p}_2p_3 + p_2\widetilde{p}_3), \\ \widetilde{m}_2 - m_2 = \epsilon(a_1 - a_3)(\widetilde{m}_1m_3 + m_1\widetilde{m}_3) + \epsilon(b_1 - b_3)(\widetilde{p}_1p_3 + p_1\widetilde{p}_3), \\ \widetilde{m}_3 - m_3 = 0, \\ \widetilde{p}_1 - p_1 = \epsilon a_3(\widetilde{p}_2m_3 + p_2\widetilde{m}_3) - \epsilon a_1(\widetilde{p}_3m_2 + p_3\widetilde{m}_2), \\ \widetilde{p}_2 - p_2 = \epsilon a_1(\widetilde{p}_3m_1 + p_3\widetilde{m}_1) - \epsilon a_3(\widetilde{p}_1m_3 + p_1\widetilde{m}_3), \\ \widetilde{p}_3 - p_3 = \epsilon a_1(\widetilde{p}_1m_2 + p_1\widetilde{m}_2) - \epsilon a_1(\widetilde{p}_2m_1 + p_2\widetilde{m}_1). \end{cases}$$

As usual, these equations define a birational map $\widetilde{x} = f(x, \epsilon)$, $x = (m, p)^{\mathrm{T}}$. We will refer to this map as dK. Like in the case of dLT, m_3 is a conserved quantity of dK. A further "simple" conserved quantity can be found from the following natural discretization of the Wronskian relation (13.2).

Proposition 43. *The set* $\Gamma = (\widetilde{m}_1 p_1 - m_1\widetilde{p}_1,\, \widetilde{m}_2 p_2 - m_2\widetilde{p}_2,\, \widetilde{m}_3 p_3 - m_3\widetilde{p}_3)$ *is a HK basis for the map dK with* $\dim K_\Gamma(x) = 1$. *At each point* $x \in \mathbb{R}^6$ *we have:* $K_\Gamma(x) = [1 : 1 : -\gamma_3]$, *where* γ_3 *is a conserved quantity of dK given by*

$$\gamma_3 = \frac{\Delta_0}{a_1\Delta_1}, \tag{13.3}$$

where

$$\Delta_0 = a_1 - 2a_3 + \epsilon^2 a_1^2(a_1 - a_3)(m_1^2 + m_2^2) + \epsilon^2 a_1 a_3(b_1 - b_3)(p_1^2 + p_2^2), \tag{13.4}$$

$$\Delta_1 = 1 + \epsilon^2 a_3(a_1 - a_3)m_3^2 + \epsilon^2 a_1(b_1 - b_3)p_3^2. \tag{13.5}$$

Proof. Like in the case of dLT, we let MAPLE compute the quantity

$$\gamma_3 = \frac{(\widetilde{m}_1 p_1 - m_1\widetilde{p}_1) + (\widetilde{m}_2 p_2 - m_2\widetilde{p}_2)}{(\widetilde{m}_3 p_3 - m_3\widetilde{p}_3)},$$

which results in (13.3), an even function of ϵ and therefore a conserved quantity.

Interestingly enough, this same integral may also be obtained from another HK basis:

Proposition 44. *The set* $\Phi_0 = (m_1^2 + m_2^2, p_1^2 + p_2^2, p_3^2, 1)$ *is a HK Basis for the map dK with* $\dim K_{\Phi_0}(x) = 1$. *The linear combination of these functions vanishing along the orbits can be put as* $\Delta_0 - \gamma_3 a_1 \Delta_1 = 0$.

Proof. The statement of the Proposition deals with the solution of a linear system of equations consisting of

$$(c_1(m_1^2 + m_2^2) + c_2(p_1^2 + p_2^2) + c_3 p_3^2) \circ f^i(m, p, \epsilon) = 1 \tag{13.6}$$

for all $i \in \mathbb{Z}$. We solve this system with $i = -1, 0, 1$ (numerically or symbolically), and observe that the solutions satisfy $a_3(b_1 - b_3)c_1 = a_1(a_1 - a_3)c_2$. Then, we consider the system of three equations for c_1, c_2, c_3 consisting of the latter linear relation between c_1, c_2, and of Eqs. (13.6) for $i = 0, 1$. This system is easily solved symbolically (by MAPLE), its unique solution can be put as in the Proposition. Its components are manifestly even functions of ϵ, thus conserved quantities.

Proposition 45.

(a) *The set* $\Phi = (m_1^2 + m_2^2,\, p_1 m_1 + p_2 m_2,\, p_1^2 + p_2^2,\, p_3^2,\, p_3,\, 1)$ *is a HK basis for the map dK with* $\dim K_\Phi(x) = 3$.

(b) *The set* $\Phi_1 = (1,\, p_3,\, p_3^2,\, m_1^2 + m_2^2)$ *is a HK basis for the map dK with a one-dimensional null-space. At each point* $x \in \mathbb{R}^6$ *we have:* $K_{\Phi_1}(x) = [c_0 : c_1 : c_2 : -1]$. *The functions* c_0, c_1, c_2 *are conserved quantities of the map dK, given by*

$$c_0 = \frac{a_1(m_1^2 + m_2^2) - (b_1 - b_3)p_3^2 + \epsilon^2 c_0^{(4)} + \epsilon^4 c_0^{(6)} + \epsilon^6 c_0^{(8)} + \epsilon^8 c_0^{(10)}}{a_1\Delta_1\Delta_2},$$

$$c_1 = -\frac{2\epsilon^2 a_3(b_1 - b_3)m_3\big(C_2 + \epsilon^2 c_1^{(4)} + \epsilon^4 c_1^{(6)} + \epsilon^6 c_1^{(8)}\big)}{\Delta_1\Delta_2},$$

$$c_2 = \frac{(b_1 - b_3)\big(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)} + \epsilon^8 c_2^{(8)}\big)}{a_1\Delta_1\Delta_2},$$

where Δ_1 *is given in (13.5), and* $\Delta_2 = 1 + \epsilon^2\Delta_2^{(2)} + \epsilon^4\Delta_2^{(4)} + \epsilon^6\Delta_2^{(6)}$; *coefficients* $c_k^{(q)}$ *and* $\Delta_2^{(q)}$ *are homogeneous polynomials of degree q in the phase variables. In particular:*

$$c_2^{(2)} = -2a_1^2(m_1^2 + m_2^2) - (a_1^2 - 2a_1a_3 + 3a_3^2)m_3^2 + a_1(b_1 - b_3)(p_1^2 + p_2^2) - a_1(b_1 - b_3)p_3^2,$$

$$\Delta_2^{(2)} = a_1^2(m_1^2 + m_2^2) + (a_1^2 - 3a_1a_3 + 3a_3^2)m_3^2 - a_1(b_1 - b_3)(p_1^2 + p_2^2) + a_1(b_1 - b_2)p_3^2.$$

(c) *The set* $\Phi_2 = (1, p_3, p_3^2, m_1p_1 + m_2p_2)$ *is a HK basis for the map dK with a one-dimensional null-space. At each point* $x \in \mathbb{R}^6$ *we have:* $K_{\Phi_2}(x) = [d_0 : d_1 : d_2 : -1]$. *The functions* d_0, d_1, d_2 *are conserved quantities of the map dK, given by*

$$d_0 = \frac{C_2 + \epsilon^2 d_0^{(4)} + \epsilon^4 d_0^{(6)} + \epsilon^6 d_0^{(8)} + \epsilon^8 d_0^{(10)}}{\Delta_1\Delta_2},$$

$$d_1 = \frac{m_3\big(-1 + \epsilon^2 d_1^{(2)} + \epsilon^4 d_1^{(4)} + \epsilon^6 d_1^{(6)} + \epsilon^8 d_1^{(8)}\big)}{\Delta_1\Delta_2},$$

$$d_2 = \frac{a_1(b_3 - b_1)\epsilon^2\big(C_2 + \epsilon^2 c_1^{(4)} + \epsilon^4 c_1^{(6)} + \epsilon^6 c_1^{(8)}\big)}{\Delta_1\Delta_2},$$

where $d_k^{(q)}$ *are homogeneous polynomials of degree* q *in the phase variables. In particular,*

$$d_1^{(2)} = -a_1a_3(m_1^2 + m_2^2) - (a_1^2 - 3a_1a_3 + 3a_3^2)m_3^2 + (a_1 - a_3)(b_1 - b_3)(p_1^2 + p_2^2) - 3a_1(b_1 - b_3)p_3^2.$$

(d) *The set* $\Phi_3 = (1,\, p_3,\, p_3^2,\, p_1^2 + p_2^2)$ *is a HK basis for the map dK with a one-dimensional null-space. At each point* $x \in \mathbb{R}^6$ *we have:* $K_{\Phi_3}(x) = [e_0 : e_1 : e_2 : -1]$. *The functions* e_0, e_1, e_2 *are conserved quantities of the map dK, given by*

$$e_0 = \frac{C_1 + \epsilon^2 e_0^{(4)} + \epsilon^4 e_0^{(6)} + \epsilon^6 e_0^{(8)} + \epsilon^8 e_0^{(10)}}{\Delta_1\Delta_2},$$

$$e_1 = \frac{2\epsilon^2 a_1(a_3 - a_1)m_3\big(C_2 + \epsilon^2 c_1^{(4)} + \epsilon^4 c_1^{(6)} + \epsilon^6 c_1^{(8)}\big)}{\Delta_1\Delta_2},$$

$$e_2 = \frac{-1 + \epsilon^2 e_2^{(2)} + \epsilon^4 e_2^{(4)} + \epsilon^6 e_6^{(6)} + \epsilon^8 e_8^{(8)}}{\Delta_1\Delta_2},$$

where $e_k^{(q)}$ *are polynomials of degree* q *in the phase variables. In particular,*

$$e_2^{(2)} = -a_1^2(m_1^2 + m_2^2) - (2a_1^2 - 4a_1a_3 + 3a_3^2)m_3^2 + 2a_1(b_1 - b_3)(p_1^2 + p_2^2) - a_1(b_1 - b_3)p_3^2.$$

Proof. Statement (b) is proven using direct calculation. Statements (c) and (d) then follow analogously to Proposition 38 from the existence of linear relations between c_1 and d_2, as well as between c_1 and e_1.

One can show that each of the sets $\{c_0, c_1, c_2\}$, $\{d_0, d_1, d_2\}$, and $\{e_0, e_1, e_2\}$ consists of three independent integrals of motion. Moreover, each of the sets $\{c_0, c_1, c_2, m_3\}$ and $\{e_0, e_1, e_2, m_3\}$ consists of four independent integrals. As further important results, me mention that Propositions 40 (on the "bilinear" HK bases), 41 (on the invariant biquadratic curve for $(p_3, \widetilde{p}_3)$), and 42 (on the invariant measure) hold literally true for the map dK.

14. CLEBSCH CASE OF THE RIGID BODY MOTION IN AN IDEAL FLUID

Another famous integrable case of the Kirchhoff equations was discovered by Clebsch [22] and is characterized by the Hamilton function $H = H_1/2$, where

$$H_1 = \langle m, Am\rangle + \langle p, Bp\rangle = \frac{1}{2}\sum_{k=1}^{3}(a_k m_k^2 + b_k p_k^2),$$

where $A = \mathrm{diag}(a_1, a_2, a_3)$ and $B = \mathrm{diag}(b_1, b_2, b_3)$ satisfy the condition

$$\frac{b_1 - b_2}{a_3} + \frac{b_2 - b_3}{a_1} + \frac{b_3 - b_1}{a_2} = 0. \tag{14.1}$$

This condition is also equivalent to saying that the quantity

$$\theta = \frac{b_j - b_k}{a_i(a_j - a_k)} \tag{14.2}$$

takes one and the same value for all permutations (i, j, k) of the indices (1, 2, 3).

For an embedding of this system into the modern theory of integrable systems see [23, 24]. Note that the Kirchhoff case ($a_1 = a_2$ and $b_1 = b_2$) can be considered as a particular case of the Clebsch case, but is special in many respects (the symmetry resulting in the existence of the Noether integral m_3, solvability in elliptic functions, in contrast to the general Clebsch system being solvable in terms of theta-functions of genus 2, etc.). Equations of motion of the Clebsch case are:

$$\begin{cases} \dot{m} = m \times Am + p \times Bp\,, \\ \dot{p} = p \times Am, \end{cases} \tag{14.3}$$

or in components

$$\begin{cases} \dot{m}_1 = (a_3 - a_2)m_2m_3 + (b_3 - b_2)p_2p_3, \\ \dot{m}_2 = (a_1 - a_3)m_3m_1 + (b_1 - b_3)p_3p_1, \\ \dot{m}_3 = (a_2 - a_1)m_1m_2 + (b_2 - b_1)p_1p_2, \\ \dot{p}_1 = a_3m_3p_2 - a_2m_2p_3, \\ \dot{p}_2 = a_1m_1p_3 - a_3m_3p_1, \\ \dot{p}_3 = a_2m_2p_1 - a_1m_1p_2. \end{cases} \tag{14.4}$$

Condition (14.1) can be resolved for a_i as

$$a_1 = \frac{b_2 - b_3}{\omega_2 - \omega_3}\,, \quad a_2 = \frac{b_3 - b_1}{\omega_3 - \omega_1}\,, \quad a_3 = \frac{b_1 - b_2}{\omega_1 - \omega_2}\,.$$

For fixed values of ω_i and varying values of b_i, equations of motion of the Clebsch case share the integrals of motion: the Casimirs C_1, C_2, cf. Eq. (12.2), and the Hamiltonians

$$I_i = p_i^2 + \frac{m_j^2}{\omega_i - \omega_k} + \frac{m_k^2}{\omega_i - \omega_j}.$$

There are four independent functions among C_i, I_i, because of $C_1 = I_1 + I_2 + I_3$. Note that $H_1 = b_1I_1 + b_2I_2 + b_3I_3$. One can denote all models with the same ω_i as a hierarchy, single flows of which are characterized by the parameters b_i. Usually, one denotes as "the first flow" of this hierarchy the one corresponding to the choice $b_i = \omega_i$, so that $a_i = 1$. Thus, the first flow is characterized by the value $\theta = \infty$ of the constant (14.2).

14.1. First Flow of the Clebsch System

The first flow of the Clebsch hierarchy is generated by the Hamilton function $H = H_1/2$, where

$$H_1 = m_1^2 + m_2^2 + m_3^2 + \omega_1p_1^2 + \omega_2p_2^2 + \omega_3p_3^2.$$

The corresponding equations of motion read:

$$\begin{cases} \dot{m} = p \times \Omega p, \\ \dot{p} = p \times m, \end{cases}$$

where $\Omega = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$ is the matrix of parameters, or in components:

$$\begin{cases} \dot{m}_1 = (\omega_3 - \omega_2)p_2p_3, \\ \dot{m}_2 = (\omega_1 - \omega_3)p_3p_1, \\ \dot{m}_3 = (\omega_2 - \omega_1)p_1p_2, \\ \dot{p}_1 = m_3p_2 - m_2p_3, \\ \dot{p}_2 = m_1p_3 - m_3p_1, \\ \dot{p}_3 = m_2p_1 - m_1p_2. \end{cases}$$

The fourth independent quadratic integral can be chosen as

$$H_2 = \omega_1 m_1^2 + \omega_2 m_2^2 + \omega_3 m_3^2 - \omega_2\omega_3 p_1^2 - \omega_3\omega_1 p_2^2 - \omega_1\omega_2 p_3^2.$$

Note that $H_1 = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3$, $H_1 = -\omega_2\omega_3 I_1 - \omega_3\omega_1 I_2 - \omega_1\omega_2 I_3$.

We mention the following Wronskian relation:

$$(\dot{m}_1p_1 - m_1\dot{p}_1) + (\dot{m}_2p_2 - m_2\dot{p}_2) + (\dot{m}_3p_3 - m_3\dot{p}_3) = 0, \tag{14.5}$$

which holds true for the first Clebsch flow.

The HK discretization of the first Clebsch flow (proposed in [25]) is:

$$\begin{cases} \widetilde{m}_1 - m_1 = \epsilon(\omega_3 - \omega_2)(\widetilde{p}_2p_3 + p_2\widetilde{p}_3), \\ \widetilde{m}_2 - m_2 = \epsilon(\omega_1 - \omega_3)(\widetilde{p}_3p_1 + p_3\widetilde{p}_1), \\ \widetilde{m}_3 - m_3 = \epsilon(\omega_2 - \omega_1)(\widetilde{p}_1p_2 + p_1\widetilde{p}_2), \\ \widetilde{p}_1 - p_1 = \epsilon(\widetilde{m}_3p_2 + m_3\widetilde{p}_2) - \epsilon(\widetilde{m}_2p_3 + m_2\widetilde{p}_3), \\ \widetilde{p}_2 - p_2 = \epsilon(\widetilde{m}_1p_3 + m_1\widetilde{p}_3) - \epsilon(\widetilde{m}_3p_1 + m_3\widetilde{p}_1), \\ \widetilde{p}_3 - p_3 = \epsilon(\widetilde{m}_2p_1 + m_2\widetilde{p}_1) - \epsilon(\widetilde{m}_1p_2 + m_1\widetilde{p}_2). \end{cases}$$

As usual, it leads to a reversible birational map $\widetilde{x} = f(x, \epsilon)$, $x = (m, p)^{\mathrm{T}}$, given by $f(x, \epsilon) = A^{-1}(x, \epsilon)x$ with

$$A(m, p, \epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & \epsilon\omega_{23}p_3 & \epsilon\omega_{23}p_2 \\ 0 & 1 & 0 & \epsilon\omega_{31}p_3 & 0 & \epsilon\omega_{31}p_1 \\ 0 & 0 & 1 & \epsilon\omega_{12}p_2 & \epsilon\omega_{12}p_1 & 0 \\ 0 & \epsilon p_3 & -\epsilon p_2 & 1 & -\epsilon m_3 & \epsilon m_2 \\ -\epsilon p_3 & 0 & \epsilon p_1 & \epsilon m_3 & 1 & -\epsilon m_1 \\ \epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1 \end{pmatrix},$$

where the abbreviation $\omega_{ij} = \omega_i - \omega_j$ is used. This map will be referred to as dC.

A "simple" conserved quantity can be found from the following natural discretization of the Wronskian relation (14.5).

Proposition 46. *The set* $\Gamma = (\widetilde{m}_1p_1 - m_1\widetilde{p}_1,\ \widetilde{m}_2p_2 - m_2\widetilde{p}_2,\ \widetilde{m}_3p_3 - m_3\widetilde{p}_3)$ *is a HK basis for the map dC with* $\dim K_\Gamma(x) = 1$. *At each point* $x \in \mathbb{R}^6$ *we have:* $K_\Gamma(x) = [e_1 : e_2 : e_3]$, *where*

$$e_i = 1 + \epsilon^2(\omega_i - \omega_j)p_j^2 + \epsilon^2(\omega_i - \omega_k)p_k^2. \tag{14.6}$$

The conserved quantities e_i/e_j can be put as $e_i/e_j = (1+\epsilon^2\omega_i J)/(1+\epsilon^2\omega_j J)$, where J is a nice and symmetric integral,

$$J = \frac{p_1^2 + p_2^2 + p_3^2}{1 - \epsilon^2(\omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2)}.$$

Remarkably, it can be obtained also from a different (monomial) HK basis, see part b) of the following statement.

Proposition 47 ([2]).

(a) *The set of functions* $\Phi = (p_1^2,\, p_2^2,\, p_3^2,\, m_1^2,\, m_2^2,\, m_3^2,\, m_1p_1,\, m_2p_2,\, m_3p_3,\, 1)$ *is a HK basis for the map dC with* $\dim K_\Phi(m,p) = 4$. *Thus, any orbit of the map dC lies on an intersection of four quadrics in* $\mathbb{R}^6$.

(b) *The set of functions* $\Phi_0 = (p_1^2, p_2^2, p_3^2, 1)$ *is a HK basis for the map dC with* $\dim K_{\Phi_0}(m,p) = 1$. *At each point* $(m,p) \in \mathbb{R}^6$ *there holds:*

$$\begin{aligned} K_{\Phi_0}(m,p) &= [e_1 : e_2 : e_3 : -(p_1^2+p_2^2+p_3^2)] \\ &= \left[\frac{1}{J} + \epsilon^2\omega_1 : \frac{1}{J} + \epsilon^2\omega_2 : \frac{1}{J} + \epsilon^2\omega_3 : -1\right], \end{aligned}$$

with the quantities e_i *given in (14.6).*

(c) *The sets of functions*

$$\Phi_1 = (p_1^2,\, p_2^2,\, p_3^2,\, m_1^2,\, m_2^2,\, m_3^2,\, m_1p_1), \tag{14.7}$$

$$\Phi_2 = (p_1^2,\, p_2^2,\, p_3^2,\, m_1^2,\, m_2^2,\, m_3^2,\, m_2p_2), \tag{14.8}$$

$$\Phi_3 = (p_1^2,\, p_2^2,\, p_3^2,\, m_1^2,\, m_2^2,\, m_3^2,\, m_3p_3), \tag{14.9}$$

are HK bases for the map dC with $\dim K_{\Phi_1}(m,p) = \dim K_{\Phi_2}(m,p) = \dim K_{\Phi_3}(m,p) = 1$. *At each point* $(m,p) \in \mathbb{R}^6$ *there holds:*

$$\begin{aligned} K_{\Phi_1}(m,p) &= [\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_5 : \alpha_6 : -1], \\ K_{\Phi_2}(m,p) &= [\beta_1 : \beta_2 : \beta_3 : \beta_4 : \beta_5 : \beta_6 : -1], \\ K_{\Phi_3}(m,p) &= [\gamma_1 : \gamma_2 : \gamma_3 : \gamma_4 : \gamma_5 : \gamma_6 : -1], \end{aligned}$$

where α_j,β_j, *and* γ_j *are rational functions of* (m,p), *even with respect to* ϵ. *They are conserved quantities of the map dCS. For* $j = 1,2,3$, *they are of the form*

$$h = \frac{h^{(2)} + \epsilon^2 h^{(4)} + \epsilon^4 h^{(6)} + \epsilon^6 h^{(8)} + \epsilon^8 h^{(10)} + \epsilon^{10} h^{(12)}}{2\epsilon^2(p_1^2+p_2^2+p_3^2)\Delta},$$

where h *stands for any of the functions* $\alpha_j, \beta_j, \gamma_j$, $j = 1,2,3$,

$$\Delta = m_1p_1 + m_2p_2 + m_3p_3 + \epsilon^2\Delta^{(4)} + \epsilon^4\Delta^{(6)} + \epsilon^6\Delta^{(8)},$$

and the corresponding $h^{(2q)}$, $\Delta^{(2q)}$ *are homogeneous polynomials in phase variables of degree* $2q$. *For instance,*

$$\begin{aligned} &\alpha_1^{(2)} = C_1 - I_1, && \alpha_2^{(2)} = -I_1, && \alpha_3^{(2)} = -I_1, \\ &\beta_1^{(2)} = -I_2, && \beta_2^{(2)} = C_1 - I_2, && \beta_3^{(2)} = -I_2, \\ &\gamma_1^{(2)} = -I_3, && \gamma_2^{(2)} = -I_3, && \gamma_3^{(2)} = C_1 - I_3. \end{aligned}$$

For $j = 4,5,6$, *the functions* $\alpha_j, \beta_j, \gamma_j$ *are given by*

$$\begin{pmatrix} \alpha_4 & \alpha_5 & \alpha_6 \\ \beta_4 & \beta_5 & \beta_6 \\ \gamma_4 & \gamma_5 & \gamma_6 \end{pmatrix} = \begin{pmatrix} D & A_1/(\omega_1-\omega_3) & A_1/(\omega_1-\omega_2) \\ A_2/(\omega_2-\omega_3) & D & A_2/(\omega_2-\omega_1) \\ A_3/(\omega_3-\omega_2) & A_3/(\omega_3-\omega_1) & D \end{pmatrix},$$

where

$$A_k = \frac{1+\epsilon^2 A_k^{(2)} + \epsilon^4 A_k^{(4)} + \epsilon^6 A_k^{(6)} + \epsilon^8 A_k^{(8)}}{2\epsilon^2\Delta},$$
$$D = \frac{p_1^2+p_2^2+p_3^2+\epsilon^2 D^{(4)} + \epsilon^4 D^{(6)} + \epsilon^6 D^{(8)}}{2\Delta},$$

and $A_k^{(2q)}$, $D^{(2q)}$ *are homogeneous polynomials of degree* $2q$ *in phase variables, for instance,*

$$A_k^{(2)} = m_1^2+m_2^2+m_3^2+(\omega_2+\omega_3-2\omega_1)p_1^2+(\omega_3+\omega_1-2\omega_2)p_2^2+(\omega_1+\omega_2-2\omega_3)p_3^2.$$

The four conserved quantities J, α_1, β_1 *and* γ_1 *are functionally independent.*

Our paper [2] contains a much more detailed information about the HK bases of the map dC, for instance, the further basis with a one-dimensional null-space: $\Theta = (p_1^2,\, p_2^2,\, p_3^2,\, m_1p_1,\, m_2p_2,\, m_3p_3)$. However, the following finding about the "bilinear" versions of the above bases is new.

Proposition 48. *Each of the sets of functions*

$$\Psi_0 = (\widetilde{p}_1p_1,\, \widetilde{p}_2p_2,\, \widetilde{p}_3p_3,\, 1), \tag{14.10}$$
$$\Psi_1 = (\widetilde{p}_1p_1,\, \widetilde{p}_2p_2,\, \widetilde{p}_3p_3,\, \widetilde{m}_1m_1,\, \widetilde{m}_2m_2,\, \widetilde{m}_3m_3,\, \widetilde{m}_1p_1+m_1\widetilde{p}_1), \tag{14.11}$$
$$\Psi_2 = (\widetilde{p}_1p_1,\, \widetilde{p}_2p_2,\, \widetilde{p}_3p_3,\, \widetilde{m}_1m_1,\, \widetilde{m}_2m_2,\, \widetilde{m}_3m_3,\, \widetilde{m}_2p_2+m_2\widetilde{p}_2), \tag{14.12}$$
$$\Psi_3 = (\widetilde{p}_1p_1,\, \widetilde{p}_2p_2,\, \widetilde{p}_3p_3,\, \widetilde{m}_1m_1,\, \widetilde{m}_2m_2,\, \widetilde{m}_3m_3,\, \widetilde{m}_3p_3+m_3\widetilde{p}_3), \tag{14.13}$$

is a HK basis for the map dC with a one-dimensional null-space.

14.2. General Flow of the Clebsch System

The HK discretization of the flow (14.3) reads

$$\begin{cases} \widetilde{m}-m = \epsilon(\widetilde{m}\times Am + m\times A\widetilde{m} + \widetilde{p}\times Bp + p\times B\widetilde{p}), \\ \widetilde{p}-p = \epsilon\,(\widetilde{p}\times Am + p\times A\widetilde{m})\,, \end{cases}$$

in components:

$$\begin{cases} \widetilde{m}_1-m_1 = \epsilon(a_3-a_2)(\widetilde{m}_2m_3+m_2\widetilde{m}_3)+\epsilon(b_3-b_2)(\widetilde{p}_2p_3+p_2\widetilde{p}_3), \\ \widetilde{m}_2-m_2 = \epsilon(a_1-a_3)(\widetilde{m}_3m_1+m_3\widetilde{m}_1)+\epsilon(b_1-b_3)(\widetilde{p}_3p_1+p_3\widetilde{p}_1), \\ \widetilde{m}_3-m_3 = \epsilon(a_2-a_1)(\widetilde{m}_1m_2+m_1\widetilde{m}_2)+\epsilon(b_2-b_1)(\widetilde{p}_1p_2+p_1\widetilde{p}_2), \\ \widetilde{p}_1-p_1 = \epsilon a_3(\widetilde{m}_3p_2+m_3\widetilde{p}_2)-\epsilon a_2(\widetilde{m}_2p_3+m_2\widetilde{p}_3), \\ \widetilde{p}_2-p_2 = \epsilon a_1(\widetilde{m}_1p_3+m_1\widetilde{p}_3)-\epsilon a_3(\widetilde{m}_3p_1+m_3\widetilde{p}_1), \\ \widetilde{p}_3-p_3 = \epsilon a_2(\widetilde{m}_2p_1+m_2\widetilde{p}_1)-\epsilon a_1(\widetilde{m}_1p_2+m_1\widetilde{p}_2). \end{cases} \tag{14.14}$$

In what follows, we will use the abbreviations $b_{ij} = b_i - b_j$ and $a_{ij} = a_i - a_j$. The linear system (14.14) defines an explicit, birational map $f : \mathbb{R}^6 \to \mathbb{R}^6$,

$$\begin{pmatrix} \widetilde{m} \\ \widetilde{p} \end{pmatrix} = f(m,p,\epsilon) = M^{-1}(m,p,\epsilon)\begin{pmatrix} m \\ p \end{pmatrix},$$

where

$$M(m,p,\epsilon)=\begin{pmatrix} 1 & \epsilon a_{23}m_3 & \epsilon a_{23}m_2 & 0 & \epsilon b_{23}p_3 & \epsilon b_{23}p_2 \\ \epsilon a_{31}m_3 & 1 & \epsilon a_{31}m_1 & \epsilon b_{31}p_3 & 0 & \epsilon b_{31}p_1 \\ \epsilon a_{12}m_2 & \epsilon a_{12}m_1 & 1 & \epsilon b_{12}p_2 & \epsilon b_{12}p_1 & 0 \\ 0 & \epsilon a_2p_3 & -\epsilon a_3p_2 & 1 & -\epsilon a_3m_3 & \epsilon a_2m_2 \\ -\epsilon a_1p_3 & 0 & \epsilon a_3p_1 & \epsilon a_3m_3 & 1 & -\epsilon a_1m_1 \\ \epsilon a_1p_2 & -\epsilon a_2p_1 & 0 & -\epsilon a_2m_2 & \epsilon a_1m_1 & 1 \end{pmatrix}.$$

This map will be denoted dGC in what follows.

A "simple" integral of the map dGC can be obtained by discretizing the following Wronskian relation with constant coefficients, which holds for the general flow of the Clebsch system (14.4):

$$A_1(\dot m_1p_1-m_1\dot p_1)+A_2(\dot m_2p_2-m_2\dot p_2)+A_3(\dot m_3p_3-m_3\dot p_3)=0,$$

with

$$A_i=a_ia_j+a_ia_k-a_ja_k.$$

Proposition 49. *The set* $\Gamma=(\widetilde m_1p_1-m_1\widetilde p_1,\ \widetilde m_2p_2-m_2\widetilde p_2,\ \widetilde m_3p_3-m_3\widetilde p_3)$ *is a HK basis for the map dGC with* $\dim K_\Gamma(x)=1$*. At each point* $x\in\mathbb{R}^6$ *we have:* $K_\Gamma(x)=[e_1:e_2:e_3]$*, where, for* $(i,j,k)=\text{c.p.}(1,2,3)$,

$$e_i=A_i+\epsilon^2a_i(b_i-b_j)A_k\Theta_j+\epsilon^2a_i(b_i-b_k)A_j\Theta_k,$$

with

$$\Theta_i=p_i^2+\frac{a_i}{\theta a_ja_k}m_i^2.$$

(Recall that θ *is defined by Eq. (14.2); we assume here that* $\theta\neq\infty$*.)*

As in the case of the first flow, the integrals e_i/e_j can be expressed through one symmetric integral: $e_i/e_j=(A_i-\theta a_iL)/(A_j-\theta a_jL)$, where

$$L=\frac{a_2a_3A_1\Theta_1+a_3a_1A_2\Theta_2+a_1a_2A_3\Theta_3}{1+\epsilon^2\theta a_1a_2a_3(\Theta_1+\Theta_2+\Theta_3)}.$$

The quantities e_i and the integral L can be also obtained from a different (monomial) HK basis, given in part b) of the following Proposition.

Proposition 50.

(a) *The set* $\Phi=(p_1^2,\ p_2^2,\ p_3^2,\ m_1^2,\ m_2^2,\ m_3^2,\ m_1p_1,\ m_2p_2,\ m_3p_3,\ 1)$ *is a HK basis for the map dGC with* $\dim K_\Phi(m,p)=4$*. Thus, any orbit of the map dGC lies on an intersection of four quadrics in* $\mathbb{R}^6$.

(b) *The set of functions* $\Phi_0=(p_1^2,\ p_2^2,\ p_3^2,\ m_1^2,\ m_2^2,\ m_3^2,\ 1)$ *is a HK basis for the map dGC with* $\dim K_{\Phi_0}(m,p)=1$*. At each point* $(m,p)\in\mathbb{R}^6$ *there holds:*

$$K_{\Phi_0}(m,p)=[a_2a_3e_1:a_3a_1e_2:a_1a_2e_3:(a_1/\theta)e_1:(a_2/\theta)e_2:(a_3/\theta)e_3:-e_0],$$

where

$$e_0=a_2a_3A_1\Theta_1+a_3a_1A_2\Theta_2+a_1a_2A_3\Theta_3$$

is an integral of motion of the continuous time flow (14.4).

(c) *The sets of functions (14.7)–(14.9) are HK bases for the map dGC with one-dimensional null-spaces.*

(d) *Each of the sets of functions* $\Psi_0=(\widetilde p_1p_1,\ \widetilde p_2p_2,\ \widetilde p_3p_3,\ \widetilde m_1m_1,\ \widetilde m_2m_2,\ \widetilde m_3m_3,\ 1)$ *and* (14.11)–(14.13) *is a HK basis for the map dGC with a one-dimensional null-space.*

15. $\mathfrak{su}(2)$ RATIONAL GAUDIN SYSTEM WITH $N = 2$ SPINS

The Gaudin system [26] describes an interaction of N quantum spins y_i, $i = 1, \dots, N$, with a homogeneous constant external field p. Its classical version is given by the following quadratic system of differential equations [27]:

$$\dot{y}_i = \left(\lambda_i \, p + \sum_{j=1}^{N} y_j \right) \times y_i, \qquad 1 \leqslant i \leqslant N, \tag{15.1}$$

where $y_i \in \mathfrak{su}(2) \simeq \mathbb{R}^3$, $p \in \mathfrak{su}(2) \simeq \mathbb{R}^3$ is a constant vector, and pairwise distinct numbers λ_i are parameters of the model. The flow (15.1) is Hamiltonian with respect to the Lie-Poisson bracket of the direct sum of N copies of $\mathfrak{su}(2)$, admits $2N$ independent conserved quantities in involution: the N Casimir functions

$$C_k = \langle y_k, y_k \rangle,$$

and the following N Hamiltonians:

$$H_k = \langle p, y_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^{N} \frac{\langle y_k, y_j \rangle}{\lambda_k - \lambda_j},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathfrak{su}(2) \simeq \mathbb{R}^3$. Note that the Hamilton function of the flow (15.1) is

$$H = \sum_{k=1}^{N} \lambda_k \, H_k = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \langle y_i, y_j \rangle + \sum_{i=1}^{N} \lambda_i \langle p, y_i \rangle.$$

In [27, 28] it has been proved that a contraction of N simple poles to one pole of order N provides the integrable flow of the so called *one-body rational* $\mathfrak{su}(2)$ *tower*, whose simplest instance with $N = 2$ describes the dynamics of the three-dimensional Lagrange top in the rest frame.

We consider here the HK discretization of this flow (15.1) with $N = 2$. We set $y_1 = (x_1, x_2, x_3)^{\mathrm{T}}$, $y_2 = (z_1, z_2, z_3)^{\mathrm{T}}$, and choose the constant gravity vector $p = (0, 0, 1)^{\mathrm{T}}$. We thus obtain the following system of differential equations:

$$\begin{cases} \dot{x}_1 = x_2 z_3 - x_3 z_2 + \lambda_1 x_2, \\ \dot{x}_2 = x_3 z_1 - x_1 z_3 - \lambda_1 x_1, \\ \dot{x}_3 = x_1 z_2 - x_2 z_1, \\ \dot{z}_1 = z_2 x_3 - z_3 x_2 + \lambda_2 z_2, \\ \dot{z}_2 = z_3 x_1 - z_1 x_3 - \lambda_2 z_1, \\ \dot{z}_3 = z_1 x_2 - z_2 x_1, \end{cases} \tag{15.2}$$

with λ_1, λ_2 being real parameters. The system (15.2) has the following four independent integrals of motion:

$$C_1 = x_2^2 + x_2^2 + x_3^2, \qquad C_2 = z_1^2 + z_2^2 + z_3^2,$$

$$H_1 = x_3 + \frac{x_1 z_1 + x_2 z_2 + x_3 z_3}{\lambda_1 - \lambda_2}, \qquad H_2 = z_3 + \frac{x_1 z_1 + x_2 z_2 + x_3 z_3}{\lambda_2 - \lambda_1}.$$

Note that the quantity $H_1 + H_2 = x_3 + z_3$ is a linear integral of motion. We mention also the following Wronskian relation with constant coefficients:

$$(x_3 + z_3)(\dot{x}_1 z_1 - x_1 \dot{z}_1 + \dot{x}_2 z_2 - x_2 \dot{z}_2) + (\lambda_1 + \lambda_2 + x_3 + z_3)(\dot{x}_3 z_3 - x_3 \dot{z}_3) = 0. \tag{15.3}$$

The HK discretization of (15.2) reads:

$$\begin{cases} \widetilde{x}_1 - x_1 = \epsilon(\widetilde{x}_2 z_3 + x_2\widetilde{z}_3 - \widetilde{x}_3 z_2 - x_3\widetilde{z}_2) + \epsilon\lambda_1(\widetilde{x}_2 + x_2), \\ \widetilde{x}_2 - x_2 = \epsilon(\widetilde{x}_3 z_1 + x_3\widetilde{z}_1 - \widetilde{x}_1 z_3 - x_1\widetilde{z}_3) - \epsilon\lambda_1(\widetilde{x}_1 + x_1), \\ \widetilde{x}_3 - x_3 = \epsilon(\widetilde{x}_1 z_2 + x_1\widetilde{z}_2 - \widetilde{x}_2 z_1 - x_2\widetilde{z}_1), \\ \widetilde{z}_1 - z_1 = \epsilon(\widetilde{z}_2 x_3 + x_2\widetilde{x}_3 - \widetilde{z}_3 x_2 - z_3\widetilde{x}_2) + \epsilon\lambda_2(\widetilde{z}_2 + z_2), \\ \widetilde{z}_2 - z_2 = \epsilon(\widetilde{z}_3 x_1 + z_3\widetilde{x}_1 - \widetilde{z}_1 x_3 - z_1\widetilde{x}_3) - \epsilon\lambda_2(\widetilde{z}_1 + z_1), \\ \widetilde{z}_3 - z_3 = \epsilon(\widetilde{z}_1 x_2 + z_1\widetilde{x}_2 - \widetilde{z}_2 x_1 - z_2\widetilde{x}_1). \end{cases} \tag{15.4}$$

The map $f : x \mapsto \widetilde{x}$ obtained by solving (15.4) for $\widetilde{x}$ is given by:

$$\widetilde{x} = f(x, \epsilon) = A^{-1}(x, \epsilon)(\mathbb{1} + \epsilon B)x,$$

where $x = (x_1, x_2, x_3, z_1, z_2, z_3)^{\mathrm{T}}$, and

$$A(x, \epsilon) = \begin{pmatrix} 1 & -\epsilon z_3 & \epsilon z_2 & 0 & \epsilon x_3 & -\epsilon x_2 \\ -\epsilon z_3 & 1 & -\epsilon z_1 & -\epsilon x_3 & 0 & \epsilon x_1 \\ -\epsilon z_2 & \epsilon z_1 & 1 & \epsilon x_2 & -\epsilon x_1 & 0 \\ 0 & \epsilon z_3 & -\epsilon z_2 & 1 & -\epsilon x_3 & \epsilon x_2 \\ -\epsilon z_3 & 0 & \epsilon z_1 & \epsilon x_3 & 1 & -\epsilon x_1 \\ \epsilon z_2 & -\epsilon z_1 & 0 & -\epsilon x_2 & \epsilon x_1 & 1 \end{pmatrix} - \epsilon B,$$

$$B = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This map will be called dG in the sequel. The quantity $x_3 + z_3$ is obviously preserved by the map dG. Other conserved quantities may now be found using the HK bases approach. A "simple" integral follows, as in the previous sections, by discretizing the Wronskian relation (15.3).

Proposition 51. *The set of functions* $\Gamma = (\widetilde{x}_1 z_1 - x_1\widetilde{z}_1,\ \widetilde{x}_2 z_2 - x_2\widetilde{z}_2,\ \widetilde{x}_3 z_3 - x_3\widetilde{z}_3)$ *is a HK basis for the map dG with the one-dimensional null-space* $K_\Gamma(x, z) = [x_3 + z_3 : x_3 + z_3 : I]$, *with*

$$I(x, z) = \frac{\lambda_1 + \lambda_2 + x_3 + z_3 + \epsilon^2\lambda_1(x_1^2 + x_2^2) + \epsilon^2\lambda_2(z_1^2 + z_2^2) + \epsilon^2(\lambda_1 + \lambda_2)(x_1 z_1 + x_2 z_2)}{1 - \epsilon^2(\lambda_1 x_3 + \lambda_2 z_3 + \lambda_1\lambda_2)}.$$

A full set of integrals is found in the following Proposition. The roles of the variables x_i and z_i are not quite symmetric there, and interchanging them is of course admissible but does not lead to new integrals of motion.

Proposition 52.

(a) *The set* $\Phi = (x_1^2 + x_2^2,\ z_1^2 + z_2^2,\ z_3^2,\ x_1 z_1 + x_2 z_2,\ z_3,\ 1)$ *is a HK basis for the map dG with* $\dim K_\Phi(x) = 3$.

(b) *The set* $\Phi_1 = (1,\, z_3,\, z_3^2,\, x_1^2 + x_2^2)$ *is a HK basis for the map dG with a one-dimensional null-space. At each point* $(x, z) \in \mathbb{R}^6$ *we have:* $K_{\Phi_1}(x, z) = [c_0 : c_1 : c_2 : -1]$. *The functions* c_0, c_1, c_2 *are conserved quantities of the map dG, given by*

$$c_0 = \frac{x_1^2 + x_2^2 - 2x_3z_3 - z_3^2 + \epsilon^2 c_0^{(4)} + \epsilon^4 c_0^{(6)} + \epsilon^6 c_0^{(8)} + \epsilon^8 c_0^{(10)}}{\Delta_1\Delta_2},$$

$$c_1 = \frac{2(x_3 + z_3)\left(1 + \epsilon^2 c_1^{(3)} + \epsilon^4 c_1^{(5)} + \epsilon^6 c_1^{(7)} + \epsilon^8 c_1^{(9)}\right)}{\Delta_1\Delta_2},$$

$$c_2 = -\frac{\left(1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)} + \epsilon^8 c_2^{(10)}\right)}{\Delta_1\Delta_2},$$

where

$$\Delta_1 = 1 - \epsilon^2(\lambda_1 x_3 + \lambda_2 z_3 + \lambda_1\lambda_2), \quad \Delta_2 = 1 + \epsilon^2\Delta_2^{(2)} + \epsilon^4\Delta_2^{(4)} + \epsilon^6\Delta_2^{(6)}.$$

Here $\Delta^{(q)}$ *and* $c_k^{(q)}$ *are polynomials of degree q in the phase variables. In particular:*

$$c_2^{(2)} = -(x_1^2 + x_2^2 + x_3^2) - (z_1^2 + z_2^2 + z_3^2) - 2(x_1z_1 + x_2z_2 + x_3z_3) - 2(\lambda_2 x_3 + \lambda_1 z_3) - (\lambda_1^2 + \lambda_2^2),$$

and $\Delta_2^{(2)} = c_2^{(2)} + \lambda_1 x_3 + \lambda_2 z_3 + \lambda_1\lambda_2$.

(c) *The set* $\Phi_2 = (1, z_3, z_3^2, x_1z_1 + x_2z_2)$ *is a HK basis for the map dG with a one-dimensional null-space. At each point* $(x, z) \in \mathbb{R}^6$ *we have:* $K_{\Phi_2}(x, z) = [d_0 : d_1 : d_2 : -1]$. *The functions* d_0, d_1, d_2 *are conserved quantities of the map dG, given by*

$$d_0 = \frac{x_1z_1 + x_2z_2 + x_3z_3 + (\lambda_2 - \lambda_1)z_3 + \epsilon^2 d_0^{(4)} + \epsilon^4 d_0^{(6)} + \epsilon^6 d_0^{(8)} + \epsilon^8 d_0^{(10)}}{\Delta_1\Delta_2},$$

$$d_1 = \frac{\lambda_1 - \lambda_2 - x_3 - z_3 + \epsilon^2 d_1^{(3)} + \epsilon^4 d_1^{(5)} + \epsilon^6 d_1^{(7)} + \epsilon^8 d_1^{(9)}}{\Delta_1\Delta_2},$$

$$d_2 = -\frac{1 + \epsilon^2 c_2^{(2)} + \epsilon^4 c_2^{(4)} + \epsilon^6 c_2^{(6)} + \epsilon^8 c_2^{(8)}}{\Delta_1\Delta_2},$$

where $d_k^{(q)}$ *are polynomials of degree q in the phase variables.*

(d) *The set* $\Phi_3 = (1,\, z_3,\, z_3^2,\, z_1^2 + z_2^2)$ *is a HK basis for the map dG with a one-dimensional null-space. At each point* $(x, z) \in \mathbb{R}^6$ *we have:* $K_{\Phi_3}(x, z) = [e_0 : e_1 : e_2 : -1]$. *The functions* e_0, e_1, e_2 *are conserved quantities of the map dG, given by*

$$e_0 = \frac{z_1^2 + z_2^2 + z_3^2 + \epsilon^2 e_0^{(4)} + \epsilon^4 e_0^{(6)} + \epsilon^6 e_0^{(8)} + \epsilon^8 e_0^{(10)}}{\Delta_1\Delta_2},$$

$$e_1 = \frac{2\epsilon^2\left(e_1^{(2)} + \epsilon^2 e_1^{(4)} + \epsilon^4 e_1^{(6)} + \epsilon^6 e_1^{(8)}\right)}{\Delta_1\Delta_2},$$

$$e_2 = -\frac{1 + \epsilon^2 e_2^{(2)} + \epsilon^4 e_2^{(4)} + \epsilon^6 e_2^{(6)} + \epsilon^8 e_2^{(8)}}{\Delta_1\Delta_2},$$

where $e_k^{(q)}$ *are polynomials of degree q in the phase variables.*

It can be shown that each of the sets $\{c_0, c_1, c_2, x_3 + z_3\}$, $\{d_0, d_1, d_2, x_3 + z_3\}$ and $\{e_0, e_1, e_2, x_3 + z_3\}$ contains four independent integrals.

Analogously to the situation for the map dLT, it is possible to obtain a polynomial integral and an invariant volume form for the map dG.

Proposition 53. *The function*

$$G = \frac{1}{2}(x_1 + z_1)^2 + \frac{1}{2}(x_2 + z_2)^2 + \lambda_1 x_3 + \lambda_2 z_3 - \frac{\epsilon^2}{2}\big((\lambda_1 x_1 + \lambda_2 z_1)^2 + (\lambda_1 x_2 + \lambda_2 z_2)^2\big),$$

is a conserved quantity for the map dG.

Proposition 54. *The map dG possesses an invariant volume form:*

$$\det \frac{\partial(\widetilde{x}, \widetilde{z})}{\partial(x, z)} = \frac{\phi(\widetilde{x}, \widetilde{z})}{\phi(x, z)} \quad \Leftrightarrow \quad f^*\omega = \omega, \quad \omega = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dz_1 \wedge dz_2 \wedge dz_3}{\phi(x, z)},$$

where $\phi(x, z)) = \Delta_2(x, z)$.

Explicit integration of the map dG could be based on the following claim.

Proposition 55. *The component* x_3 *of the solution of the dG map satisfies a relation of the type*

$$Q(x_3, \widetilde{x}_3) = q_0 x_3^2 \widetilde{x}_3^2 + q_1 x_3 \widetilde{x}_3 (x_3 + \widetilde{x}_3) + q_2 (x_3^2 + \widetilde{x}_3^2) + q_3 x_3 \widetilde{x}_3 + q_4 (x_3 + \widetilde{x}_3) + q_5 = 0,$$

coefficients of the biquadratic polynomial Q being conserved quantities of dG. Thus, $x_3(t)$ *is an elliptic function of degree 2. An analogous statement holds for the component* z_3.

16. CONCLUSIONS

The initial motivation for this study was the hope that HK discretization would preserve the integrability for all algebraically integrable systems. This was formulated as a conjecture in [2]. The list of integrable discretizations given in the present overview contains more than a dozen issues and is rather impressive. It includes systems integrable in terms of elliptic functions as well as those integrable in terms of theta-functions of genus $g = 2$ (Clebsch system). This list might look like a convincing argument in favor of the integrability conjecture. However, at present we have also found examples which indicate that this conjecture might be wrong (e.g., the Zhukovski–Volterra system with all $\beta_k \neq 0$, or integrable chains, Volterra and dressing ones, with a big number of particles, say $N \geqslant 5$). We do not have rigorous proofs of the non-integrability in these cases, but the numerical evidence is rather strong. Since HK discretizations are (probably) not always integrable, the big number of integrable cases becomes still more intriguing: it is hard to imagine that all their common features come as a pure coincidence. We are after a theory which would clarify the problem of integrability of HK discretizations, but at present such a theory seems to remain a rather remote goal. Still, even without a general framework, HK discretizations represent a new fascinating chapter in the theory of integrable systems: we are now in a possession of a big and potentially growing stock of birational maps, integrable in terms of Abelian functions, highly non-trivial from the point of view of algebraic geometry and very different in nature from anything known before. The immediate goal of this review will be achieved if HK discretizations will attract attention of experts in the theory of integrable systems and in algebraic geometry.

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 290–310. © Pleiades Publishing, Ltd., 2011.

Nonlinear Evolution Equations and Hyperelliptic Covers of Elliptic Curves

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Received March 4, 2010; accepted August 26, 2010

Abstract—This paper is a further contribution to the study of exact solutions to KP, KdV, sine-Gordon, 1D Toda and nonlinear Schrodinger equations. We will be uniquely concerned with algebro-geometric solutions, doubly periodic in one variable. According to (so-called) Its-Matveev's formulae, the Jacobians of the corresponding spectral curves must contain an elliptic curve X, satisfying suitable geometric properties. It turns out that the latter curves are in fact contained in a particular algebraic surface $S\perp$, projecting onto a rational surface $\widetilde{S}$. Moreover, all spectral curves project onto a rational curve inside $\widetilde{S}$. We are thus led to study all rational curves of $\widetilde{S}$, having suitable numerical equivalence classes. At last we obtain d - 1-dimensional of spectral curves, of arbitrary high genus, giving rise to KdV solutions doubly periodic with respect to the d-th KdV flow ($d \geq 1$). Analogous results are presented, without proof, for the 1D Toda, NL Schrodinger an sine-Gordon equation.

MSC2010 numbers: 14E05, 14H30, 14H40, 14H55, 14H70, 14H81, 14C20, 35C08, 35Q51, 35Q53, 35Q55, 37K20
DOI: 10.1134/S1560354711030063

Keywords: elliptic and hyperelliptic curves, Jacobian variety, ruled and rational surfaces, exceptional curve, elliptic soliton.

1. INTRODUCTION

1.1. A huge variety of nonlinear integrable processes and phenomena in physics and mathematics can be described by a few nonlinear partial derivative equations (e.g., *Korteweg–de Vries* and *Kadomtsev–Petviashvili, 1D and 2D Toda, sine-Gordon, nonlinear Schrödinger* equations). For almost 40 years a full range of methods coming from distinct areas were developed in order to deal and present exact solutions of the latter equations (e.g., [1–37] and references therein). Zero-curvature equations, Lax pair's presentation and inverse scattering methods revolutionized the whole domain ([21, 37]). Rational and trigonometric exact solutions ([1, 6, 13]) were followed by quasi-periodic ones, also called *finite-gap*, given in terms of the theta function of an arbitrary hyperelliptic curve, via the Its–Matveev formula or its variants ([7, 14]). A few years later I.M. Krichever made a major contribution in [17], extending the latter results to finite-gap solutions of the *KP equation* associated to an arbitrary compact Riemann surface. M. Sato's infinite dimensional approach, developed in the beginning of the 1980s ([15, 25, 26]), further generalized

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Krichever's dictionary as well as the classical theta and Baker–Akhiezer function. From then on, all previously studied nonlinear evolution equations were reconsidered, and considerable effort was made in order to find doubly periodic solutions to each one of them. The starting point to this new trend was Krichevers's seminal article [18]. The first doubly periodic solutions to the *KdV equation* and a remarkable connection with the elliptic *Calogero–Moser integrable system* had already been found (e.g., [1] and [8], as well as [6] for the rational/trigonometric case), but [18] generalizes to an equivalence between the elliptic C-M integrable system and the KP solutions, doubly periodic in x. More precisely, given $n \geqslant 1$ and the lattice $L \subset \mathbb{C}$, the corresponding elliptic Calogero–Moser integrable system is solved. Its ($2n$-dimensional) phase space is cut out by the Jacobian Varieties of an n-dimensional family of genus n marked compact Riemann surfaces, each one of which (is effectively constructed and) gives rise to KP solutions L-periodic in $x \in \mathbb{C}$. The analogous problems for the *KdV*, *1D Toda*, *NL Schrödinger*, *sine-Gordon* equations and related problems ([22–24]) amount to finding hyperelliptic curves equipped with a projection onto X, satisfying specific geometrical properties, as briefly explained hereafter.

Let indeed $\pi : (\Gamma, p) \to (X, q)$ be an arbitrary ramified cover, where $\pi(p) = q$ and (X, q) is the elliptic curve $(\mathbb{C}/L, 0)$. Up to a translation, there exist canonical copies of Γ and X inside $Jac\,\Gamma$, the Jacobian variety of Γ. Consider the flag $\{0\} \subsetneq V^1_{\Gamma,p} \ldots \subsetneq V^g_{\Gamma,p}$, of hyperosculating spaces to Γ at p, and T_oX the tangent line to (the copy of) X, inside $Jac\,\Gamma$.

The d-th case of the KP equation: $\frac{3}{4}u_{yy} + \frac{\partial}{\partial x}\big(u_t + \frac{1}{4}(6uu_x - u_{xxx})\big)$.

We will call $\pi : (\Gamma, p) \to (X, q)$ a *d-osculating cover* if $T_oX \subset V^d_{\Gamma,p} \backslash V^{d-1}_{\Gamma,p}$. Such covers, studied and constructed for any $d \geqslant 1$, give rise to KP solutions L-periodic with respect to the d-th KP flow (cf. [35] for $d = 1$ and [33] for any other d).

The d-th case of the KdV equation: $u_t + \frac{1}{4}(6uu_x - u_{xxx})$.

Recall that $p \in \Gamma$ is a *Weierstrass point* of the *hyperelliptic curve* Γ, if and only if there exists a degree-2 projection $\Gamma \to \mathbb{P}^1$, ramified at p. Or in other words, if and only if there exists an involution, say $\tau_\Gamma : \Gamma \to \Gamma$, fixing p and such that the quotient curve Γ/τ_Γ is isomorphic to $\mathbb{P}^1$. Let $\pi : (\Gamma, p) \to (X, q)$ be a *d-osculating cover* such that Γ is *hyperelliptic* and $p \in \Gamma$ a *Weierstrass point*. Then, all KdV solutions classically associated to (Γ, p) are L-periodic with respect to the d-th KdV flow.

The nonlinear Schrödinger: $ip_y + p_{xx} \mp 8|p|^2p = 0$

and the 1D Toda case: $\frac{\partial^2}{\partial t^2}\varphi_n = exp(\varphi_n - \varphi_{n\text{-}1}) - exp(\varphi_{n+1} - \varphi_n)$.

Let $\pi : (\Gamma, p^+) \to (X, q)$ be a *1-osculating cover* (i.e., also called a *tangential cover* in [32]) such that Γ is *hyperelliptic* and $p^+ \in \Gamma$ is not a *Weierstrass point*. Then, all nonlinear Schrödinger and 1D Toda solutions classically associated to $(\Gamma, p^+, \tau_\Gamma(p^+))$, are L-periodic in x and in t, respectively.

The sine-Gordon case: $u_{xx} - u_{tt} = \sin u$.

Let Γ be a *hyperelliptic* curve, equipped with a projection $\pi : \Gamma \to X$ and two *Weierstrass points*, say $p, p' \in \Gamma$, such that the tangent line T_oX is contained in the plane $V^1_{\Gamma,p} + V^1_{\Gamma,p'}$, generated by the tangents to Γ at p and p' (inside $Jac\,\Gamma$). Then, up to choosing suitable local coordinates of Γ at p and p', the sine-Gordon solutions classically associated to (Γ, p, p') are L-periodic in x.

The KP case being rather well understood, we will focus on the three other cases, and in particular, on ramified projections $\pi : \Gamma \to X$, of a hyperelliptic curve onto the fixed elliptic one, marked at, either one or two *Weierstrass points* KdV and sine-Gordon cases), or two points exchanged by the hyperelliptic involution. Studying the tangent and osculating spaces at the marked points (in $Jac\,\Gamma$) is an interesting geometric problem which, I believe, does not need any further motivation. It was first considered, however, through its links with L-periodic solutions of the *Korteweg–de Vries* equation (e.g., [1, 8, 14, 18, 27, 35] for $d = 1$ and [2, 10, 11, 29] for $d = 2$), as well as the Toda, sine-Gordon and nonlinear Schrödinger equations (e.g., [5, 28, 30]). Studying their general properties (such as the relations between the genus and the degree of the cover), and constructing examples in any genus, will be the main issues of this article.

After fixing a lattice $L \subset \mathbb{C}$ defining the marked elliptic curve $(X, q) := (\mathbb{C}/L, 0)$, we will develop in Section 3 a well suited algebraic-surface approach, for studying the structure of all ramified covers of X we are interested in, and their canonical factorization through a particular algebraic surface. Natural numerical invariants will then be defined, in terms of which we will characterize the latter covers and, ultimately, construct arbitrarily high genus examples to each case.

1.2. We sketch hereafter the structure and main results of our article.

1. We start Section 2 defining the Abel rational embedding of a curve Γ, of positive genus g, into its *generalized Jacobian*, $Jac\,\Gamma$, and construct the *flag of hyperosculating spaces* $\{0\} \subsetneq V_{1,p} \ldots \subsetneq V_{g,p} = H^1(\Gamma, O_\Gamma)$, at the image of any smooth point $p \in \Gamma$. From then on, we restrict to Jacobians of hyperelliptic curves such that $Jac\,\Gamma$ contains the elliptic curve $(X, q) = (\mathbb{C}/L, 0)$, or equivalently, to any *hyperelliptic cover* $\pi : (\Gamma, p) \to (X, q)$. Dualizing such a cover π, we obtain a homomorphism $\iota_\pi : X \to Jac\,\Gamma$, with image an elliptic curve isogeneous to X. Let d be the smallest positive integer, called the *osculating order* of π, such that the tangent line defined by $\iota_\pi(X)$ is contained in $V_{d,p} \subset H^1(\Gamma, O_\Gamma)$. Whenever $p \in \Gamma$ is a *Weierstrass* point, π is called a *hyperelliptic d-osculating cover*, and gives rise to KdV solutions, L-periodic with respect to the d-th KdV flow. Such covers are characterized by the existence of a particular projection $\kappa : \Gamma \to \mathbb{P}^1$ (2.6). Given any *hyperelliptic cover* π, marked at, either two points exchanged by the hyperelliptic involution, or two *Weierstrass* points, we also find analogous characterizations for π to solve, the NL Schrödinger and 1D Toda or the sine-Gordon case (2.9, 2.10).
2. The characterizations 2.6 pave the way to the algebraic surface approach developed in the remaining sections. The main characters are played by three projective surfaces and corresponding morphisms, canonically associated to X:
   - $\pi_S : S \to X$: a particular ruled surface;
   - $e : S^\perp \to S$: the blow-up of S, at the 8 fixed points of its involution;
   - $\varphi : S^\perp \to \widetilde{S}$: a projection onto an anticanonical rational surface.
3. We construct in Section 3 the projective surfaces S and $S^\perp$, equipped with natural involutions τ and $\tau^\perp$, as well as $\widetilde{S}$, the quotient of $S^\perp$ by $\tau^\perp$. We then prove that any *hyperelliptic d-osculating cover* $\pi : (\Gamma, p) \to (X, q)$ factors through $S^\perp$, and projects onto a rational irreducible curve in $\widetilde{S}$ (3.7 and 3.8). An analogous characterization is in order, for π to solve the NL Schrödinger and 1D Toda or the sine-Gordon case (3.9).
4. In Section 4 we fix a complex elliptic curve $(X, q) = (\mathbb{C}/L, 0)$, and give the original motivation for studying *hyperelliptic d-osculating covers* of X. We start recalling the definition of the *Baker–Akhiezer* function ψ_D, associated to the data (Γ, p, λ, D), where Γ is a smooth complex projective curve of positive genus g, λ a local parameter at $p \in \Gamma$ and D a non-special effective divisor of Γ. In case (Γ, p) is a hyperelliptic curve marked at a Weierstrass point, we give the Its–Matveev (I-M) exact formula for the KdV solution associated to ψ_D, as a function of infinitely many variables $\{t_{2j-1},\, j \in \mathbb{N}^*\}$. We end up Section 4 proving that any *hyperelliptic d-osculating cover* of $\mathbb{C}/L$, gives rise to KdV solutions L-periodic in t_{2d-1}.
5. In Section 5 we take up again the algebraic surface set up developed in Section 3, recalling that any *hyperelliptic d-osculating cover* $\pi : (\Gamma, p) \to (X, q)$ factors through an equivariant morphism $\iota^\perp : \Gamma \to \iota^\perp(\Gamma) \subset S^\perp$, before projecting onto the rational irreducible curve $\widetilde{\Gamma} := \varphi\big(\iota^\perp(\Gamma)\big) \subset \widetilde{S}$. The ramification index of π at p and the degree of $\iota^\perp : \Gamma \to \iota^\perp(\Gamma) \subset S^\perp$, say ρ and m, are natural numerical invariants attached to π. We also define its *type*, $\gamma = (\gamma_i) \in \mathbb{N}^4$, by intersecting $\iota^\perp_*(\Gamma)$ with four suitably chosen exceptional divisors (5.2). We assume henceforth that $m = 1$ and calculate the linear equivalence class of $\Gamma^\perp \subset S^\perp$. Basic congruences and inequalities for the latter invariants follow (5.4 and 5.5). For example,

the genus of Γ satisfies $(2g+1)^2 \leqslant (2d-1)\big(8n+2d-1\big)$. Any *hyperelliptic cover* solving the other three cases also factors through $S^{\perp}$ and projects onto a rational irreducible curve in $\widetilde{S}$. Similar congruences and inequalities for their invariants follow as well (5.6, 5.7 and 5.8)

6. At last, in Section 6 we focus on $MH_X(n,d,1,1,\gamma)$, the set of of degree-n *hyperelliptic d-osculating covers*, of type γ, not ramified at the marked point and birational to their natural images in $S^{\perp}$ (i.e., such that $\rho = m = 1$). For any given $(n,d) \in \mathbb{N}^* \times \mathbb{N}^*$, we find explicit types $\gamma \in \mathbb{N}^4$ satisfying $\gamma^{(2)} = (2d-1)(2n-2)+3$, for which we give an effective construction (leading ultimately to explicit equations) of the corresponding covers. We thus obtain $(d-1)$-dimensional families of arbitrarily high genus marked curves, solving the d-th KdV case. A completely analogous constructive approach can be worked out for the other three cases.

2. JACOBIANS OF CURVES AND HYPERELLIPTIC d-OSCULATING COVERS

2.1. Let $\mathbb{P}^1$ denote the projective line over $\mathbb{C}$ and (X,q) the elliptic curve $(\mathbb{C}/L, 0)$, where L is a fixed lattice of $\mathbb{C}$. By a curve we will mean hereafter a complete integral curve over $\mathbb{C}$, say Γ, of positive arithmetic genus $g > 0$. If Γ is smooth, its Jacobian variety is a complete connected commutative algebraic group of dimension g. For a singular irreducible curve of arithmetic genus g instead, the analogous picture decouples into canonically related pieces, as briefly explained hereafter. We have, on the one hand, the moduli space of degree-0 invertible sheaves over Γ, still denoted by $Jac\,\Gamma$ and called the *generalized Jacobian* of $Jac\,\Gamma$. It is a connected commutative algebraic group, canonically identified to $H^1(\Gamma, O_\Gamma^*)$, with tangent space at its origin equal to $H^1(\Gamma, O_\Gamma)$. In particular, it is g-dimensional, although not a complete variety any more.

The latter is related to the Jacobian variety of the smooth model of Γ. More generally, let $j : \hat{\Gamma} \to \Gamma$ be any partial desingularization and consider the natural injection $O_\Gamma \to j_*(O_{\hat{\Gamma}}^*)$, with quotient N_j , a finite support sheaf of abelian groups. From the corresponding exact cohomology sequence we can then extract

$$0 \to H^0(\Gamma, N_j) \to H^1(\Gamma, O_\Gamma^*) \xrightarrow{j^*} H^1(\hat{\Gamma}, O_{\hat{\Gamma}}^*) \to 0$$

or

$$0 \to H^0(\Gamma, N_j) \to Jac\,\Gamma \xrightarrow{j^*} Jac\,\hat{\Gamma} \to 0.$$

Hence, the homomorphism $j^* : Jac\,\Gamma \to Jac\,\hat{\Gamma}, L \mapsto j^*(L)$, is surjective, with kernel the affine algebraic group $H^0(\Gamma, N_j)$.

On the other hand, we have the moduli space $W(\Gamma)$, of torsionless, zero Euler characteristic, coherent sheaves over Γ, also called *compactified Jacobian* of Γ, on which $Jac\,\Gamma$ acts by tensor product. Taking direct images by any partial desingularization $j : \hat{\Gamma} \to \Gamma$, defines an equivariant embedding $j_* : W(\hat{\Gamma}) \to W(\Gamma)$, such that $\forall \hat{F} \in W(\hat{\Gamma}), \forall L \in Jac\,\Gamma$, we have the projection formula $j_*(\hat{F} \otimes j^*(L)) = j_*(\hat{F}) \otimes L$. Hence, a $Jac\,\Gamma$-invariant stratification of $W(\Gamma)$, encoding the web of different partial desingularizations between Γ and its smooth model. Let me stress that, up to choosing the marked points, any singular irreducible hyperelliptic curves gives rise to KdV, 1D Toda and NL Schrödinger solutions, parameterized by the compactified Jacobian $W(\Gamma)$ (cf. [26]).

For any curve Γ, let Γ^0 and $Jac\,\Gamma$ denote, respectively, the open subset of smooth points of Γ and its *generalized Jacobian.* Recall that for any smooth point $p \in \Gamma^0$, the Abel morphism, $A_p : \Gamma^0 \to Jac\,\Gamma$, $p' \mapsto O_\Gamma(p'-p)$, is an embedding and $A_p(\Gamma^0)$ generates the whole jacobian. For any marked curve (Γ, p) as above, and any positive integer j, let us consider the exact sequence of O_Γ-modules $0 \to O_\Gamma \to O_\Gamma(jp) \to O_{jp}(jp) \to 0$, as well as the corresponding long exact cohomology sequence:

$$0 \to H^0(\Gamma, O_\Gamma) \to H^0\big(\Gamma, O_\Gamma(jp)\big) \to H^0\big(\Gamma, O_{jp}(jp)\big) \xrightarrow{\delta} H^1(\Gamma, O_\Gamma) \to \dots,$$

where $\delta : H^0\big(\Gamma, O_{jp}(jp)\big) \to H^1(\Gamma, O_\Gamma)$ is the cobord morphism and $H^1(\Gamma, O_\Gamma)$ is canonically identified with the tangent space to $Jac\,\Gamma$ at 0.

According to the Weierstrass gap Theorem, for any $d = 1, \dots, g := genus(\Gamma)$, there exists $0 < j < 2g$ such that $\delta\Big(H^0\big(\Gamma, O_{jp}(jp)\big)\Big)$ is a d-dimensional subpace, denoted hereafter by $V_{d,p}$.

For a generic point p of Γ we have $V_{d,p} = \delta\Big(H^0\big(\Gamma, O_{dp}(dp)\big)\Big)$(*i.e.* : $j = d$).

In any case, the above filtration $\{0\} \subsetneq V_{1,p} \dots \subsetneq V_{g,p} = H^1(\Gamma, O_\Gamma)$ is the, so-called, *flag of hyperosculating spaces* to $A_p(\Gamma)$ at 0. For example, $V_{1,p}$ is equal to $\delta\Big(H^0\big(\Gamma, O_p(p)\big)\Big)$, the tangent to $A_p(\Gamma)$ at 0.

Proposition 2.2. ([33]1.6.) *Let (Γ, p, λ) be a hyperelliptic curve, equipped with a local parameter λ at a smooth Weierstrass point $p \in \Gamma^0$, and consider, for any odd integer $j = 2d - 1 \geqslant 1$, the exact sequence of O_Γ-modules:*

$$0 \to O_\Gamma \to O_\Gamma(jp) \to O_{jp}(jp) \to 0,$$

as well as its long exact cohomology sequence

$$0 \to H^0(\Gamma, O_\Gamma) \to H^0\big(\Gamma, O_\Gamma(jp)\big) \to H^0\big(\Gamma, O_{jp}(jp)\big) \xrightarrow{\delta} H^1(\Gamma, O_\Gamma) \to \dots,$$

δ being the cobord morphism.
For any, $m \geqslant 1$, we also let $[\lambda^{-m}]$ denote the class of λ^{-m} in $H^0\big(\Gamma, O_{mp}(mp)\big)$. Then $V_{d,p}$ is generated by $\Big\{\delta\big([\lambda^{2l-1}]\big), l = 1, \dots, d\Big\}$. In other words, the d-th osculating subspace to $A_p(\Gamma)$ at 0 is equal to $\delta\Big(H^0\big(\Gamma, O_{jp}(jp)\big)\Big)$, for $j = 2d - 1$.

Definition 2.3. *A finite marked morphism $\pi : (\Gamma, p) \to (X, q)$, such that Γ is a hyperelliptic curve and $p \in \Gamma$ a smooth Weierstrass point, will be called a hyperelliptic cover. Let $[-1] : (X, q) \to (X, q)$ denote the canonical symmetry, fixing the origin $q \in X$, as well as the three other half-periods $\{\omega_j, j = 1, 2, 3\}$, and $\tau_\Gamma : (\Gamma, p) \to (\Gamma, p)$ the hyperelliptic involution. Let us recall that the quotient curve Γ/τ_Γ is isomorphic to $\mathbb{P}^1$ and $[-1] \circ \pi = \pi \circ \tau_\Gamma$.*

Definition 2.5. *Let $\pi : (\Gamma, p) \to (X, q)$ be a finite marked morphism and let $\iota_\pi : X \to Jac\,\Gamma$ denote the group homomorphism $q' \mapsto A_p\big(\pi^*(q' - q)\big)$. We will say that π has osculating order d, or equivalently, that it is a d-osculating cover, if $T_oX \subset H^1(\Gamma, O_\Gamma)$, the tangent to $\iota_\pi(X)$ at 0 is contained in $V_{d,p} \backslash V_{d-1,p}$. If π also happens to be a hyperelliptic cover, we will simply say that it is a hyperelliptic d-osculating cover.*

The *osculating order* of π is a geometrical invariant, bounded by the arithmetic genus of Γ, which we may want to know. The following hyperelliptic d-osculating criterion, analog to Krichever's tangential one (cf. [18, p. 289]), will be instrumental for its calculation, as well as for further development in Section 5.

Theorem 2.6.
Let $\pi : (\Gamma, p) \to (X, q)$ be an arbitrary hyperelliptic cover of arithmetic genus g. Then its osculating order $d \in \{1, \dots, g\}$ is characterized by the existence of a projection $\kappa : \Gamma \to \mathbb{P}^1$ such that:

(1) *the poles of κ lie along $\pi^{-1}(q)$;*

(2) *$\kappa + \pi^*(z^{-1})$ has a pole of order $2d - 1$ at p, and no other pole along $\pi^{-1}(q)$.*

Furthermore, if $\tau_\Gamma : \Gamma \to \Gamma$ denotes the hyperelliptic involution of Γ, there exists a unique projection $\kappa : \Gamma \to \mathbb{P}^1$ satisfying properties (1) & (2) *above, as well as*:

(3) *$\tau_\Gamma^*(\kappa) = -\kappa$.*

Proof. According to 2.2, $\forall k \in \{1, \dots, g\}$ the k-th osculating subspace $V_{k,p}$ is generated by $\Big\{\delta\big([\lambda^{-(2l-1)}]\big), l = 1, \dots, k\Big\}$. On the other hand, the tangent to $\iota_\pi(X) \subset Jac\,\Gamma$ at 0 is equal to $\pi^*\big(H^1(X, O_X)\big)$ and generated by $\delta\big([\pi^*(z^{-1})]\big)$. In other words, the *osculating order* d is the smallest positive integer such that $\delta\big([\pi^*(z^{-1})]\big)$ is a linear combination $\sum_{l=1}^d a_l \delta\big([\lambda^{-(2l-1)}]\big)$, with $a_d \neq 0$. Or equivalently, thanks to the Mittag–Leffler Theorem, if and only if there exists a projection $\kappa : \Gamma \to \mathbb{P}^1$, with polar parts equal to $\pi^*(z^{-1}) - \sum_{l=1}^d a_l \lambda^{-(2l-1)}$. The latter conditions on κ are equivalent to 2.6. (1) and (2). Moreover, up to replacing κ by $\frac{1}{2}\big(\kappa - \tau_\Gamma^*(\kappa)\big)$, we can assume κ is τ_Γ-anti-invariant. Now, the difference of two such functions should be τ_Γ-anti-invariant, while having a unique pole at p, of order strictly smaller than $2d - 1 \leqslant 2g - 1$. But the latter functions are all τ_Γ-invariant, implying that the projection κ (satisfying conditions 2.6 (1)–(3)) is unique. □

Definition 2.7. *The pair of marked projections (π, κ), satisfying* 2.6 (1)–(3), *will be called a hyperelliptic d-osculating pair, and κ the hyperelliptic d-osculating function associated to π. In the latter case, π gives rise to solutions of the KdV hierarchy, L periodic in the d-th KdV flow, as will be proved in Section 4.*

The following Proposition calculates the tangent at any point of the curve $A_p(\Gamma) \subset Jac\,\Gamma$, and leads to a useful characterization of the hyperelliptic covers solving the other cases. Its proof follows along the same lines as 2.2's proof.

Proposition 2.8. *Let (Γ, r, λ) be a hyperelliptic curve equipped with a local parameter at an arbitrary smooth point $r \in \Gamma$. Then $V^1_{\Gamma,r} \subset H^1(\Gamma, O_\Gamma)$, the tangent line to $A_p(\Gamma)$ at $A_p(r)$, is generated by $\delta\big([\lambda^{-1}]\big)$.*

Corollary 2.9. *Let $\pi : (\Gamma, p) \to (X, q)$ be an arbitrary hyperelliptic cover, $p^+ \in \Gamma$ a non-Weierstrass point, $p^- := \tau_\Gamma(p^+)$, and let $T_oX \subset H^1(\Gamma, O_\Gamma)$ denote the tangent line defined by the elliptic curve $\iota_\pi(X) \subset Jac\,\Gamma$. Then, the data (π, p^+, p^-) solves the NL Schrödinger and 1D Toda case (i.e., $T_oX = V^1_{\Gamma,p^+} = V^1_{\Gamma,p^-}$), if and only if there exists a projection $\kappa : \Gamma \to \mathbb{P}^1$ such that:*

(1) *the poles of κ lie in $\pi^{-1}(q) \cup \{p^+, p^-\}$.*

(2) *$\kappa + \pi^*(z^{-1})$ has simple poles at $\{p^+, p^-\}$, and no other pole along $\pi^{-1}(q)$.*

(3) $\tau_\Gamma^*(\kappa) = -\kappa$.

Corollary 2.10. *Let $\pi : (\Gamma, p) \to (X, q)$ be an arbitrary hyperelliptic cover equipped with two Weierstrass points p_o, p_1, and let $T_oX \subset H^1(\Gamma, O_\Gamma)$ denote the tangent line defined by the elliptic curve $\iota_\pi(X) \subset Jac\,\Gamma$. Then, the data (π, p_o, p_1) solves the sine-Gordon case (i.e., $T_oX \subset V^1_{\Gamma,p_o} + V^1_{\Gamma,p_1}$), if and only if there exists a projection $\kappa : \Gamma \to \mathbb{P}^1$ such that:*

(1) *the poles of κ lie in $\pi^{-1}(q) \cup \{p_o, p_1\}$.*

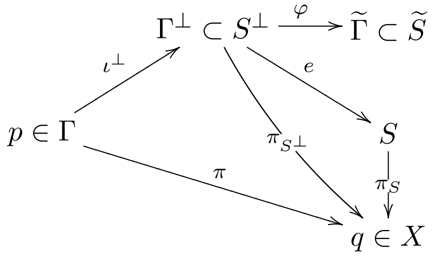
(2) *$\kappa + \pi^*(z^{-1})$ has simple poles at $\{p_1, p_2\}$, and no other pole along $\pi^{-1}(q)$.*

(3) $\tau_\Gamma^*(\kappa) = -\kappa$.

3. THE ALGEBRAIC SURFACE SET UP

3.1. We will construct hereafter a ruled surface $\pi_S : S \to X$, as well as a blowing-up $e : S^\perp \to S$, having a natural involution $\tau^\perp : S^\perp \to S^\perp$, such that any *hyperelliptic osculating cover* $\pi : (\Gamma, p) \to (X, q)$ factors through $\pi_{S^\perp}$, via an equivariant morphism $\iota^\perp : \Gamma \to \Gamma^\perp := \iota^\perp(\Gamma) \subset S^\perp$ (i.e., $\iota^\perp \circ \tau_\Gamma = \tau^\perp \circ \iota^\perp$). We will also prove that $\widetilde{\Gamma} := \varphi(\Gamma^\perp)$, its image in the quotient surface $\widetilde{S} := S^\perp/\tau^\perp$, is an irreducible rational curve. Generally speaking, our main strategy, fully developed in Section 5, will consist in translating numerical invariants of $\pi : (\Gamma, p) \to (X, q)$, in terms of the numerical equivalence class of the corresponding rational irreducible curve $\widetilde{\Gamma} \subset \widetilde{S}$ and its geometric properties.

The whole relationship is sketched in the diagram below.



Definition 3.3.

1. *Besides the origin $\omega_o := q \in X$, there are three other half-periods, say $\{\omega_1, \omega_2, \omega_3\} \subset X$, fixed by the canonical symmetry $[-1] : (X, q) \to (X, q)$.*

2. *Consider the open affine subsets $U_o := X \setminus \{q\}$ and $U_1 := X \setminus \{\omega_1\}$ and fix an odd meromorphic function $\zeta : X \to \mathbb{P}^1$, with divisor of poles equal to $(\zeta) = q + \omega_1 - \omega_2 - \omega_3$. Let $\pi_S : S \to X$ denote the ruled surface obtained by identifying $\mathbb{P}^1 \times U_o$ with $\mathbb{P}^1 \times U_1$ over $X \setminus \{q, \omega_1\}$ as follows:*

$$\forall q' \neq q, \omega_1, \quad (T_o\,, q') \in \mathbb{P}^1 \times U_o \text{ is glued with } (T_1 + \frac{1}{\zeta(q')}\,, q') \in \mathbb{P}^1 \times U_1.$$

In other words, we glue the fibers of $\mathbb{P}^1 \times U_0$ and $\mathbb{P}^1 \times U_1$, over any $q' \neq q, \omega_1$, by means of a translation. In particular the constant sections $q' \in U_i \mapsto (\infty, q') \in \mathbb{P}^1 \times U_i$, for $i \in \{0, 1\}$, get glued together, defining a particular one of zero self-intersection, denoted by $C_o \subset S$.

3. *The meromorphic differentials dT_o and dT_1 get also glued together, implying that K_S, the canonical divisor of S is represented by -2C_o. Any section of $\pi_S : S \to X$, other than C_o, is given by two non-constant morphisms $f_i : U_i \to \mathbb{P}^1$ $(i = 1, 2)$, such that $f_o = f_1 - \frac{1}{\zeta}$ outside $\{q, \omega_1\}$. A straightforward calculation shows that any such a section intersects C_o, while having self-intersection number greater or equal to 2. It follows from the general Theory of Ruled Surfaces (cf. [12, V. 2]) that C_o must be the unique section with zero self-intersection.*

4. *The only irreducible curve linearly equivalent to a multiple of C_o is C_o itself (cf.[35, 3.2(1)]).*

5. *The involutions $\mathbb{P}^1 \times U_i \to \mathbb{P}^1 \times U_i$, $\quad (T_i, q') \mapsto \big(-T_i, [-1](q')\big)$ $(i = 0, 1)$, get glued under the above identification and define the involution $\tau : S \to S$, such that $\pi_S \circ \tau = [-1] \circ \pi_S$, already mentioned in* 3.1. *In particular, τ has two fixed points over each half-period ω_i, one in C_o, denoted by s_i, and the other one denoted by r_i $(i = 0, \ldots, 3)$.*

6. Let *$e : S^\perp \to S$ denote hereafter the blow-up of S at $\{s_i, r_i, i = 0, \ldots, 3\}$, the eight fixed points of τ, and $\tau^\perp :\ S^\perp \to S^\perp$ its lift to an involution fixing the corresponding exceptional divisors $\big\{s_i^\perp := e^{-1}(s_i), r_i^\perp := e^{-1}(r_i), i = 0, \ldots, 3\big\}$. Taking the quotient of $S^\perp$ with respect to $\tau^\perp$, we obtain a degree-2 projection $\varphi : S^\perp \to \widetilde{S}$ onto a smooth rational surface $\widetilde{S}$, ramified*

along the exceptional curves $\{s_i^\perp, r_i^\perp, i = 0, \ldots, 3\}$. *Let* $C_o^\perp$ *and* $\widetilde{C}_o$ *denote, respectively, the strict transform in* $S^\perp$ *of* $C_o \subset S$ *(respectively: the corresponding projections in* $\widetilde{S}$*). For any* $i = 0, \ldots, 3$*, let also* $\widetilde{s}_i$ *and* $\widetilde{r}_i$ *denote the projections in* $\widetilde{S}$ *of* $s_i^\perp$ *and* $r_i^\perp$*, respectively. The canonical divisor of* $\widetilde{S}$*, say* $\widetilde{K}$*, satisfies* $\varphi^*(\widetilde{K}) = e^*(-2C_o)$ *and is linearly equivalent to* $-2\widetilde{C}_o - \sum_{i=0}^{3} \widetilde{s}_i$.

The lemma and propositions hereafter, proved in [33, see 2.3, 2.4 and 2.5], will be instrumental in constructing the equivariant factorization $\iota^\perp : \Gamma \to S^\perp$ (3.1).

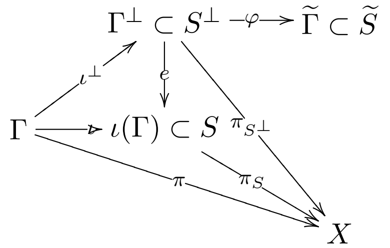
Lemma 3.6. *There exists a unique,* τ*-anti-invariant, rational morphism* $\kappa_s : S \to \mathbb{P}^1$*, with poles over* $C_o + \pi_S^{-1}(q)$*, such that over a suitable neighborhood* U *of* $q \in X$*, the divisor of poles of* $\kappa_s + \pi_S^*(z^{-1})$ *is reduced and equal to* $C_o \cap \pi_S^{-1}(U)$.

Proposition 3.7. *For any hyperelliptic cover* $\pi : (\Gamma, p) \to (X, q)$*, the following conditions are equivalent:*

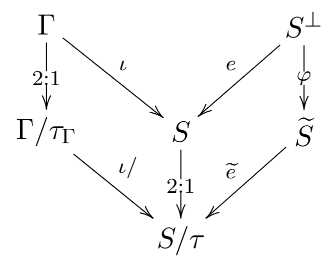
1. *there is a projection* $\kappa : \Gamma \to \mathbb{P}^1$*, satisfying properties* 2.6 (1), (2) and (3);
2. *there is a morphism* $\iota : \Gamma \to S$ *such that* $\pi = \pi_S \circ \iota$*,* $\iota \circ \tau_\Gamma = \tau \circ \iota$ *and* $\iota^*(C_o) = (2d - 1)p$.

In the latter case, π *is a hyperelliptic d-osculating morphism (2.5) and solves the d-th KdV case.*

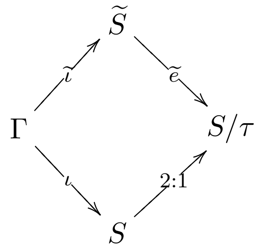
Proposition 3.8. *For any hyperelliptic d-osculating pair* (π, κ)*, the above morphism* $\iota : \Gamma \to S$ *lifts to a unique equivariant morphism* $\iota^\perp : \Gamma \to S^\perp$ *(i.e.,* $\tau^\perp \circ \iota^\perp = \iota^\perp \circ \tau_\Gamma$*). In particular,* (π, κ) *is the pullback of* $(\pi_{S^\perp}, \kappa_{s^\perp}) = (\pi_S \circ e, \kappa_s \circ e)$*, and* Γ *lifts to a* $\tau^\perp$*-invariant curve,* $\Gamma^\perp := \iota^\perp(\Gamma) \subset S^\perp$*, which projects onto the rational irreducible curve* $\widetilde{\Gamma} := \varphi(\Gamma^\perp) \subset \widetilde{S}$.



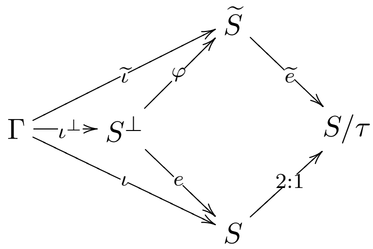
Proof. The blow-up $e : S^\perp \to S$, as well as $\iota : \Gamma \to S$, can be pushed down to the corresponding quotients, making up the following diagram:



Moreover, since $\widetilde{e} : \widetilde{S} \to S/\tau$ is a birational morphism and Γ/τ_Γ is a smooth curve (in fact isomorphic to $\mathbb{P}^1$), we can lift $\iota/ : \Gamma/\tau_\Gamma \to S/\tau$ to $\widetilde{S}$, obtaining a morphism $\widetilde{\iota} : \Gamma \to \widetilde{S}$, fitting in the diagram:



Recall now that $S^{\perp}$ is the fibre product of $\widetilde{e} : \widetilde{S} \to S/\tau$ and $S \to S/\tau$ (cf. [35, 4.1]). Hence, ι and $\widetilde{\iota}$ lift to a unique equivariant morphism $\iota^{\perp} : \Gamma \to S^{\perp}$, fitting in



Furthermore, since $\widetilde{\iota} : \Gamma \to \widetilde{S}$ factors through $\Gamma \to \Gamma/\tau_{\Gamma} \cong \mathbb{P}^1$, its image $\widetilde{\Gamma} := \varphi\big(\iota^{\perp}(\Gamma)\big) = \widetilde{\iota}(\Gamma) \subset \widetilde{S}$ is a rational irreducible curve as claimed. □

Analogously to the KdV case, any data (π, p^+, p^-) or (π, p_1, p_2), solving the NL Schrödinger and 1D Toda or the sine-Gordon case, factors through an equivariant morphism $\iota^{\perp} : \Gamma \to S^{\perp}$, and its image $\Gamma^{\perp} := \iota^{\perp}(\Gamma)$ projects onto a rational irreducible curve in $\widetilde{S}$.

Proposition 3.9. *Let $\pi : (\Gamma, p) \to (X, q)$ be an arbitrary hyperelliptic cover equipped with two points $p' \neq p'' \in \Gamma$ such that the (divisor) sum $p' + p''$ is τ_{Γ}-invariant. Then, the following conditions are equivalent:*

1. *there is a projection $\kappa : \Gamma \to \mathbb{P}^1$, satisfying properties* 2.9 (1), (2) and (3) *or* 2.10 (1)–(3);

2. *there is a morphism $\iota : \Gamma \to S$ such that $\pi = \pi_S \circ \iota$, $\iota \circ \tau_{\Gamma} = \tau \circ \iota$ and $\iota^*(C_o) = p' + p''$.*

In the latter case, (π, p', p'') solves, either the NL Schrödinger and 1D Toda case, if $\tau_{\Gamma}(p') = p''$, or the sine-Gordon case, if p' and p'' are Weierstrass points.

Proposition 3.10. *For any data (π, p', p'', κ) as in 3.9, the morphism $\iota : \Gamma \to S$ lifts to a unique equivariant morphism $\iota^{\perp} : \Gamma \to S^{\perp}$ (i.e., $\tau^{\perp} \circ \iota^{\perp} = \iota^{\perp} \circ \tau_{\Gamma}$). In particular, (π, κ) is the pullback of $(\pi_{S^{\perp}}, \kappa_{s^{\perp}}) = (\pi_S \circ e, \kappa_s \circ e)$, and Γ lifts to a $\tau^{\perp}$-invariant curve, $\Gamma^{\perp} := \iota^{\perp}(\Gamma) \subset S^{\perp}$, which projects onto the rational irreducible curve $\widetilde{\Gamma} := \varphi\big(\Gamma^{\perp}\big) \subset \widetilde{S}$.*

4. COMPLEX HYPERELLIPTIC CURVES AND ELLIPTIC KDV SOLITONS

4.1. Let Γ be a smooth complex projective curve of positive genus g, equipped with a local coordinate at $p \in \Gamma$, say λ, as well as a non-special degree-g effective divisor D with support disjoint from p. Then the so-called *Baker–Akhiezer function* associated to the spectral data (Γ, p, λ, D) and denoted by ψ_D, is the unique meromorphic function on $\mathbb{C}^{\infty} \times (\Gamma \backslash \{p\})$ such that for any $\vec{t} = (t_1, t_2, \dots) \in \mathbb{C}^{\infty}$:

1. the divisor of poles of $\psi_D(\vec{t},)$, on $\Gamma \backslash \{p\}$, is bounded by D;

2. in a neighbourhood of p, $\psi_D(\vec{t}, \lambda)$ has an essential singularity of type:

$$\psi_D(\vec{t}, \lambda) = \exp\Big(\sum_{0<i}^{\infty} t_i \lambda^{-i}\Big)\Big(1 + \sum_{0<i}^{\infty} \xi_i^D(\vec{t})\lambda^i\Big).$$

For any $i \geqslant 1$, differentiating ψ_D, either with respect to t_i, or i times with respect to $x := t_1$, we obtain a meromorphic function with divisor of poles $D + ip$ and same type of essential singularity at p as ψ_D. We can therefore construct a differential polynomial of degree i in $\frac{\partial}{\partial x}$, with functions of $\vec{t}$ as coefficients, say $P_i(\frac{\partial}{\partial x})$, such that $\frac{\partial}{\partial t_i}\psi_D - P_i(\frac{\partial}{\partial x})\psi_D$ has the same properties as ψ_D. The uniqueness of the latter BA function implies that $\psi_D(\vec{t}, \lambda)$ satisfies the (so-called KP) hierarchy of partial derivatives equations $\frac{\partial}{\partial t_i}\psi_D = P_i(\frac{\partial}{\partial x})\psi_D$, $i \in \mathbb{N}^*$.

4.2. Let us suppose in the sequel that (Γ, p) is a hyperelliptic curve, marked at a Weierstrass point, and λ an odd local parameter at $p \in \Gamma$. Or in other words, that there exists a degree-2 projection $f : \Gamma \to \mathbb{P}^1$, with a double pole at p, and $f(\lambda) = \frac{1}{\lambda^2} + O(\lambda^2)$. It is classically known then that the BA function $\psi_D(\vec{t}, \lambda)$, corresponding to any non-special degree-g effective divisor D of Γ, does not depend, up to an exponential, on the even variables $\{t_{2j},\ j \in \mathbb{N}^*\}$.

For example, choosing λ such that $f(\lambda) = \frac{1}{\lambda^2}$, we will have $\psi_D = \exp\big(\sum_j t_{2j} f^j\big)\psi_{D\big|\{t_{2j}=0\}}$.

It then follows that $\psi_{D\big|\{t_{2j}=0\}}$ solves the KdV hierarchy and $u := -2\frac{\partial}{\partial x}\xi_1^D$ the *Korteweg–de Vries* equation:

$$u_{t_3} = \frac{1}{4}(6u \cdot u_x + u_{xxx}) \quad (x := t_1).$$

A more concrete formula, (due to A. Its and V. Matveev, cf. [14]), is in order:

$$(I - M) \quad u(t_1, t_3, t_5, \dots) = 2\frac{\partial^2}{\partial x^2}\Big(\log\theta_\Gamma(Z - \sum_{0<j}^{\infty} t_{2j-1}U_j\,)\Big) + c,$$

where

i) $\theta_\Gamma : \mathbb{C}^g \to \mathbb{C}$ denotes the Riemann theta-function of Γ;

ii) $Z \in \mathbb{C}^g$ projects onto $A_p(D)$ and $c \in \mathbb{C}$;

iii) $\forall j \geqslant 1$, $(2j)! \cdot U_j = A_p^{(2j-1)}(\lambda)_{\big|\lambda=0}$, the $(2j-1)$-th derivative of $A_p(\lambda)$ at $\lambda = 0$.

Remark 4.3.

1. The vectors $\{U_k, 1 \leqslant k \leqslant j\}$ generate $V_{j,p}$, the j-th *hyperosculating space* to $A_p(\Gamma)$ at $A_p(p)$ (see 2.1).

2. The above construction of KdV solutions can be generalized to any singular marked hyperelliptic curve (Γ, p), as recalled in [26]. The corresponding solutions are then parameterized by $W(\Gamma)$, the *compactified jacobian* of Γ. Roughly speaking, any $L \in W(\Gamma)$, in the complement of the theta divisor, corresponds to a non-special degree-g effective divisor, with support at the smooth points of Γ. Working in the frame of Sato's Grassmannian (cf. [25, 26]),one can still define an analogous BA function, as well as a KdV solution. Hence, the highest the arithmetic genus, the biggest the family of KdV solutions. We are thus naturally led to allow singular marked hyperelliptic curves.

3. According to the (I-M) formula, the KdV solution $u = -2\frac{\partial}{\partial x}\xi_1^D$ is a t_{2d-1}-*elliptic KdV soliton* (i.e., doubly periodic in t_{2d-1}), if and only if U_d generates an elliptic curve $X \subset Jac\,\Gamma$. Or in other words, if $(\Gamma, p) \to (X, q)$ is a smooth *hyperelliptic d-osculating cover*.

4. We will actually prove that any KdV solution associated to a *hyperelliptic d-osculating cover*, is doubly periodic in t_{2d-1}, without assuming the above (I-M) formula, or that Γ is a smooth curve (see 4.5). The original idea goes back to [18, pp. 288–289].

Notations 4.4. Choose a lattice $L \subset \mathbb{C}$, equipped with a $\mathbb{Z}$-basis $(2\omega_1, 2\omega_2)$, such that the elliptic curve (X, q) is isomorphic to the complex torus $(\mathbb{C}/L, 0)$, and let $\zeta(z) : \mathbb{C} \to \mathbb{P}^1$, denote the ζ-Weierstrass meromorphic function. Recall (cf. [18, p. 283]) that ζ is holomorphic outside L and characterized by the following properties:

$$\forall z \in \mathbb{C} \setminus L \quad \begin{cases} \zeta(z) = z^{-1} + O(z), & \text{in a neighborhood of } 0 \in \mathbb{C}, \\ \zeta(z + 2\omega_j) = \zeta(z) + \eta_j, & j = 1, 2, \end{cases}$$

for some $\eta_1, \eta_2 \in \mathbb{C}$, satisfying Legendre's relation: $\eta_1 \, 2\omega_2 - \eta_2 \, 2\omega_1 = 2\pi\sqrt{-1}$.

Proposition 4.5. *Let $\pi : (\Gamma, p) \to (X, q)$ be a genus-g, hyperelliptic d-osculating cover, κ the unique hyperelliptic d-osculating function associated to π, and choose λ, an odd local parameter at p, such that $\kappa + \pi^*(z^{-1}) = \lambda^{-(2d-1)}$. Then, for any non-special degree-g effective divisor D, with support disjoint from p, the KdV solution $u = -2\frac{\partial}{\partial x}\xi_1^D$ associated to (Γ, p, λ, D) (see 4.2), is L-periodic in t_{2d-1}.*

Proof. Denote again by $\psi_D(\vec{t}, \lambda)$ the *BA* function associated to D. Recall (see 2.4) that κ has poles only over $\pi^{-1}(q)$, and

$$\kappa + \pi^*\big(\zeta(z)\big) = \kappa + \pi^*\big(z^{-1} + O(z)\big) = \lambda^{-(2d-1)} + O(\lambda)$$

has a pole of order $2d - 1$ at p. We then prove, coupling the properties of ζ and κ, that for $j = 1, 2$, the function

$$\phi_j(p') = exp\left(2\omega_j\Big(\kappa(p') + \zeta\big(\pi(p')\big)\Big) - \eta_j \pi(p')\right)$$

is well defined and holomorphic all over $\Gamma \setminus \{p\}$, thanks to Legendre's relations, and has an essential singularity at p of the following type:

$$\phi_j(p') = exp\Big(2\omega_j \lambda^{-(2d-1)} + O(\pi(p'))\Big) = exp\big(2\omega_j \lambda^{-(2d-1)}\big)\big(1 + O(\lambda)\big).$$

The main final argument run as follows. The uniqueness of the *BA* function $\psi_D(\vec{t}, \lambda)$ implies that

$$\psi_D(\vec{t} + 2\omega_j \vec{e}_{2d-1}, \lambda) = \phi_j(\lambda) \cdot \psi_D(\vec{t}, \lambda)\,,$$

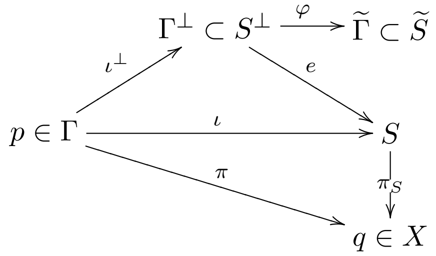
where $\vec{e}_{2d-1} = (0, \ldots 0, 1, 0 \ldots) \in \mathbb{C}^\infty$ is the vector having a 1 at the $(2d-1)$-th place and 0 everywhere else. At last, comparing their developments around p we obtain the following equality:

$$\frac{\partial}{\partial x}\xi_1^D(\vec{t} + 2\omega_j \vec{e}_{2d-1}) = \frac{\partial}{\partial x}\xi_1^D(\vec{t})\ , \quad j = 1, 2.$$

In other words, the *KdV* solution $u = -2\frac{\partial}{\partial x}\xi_1^D$ associated to the data (Γ, p, λ, D), is L-periodic in t_{2d-1}. □

5. THE HYPERELLIPTIC d-OSCULATING COVERS AS DIVISORS OF A SURFACE

5.1. Let us consider again the algebraic surface set up constructed in Section 3, with the equivariant factorization of any hyperelliptic d-osculating cover through $S^\perp$, and its projection onto a rational irreducible curve $\widetilde{\Gamma} \subset \widetilde{S}$. The corresponding diagram of morphisms, given hereafter, will also be useful for the NL Schrödinger and 1D Toda and sine-Gordon cases.



Definition 5.2. *For any $i = 0, \dots, 3$, the intersection number between the divisors $\iota_*^\perp(\Gamma)$ and $r_i^\perp$ will be denoted by γ_i, and the corresponding vector $\gamma = (\gamma_i) \in \mathbb{N}^4$ called the type of π. Furthermore, $\gamma^{(1)}$ and $\gamma^{(2)}$ will denote, respectively, the sums*

$$\gamma^{(1)} := \sum_{i=0}^{3} \gamma_i \qquad \textit{and} \qquad \gamma^{(2)} := \sum_{i=0}^{3} \gamma_i^2.$$

Remark 5.3. The next step concerns studying the above rational irreducible curves $\widetilde{\Gamma} \subset \widetilde{S}$. We will characterize their linear equivalence classes, and dress the basic relations between them and the numerical invariants of the corresponding *hyperelliptic d-osculating covers*. These results, already known for $d = 1$ ([35]) and $d = 2$ ([10]), can be proven within the same framework for any other $d > 2$.

Lemma 5.4. *Let $\pi : (\Gamma, p) \to (X, q)$ be a degree-n hyperelliptic d-osculating cover, $\iota^\perp : \Gamma \to \Gamma^\perp$ its unique equivariant factorization through $S^\perp$ and $\iota := e \circ \iota^\perp$. We let again γ denote the type of π, ρ its ramification index at p and m the degree of $\iota^\perp : \Gamma \to \iota^\perp(\Gamma)$. Then*

1. *$\iota_*(\Gamma)$ is equal to $m.\iota(\Gamma)$ and linearly equivalent to nC_o+ $(2d-1)S_o$;*
2. *$\iota_*(\Gamma)$ is unibranch, and transverse to the fiber $S_o := \pi_S^*(q)$ at $s_o = \iota(p)$;*
3. *ρ is odd, bounded by $2d - 1$ and equal to the multiplicity of $\iota_*(\Gamma)$ at s_o;*
4. *the degree m divides n, $2d - 1$ and ρ, as well as γ_i, $\forall i = 0, \dots, 3$;*
5. *$\gamma_o + 1 \equiv \gamma_1 \equiv \gamma_2 \equiv \gamma_3 \equiv n(\mathrm{mod}.2)$;*
6. *$\iota_*^\perp(\Gamma)$ is linearly equivalent to $e^*\big(nC_o$+$(2d-1)S_o\big) - \rho\, s_o^\perp$ - $\sum_{i=0}^{3} \gamma_i\, r_i^\perp$.*

Proof. (1) Checking that $\iota_*(\Gamma)$ is numerically equivalent to $nC_o + (2d-1)S_o$ amounts to proving that the intersections numbers $\iota_*(\Gamma) \cdot S_o$ and $\iota_*(\Gamma) \cdot C_o$ are equal to n and $2d-1$. The latter numbers are equal, respectively, to the degree of $\pi : \Gamma \to X$ and the degree of $\iota^*(C_o) = (2d-1)p$, hence the result. Finally, since $\iota_*(\Gamma)$ and C_o only intersect at $s_o \in S_o$, we also obtain their linear equivalence.

(2) and (3) Let $\kappa : \Gamma \to \mathbb{P}^1$ be the *hyperelliptic d-osculating function* associated to π, uniquely characterized by properties 2.6 (1)–(3), and $U \subset X$ a symmetric neighborhood of $q := \pi(p)$. Recall that $\kappa + \pi^*(z^{-1})$ is τ_Γ-anti-invariant and well defined over $\pi^{-1}(U)$, and has a (unique) pole of order $2d-1$ at p. Studying its trace with respect to π we can deduce that ρ must be odd and bounded by $2d-1$.

On the other hand, let $\big(\iota_*(\Gamma), S_o\big)_{s_o}$ and $\big(\iota_*(\Gamma), C_o\big)_{s_o}$ denote the intersection multiplicities at s_o, between $\iota_*(\Gamma)$ and the curves S_o and C_o. They are respectively equal, via the projection formula for ι, to ρ and $2d-1$. At last, since $\iota_*(\Gamma)$ is unibranch at s_o and $\big(\iota_*(\Gamma), S_o\big)_{s_o} = \rho \leqslant 2d-1 = \big(\iota_*(\Gamma), C_o\big)_{s_o}$, we immediately deduce that ρ is the multiplicity of $\iota_*(\Gamma)$ at s_o (and S_o is transverse to $\iota_*(\Gamma)$ at s_o).

(4) By definition of m, we clearly have $\iota_*(\Gamma) = m.\iota(\Gamma)$, while $\{\rho, \gamma_i, i = 0, \dots, 3\}$ are the multiplicities of $\iota_*(\Gamma)$ at different points of S. Hence, m divides n and $2d-1$, as well as all integers $\{\rho, \gamma_i, i = 0, \dots, 3\}$.

(5) For any $i = 0, \dots, 3$, the strict transform of the fiber $S_i := \pi_S^{-1}(\omega_i)$, by the blow-up $e : S^\perp \to S$, is a $\tau^\perp$-invariant curve, equal to $S_i^\perp := e^*(S_i) - s_i^\perp - r_i^\perp$, but also to $\varphi^*(\widetilde{S}_i)$, where $\widetilde{S}_i := \varphi(S_i^\perp)$. Hence, the intersection number $\iota_*^\perp(\Gamma) \cdot S_i^\perp$ is equal to the even integer

$$\iota_*^\perp(\Gamma) \cdot S_i^\perp = \iota_*^\perp(\Gamma) \cdot \varphi^*(\widetilde{S}_i) = \varphi_*\big(\iota_*^\perp(\Gamma)\big) \cdot \widetilde{S}_i = 2\widetilde{\Gamma} \cdot \widetilde{S}_i,$$

implying that $n=\iota^{\perp}_*(\Gamma)\cdot e^*(S_i)$ is congruent mod.2 to

$$\iota^{\perp}_*(\Gamma)\cdot S^{\perp}_i+\iota^{\perp}_*(\Gamma)\cdot(s^{\perp}_i+r^{\perp}_i)\equiv\iota^{\perp}_*(\Gamma)\cdot(s^{\perp}_i+r^{\perp}_i)(\text{mod}.2).$$

We also know, by definition, that $\gamma_i:=\iota^{\perp}_*(\Gamma)\cdot r^{\perp}_i$, while $\iota^{\perp}_*(\Gamma)\cdot s^{\perp}_o=\rho$, the multiplicity of $\iota_*(\Gamma)$ at s_o, and $\iota^{\perp}_*(\Gamma)\cdot s^{\perp}_i=0$ if $i\neq 0$, because $s_i\notin\iota(\Gamma)$. Hence, n is congruent mod.2, to $\rho+\gamma_o\equiv 1+\gamma_o\,(\text{mod}.2)$, as well as to γ_i, if $i\neq 0$.

(6) The Picard group $Pic(S^{\perp})$ is the direct sum of $e^*(Pic(S))$ and the rank-8 lattice generated by the exceptional curves $\{s^{\perp}_i,r^{\perp}_i,i=0,\dots,3\}$. In particular, knowing that $\iota_*(\Gamma)$ is linearly equivalent to $nC_o+(2d-1)S_o$, and having already calculated $\iota^{\perp}_*(\Gamma)\cdot s^{\perp}_i$ and $\iota^{\perp}_*(\Gamma)\cdot r^{\perp}_i$, for any $i=0,\dots,3$, we can finally check that $\iota^{\perp}_*(\Gamma)$ is linearly equivalent to $e^*\big(nC_o+(2d-1)S_o\big)-\rho\,s^{\perp}_o-\sum_0^3\gamma_i\,r^{\perp}_i$. □

We are now ready to deduce the basic inequalities relating the numerical invariants, associated so far to any such cover π (i.e., $\{n,d,g,\rho,m,\gamma\}$). The arithmetic genus of the irreducible curve $\widetilde{\Gamma}:=\varphi(\Gamma^{\perp})\subset\widetilde{S}$, say $\widetilde{g}$, can be deduced from 5.4 (6) via the projection formula for $\varphi:S^{\perp}\to\widetilde{S}$. We start proving the inequality $2g+1\leqslant\gamma^{(1)}$, before deducing the main one (5.5 (4)) from $\widetilde{g}\geqslant 0$.

Theorem 5.5. *Consider any hyperelliptic d-osculating cover $\pi:(\Gamma,p)\to(X,q)$, of degree n, type γ, arithmetic genus g and ramification index ρ at p, and let m denote the degree of its canonical equivariant factorization $\iota^{\perp}:\Gamma\to\iota^{\perp}(\Gamma)\subset S^{\perp}$. Then the numerical invariants $\{n,d,g,\rho,m,\gamma\}$ satisfy the following inequalities:*

1. $2g+1\leqslant\gamma^{(1)}$;
2. $\rho=1$ *implies* $m=1$;
3. $\gamma^{(2)}\leqslant 2(2d-1)(n-m)+4\,m^2-\rho^2$;
4. $(2g+1)^2\leqslant 8(2d-1)(n-m)+13\,m^2-4\rho^2\leqslant 8(2d-1)n+(2d-1)^2$.
   Hence, if π is not ramified at p, we must have $m=1$, as well as:
5. $(2g+1)^2\leqslant 8(2d-1)(n-1)+9$.

Proof. (1) For any $i=0,\dots,3$, the fiber of $\pi_{S^{\perp}}:=\pi_S\circ e:S^{\perp}\to X$ over the half-period ω_i, decomposes as $s^{\perp}_i+r^{\perp}_i+S^{\perp}_i$, where $S^{\perp}_i$ is a $\tau^{\perp}$-invariant divisor and $s^{\perp}_i$ is disjoint with $\iota^{\perp}_*(\Gamma)$, if $i\neq 0$, while ${\iota^{\perp}}^*(s^{\perp}_i)=\rho\,p$, by 5.4 (2). Hence, the divisor $R_i:={\iota^{\perp}}^*(r^{\perp}_i)$ of Γ is linearly equivalent to $R_i\equiv\pi^{-1}(\omega_i)-(n-\gamma_i)\,p$ (and also $2R_i\equiv 2\gamma_i\,p$). Recalling at last, that $\sum_{j=1}^3\omega_j\equiv 3\,\omega_o$, and taking inverse image by π, we finally obtain that $\sum_{i=0}^3 R_i\equiv\gamma^{(1)}\,p$. In other words, there exists a well defined meromorphic function, (i.e., a morphism), from Γ to $\mathbb{P}^1$, with a pole of (odd!) degree $\gamma^{(1)}$ at the Weierstrass point p. The latter can only happen (by the Riemann–Roch Theorem) if $2g+1\leqslant\gamma^{(1)}$, as asserted.

(2) According to 5.4 (4), m divides ρ. Hence, $\rho=1$ implies $m=1$.

(3) The curve $\iota^{\perp}(\Gamma)$ is $\tau^{\perp}$-invariant and linearly equivalent (5.4 (4)–(6)) to:

$$\iota^{\perp}(\Gamma)\sim\frac{1}{m}\Big(e^*\big(nC_o+(2d-1)S_o\big)-\rho s^{\perp}_o-\sum_{i=0}^{3}\gamma_i\,r^{\perp}_i\Big).$$

Recall also that $\varphi^*(\widetilde{K})$, the inverse image by φ of the canonical divisor of $\widetilde{S}$, is linearly equivalent to $\varphi^*(\widetilde{K})\sim e^*(-2C_0)$. Applying the projection formula for $\varphi:S^{\perp}\to\widetilde{S}$, to the divisor $\iota^{\perp}(\Gamma)$, we calculate $g(\widetilde{\Gamma})$, the arithmetic genus of $\widetilde{\Gamma}:=\varphi\left(\iota^{\perp}(\Gamma)\right)\subset\widetilde{S}$:

$$0\leqslant g(\widetilde{\Gamma})=\frac{1}{4m^2}\Big((2d-1)(2n-2m)+4m^2-\rho^2-\gamma^{(2)}\Big),$$

implying

$$\gamma^{(2)} \leqslant (2d-1)(2n-2m) + 4m^2 - \rho^2,$$

as claimed.

(4) and (5) We start remarking that, for any $j = 1, 2, 3$, $(\gamma_o - \gamma_j)$ is a non-zero multiple of m. Hence, $\sum_{i<j}(\gamma_i - \gamma_j)^2 \geqslant 3m^2$, and replacing in 5.5 (1) we get:

$$(2g+1)^2 \leqslant (\gamma^{(1)})^2 = 4\gamma^{(2)} - \sum_{i<j}(\gamma_i - \gamma_j)^2 \leqslant 4\gamma^{(2)} - 3m^2.$$

Taking into account 5.5 (3), we obtain the inequality 5.5 (4), as well as 5.5 (5), which corresponds to the particular case $\rho = m = 1$. □

Lemma 5.6. *Let $\pi : (\Gamma, p) \to (X, q)$ be an arbitrary degree-n hyperelliptic cover, equipped with two points $p' \neq p'' \in \Gamma$ such that the (divisor) sum $p' + p''$ is τ_Γ-invariant. Assume the data $(\pi,\, p', p'')$ solves the NL Schrödinger and 1D Toda or the sine-Gordon case, i.e., $T_oX = V^1_{\Gamma,p'} + V^1_{\Gamma,p''}$ (2.9 and 2.10.) We let again $\iota : \Gamma \to S$ denote the corresponding morphism (3.10), $\Gamma^\perp$ the image of its lift $\iota^\perp : \Gamma \to S^\perp$, and $\gamma = (\gamma_i) \in \mathbb{N}^4$ its type, obtained by intersecting $\Gamma^\perp$ with the curves $\{r_i^\perp\}$. Then*

1. *$\iota(\Gamma)$ is birational to Γ and numerically equivalent to $nC_o + 2S_o$;*
2. *$\iota(\Gamma)$ intersects C_o at $\{\iota(p'), \iota(p'')\}$, with multiplicity 1 at each point, if $\pi(p') \neq \pi(p'')$, and with multiplicity 2 if $\pi(p') = \pi(p'')$;*
3. *if $\pi(p') = \pi(p'')$, then $\gamma_o \equiv \gamma_1 \equiv \gamma_2 \equiv \gamma_3 \equiv n(\mathrm{mod.}2)$, $\pi(p') = \omega_{i_o}$ is a half-period and $\iota^\perp_*(\Gamma)$ is linearly equivalent to $e^*\big(nC_o + 2S_o\big) - 2s^\perp_{i_o}$ - $\sum_{i=0}^3 \gamma_i\, r_i^\perp$;*
4. *if $\pi(p') \neq \pi(p'') \notin \{\omega_i\}$, then $\gamma_o \equiv \gamma_1 \equiv \gamma_2 \equiv \gamma_3 \equiv n(\mathrm{mod.}2)$ and $\iota^\perp_*(\Gamma)$ is linearly equivalent to $e^*\big(nC_o + 2S_o\big) - \sum_{i=0}^3 \gamma_i\, r_i^\perp$;*
5. *if $\pi(p') \neq \pi(p'')$ are two half-periods of (X, q), say $\{\omega_k, \omega_j\}$, for some $k \neq j$, then $\gamma_k + 1 \equiv \gamma_j + 1 \equiv \gamma_i \equiv \gamma_l \equiv n(\mathrm{mod.}2)$, where $\{j, k, i, l\} = \{0, 1, 2, 3\}$ and $\iota^\perp_*(\Gamma)$ is linearly equivalent to $e^*\big(nC_o + S_k + S_j\big) - s_k^\perp$ - $s_j^\perp - \sum_{i=0}^3 \gamma_i\, r_i^\perp$.*

Analogously to what we proved for the d-th KdV case (5.5), we obtain the following relations between the degree and arithmetic genus of the other cases.

Theorem 5.7 (NL Schrödinger and 1D Toda case). *Let $\pi : (\Gamma, p) \to (X, q)$ be an arbitrary degree-n hyperelliptic cover of arithmetic genus g, equipped with two points $p^+ \neq p^- \in \Gamma$ exchanged by the hyperelliptic involution τ_Γ. Assume $(\pi,\, p^+, p^-)$ solves the NL Schrödinger and 1D Toda case and let $\gamma \in \mathbb{N}^4$ denote its type (5.6). Then, $\gamma_i \equiv n(\mathrm{mod.}2)$, for any i, and*

1. *$2g + 2 \leqslant \gamma^{(1)}$;*
2. *$\pi(p^+) \neq \pi(p^-)$ implies $\gamma^{(2)} \leqslant 4n$, as well as $(g+1)^2 \leqslant 4n$;*
3. *$\pi(p^+) = \pi(p^-)$ and $n \equiv 0(\mathrm{mod.}2)$ imply $\gamma^{(2)} \leqslant 4n - 4$ and $(g+1)^2 \leqslant 4n - 4$;*
4. *$\pi(p^+) = \pi(p^-)$ and $n \equiv 1(\mathrm{mod.}2)$ imply $\gamma^{(2)} \leqslant 4n - 8$ and $(g+1)^2 \leqslant 4n - 8$.*

Theorem 5.8 (sine-Gordon case). *Let $\pi : (\Gamma, p) \to (X, q)$ be an arbitrary degree-n hyperelliptic cover of arithmetic genus g, equipped with two Weierstrass points $p_1, p_2 \in \Gamma$. Assume $(\pi,\, p_1, p_2)$ solves the sine-Gordon case and let $\gamma \in \mathbb{N}^4$ denote its type (5.6). Then*

1. $2g \leqslant \gamma^{(1)}$;
2. $\pi(p_1) \neq \pi(p_2)$ *implies* $\gamma^{(2)} \leqslant 4n$, *as well as* $g^2 \leqslant 4n$;
3. $\pi(p_1) = \pi(p_2)$ *and* $n \equiv 0(\text{mod.}2)$ *imply* $\gamma^{(2)} \leqslant 4n - 4$ *and* $g^2 \leqslant 4n - 4$;
4. $\pi(p^+) = \pi(p^-)$ *and* $n \equiv 1(\text{mod.}2)$ *imply* $\gamma^{(2)} \leqslant 4n - 8$ *and* $g^2 \leqslant 4n - 8$.

6. ON HYPERELLIPTIC d-OSCULATING COVERS OF ARBITRARY HIGH GENUS

6.1. Let $C_o^{\perp}$ denote the strict transform of C_o in $S^{\perp}$, $\widetilde{C}_o := \varphi(C_o^{\perp})$ its projection in $\widetilde{S}$ and consider an arbitrary degree-n *hyperelliptic* d*-osculating cover* of *type* γ, say $\pi : (\Gamma, p) \to (X, q)$, with ramification index ρ at p. We will let $\iota^{\perp} : \Gamma \to S^{\perp}$ denote its unique equivariant factorization through $\pi_{S^{\perp}} : S^{\perp} \to X$ (5.1), $\Gamma^{\perp} := \iota^{\perp}(\Gamma)$ its image in $S^{\perp}$ and $\widetilde{\Gamma}$ the corresponding projection into $\widetilde{S}$. Recall (5.4 and 5.5) that the above numerical invariants must satisfy the following restrictions

1. ρ is an odd integer bounded by $2d - 1$;
2. $\gamma_o + 1 \equiv \gamma_1 \equiv \gamma_2 \equiv \gamma_3 \equiv n(\text{mod.}2)$.

   Furthermore, whenever $m := deg(\iota^{\perp} : \Gamma \to \Gamma^{\perp})$ is equal to 1 (i.e., Γ is birational to $\Gamma^{\perp}$), π can be canonically recovered from $\widetilde{\Gamma} := \varphi(\Gamma^{\perp})$, and they all satisfy the following properties:
3. $\widetilde{\Gamma}$ is an irreducible rational curve of non-negative arithmetic genus equal to $\widetilde{g} := \frac{1}{4}((2d - 1)(2n - 2) + 4 - \rho^2 - \gamma^{(2)}) \geqslant 0$;
4. $\Gamma^{\perp}$ is linearly equivalent to $e^*(nC_o + (2d - 1)S_o) - \rho {s_o}^{\perp} - \sum_{i=0}^{3} \gamma_i {r_i}^{\perp}$;
5. $\widetilde{\Gamma}$ intersects $\widetilde{s}_o := \varphi({s_o}^{\perp})$ at a unique point, where it is unibranch and has multiplicity ρ;
6. $\widetilde{\Gamma}$ intersects $\widetilde{C}_o$ (at most) at $\widetilde{p}_o := \widetilde{C}_o \cap \widetilde{s}_o$ (i.e., $\widetilde{\Gamma} \cap \widetilde{C}_o \subset \widetilde{C}_o \cap \widetilde{s}_o$), with multiplicity $\frac{1}{2}(2d - 1 - \rho)$. In particular, if $\rho = 2d - 1$, $\widetilde{\Gamma}$ and $\widetilde{C}_o$ are disjoint curves.

Definition 6.2. *For any* $(n, d, \rho, \gamma) \in \mathbb{N}^7$ *satisfying the above restrictions, we let* $\Lambda(n, d, \rho, \gamma)$ *denote the unique element of* $Pic(\widetilde{S})$ *such that* $\varphi^*(\Lambda(n, d, \rho, \gamma))$ *is linearly equivalent to* $e^*\big(nC_o + (2d\text{-}1)S_o\big) - \rho {s_o}^{\perp} - \sum_{i=0}^{3} \gamma_i {r_i}^{\perp}$, *and* $MH_X(n, d, \rho, 1, \gamma)$ *denote the moduli space of degree-n hyperelliptic d-osculating covers of type* γ, *ramification index* ρ *at their marked point, and birational to their canonical images in* $S^{\perp}$.

Remark 6.3. We will restrict to the simpler case where $\rho = 1$, Γ is isomorphic to $\Gamma^{\perp}$ and $\widetilde{\Gamma}$ is isomorphic to $\mathbb{P}^1$. In other words, we will focus on degree-n *hyperelliptic* d*-osculating covers* with $\rho = m = 1$, and of type γ satisfying $\gamma^{(2)} = (2d - 1)(2n - 2) + 3$. We will actually choose $\gamma = (2d - 1)\mu + 2\varepsilon$, where μ is an arbitrary $\mu \in \mathbb{N}^4$ satisfying $\mu_o + 1 \equiv \mu_1 \equiv \mu_2 \equiv \mu_3(\text{mod.}2)$ and $\varepsilon \in \mathbb{Z}^4$ is equal to $\varepsilon = (d - 1, d - 1, d - 1, 0)$. Given such triplet (n, d, γ) we give a straightforward construction of $MH_X(n, d, 1, 1, \gamma)$ as a $(d - 1)$-dimensional family of curves, embedded in $S^{\perp}$ (6.9). Moreover, it can also be proved that any $\pi \in MH_X(n, d, 1, 1, \gamma)$ has a unique birational model in $\mathbb{P}^1 \times X$, as a linear combination of d specific polynomials with elliptic coefficients. The same can be done for $2\varepsilon = (d + 1, d - 1, d - 1, d - 1)$ if d is odd, or for $2\varepsilon = (d - 2, d, d, d)$ if d is even; or when permuting and/or changing the signs of their coefficients.

We will need the following existence and irreducibility criteria.

Proposition 6.4. ([33, see 3.4]). *Any curve* $\Gamma \subset S$ *intersecting* C_o *at a unique smooth point* $p \in \Gamma$ *is irreducible.*

Proposition 6.5. *Let* $\Gamma^{\perp} \subset S^{\perp}$ *be a curve with no irreducible component in* $\{r_i^{\perp}, i = 0, \ldots, 3\}$, *and intersecting* $C_o^{\perp}$ *(at most) at a unique smooth point* $p^{\perp} \in \Gamma^{\perp}$. *Then* $\Gamma^{\perp}$ *is an irreducible curve.*

Proof. The properties satisfied by $\Gamma^\perp$ assure us that it is the strict transform of its direct image by $e : S^\perp \to S$, $\Gamma := e_*(\Gamma^\perp)$, and that the latter does not contain C_o. We can also check, that Γ is smooth at $p := e(p^\perp)$ and $\Gamma \cap C_o = \{p\}$. It follows, by 6.4, that (Γ, as well as its strict transform) $\Gamma^\perp$ is an irreducible curve. $\square$

Proposition 6.6. ([35, see 6.2]). *Any $\alpha = (\alpha_i) \in \mathbb{N}^4$ such that $\alpha^{(2)} = 2n + 1$ is odd gives rise to an exceptional curve of the first kind $\widetilde{\Gamma}_\alpha \subset \widetilde{S}$. More precisely, let $k \in \{0, 1, 2, 3\}$ denote the index satisfying $\alpha_k + 1 \equiv \alpha_j(\text{mod.}2)$, for any $j \neq k$, and $S_k := \pi_S^{-1}(s_k)$, then $\widetilde{\Gamma}_\alpha$ has self-intersection -1 and $\varphi^*(\widetilde{\Gamma}_\alpha) \subset S^\perp$ is the unique $\tau^\perp$-invariant irreducible curve linearly equivalent to $e^*(nC_o + S_k) - s_k^\perp - \sum_{i=0}^{3} \alpha_i r_i^\perp$.*

Proof. Let Λ denote the unique numerical equivalence class of $\widetilde{S}$ satisfying $\varphi^*(\Lambda) = e^*(nC_o + S_k) - s_k^\perp - \sum_{i=0}^{3} \alpha_i r_i^\perp$. It has self-intersection $\Lambda \cdot \Lambda = -1$, and $\Lambda \cdot \widetilde{K} = -1$ as well. It follows that $h^o(\widetilde{S}, O_{\widetilde{S}}(\Lambda)) \geqslant \chi(O_{\widetilde{S}}(\Lambda)) = 1$, hence there exists an effective divisor $\widetilde{\Gamma} \in |\Lambda|$. Such a divisor is known to be unique and irreducible ([35] 6.2.). $\square$

Corollary 6.7. ([35]). *Let $\alpha \in \mathbb{N}^4$ be such that $\alpha_o + 1 \equiv \alpha_j(\text{mod.}2)$, $\widetilde{\Gamma}_\alpha$ the corresponding exceptional curve (see 6.6), and $\Gamma_\alpha^\perp := \varphi^*(\widetilde{\Gamma}_\alpha)$ its inverse image in $S^\perp$, marked at its Weierstrass point $p_\alpha := \Gamma_\alpha^\perp \cap s_o^\perp$. Then, $(\Gamma_\alpha^\perp, p_\alpha)$ gives rise to KdV solutions, L-periodic in $x = t_1$ (the first KdV flow).*

The latter corollary will be generalized as follows: given any $n, d \in \mathbb{N}^*$, we will construct *types* $\gamma = (2d-1)\mu + 2\varepsilon \in \mathbb{N}^4$, such that $\gamma_o + 1 \equiv \gamma_1 \equiv \gamma_2 \equiv \gamma_3(\text{mod.}2)$ and $\gamma^{(2)} = (2n-2)(2d-1) + 3$, for which the linear system $|\Lambda(n, d, 1, \gamma)|$ (see 6.2) has dimension $d - 1$ and a generic element isomorphic to $\mathbb{P}^1$. Hence, they will give rise to $(d-1)$-dimensional families of marked curves solving the d-th KdV case.

Theorem 6.8. *Let us fix $d \geqslant 2$, $k \in \{0, 1, 2, 3\}$, and $\mu \in \mathbb{N}^4$ such that $\mu_o + 1 \equiv \mu_j(\text{mod.}2)$ (for $j = 1, 2, 3$). Pick any vector $2\varepsilon = (2\varepsilon_i) \in 2\mathbb{Z}^4$, satisfying $(\forall i = 0, \dots, 3)$, either*

$$|2\varepsilon_i| = (2d-2)(1 - \delta_{i,k}),$$

$$\text{or} \quad \begin{cases} |2\varepsilon_i| = d - (-1)^{\delta_{i,k}} & \text{if } d \text{ is odd,} \\ |2\varepsilon_i| = d - 2\delta_{i,k} & \text{if } d \text{ is even,} \end{cases}$$

as long as $\gamma := (2d-1)\mu + 2\varepsilon \in \mathbb{N}^4$, and let n satisfy $\gamma^{(2)} = (2d-1)(2n-2) + 3$. Then $|\varphi^*(\Lambda(n, d, 1, \gamma))|$ contains a $(d-1)$-dimensional subspace such that its generic element, say $\Gamma^\perp$, satisfies the following properties:

1. *$\Gamma^\perp$ is a $\tau^\perp$-invariant smooth irreducible curve of genus $g := \frac{1}{2}(-1 + \gamma^{(1)})$;*
2. *$\Gamma^\perp$ can only intersect $C_o^\perp$ at $p_o^\perp := C_o^\perp \cap s_o^\perp$;*
3. *$\varphi(\Gamma^\perp) \subset \widetilde{S}$ is isomorphic to $\mathbb{P}^1$.*

Corollary 6.9. *Given $(n, d, \gamma) \in \mathbb{N}^* \times \mathbb{N}^* \times \mathbb{N}^4$ as above, the moduli space $MH_X(n, d, 1, 1, \gamma)$ (6.2) has dimension $d - 1$, and a smooth generic element of genus $g := \frac{1}{2}(-1 + \gamma^{(1)})$.*

Proof of Theorem 6.8. We will only work out the case $\gamma := (2d-1)\mu + 2\varepsilon$, with $\varepsilon = (0,\, d-1,\, d-1,\, d-1)$. For any other choice of ε, the corresponding proof runs along the same lines and will be skipped. In our case, the arithmetic genus g and the degree n satisfy:

$$2g+1 = (2d-1)\mu^{(1)} + 6(d-1) \quad \text{and} \quad 2n = (2d-1)\mu^{(2)} + 4(d-1)(\mu_1+\mu_2+\mu_3) + 6d - 7.$$

Consider $\overline{\mu} := \mu + (1,1,1,1)$, $\mu' := \mu + (0,2,1,1)$, $\mu'' = \mu + (0,0,1,1)$, and let $\overline{Z}^{\perp}, Z'^{\perp}, Z''^{\perp} \subset S^{\perp}$ denote the unique $\tau^{\perp}$-invariant curves linearly equivalent to:

1) $\overline{Z}^{\perp} \sim e^*(\overline{m}\, C_o + S_o) - s_o^{\perp} - \sum_i \overline{\mu}_i r_i^{\perp}$, where $2\overline{m}+1 = \overline{\mu}^{(2)}$;

2) $Z'^{\perp} \sim e^*(m' C_o + S_1) - s_1^{\perp} - \sum_i \mu'_i r_i^{\perp}$, where $2m'+1 = \mu'^{(2)}$;

3) $Z''^{\perp} \sim e^*(m'' C_o + S_1) - s_1^{\perp} - \sum_i \mu''_i r_i^{\perp}$, where $2m''+1 = \mu''^{(2)}$.

Moreover, if $\mu_o \neq 0$ we choose $\underline{\mu} = \mu + (-1,1,1,1)$ and $2\underline{m}+1 = \underline{\mu}^{(2)}$, and let $\underline{Z}^{\perp} \subset S^{\perp}$ denote the unique $\tau^{\perp}$-invariant curve $\underline{Z}^{\perp} \sim e^*(\underline{m} C_o + S_o) - s_o^{\perp} - \sum_i \underline{\mu}_i r_i^{\perp}$.

However, if $\mu_o = 0$ we will simply put $\underline{Z}^{\perp} := \overline{Z}^{\perp} + 2r_o^{\perp}$, so that in both cases, the divisors $D_0^{\perp} := \overline{Z}^{\perp} + \underline{Z}^{\perp} + 2s_0^{\perp}$ and $D_1^{\perp} := Z'^{\perp} + Z''^{\perp} + 2s_1^{\perp}$ will be linearly equivalent. Let us also define,

$$\mu_{(1)} := \mu'' = \mu + (0,0,1,1),$$
$$\mu_{(2)} := \mu + (0,1,0,1),$$
$$\mu_{(3)} := \mu + (0,1,1,0),$$

and let $Z_{(k)}^{\perp} (k = 1,2,3)$ be the $\tau^{\perp}$-invariant curve of $S^{\perp}$, linearly equivalent to $e^*(m_{(k)} C_o + S_k) - s_k^{\perp} - \sum_i \mu_{(k)i} r_i^{\perp}$, where $2m_{(k)}+1 = \sum_i \mu_{(k)i}^2$.

At last, consider $Z^{\perp} \sim e^*(mC_o + S_o) - s_o^{\perp} - \sum_i \mu_i r_i^{\perp}$, where $2m+1 = \sum_i \mu_i^2$ (6.2). The $(d-1)$-dimensional subspace of $\left|\varphi^*(\Lambda(n,d,1,\gamma))\right|$ we are looking for, will be made of all above curves. We first remark the following facts:

a) we can check, via the adjunction formula, that any $\tau^{\perp}$-invariant element of $|\varphi^*(\Lambda(n,d,1,\gamma))|$ has arithmetic genus $g := \frac{1}{2}(-1+\gamma^{(1)})$, and is the pull-back by $\varphi : S^{\perp} \to \widetilde{S}$, of a divisor of zero arithmetic genus of $\widetilde{S}$;

b) the following $d-1$ divisors

$$\Big\{F_j^{\perp} := C_o^{\perp} + \sum_{k=1}^{3}(Z_{(k)}^{\perp} + 2s_k^{\perp}) + jD_o^{\perp} + (d-2-j)D_1^{\perp},\ j = 0,\ldots,d-2\Big\},$$

as well as

$$G^{\perp} := Z^{\perp} + (d-1)D_o^{\perp},$$

are $\tau^{\perp}$-invariant, belong to $|\varphi^*(\Lambda(n,d,1,\gamma))|$ and have $p_o^{\perp} := C_o^{\perp} \cap s_o^{\perp}$ as their unique common point;

c) the curve $F_o^{\perp}$ is smooth at $p_o^{\perp}$, while any other $F_j^{\perp}$ has multiplicity $1 < 2j+1 < 2d$ at $p_o^{\perp}$. In particular, they span a $(d-2)$-subspace of $|\varphi^*(\Lambda(n,d,1,\gamma))|$, having a generic element smooth and transverse to $s_o^{\perp}$ at $p_o^{\perp}$;

d) the curve $G^{\perp}$ has multiplicity $2d$ at $p_o^{\perp}$, and no common irreducible component with any $F_j^{\perp} (\forall j = 0,\ldots,d-2)$, implying that $\langle\, G^{\perp},\, F_j^{\perp},\, j = 0,\ldots,d-2\,\rangle \subset \left|\varphi^*(\Lambda)\right|$, the $(d-1)$-subspace they span, is fixed component-free;

e) any irreducible curve $\Gamma^{\perp} \in \langle G^{\perp}, F_j^{\perp}, j = 0, \dots, d-2 \rangle$ projects onto a smooth irreducible curve (isomorphic to $\mathbb{P}^1$). In particular $\Gamma^{\perp}$ must be smooth outside $\cup_{i=0}^{3} r_i^{\perp}$.

f) the curves $G^{\perp}$ and $F_o^{\perp}$ have no common point on any $r_i^{\perp}$ $(i = 0, \dots, 3)$, implying that $\Gamma^{\perp}$, the generic element of $\langle G^{\perp}, F_j^{\perp}, j = 0, \dots, d-2 \rangle$, is smooth at any point of $\cup_{i=0}^{3} r_i^{\perp}$ and satisfies the announced properties, i.e.,

(1) $\Gamma^{\perp}$ is $\tau^{\perp}$-invariant, smooth and satisfies the irreducibility criterion 6.5;

(2) $p_o^{\perp}$ is the unique base point of the linear system and $\Gamma^{\perp} \cap C_o^{\perp} = \{p_o^{\perp}\}$;

(3) its image $\varphi(\Gamma^{\perp}) \subset \widetilde{S}$ is irreducible, linearly equivalent to $\Lambda(n, d, 1, \gamma)$ and of arithmetic genus $\frac{1}{4}\big((2d-1)(2n-2) + 3 - \gamma^{(2)}\big) = 0$; hence, isomorphic to $\mathbb{P}^1$. □

Proof of Corollary 6.9. The degree-2 projection $\varphi : \Gamma^{\perp} \to \varphi(\Gamma^{\perp})$ is ramified at $p_o^{\perp}$ and $\varphi(\Gamma^{\perp})$ is isomorphic to $\mathbb{P}^1$. Moreover, $\Gamma^{\perp}$ is a smooth irreducible curve linearly equivalent to $|\varphi^*(\Lambda(n, d, 1, \gamma))|$, of arithmetic genus $g := \frac{1}{2}(\gamma^{(1)} - 1)$.

In other words, the natural projection $(\Gamma^{\perp}, p_o^{\perp}) \subset (S^{\perp}, p_o^{\perp}) \xrightarrow{\pi_{S^{\perp}}} (X, q)$ is a smooth degree-n *hyperelliptic d-osculating cover* of type γ, and genus g, such that $(2n-2)(2d-1) + 3 = \gamma^{(2)}$ and $2g + 1 = \gamma^{(1)}$. □

Remark 6.10.

1. The irreducible components of the d generators $\langle G^{\perp}, F_j^{\perp}, j = 0, \dots, d-2 \rangle$ are well known curves, for which one can provide explicit equations in $\mathbb{P}^1 \times X$. Hence, any element of $MH_X(n, d, 1, 1, \gamma)$ is birational to the zero set of a linear combination of d specific degree-n polynomials with coefficients in *K(X)*, the field of meromorphic functions on X.
2. Effective solutions to the NL Schrödinger and 1D Toda and sine-Gordon cases can also been found through an analogous method. Roughly speaking, we construct infinitely many 1-dimensional families of solutions (for both cases), having arbitrary degree n, and arbitrary genus g. As we shall see, the results differ on whether the pair of marked points have same projection in X or not (and depend on the parity of n as well). The main results are given below (detailed proofs will be given elsewhere).

Proposition 6.11 (NL Schrödinger and 1D Toda restrictions). *Let $\pi : (\Gamma, p) \to (X, q)$ be an arbitrary hyperelliptic cover, equipped with two non-Weierstrass points $p^+, p^- \in \Gamma$, such that (π, p^+, p^-) solves the NL Schrödinger and 1D Toda case. Then, the arithmetic genus of Γ and the degree of π, say g and n, satisfy*:

1. $(g+1)^2 \leqslant 4n - 4$, *if* $\pi(p^+) = \pi(p^-)$ *and* $n \equiv 0 \pmod 2$;
2. $(g+1)^2 \leqslant 4n - 8$, *if* $\pi(p^+) = \pi(p^-)$ *and* $n \equiv 1 \pmod 2$;
3. $(g+1)^2 \leqslant 4n$, *if* $\pi(p^+) \neq \pi(p^-)$.

Proposition 6.12 (sine-Gordon restrictions). *Let $\pi : (\Gamma, p) \to (X, q)$ be an arbitrary hyperelliptic cover, equipped with two Weierstrass points $p_o, p_1 \in \Gamma$, such that (π, p_o, p_1) solves the sine-Gordon case. Then, the arithmetic genus of Γ and the degree of π, say g and n, satisfy*:

1. $g^2 \leqslant 4n - 4$, *if* $\pi(p_o) = \pi(p_1)$ *and* $n \equiv 0 \pmod 2$;
2. $g^2 \leqslant 4n - 8$, *if* $\pi(p_o) = \pi(p_1)$ *and* $n \equiv 1 \pmod 2$;
3. $g^2 \leqslant 4n - 3$, *if* $\pi(p_o) \neq \pi(p_1)$.

Along with the latter restrictions we have the following effective results.

Theorem 6.13 (odd degree NL Schrödinger and 1D Toda case). *For any* $\alpha \in \mathbb{N}^4$ *and* $a \in X$ *there exists a hyperelliptic cover* $\pi : (\Gamma, p) \to (X, q)$, *equipped with two non-Weierstrass points* $p^+, p^- \in \Gamma$ *such that:*

1. $\pi(p^+) = a$, $p^+ = \tau_\Gamma(p^-)$ and (π, p^+, p^-) *solves the NL Schrödinger case*;
2. Γ *has arithmetic genus* $g := \alpha^{(1)} + 1$;
3. $deg(\pi) = \alpha^{(2)} + \alpha^{(1)} + 1$ *if* $a \notin \{\omega_i\}$, *hence* $\pi(p^+) \neq \pi(p^-)$;
4. $deg(\pi) = \alpha^{(2)} + \alpha^{(1)} + 3$ *if* $a \in \{\omega_i\}$, *hence* $\pi(p^+) = \pi(p^-)$.

Theorem 6.14 (even degree NL Schrödinger and 1D Toda case). *For any* $\alpha \in \mathbb{N}^4 \setminus \{0\}$ *and* $a \in X$ *such that, either* $\alpha^{(1)} \equiv 0 (\mathrm{mod}\ 2)$ *and* $a \notin \{\omega_i\}$, *or* $\alpha^{(1)} \equiv 1 (\mathrm{mod}\ 2)$ *and* $a \in \{\omega_i\}$, *there exists a hyperelliptic cover* $\pi : (\Gamma, p) \to (X, q)$, *equipped with two non-Weierstrass points* $p^+, p^- \in \Gamma$ *such that:*

1. $\pi(p^+) = a$, $p^+ = \tau_\Gamma(p^-)$ *and* (π, p^+, p^-) *solves the NL Schrödinger case;*
2. Γ *has arithmetic genus* $g := \alpha^{(1)} - 1$;
3. $deg(\pi) = \alpha^{(2)}$ *if* $a \notin \{\omega_i\}$, *and* $deg(\pi) = \alpha^{(2)} + 1$ *otherwise.*

For a better presentation of our sine-Gordon's results, we must also take in account the projections of (p_o, p_1), the pair of Weierstrass points (see 2.10). They either project onto the same point, which can be chosen equal to $\pi(p_o) = \pi(p_1) = \omega_o$, or their projections differ by a non-zero half-period, say $\pi(p_o) = \omega_o$ and $\pi(p_1) = \omega_1$. In all four cases we find 1-dimensional families of solutions. Additional properties, such as the existence of a fixed point free involution or a real structure can also be found. For example, if (X, q) has a real structure, we can extract from the first three sine-Gordon cases a real 1-dimensional family having a real structure fixing the Weierstrass points.

Theorem 6.15 (even degree sine-Gordon with distinct projections). *Pick any* $\alpha \in \mathbb{N}^4$ *satisfying* $\alpha_2 + \alpha_3 \equiv 1 (\mathrm{mod}\ 2)$. *Then, there exists a* 1*-dimensional family of hyperelliptic covers* $\pi : (\Gamma, p) \to (X, q)$, *equipped with a pair of distinct Weierstrass points* $\{p_o, p_1\} \in \Gamma$, *such that:*

1. $\pi(p_j) = \omega_j$, *for* $j = 0, 1$ *and* (π, p_o, p_1) *solves the sine-Gordon case*;
2. Γ *has arithmetic genus* $g := \alpha^{(1)} + 1$ *and* $deg(\pi) = \alpha^{(2)} + \alpha_o + \alpha_1 + 1$.

Theorem 6.16 (odd degree sine-Gordon with distinct projections). *Pick any* $\alpha \in \mathbb{N}^4$ *satisfying* $\alpha_o + \alpha_1 \equiv 0 (\mathrm{mod}\ 2)$. *Then, there exists a* 1*-dimensional family of hyperelliptic covers* $\pi : (\Gamma, p) \to (X, q)$, *equipped with a pair of distinct Weierstrass points* $\{p_o, p_1\} \in \Gamma$, *such that:*

1. $\pi(p_j) = \omega_j$, *for* $j = 0, 1$ *and* (π, p_o, p_1) *solves the sine-Gordon case*;
2. Γ *has arithmetic genus* $g := \alpha^{(1)} + 1$ *and* $deg(\pi) = \alpha^{(2)} + \alpha_2 + \alpha_3 + 1$.

Theorem 6.17 (even degree sine-Gordon with same projection). *Fix* $j_o \in \{1, 2, 3\}$ *and pick any* $\alpha \in \mathbb{N}^4$ *satisfying* $\alpha_{j_o} + 1 \equiv \alpha_i (\mathrm{mod}\ 2)$ *for any* $i \neq j_o$. *Then, there exists a* 1*-dimensional family of hyperelliptic covers* $\pi : (\Gamma, p) \to (X, q)$, *equipped with a pair of distinct Weierstrass points* $\{p_o, p_1\} \in \Gamma$, *such that:*

1. $\pi(p_o) = \pi(p_1) = \omega_o$ *and* (π, p_o, p_1) *solves the sine-Gordon case*;
2. Γ *has arithmetic genus* $g := \alpha^{(1)}$ *and* $deg(\pi) = \alpha^{(2)} + 1$.

Theorem 6.18 (odd degree sine-Gordon with same projection). *For any* $\alpha \in \mathbb{N}^4$ *there exists a* 1*-dimensional family of hyperelliptic covers* $\pi : (\Gamma, p) \to (X, q)$, *equipped with a pair of distinct Weierstrass points* $\{p_o, p_1\} \in \Gamma$, *such that:*

1. $\pi(p_o) = \pi(p_1) = \omega_o$ *and* (π, p_o, p_1) *solves the sine-Gordon case*;
2. Γ *has arithmetic genus* $g := \alpha^{(1)} + 2$ *and* $deg(\pi) = \alpha^{(2)} + \alpha^{(1)} + 3$.

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 311–329. © Pleiades Publishing, Ltd., 2011.

Lotka–Volterra Equations in Three Dimensions Satisfying the Kowalevski–Painlevé Property

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Received October 2, 2010; accepted November 22, 2010

Abstract—We examine a class of Lotka–Volterra equations in three dimensions which satisfy the Kowalevski–Painlevé property. We restrict our attention to Lotka–Volterra systems defined by a skew symmetric matrix. We obtain a complete classification of such systems. The classification is obtained using Painlevé analysis and more specifically by the use of Kowalevski exponents. The imposition of certain integrality conditions on the Kowalevski exponents gives necessary conditions. We also show that the conditions are sufficient.

MSC2010 numbers: 34G20, 34M55, 37J35

DOI: 10.1134/S1560354711030075

Keywords: Lotka–Volterra equations, Kowalevski exponents, Painlevé analysis

1. INTRODUCTION

The Lotka–Volterra model is a basic model of predator-prey interactions. The model was developed independently by Alfred Lotka (1925), and Vito Volterra (1926). It forms the basis for many models used today in the analysis of population dynamics. In three dimensions it describes the dynamics of a biological system where three species interact.

The most general form of Lotka–Volterra equations is

$$\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{ij} x_i x_j, \quad i = 1, 2, \ldots, n. \tag{1.1}$$

We consider Lotka–Volterra equations without linear terms ($\varepsilon_i = 0$), and where the matrix of interaction coefficients $A = (a_{ij})$ is skew-symmetric. This is a natural assumption related to the principle that crowding inhibits growth. The special case of Kac–van Moerbeke system (KM-system) was used to describe population evolution in a hierarchical system of competing individuals. The KM-system has close connection with the Toda lattice. The Lotka–Volterra equations were studied by many authors in its various aspects, e.g. complete integrability [1] Poisson and bi-Hamiltonian formulation ([2–5]), stability of solutions and Darboux polynomials ([6, 7]).

In this paper we examine such Lotka–Volterra equations in three dimensions satisfying the Kowalevski–Painlevé property. The basic tools for the required classification are, the use of Painlevé analysis, the examination of the eigenvalues of the Kowalevski matrix and other standard Lax pair and Poisson techniques. The Kowalevski exponents are useful in establishing integrability or non-integrability of Hamiltonian systems; see [8–14]. The first step is to impose certain conditions on the exponents, i.e., we require that all the Kowalevski exponents be integers for every solution of the indicial equation. This gives a finite list of values of the parameters satisfying such conditions. This step requires some elementary number theoretic techniques as is usual with such type of

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classification. In the three-dimensional case the general expressions for the Kowalevski exponents are rational and therefore the number theoretic analysis is manageable.

The second step is to check that the leading behavior of the Laurent series solutions agrees with the weights of the corresponding homogeneous vector field defining the dynamical system. In our case the weights are all equal to one and therefore we must exclude the possibility that some of the Laurent series have leading terms with poles of order greater than one. This is a step usually omitted by some authors due to its complexity, but in this paper we analyze this in detail. To accomplish this step we use the old-fashioned Painlevé Analysis, i.e., Laurent series. The application of Painlevé analysis and especially of the ARS algorithm (see [1, 12, 15–18]) is useful in calculating the Laurent solution of a system and check if there are $(n-1)$ free parameters.

In performing Painlevé analysis we use the fact that the sum of the variables is always a first integral. Surprisingly the Painlevé analysis does not reveal any additional cases besides the ones already found by using the Kowalevski exponents. In this classification of Lotka–Volterra systems we discover, as expected, some well known integrable systems like the open and periodic Kac–van Moerbeke systems.

To make sure that our conditions are not only necessary but also sufficient we verify that the systems obtained indeed satisfy the Kowalevski–Painlevé property by checking the number of free parameters. It is shown in this paper that all the solutions to the indicial equations extend to full convergent Laurent solutions depending on two free parameters; moreover it is shown that there are no other Laurent solutions (except for Taylor solutions) so the description is complete.

We also have to point out that our classification is up to isomorphism. In other words, if one system is obtained from another by an invertible linear change of variables, we do not consider them as different. Modulo this identification we obtain only six classes of solutions.

The Lotka–Volterra system can be expressed in hamiltonian form as follows: Define a quadratic Poisson bracket by the formula

$$\{x_i, x_j\} = a_{ij} x_i x_j, \quad i, j = 1, 2, \ldots, n. \tag{1.2}$$

Then the system can be written in the form $\dot{x}_i = \{x_i, H\}$, where $H = \sum_{i=1}^{n} x_i$. The Louville integrability in the three-dimensional case can be easily established. In addition to the Hamiltonian function H, there exists a second integral, in fact a Casimir F. The formula for this Casimir is given afterwards.

In this paper we restrict our attention to the three dimensional case. For $n = 3$ the system is defined by the matrix

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

where a, b, c are constants. We use the notation (a, b, c) to denote this system. It turns out that the Lotka–Volterra systems which possess the Kowalevski–Painlevé property fall either into two infinite families or four exceptional cases:

Theorem 1. *The Lotka–Volterra equations in three dimensions satisfy the Kowalevski–Painlevé property if and only if* (a, b, c) *is in the class of*

(l_2) $(1, 0, 1)$

(l_3) $(1, -1, 1)$

(l_4) $(1, -1, 2)$

(l_6) $(1, -2, 3)$

(l_λ) $(1, 1, \lambda)$ $\lambda \in \mathbf{Z} \setminus 0$.

(l_0) $(1, 1 + \mu, \mu)$ $\mu \in \mathbf{R}$.

We use the notation l_j to indicate that the system has an invariant of degree j.

In Section 2 we give the basic definitions of weight-homogeneous vector fields and the Kowalevski matrix. The definition of Kowalevski exponents and relevant results follow the recent book [9]. See also the review article of Goriely [11] where one can find many more properties of these exponents. In Section 3 we give some related properties of the Kowalevski exponents. In Section 4 we define the three dimensional Lotka–Volterra systems and find necessary conditions for ensuring the Kowalevski–Painlevé property by analyzing the corresponding Kowalevski exponents. In Section 5 we show that our classification is complete. Finally, in Section 6 we exclude any cases that may exist due to higher order poles.

2. BASIC DEFINITIONS

We begin by defining what is a weight homogeneous polynomial. We follow the notation from [9].

Definition 1. *A polynomial $f \in \mathbf{C}\,[x_1, x_2, \ldots, x_n]$ is called a* weight-homogeneous polynomial of weight k *with respect to a vector $v = (v_1, v_2, \ldots, v_n)$ if*

$$f(t^{v_1}x_1, \ldots, t^{v_n}x_n) = t^k f(x_1, x_2, \ldots, x_n).$$

The vector v is called the weight vector. *The v_i are all positive integers without a common divisor. The weight k is denoted by $\varpi(f)$.*

Definition 2. *A polynomial vector field on $\mathbf{C}^n$,*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) \end{aligned} \tag{2.1}$$

is called a weight-homogeneous vector field *of weight k (with respect to a weight vector v), if $\varpi(f_i) = v_i + k = \varpi(x_i) + k$ for $i = 1, 2, \ldots, n$. A weight-homogeneous vector field of weight 1 is called* weight-homogeneous vector field. *Furthermore, when all the weights are equal to 1, this is simply called a* homogeneous vector field.

Example 1. We consider the periodic 5-particle Kac–van Moerbeke lattice that is given by the quadratic vector field

$$\dot{x}_i = x_i(x_{i-1} - x_{i+1}), \quad i = 1, \ldots, 5, \tag{2.2}$$

with $x_i = x_{i+5}$. This system has three independent constants of motion,

$$\begin{aligned} F_1 &= x_1 + x_2 + x_3 + x_4 + x_5, \\ F_2 &= x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2, \\ F_3 &= x_1x_2x_3x_4x_5. \end{aligned} \tag{2.3}$$

Taking $v = (1, 1, 1, 1, 1)$, (2.2) becomes a homogeneous vector field and the weights of the integrals of motion are $\varpi(F_1)$=1, $\varpi(F_2) = 2$ and $\varpi(F_3) = 5$.

Definition 3. *Consider a homogeneous vector field of the form* (2.1)*. Then a Laurent solution to* (2.1) *of the form*

$$x_i(t) = \frac{1}{t^{v_i}} \sum_{k=0}^{\infty} x_i^{(k)} t^k, \quad i = 1, 2, \ldots, n,$$

with $x^{(0)} \neq 0$, is called a homogeneous Laurent solution. The homogeneous vector field (2.1) *is said to satisfy the* Kowalevski–Painlevé property *if to each non trivial solution to the indicial equations there corresponds a homogeneous Laurent series solution depending on $n - 1$ free parameters.*

In our case since the $n = 3$ we impose the requirement that the Laurent solutions should involve two free parameters.

2.1. Kowalevski Exponents

The following proposition is important for two reasons. First, it gives an induction formula for finding the Laurent solution of a weight-homogeneous vector field and second it defines the Kowalevski exponents which is an important tool for our classification.

Proposition 1. *Suppose that we have a weight-homogeneous vector field on* $\mathbf{C}^n$ *given by*

$$\dot{x}_i = f_i(x_1, \ldots, x_n), \qquad i = 1, 2, \ldots, n,$$

and suppose that

$$x_i(t) = \frac{1}{t^{v_i}} \sum_{k=0}^{\infty} x_i^{(k)} t^k, \qquad i = 1, 2, \ldots, n \tag{2.4}$$

is a weight-homogeneous Laurent solution for this vector field. Then the leading coefficients, $x_i^{(0)}$*, satisfy the non linear algebraic equations*

$$\begin{gathered} v_1 x_1^{(0)} + f_1(x_1^{(0)}, \ldots, x_n^{(0)}) = 0, \\ \vdots \\ v_n x_n^{(0)} + f_n(x_1^{(0)}, \ldots, x_n^{(0)}) = 0, \end{gathered} \tag{2.5}$$

while the subsequent terms $x_i^{(k)}$ *satisfy*

$$\left(k\mathrm{Id}_n - \mathcal{K}\left(x^{(0)}\right)\right) x^{(k)} = R^{(k)}, \tag{2.6}$$

where $x^{(k)} = \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}$ *and* $R^{(k)} = \begin{pmatrix} R_1^{(k)} \\ \vdots \\ R_n^{(k)} \end{pmatrix}$. $R^{(k)}$ *is a polynomial, which depends on the variables* $x_1^{(l)}, \ldots, x_n^{(l)}$ *with* $0 \leqslant l < k$ *only. The elements of the* $n \times n$ *matrix* $\mathcal{K}$ *are given by*

$$\mathcal{K}_{i,j} := \frac{\partial f_i}{\partial x_j} + v_i \delta_{ij}, \tag{2.7}$$

where δ *is the Kronecker delta.*

Remark 1. The number, v_i, is not necessarily the pole order of x_i because some of the $x_i^{(0)}$ that can be calculated solving (2.5) may be equal to zero.

Definition 4.
The system (2.5) *is called the* indicial equation *and its solution set is called the* indicial locus *and it is denoted by* $\mathcal{I}$*. The* $n \times n$ *matrix* $\mathcal{K}$*, defined by* (2.7)*, is called the* Kowalevski matrix *and its eigenvalues are called* Kowalevski exponents *(a terminology due to Yoshida).*

A necessary condition for a vector field to satisfy the Kowalevski–Painlevé property is that $n-1$ eigenvalues of $\mathcal{K}$ should be integers for every solution of the indicial equation. It turns out that the last eigenvalue is always -1. The eigenvector that corresponds to -1 is also known. We have the following Proposition which can be found in [9].

Proposition 2.
For any m *which belongs to the indicial locus* $\mathcal{I}$*, except for the trivial element, the Kowalevski matrix* $\mathcal{K}(m)$ *of a weight homogeneous vector field always has* -1 *as an eigenvalue. The corresponding eigenspace contains* $(v_1 m_1, \ldots, v_n m_n)^T$ *as an eigenvector.*

3. PROPERTIES OF KOWALEVSKI EXPONENTS

In this section we state some properties of Kowalevski exponents clarifying the connection with the degrees of the first integrals. We also give a necessary condition for a system to posses the Kowalevski–Painlevé property. The following results can be found in [11, 13, 14, 19, 20].

Theorem 2. *If the weight-homogeneous system $\dot{x} = f(x)$ has k independent algebraic first integrals $I_1, \ldots, I_k$ of weighted degrees $d_1, \ldots, d_k$ and Kowalevski exponents $\rho_2, \ldots, \rho_n$, then there exists a $k \times (n-1)$ matrix $\mathcal{N}$ with integer entries, such that*

$$\sum_{j=2}^{n} \mathcal{N}_{ij}\, \rho_j = d_i, \qquad i = 1, \ldots, k.$$

From this theorem we have the two following corollaries:

Corollary 1.
If the Kowalevski exponents are $\mathbf{Z}$-independent, then there is no rational first integrals.

Corollary 2. *If the Kowalevski exponents are $\mathbf{N}$-independent, then there is no polynomial first integrals.*

We also have the following theorem, see [11, 19, 21].

Theorem 3. *Suppose that the system (1.1) possesses a homogeneous first integral F_m of degree m. Then there exists a set of non-negative integers $k_2, \ldots, k_n$ such that*

$$\sum_{j=2}^{n} k_j \rho_j = m \qquad k_2 + k_3 + \cdots + k_n \leqslant m.$$

The next theorem which can be found in [9] gives us a necessary condition for a system to satisfy the Kowalevski–Painlevé property. This criterion can be checked easily simply by computing the Kowalevski exponents.

Theorem 4.
Let $\rho_1 = -1$. A necessary condition for a system of the form (2.1) to satisfy the Kowalevski–Painlevé property is that all the Kowalevski exponents $\rho_2, \ldots, \rho_n$ should be integers for every solution of the indicial equation.

4. LOTKA–VOLTERRA SYSTEMS

4.1. Hamiltonian Formulation

Consider a Lotka–Volterra system of the form

$$\dot{x}_j = \sum_{k=1}^{n} a_{jk} x_j x_k, \quad \text{for} \quad j = 1, 2, \ldots, n, \tag{4.1}$$

where the matrix $A = (a_{ij})$ is constant and skew symmetric.

There is a symplectic realization of the system which goes back to Volterra. In other words a projection from $\mathbf{R}^{2n} \mapsto \mathbf{R}^n$ from a symplectic space to a Poisson space. Volterra defined the variables

$$q_i(t) = \int_0^t u_i(s)ds$$

and

$$p_i(t) = \ln(\dot{q}_i) - \frac{1}{2}\sum_{k=1}^{n} a_{ik} q_k,$$

for $i = 1, 2, \ldots, n$. Now the number of variables is doubled and Volterra's transformation is given explicitly by

$$x_i = e^{p_i + \frac{1}{2}\sum_{k=1}^{n} a_{ik} q_k} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

The Hamiltonian in these coordinates becomes

$$H = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \dot{q}_i = \sum_{i=1}^{n} \mathbf{e}^{p_i + \frac{1}{2}\sum_{k=1}^{n} a_{ik} q_k}.$$

The equations (4.1) can be written in Hamiltonian form

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} = \{q_i, H\}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} = \{p_i, H\}, \end{aligned}$$

$i = 1, 2, \ldots, n$, and the bracket $\{\cdot, \cdot\}$ is the standard symplectic bracket on $\mathbf{R}^{2n}$:

$$\{q_i, p_j\} = \delta_{ij} = \begin{cases} 1, \text{ if } & i = j \\ 0, \text{ if } & i \neq j \end{cases}, \quad i, j = 1, 2, \ldots, n\ ;$$

all other brackets are zero. The corresponding Poisson bracket in x coordinates is quadratic

$$\{x_i, x_j\} = a_{ij} x_i x_j, \quad i, j = 1, 2, \ldots, n\ .$$

This argument shows that the bracket (1.2) is Poisson.

Equations (4.1) in x coordinates are obtained by using this Poisson bracket and the Hamiltonian, $H = x_1 + x_2 + \cdots + x_n$.

4.2. The Three-dimensional Case

In this paper we restrict our attention to the three dimensional case. For $n = 3$ the system is defined by the matrix

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \tag{4.2}$$

where a, b, c are real constants.
Using equations (2.7) we obtain the Kowalevski matrix

$$\begin{pmatrix} ax_2^{(0)} + bx_3^{(0)} + 1 & ax_1^{(0)} & bx_1^{(0)} \\ -ax_2^{(0)} & -ax_1^{(0)} + cx_3^{(0)} + 1 & cx_2^{(0)} \\ -bx_3^{(0)} & -cx_3^{(0)} & -bx_1^{(0)} - cx_2^{(0)} + 1 \end{pmatrix}, \tag{4.3}$$

where $x^{(0)} = \left(x_1^{(0)},\ x_2^{(0)},\ x_3^{(0)}\right)$ is an element of the indicial locus, i.e., a solution of the simultaneous equation (2.5), which in this case is written as

$$\begin{aligned} x_1^{(0)} + ax_1^{(0)}x_2^{(0)} + bx_1^{(0)}x_3^{(0)} &= 0, \\ x_2^{(0)} - ax_1^{(0)}x_2^{(0)} + cx_2^{(0)}x_3^{(0)} &= 0, \\ x_3^{(0)} - bx_1^{(0)}x_3^{(0)} - cx_2^{(0)}x_3^{(0)} &= 0. \end{aligned} \tag{4.4}$$

The system is equivalent to

$$\begin{aligned} x_1^{(0)}\left(1+ax_2^{(0)}+bx_3^{(0)}\right) &= 0,\\ x_2^{(0)}\left(1-ax_1^{(0)}+cx_3^{(0)}\right) &= 0,\\ x_3^{(0)}\left(1-bx_1^{(0)}-cx_2^{(0)}\right) &= 0. \end{aligned}$$

One solution is

$$x_1^{(0)}=x_2^{(0)}=x_3^{(0)}=0.$$

This a trivial solution and the eigenvalues are all equal to 1.

If $x_i^{(0)} \neq 0$ for $i=1,2,3$. Then the system has a solution only if $b=a+c$. In this case the eigenvalues are $-1,0,1$. We will comment on this case at the end and for now we assume $b \neq a+c$.

If $x_1^{(0)}=0$, $x_2^{(0)} \neq 0$, $x_3^{(0)} \neq 0$ then we easily find that $x_2^{(0)}=\frac{1}{c}$, $x_3^{(0)}=-\frac{1}{c}$ and the eigenvalues of the Kovalevski matrix are

$$-1, 1, \frac{a-b+c}{c}.$$

We have two other similar cases corresponding to $x_2^{(0)}=0$ and $x_3^{(0)}=0$ which are summarized in Table 1. In the Table we list the corresponding Kowalevski exponents for each element of the indicial locus.

Table 1. Kowalevski exponents of 3x3 Lotka–Volterra equations

Vector $x^{(0)}$	Kowalevski exponents	Vector $x^{(0)}$	Kowalevski exponents
(0,0,0)	1,1,1	$(0,\frac{1}{c},-\frac{1}{c})$	$-1,1,\frac{a-b+c}{c}$
$(\frac{1}{b},0,-\frac{1}{b})$	$-1,1,-\frac{a-b+c}{b}$	$(\frac{1}{a},-\frac{1}{a},0)$	$-1,1,\frac{a-b+c}{a}$

A necessary condition for the system to have the Kowalevski–Painlevé property is that all the Kowalevski exponents must be integers for every solution of the indicial equation. So we have to solve the simultaneous Diophantine equations

$$\frac{a-b+c}{a}=k_1, \quad \frac{a-b+c}{c}=k_2, \quad -\frac{a-b+c}{b}=k_3, \tag{4.5}$$

where $k_1,\ k_2,\ k_3 \in \mathbf{Z}$. The case $b=a+c$ for which $k_1=k_2=k_3=0$ is investigated below. Solving (4.5) we find that

$$k_3=\frac{k_1k_2}{k_1k_2-k_1-k_2}, \quad \begin{cases} c=\frac{k_1}{k_2}a,\ b=\frac{k_1+k_2-k_1k_2}{k_2}a \\ a=\frac{k_2}{k_1}c,\ b=\frac{k_1+k_2-k_1k_2}{k_1}c \end{cases} \tag{4.6}$$

$$k_2=\frac{k_1k_3}{k_1k_3-k_1-k_3}, \quad \begin{cases} b=-\frac{k_1}{k_3}a,\ c=\frac{k_1+k_3-k_1k_3}{k_3}a \\ a=-\frac{k_3}{k_1}b,\ c=\frac{k_1+k_3-k_1k_3}{k_1}b \end{cases} \tag{4.7}$$

$$k_1 = \frac{k_2 k_3}{k_2 k_3 - k_2 - k_3}, \quad \begin{cases} b = -\frac{k_2}{k_3}c, \ a = \frac{k_2k_3-k_2-k_3}{k_3}c \\ c = -\frac{k_3}{k_2}b, \ a = \frac{k_2+k_3-k_2k_3}{k_2}b \end{cases} \tag{4.8}$$

We assume first, that the Kowalevski exponents are not zero. We examine the solution

$$k_3 = \frac{k_1 k_2}{k_1 k_2 - k_1 - k_2}, \quad b = \frac{k_1 + k_2 - k_1 k_2}{k_2} a, \quad c = \frac{k_1}{k_2} a.$$

We determine the values of k_1 and k_2 so that the fraction,

$$k_3 = \frac{k_1 k_2}{k_1 k_2 - k_1 - k_2}, \tag{4.9}$$

is an integer. We first consider the case $k_1 k_2 - k_1 - k_2 \neq 0$.

Case I. Assume positive values for both k_1 and k_2.
Since

$$\frac{k_1k_2}{k_1k_2-k_1-k_2} = 1 + \frac{k_1+k_2}{k_1k_2-k_1-k_2}$$

it is enough to to solve the Diophantine equation

$$\frac{x+y}{xy-x-y} = z$$

for x, y positive integers and $z \in \mathbf{Z}$.

Lemma 1. *Let $x, y \in \mathbf{Z}^+$ with $x \leqslant y$. Then*

$$\frac{x+y}{xy-x-y} \in \mathbf{Z}$$

if and only if (x, y) is one of the following: $(1, \lambda)$, $\lambda \in \mathbf{Z}^+$, $(2, 3)$, $(2, 4)$, $(2, 6)$, $(3, 3)$, $(3, 6)$, $(4, 4)$.

Proof. Since

$$xy - x - y \leqslant x + y$$

we have

$$xy \leqslant 2(x + y) \leqslant 4y.$$

Since $y \neq 0$ we get $x \leqslant 4$. Therefore $x = 1, 2, 3, 4$. We examine each case separately.

- If $x = 1$

$$\frac{x+y}{xy-x-y} = \frac{1+y}{-1} = -1 - y \in \mathbf{Z}.$$

  Therefore $(1, \lambda)$, $\lambda \in \mathbf{Z}^+$ is always a solution.

- Suppose $x = 2$. Then

$$\frac{x+y}{xy-x-y} = \frac{2+y}{y-2} = 1 + \frac{4}{y-2}$$

  should be an integer. Therefore $y - 2 = \pm 1, \pm 2, \pm 4$. We obtain the solutions $(2, 3)$, $(2, 4)$ and $(2, 6)$.

- Suppose $x = 3$. Then

$$\frac{x+y}{xy-x-y} = \frac{y+3}{2y-3}$$

  should be an integer. Therefore

$$2y - 3 \leqslant y + 3$$

  and we obtain $y \leqslant 6$. We obtain the solutions $(3, 3)$ and $(3, 6)$.

- Suppose $x = 4$. Then
$$\frac{x+y}{xy-x-y} = \frac{y+4}{4y-4}$$
should be an integer. Therefore
$$3y - 4 \leqslant y + 4$$
and we obtain $y \leqslant 4$. We obtain the solution $(4, 4)$.

Of course, since the fraction
$$\frac{x+y}{xy-x-y}$$
is symmetric with respect to x and y, we easily obtain all solutions in positive integers.

We summarize:

$$\text{For} \quad 1 \leqslant k_1 \leqslant k_2 \quad \begin{cases} k_1 = 1 & \Longrightarrow k_2 = \lambda \in \mathbf{Z}^+ \\ k_1 = 2 & \Longrightarrow k_2 \in \{3, 4, 6\} \\ k_1 = 3 & \Longrightarrow k_2 \in \{3, 6\} \\ k_1 = 4 & \Longrightarrow k_2 = 4. \end{cases} \tag{4.10}$$

Note that the case $k_1 = 3$, $k_2 = 3$ implies $k_3 = 3$ and we obtain the periodic KM-system $(1, -1, 1)$.

Case II.
Suppose one of them, say k_1, is positive while the other, k_2, is negative. Let $k_2 = -x,\ x > 0$. Then
$$k_3 = \frac{-k_1 x}{-k_1 x - k_1 + x} = \frac{k_1 x}{k_1 x + k_1 - x} = 1 + \frac{x - k_1}{k_1 x + k_1 - x}$$

It is enough to to solve the Diophantine equation
$$\frac{x-y}{xy+y-x} = z$$
for x, y positive integers and $z \in \mathbf{Z}$.

Lemma 2. *Let $x, y \in \mathbf{Z}^+$. Then*
$$\frac{x-y}{xy+y-x} \in \mathbf{Z}$$
if and only if (x, y) is of the form $(\lambda, 1)$ or (λ, λ) with $\lambda \in \mathbf{Z}^+$.

Proof. If $y = 1$ then
$$\frac{x-y}{xy+y-x} = x \in \mathbf{Z}.$$
Therefore a pair of the form $(\lambda, 1)$ is always a solution.

Assume $y > 1$. We note that
$$\frac{xy}{xy+y-x} = 1 + \frac{x-y}{xy+y-x}$$
and therefore $xy + y - x \leqslant xy$ implies $y \leqslant x$. If $x = y$ then our fraction is clearly an integer. On the other hand, if $y < x$, then the fraction
$$\frac{x-y}{xy+y-x} \notin \mathbf{Z}$$
since
$$(y-1)x + y \geqslant x + y > x - y.$$

If $k_1 = 1$ then $k_2 = -\lambda$ and $k_3 = \lambda$. Similarly, if $k_1 = \lambda$ then $k_2 = -\lambda$ and $k_3 = 1$. The two cases are isomorphic and correspond to Case 5 in Table 3.

Case III.

If we take negative values for both k_1 and k_2, then

$$k_3 = \frac{xy}{xy + x + y} = 1 - \frac{x + y}{xy + x + y},$$

where $k_1 = -x$ and $k_2 = -y$ with $x, y > 0$. We have that $xy > 0$ and $xy + x + y > 0$ so that the Kowalevski exponent is an integer if

$$xy + x + y \leqslant x + y$$

which implies $xy \leqslant 0$, a contradiction. Therefore, in this case k_3 cannot be an integer.

This completes the analysis of the case $k_1k_2 - k_1 - k_2 \neq 0$.

Now suppose $k_1k_2 - k_1 - k_2 = 0$.

In this case we have $k_1 + k_2 = k_1k_2$ and obviously (since we assume non-zero Kowalevski exponents) we must have $k_1 = k_2 = 2$. We easily obtain $a = c$ and $b = 0$. This system is equivalent to the open KM-system (also known as the Volterra lattice). This is Case 1 in Table 3.

This concludes our analysis. The results are summarized in Table 2. In Table 3 we also include the case of a zero exponent i.e. $b = a + c$. Note that the case $b = a + c$ which is equivalent to $(1, 1 + \mu, \mu)$ for $\mu \in \mathbf{R}$ was also considered in [1] from a different point of view.

4.3. Equivalence

In order to have a more compact classification, we define an equivalence between two Lotka–Volterra systems. To begin with, common factors can be removed. In other words, suppose that matrix $A = (a_{ij})$ in (4.1) has a common factor a. Precisely, if

$$a_{ij} = C_{ij}a, \quad \text{where } C_{ij} \in \mathbf{R}, \quad i, j = 1, 2, \ldots, n,$$

then the Lotka–Volterra system (4.1) can be simplified to

$$\dot{u}_i = \sum_{j=1}^{n} C_{ij}u_iu_j, \quad i = 1, 2 \ldots, n,$$

using the transformation

$$u_i = ax_i, \quad i = 1, 2 \ldots, n.$$

More generally, we consider two systems to be isomorphic if there exists an invertible linear transformation mapping one to the other. Special cases of isomorphic systems are those that are obtained from a given system by applying a permutation of the coordinates. Let $\sigma \in S_n$, and define a transformation

$$X_i \longmapsto x_{\sigma(i)}, \quad i = 1, 2, \ldots, n.$$

The transformed system is then considered equivalent to the original system. We illustrate with an example for $n = 3$.

Example 2. We prove that the system

$$\begin{aligned} \dot{x}_1 &= ax_1x_2 - \tfrac{a}{3}x_1x_3 \\ \dot{x}_2 &= -ax_1x_2 + \tfrac{2a}{3}x_2x_3 \\ \dot{x}_3 &= \tfrac{a}{3}x_1x_3 - \tfrac{2a}{3}x_2x_3 \end{aligned} \quad \xrightarrow[u_i = ax_i]{} \quad \begin{aligned} \dot{x}_1 &= 3x_1x_2 - x_1x_3 \\ \dot{x}_2 &= -3x_1x_2 + 2x_2x_3 \\ \dot{x}_3 &= x_1x_3 - 2x_2x_3 \end{aligned}$$

is isomorphic to the system

$$\begin{aligned} \dot{x}_1 &= ax_1x_2 - 2ax_1x_3 \\ \dot{x}_2 &= -ax_1x_2 + 3ax_2x_3 \\ \dot{x}_3 &= 2ax_1x_3 - 3ax_2x_3 \end{aligned} \quad \xrightarrow{\ u_i = ax_i\ } \quad \begin{aligned} \dot{x}_1 &= x_1x_2 - 2x_1x_3 \\ \dot{x}_2 &= -x_1x_2 + 3x_2x_3 \\ \dot{x}_3 &= 2x_1x_3 - 3x_2x_3 \end{aligned}$$

Applying $\sigma = (1\ 3\ 2)$ we have that

$$\begin{aligned} \dot{X}_1 = \dot{x}_{\sigma(1)} = \dot{x}_3 &= \quad x_1x_3 - 2x_2x_3 &&= \quad X_2X_1 - 2X_3X_1 \\ \dot{X}_2 = \dot{x}_{\sigma(2)} = \dot{x}_1 &= \quad 3x_1x_2 - x_1x_3 &&= \quad 3X_2X_3 - X_2X_1 \\ \dot{X}_3 = \dot{x}_{\sigma(3)} = \dot{x}_2 &= -3x_1x_2 + 2x_2x_3 &&= -3X_2X_3 + 2X_3X_1, \end{aligned}$$

which is the second vector field.

Example 3. Note that the system

$$\begin{aligned} \dot{x}_1 &= -x_2x_3 \\ \dot{x}_2 &= x_2x_3 \\ \dot{x}_3 &= x_1x_3 - x_2x_3 - x_3^2 \end{aligned}$$

is equivalent to the open KM-system $(1, 0, 1)$ under the transformation

$$(x_1, x_2, x_3) \to (x_2 + x_3, x_1, x_2)$$

but it is not a Lotka–Volterra system.

In Table 2 we display the different values of (a, b, c) of the solutions (4.6), (4.7) and (4.8) of the simultaneous equations (4.5) which ensure integer Kowalevski exponents for the Lotka–Volterra system in three dimensions. We also list the elements of the symmetric group S_3 which realize the isomorphism. Note that $\lambda \in \mathbf{Z} \setminus 0$. The final six non-isomorphic systems are displayed in Table 3.

Example 4. The periodic KM system ([22]) in three dimensions is the system

$$\dot{x}_i = \sum_{i=1}^{3} a_{ij}x_ix_j, \quad i = 1, 2, 3,$$

where A is the 3×3 skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

This system is a special case of the system (4.1) where $(a, b, c) = (-1, 1, -1)$. This is Case 2 in Table 3. The Kowalevski exponents of this system are $-1, 1, 3$. The system can be written in the Lax-pair form $\dot{L} = [L, B]$, where

$$L = \begin{pmatrix} 0 & x_1 & 1 \\ 1 & 0 & x_2 \\ x_3 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & x_1x_2 \\ x_2x_3 & 0 & 0 \\ 0 & x_1x_3 & 0 \end{pmatrix}.$$

We have the constants of motion

$$H_k = \operatorname{trace}\left(L^k\right), \quad k = 1, 2, \ldots$$

Table 2. Systems with integer Kowalevski exponents

Vector $(a,\ b,\ c)$	Kowalevski exponents	σ
$\left(a, \frac{a}{\lambda}, \frac{a}{\lambda}\right)$	$-1, 1, 1$	
$(a, a, \lambda a)$	$-1, 1, \lambda$	$\sigma = (1\ 3)$
$(a, \lambda a, -a)$	$-1, 1, -\lambda$	$\sigma = (2\ 3)$
$\left(a, -\frac{a}{2}, \frac{a}{2}\right)$	$-1, 1, 2$	
$(a, -a, 2a)$	$-1, 1, 4$	$\sigma = (1\ 3)$
$(a, -2a, a)$		$\sigma = (1\ 3\ 2)$
$(a, -a, a)$	$-1, 1, 3$	
$\left(a, -\frac{a}{3}, \frac{2a}{3}\right)$		$\sigma = (1\ 3\ 2)$
$\left(a, -\frac{2a}{3}, \frac{a}{3}\right)$	$-1, 1, 2$	$\sigma = (1\ 3)$
$\left(a, -\frac{3a}{2}, \frac{a}{2}\right)$	$-1, 1, 3$	$\sigma = (1\ 2\ 3)$
$\left(a, -\frac{a}{2}, \frac{3a}{2}\right)$	$-1, 1, 6$	$\sigma = (2\ 3)$
$(a, -2a, 3a)$		
$(a, -3a, 2a)$		$\sigma = (1\ 2)$
$(a, 0, a)$		
$(a, -a, 0)$	$-1, 1, 2$	$\sigma = (1\ 3\ 2)$
$(0, b, -b)$		$\sigma = (1\ 2\ 3)$

The functions

$$\begin{aligned} H_2 &= x_1 + x_2 + x_3 \\ H_3 &= 1 + x_1 x_2 x_3 \end{aligned}$$

are independent constants of motion in involution with respect to the Poisson bracket

$$\pi = \begin{pmatrix} 0 & -x_1x_2 & x_1x_3 \\ x_1x_2 & 0 & -x_2x_3 \\ -x_1x_3 & x_2x_3 & 0 \end{pmatrix}.$$

We note that the positive Kowalevski exponents, 1 and 3, correspond to the degrees of the constants of motion.

We have to point out that all Lotka–Volterra systems in three dimensions are integrable in the sense of Liouville since there exist two constants of motion which are independent and in involution. The function

$$H = x_1 + x_2 + x_3$$

is the Hamiltonian for these systems using the quadratic Poisson bracket

$$\pi = \begin{pmatrix} 0 & ax_1x_2 & bx_1x_3 \\ -ax_1x_2 & 0 & cx_2x_3 \\ -bx_1x_3 & -cx_2x_3 & 0 \end{pmatrix}.$$

Table 3. Free parameters of Lotka–Volterra systems

Vector	**Kowalevski exponents**	**Free parameters**	Degree of invariant
$(a, 0, a)$	$-1, 1, 2$	$x_3^{(1)},\ \ x_3^{(2)}$	2
$(a, -a, a)$	$-1, 1, 3$	$x_3^{(1)},\ \ x_3^{(3)}$	3
$\left(a, -\frac{a}{2}, \frac{a}{2}\right)$	$-1, 1, 2$ $-1, 1, 4$	$x_3^{(1)},\ \ x_3^{(2)}$	4
$(a, -2a, 3a)$	$-1, 1, 2$ $-1, 1, 3$ $-1, 1, 6$	$x_3^{(1)},\ \ x_3^{(2)}$	6
$\left(a, \frac{a}{\lambda}, \frac{a}{\lambda}\right)$	$-1, 1, 1$ $-1, 1, \lambda$ $-1, 1, -\lambda$	$x_1^{(1)},\ \ x_2^{(1)}$	λ
$(a, a+c, c)$	$-1, 1, 0$	$x_1^{(1)},\ \ x_3^{(0)}$	0

The equations of motion can be written in Hamiltonian form

$$\dot{x}_i = \{x_i, H\}, \quad i = 1, 2, 3.$$

The second constant of motion, independent of H always exists. It is straightforward to check that the function

$$F = x_1^c x_2^{-b} x_3^a$$

is always a Casimir. Therefore the system is Liouville integrable for any value of a, b, c. This is not the case if $n \geqslant 4$.

5. FREE PARAMETERS

We would like to classify the Lotka–Volterra equations in three dimensions which satisfy the Kowalevski–Painlevé property. In order to use Proposition 1 we have to assume Laurent solutions of the form

$$x_i(t) = \frac{1}{t^{v_i}} \sum_{k=0}^{\infty} x_i^{(k)} t^k, \qquad i = 1, 2, 3 \tag{5.1}$$

where v_i are the components of the weight vector v that makes the vector field

$$\dot{x}_i = f_i(x_1, x_2, x_3), \qquad i = 1, 2, 3,$$

to be weight homogeneous. In our case the weight vector is $v = (1, 1, 1)$.

To make sure that our classification is complete we must check that each system obtained satisfies the Kowalevski–Painlevé property. This means that the Laurent series of the solutions $x_1,\ x_2$ and x_3 must have $n - 1 = 2$ free parameters. Using the results in [9], the free parameters appear in a finite number of steps of calculation. The first thing to do is to substitute (5.1) into equations (4.1). After that we equate the coefficients of t^k. We have already equated the coefficients of t^{-v_i-1} by solving the indicial equation to find $x_i^{(0)}$. Then we call *Step m* ($m \in \mathbf{N}$) when we equate the coefficients of

t^{-v_i-1+m} to find $x_i^{(m)}$. According to [9] all the free parameters appear in the first k_p *Steps*, where k_p is the largest (positive) Kowalevski exponent of the system. The calculations are straightforward and we omit the details.

All the systems that we have obtained turn out to satisfy the Kowalevski–Painlevé property. We summarize the results in Table 3 where we display the 2 free parameters in each case. We note that the six Cases of Theorem 1 are non-isomorphic by examining the degree of the Casimir.

6. HIGHER ORDER POLES

In our classification, using the Kowalevski exponents, we assume that the order of the poles agrees with the components of the weight vector, in our case all equal to 1. We have to exclude the possibility of missing some cases due to solutions with higher order poles. We show that no such new cases appear.

Suppose that the Laurent solution of the system is

$$\begin{aligned} x_1(t) &= \frac{1}{t^{\nu_1}} \sum_{k=0}^{\infty} x_1^{(k)} t^k, \text{ with } x_1^{(0)} \neq 0, \\ x_2(t) &= \frac{1}{t^{\nu_2}} \sum_{k=0}^{\infty} x_2^{(k)} t^k, \text{ with } x_2^{(0)} \neq 0, \\ x_3(t) &= \frac{1}{t^{\nu_3}} \sum_{k=0}^{\infty} x_3^{(k)} t^k, \text{ with } x_3^{(0)} \neq 0. \end{aligned} \tag{6.1}$$

If $\nu_1, \nu_2, \nu_3 \leqslant 1$, then these systems have been already investigated using Proposition 1. On the other hand, keeping in mind that $H = x_1 + x_2 + x_3$ is always a constant of motion, we end-up with the following four cases to consider:

(i) $\nu_1 = \nu_2 = \nu > 1$ and $\nu_3 < \nu$, or

(ii) $\nu_1 = \nu_3 = \nu > 1$ and $\nu_2 < \nu$, or

(iii) $\nu_2 = \nu_3 = \nu > 1$ and $\nu_1 < \nu$, or

(iv) $\nu_1 = \nu_2 = \nu_3 = \nu > 1$.

Recall that equations (4.1) in three dimensions are:

$$\dot{x}_1 = \quad a x_1 x_2 + b x_1 x_3, \tag{6.2}$$

$$\dot{x}_2 = -a x_1 x_2 + c x_2 x_3, \tag{6.3}$$

$$\dot{x}_3 = -b x_1 x_3 - c x_2 x_3. \tag{6.4}$$

We examine each of the four cases:

(*i*) $\boxed{\nu_1 = \nu_2 = \nu > 1 \text{ and } \nu_3 < \nu}$ Since $\nu_1 = \nu_2 = \nu$ and using the fact that $H = x_1 + x_2 + x_3$ is a constant of motion we have that

$$x_1^{(0)} = -x_2^{(0)} = \alpha \neq 0.$$

We also note that $\nu + \nu_3 < 2\nu$ and $x_1^{(0)} x_2^{(0)} \neq 0$. Equating the coefficients of $t^{2\nu}$ of the LHS and RHS of (6.2) or (6.3), we are led to $a\, x_1^{(0)} x_2^{(0)} = 0$. Therefore $a = 0$.

As we know that $\nu_3 + 1 < \nu + \nu_3$, the coefficient of $t^{\nu+\nu_3}$ of the RHS of (6.4) must be equal to zero. So

$$x_3^{(0)}\left(-bx_1^{(0)} - cx_2^{(0)}\right) = 0,$$

but $x_3^{(0)} \neq 0$ and $x_2^{(0)} = -x_1^{(0)} \neq 0$; therefore $b = c$.

If $b = 0$, then from (6.2) and (6.3) we have that

$$\dot{x}_1 = \dot{x}_2 = 0 \Longrightarrow x_1,\ x_2 \text{ are constant functions},$$

that is a contradiction because $\nu_1 = \nu_2 = \nu > 1$.

If b and c are non-zero, then the equations (6.2) and (6.3) become

$$\dot{x}_1 = bx_1x_3, \tag{6.5}$$

$$\dot{x}_2 = bx_2x_3. \tag{6.6}$$

Using equation (6.2) we obtain

$$\nu + 1 = \nu + \nu_3$$

therefore $\nu_3 = 1$, since $x_1^{(0)}x_3^{(0)} \neq 0$ and $x_2^{(0)}x_3^{(0)} \neq 0$.

It follows from (6.5) and (6.6) that

$$\frac{\dot{x}_1}{x_1} = \frac{\dot{x}_2}{x_2} = bx_3 \Longrightarrow x_1 = \kappa x_2, \quad \kappa \text{ is a constant.}$$

However, we know that

$$x_1^{(0)} = -x_2^{(0)} \Longrightarrow \kappa = -1 \Rightarrow x_1 = -x_2.$$

Equation (6.4) becomes

$$\dot{x}_3 = -b(-x_2)x_3 - bx_2x_3 = 0 \Longrightarrow x_3 = c,\ c \text{ is a constant.}$$

This is a contradiction since $\nu_3 = 1$ and $x_3^{(0)} \neq 0$.

(*ii*) $\boxed{\nu_1 = \nu_3 = \nu > 1 \text{ and } \nu_2 < \nu}$

It leads to a contradiction, as in case (*i*).

(*iii*) $\boxed{\nu_2 = \nu_3 = \nu > 1 \text{ and } \nu_1 < \nu}$

It leads to a contradiction, as in case (*i*).

(*iv*) $\boxed{\nu_1 = \nu_2 = \nu_3 = \nu > 1}$

In this case, for $i = 1, 2, 3$,

$$x_i(t) = \frac{1}{t^\nu}\sum_{k=0}^{\infty} x_i^{(k)} t^k,$$

we have that the degrees of the leading term of the LHS of the equations (6.2), (6.3) and (6.4) are equal to $\nu + 1$, but the degrees of the leading term RHS of these equations are equal to 2ν and so the coefficients of $\frac{1}{t^{\nu+k}}$ of the RHS of these equations must be zero for $k = 2, 3, \ldots, \nu$.

The coefficients of $\frac{1}{t^{\nu+k}}$, $k = 1, 2, \ldots, \nu$, are given by the sums

$$S_{i,k} = \sum_{\lambda=0}^{\nu-k} x_i^{(\lambda)} u_{i,k}^{(\lambda)}, \text{ for } i = 1, 2, 3, \tag{6.7}$$

where

$$\begin{aligned} u_{1,k}^{(\lambda)} &= ax_2^{(\nu-k-\lambda)} + bx_3^{(\nu-k-\lambda)}, \\ u_{2,k}^{(\lambda)} &= -ax_1^{(\nu-k-\lambda)} + cx_3^{(\nu-k-\lambda)}, \\ u_{3,k}^{(\lambda)} &= -bx_2^{(\nu-k-\lambda)} - cx_3^{(\nu-k-\lambda)}. \end{aligned}$$

Note that

$$u_{i,k}^{(\lambda)} = u_{i,j}^{(m)}, \text{ if } \ k + \lambda = j + m. \tag{6.8}$$

In addition

$$S_{i,k} = 0, \text{ for } i = 1, 2, 3 \text{ and } k = 2, 3, \ldots, \nu.$$

For $k = n$ sum (6.7) becomes

$$S_{i,\nu} = x_i^{(0)} u_{i,\nu}^{(0)} = 0 \Longrightarrow u_{i,\nu}^{(0)} = 0$$

since $x_i^{(0)} \neq 0$.

For $k = \nu - 1$ we have that

$$\begin{aligned} S_{i,\nu-1} &= x_i^{(0)} u_{i,\nu-1}^{(0)} + x_i^{(1)} u_{i,\nu-1}^{(1)} = 0 \\ (6.8) &\Rightarrow x_i^{(0)} u_{i,\nu-1}^{(0)} + x_i^{(1)} u_{i,\nu}^{(0)} = x_i^{(0)} u_{i,\nu-1}^{(0)} = 0 \\ &\Rightarrow u_{i,\nu-1}^{(0)} = 0 \text{ because } \ x_i^{(0)} \neq 0. \end{aligned}$$

Let $m \in \{1, 2, \ldots, \nu - 1\}$ and assume that $u_{i,k}^{(0)} = 0$ for $k > m$.

For $k = m$ we have that

$$\begin{aligned} S_{i,m} &= \sum_{\lambda=0}^{\nu-m} x_i^{(\lambda)} u_{i,m}^{(\lambda)} = x_i^{(0)} u_{i,m}^{(0)} + \sum_{\lambda=1}^{\nu-m} x_i^{(\lambda)} u_{i,m}^{(\lambda)} \\ &= x_i^{(0)} u_{i,m}^{(0)} + \sum_{\lambda=1}^{\nu-m} x_i^{(\lambda)} u_{i,m+\lambda}^{(0)} = x_i^{(0)} u_{i,m}^{(0)}. \end{aligned}$$

Since $S_{i,m} = 0$ for $m > 1$ and, since $x_i^{(0)} \neq 0$, then $u_{i,m}^{(0)} = 0$.

Now we equate the coefficients of $\frac{1}{t^{\nu+1}}$ on both sides of the equations (6.2)–(6.4) to obtain

$$S_{i,1} = x_i^{(0)} u_{i,1}^{(0)} = -\nu x_i^{(0)}.$$

Therefore, $\nu + u_{i,1}^{(0)} = 0$.

Therefore we have that

$$\begin{aligned} ax_2^{(\nu-1)} + bx_3^{(\nu-1)} &= -\nu, \\ -ax_1^{(\nu-1)} + cx_3^{(\nu-1)} &= -\nu, \\ -bx_2^{(\nu-1)} - cx_3^{(\nu-1)} &= -\nu. \end{aligned} \tag{6.9}$$

These simultaneous equations have solutions only if

$$b = a + c.$$

If $a = 0$, Then $b = c$ (obviously $b = c \neq 0$). Then the system is isomorphic to the following $(0, 1, 1)$ system:

$$\begin{aligned} \dot{x}_1 &= x_1x_3 \\ \dot{x}_2 &= x_2x_3 \\ \dot{x}_3 &= -x_1x_3 - x_2x_3 \end{aligned}$$

Equating the coefficients of $t^{-2\nu}$ $(\nu > 1)$ in the first and second equations we have that

$$x_1^{(0)}x_3^{(0)} = x_2^{(0)}x_3^{(0)} = 0.$$

This is impossible because $x_i^{(0)} \neq 0$, for $i = 1, 2, 3$.

The same happens if $bc = 0$. So in the following calculations we assume that $abc \neq 0$.

We will show that there exists no such solution with $\nu \geqslant 2$. Since $b = a + c$ and the function $H = x_1 + x_2 + x_3$ is a constant of motion, the Lotka–Volterra equations in three dimensions can be written in the form

$$\begin{aligned} \dot{x}_1 &= \quad akx_1 - ax_1^2 + cx_1x_3, \\ \dot{x}_2 &= -\dot{x}_1 - \dot{x}_3, \\ \dot{x}_3 &= -ckx_3 + cx_3^2 - ax_1x_3, \end{aligned} \tag{6.10}$$

where k is the constant value of the function H. It is straightforward to see that if $k \neq 0$, then the solution is

$$x_1 = \frac{kC_1\mathbf{e}^{akt}}{C_1\mathbf{e}^{akt} + a\mathbf{e}^{-ckt} - C_2}, \; x_3 = \frac{ka\mathbf{e}^{-ckt}}{C_1\mathbf{e}^{akt} + a\mathbf{e}^{-ckt} - C_2},$$

$$x_2 = k - x_1 - x_3 = -\frac{kC_2}{C_1\mathbf{e}^{akt} + a\mathbf{e}^{-ckt} - C_2}. \tag{6.11}$$

Obviously $C_2 \neq 0$. The pole t_* satisfies

$$C_1\mathbf{e}^{akt_*} + a\mathbf{e}^{-ckt_*} - C_2 = 0 \Rightarrow C_2 = C_1\mathbf{e}^{akt_*} + a\mathbf{e}^{-ckt_*} \neq 0$$

Hence using De l' Hôpital Rule we are led to the fact that

$$\lim_{t\to t_*}(t - t_*)x_2(t) = \frac{C_2}{aC_1\mathbf{e}^{akt_*} - ac\mathbf{e}^{-ckt_*}}.$$

Since the pole order is greater than 1, we have that

$$\lim_{t\to t_*}(t - t_*)x_2(t) = \infty.$$

Therefore

$$C_1 = c\mathbf{e}^{-(a+c)kt} = c\mathbf{e}^{-bkt}$$

The solution (6.11) possesses only one arbitrary constant k, but we need 2.

Now if $k = 0$ the solutions of (6.10) are

$$x_3(t) = 0, \quad x_1(t) = \frac{1}{at + C_1}\ ,$$

or

$$x_1(t) = \frac{C_1 - c}{a(C_1 t + C_2)}, \quad x_3(t) = -\frac{1}{C_1 t + C_2}.$$

Both solutions lead to a contradiction since the pole order of x_1 and x_3 is assumed to be greater than 1.

Therefore the case $\nu_1 = \nu_2 = \nu_3 = \nu > 1$ does not give us any new cases. The conclusion is that the case $b = a + c$ gives a system satisfying the Kowalevski–Painlevé property only when $\nu_1 = \nu_2 = \nu_3 = 1$.

ACKNOWLEDGMENTS

We thank Tassos Bountis for introducing us to this area of research during a short course he gave at the University of Cyprus. We thank Pol Vanhaecke for useful discussions and for pointing out the need to exclude higher order poles.

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 330–355. © Pleiades Publishing, Ltd., 2011.

Algebraic Integrability and Geometry of the $\mathfrak{d}_3^{(2)}$ Toda Lattice

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Received December 15, 2009; accepted August 21, 2010

Abstract—In this paper, we consider the Toda lattice associated to the twisted affine Lie algebra $\mathfrak{d}_3^{(2)}$. We show that the generic fiber of the momentum map of this system is an affine part of an Abelian surface and that the flows of integrable vector fields are linear on this surface, so that the system is algebraic completely integrable. We also give a detailed geometric description of these Abelian surfaces and of the divisor at infinity. As an application, we show that the lattice is related to the Mumford system and we construct an explicit morphism between these systems, leading to a new Poisson structure for the Mumford system. Finally, we give a new Lax equation with spectral parameter for this Toda lattice and we construct an explicit linearization of the system.

MSC2010 numbers: 34G20, 34M55, 37J35

DOI: 10.1134/S1560354711030087

Keywords: Toda lattice, integrable systems, algebraic integrability, Abelian surface

Contents

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1. INTRODUCTION

The classical Toda lattice is a system of particles of unit mass, connected by exponential springs. It is a fundamental example of a finite-dimensional integrable Hamiltonian system. It has various connections with other parts of mathematics and physics [1]. Its equations of motion are derived from the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} \exp(q_j - q_{j+1}), \tag{1.1}$$

where q_j is the position of the j-th particle and p_j is its momentum. This type of Hamiltonian was considered first by Morikazu Toda [2]. The equation (1.1) is known as the classical finite non-periodic Toda lattice to distinguish the system from various other versions. The periodic version of (1.1) is given by

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n} \exp(q_j - q_{j+1}), \quad q_{n+1} = q_1, \tag{1.2}$$

whose equations of motion are given by

$$\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} = p_j, \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} = \exp(q_{j-1} - q_j) - \exp(q_j - q_{j+1}), \end{aligned} \qquad 1 \leqslant j \leqslant n.$$

The integrability of the periodic Toda lattice was established by Hénon [3] and Flaschka [4] using the method of Lax pairs. The equations of motion were solved by quadratures by Kac and van Moerbeke [5] and latter integrated by Krichever [6] in terms of theta functions by using algebro-geometric methods. There now exists an extensive literature on the problem. Properties of eigenvectors of Lax pairs over their associated spectral curve were studied in classical papers by Adler and van Moerbeke [7–9] and Mumford and van Moerbeke [10]. In 1976, Bogoyavlensky [11] introduced a generalization of the classical periodic Toda lattice to arbitrary affine Lie algebras. Adler and van Moerbeke [12], after some changes of variables, have shown that the Hamiltonian vector field $\mathcal{X}_H$ associated to the general form of the Hamiltonian H corresponding to an affine Lie algebra $\mathfrak{g}$ of rank l, takes the simple form

$$\begin{cases} \dot{x} = x \cdot y, \\ \dot{y} = Ax, \end{cases} \tag{1.3}$$

where $x, y \in \mathbb{C}^{l+1}$, $x \cdot y = (x_0 y_0, \ldots, x_l y_l)$ and A is the Cartan matrix of $\mathfrak{g}$.

A main tool in the study of the integrable systems is the use of formal Laurent solutions of the differential equations that describe the vector fields. Thus, S. Kowalevski discovered her top by searching families of Laurent solutions depending on a maximal number of free parameters. This technique was further developed by Adler and Moerbeke who applied it to discover several other cases of interest. At the same time, they introduced the notion of algebraic integrability. We recall that an Abelian variety is a complex torus $\mathbb{C}^n/\Lambda$, where Λ is a lattice in $\mathbb{C}^n$, that is algebraic, which means that it admits an embedding in some projective space $\mathbb{P}^N$. An integrable system $(\mathbb{C}^n, \{\cdot,\cdot\}, \mathbb{F})$, where $\mathbb{F} = (F_1, \ldots, F_s)$, will be called algebraic completely integrable (a.c.i.) if for generic $\mathbf{c} = (c_1, \ldots, c_s)$ in $\mathbb{C}^s$ the invariant manifold

$$\mathcal{A}_{\mathbf{c}} = \bigcap_{i=1}^{s} \{m \in \mathbb{C}^n : F_i(m) = c_i\},$$

thought of as a (non-compact) affine variety in $\mathbb{C}^n$, can be completed into a complex algebraic torus $\mathbb{T}^n_{\mathbf{c}}$ as follows $\mathcal{A}_{\mathbf{c}} = \mathbb{T}^n_{\mathbf{c}} \backslash \mathcal{D}_{\mathbf{c}}$, where $\mathcal{D}_{\mathbf{c}}$ is a divisor (one or several hypersurfaces) in $\mathbb{T}^n_{\mathbf{c}}$. Moreover, the Hamiltonian vector fields $\mathcal{X}_{F_i}$ are translation invariant when restricted to these tori. It turns out that many (most) of integrable systems that were known classically, turn out to be a.c.i., when complexified. This means that the powerful tools of the theory of Abelian varieties can be used to solve and study these systems.

Adler, van Moerbeke and Vanhaecke in [12] have shown that every a.c.i. system admits Laurent solutions depending on a maximal number of free parameters; which justifies the criterion used by Kowalevski to find her top. In this line, they have developped a method (algorithm) [12], based on the formal Laurent solutions, to prove the algebraic integrability of an integrable system. This algorithm is based on the Complex Liouville Theorem (see Theorem 1 below). They used it to show that a number of well-known systems are a.c.i., like the geodesic flow on $SO(4)$, the Henon-Heiles system and the famous Kowalevski top. We will use also later this algorithm to show that the Toda lattice which we consider in this paper, is an a.c.i. system. Finally, the Laurent solutions are used to further explore the geometry of the invariant manifolds.

A charaterization of periodic Toda lattices is given in [12]. Indeed, Adler and van Moerbeke have shown that if the vector field $\mathcal{X}_H$ in (1.3) is a vector field of an a.c.i. system, then A is the Cartan matrix of a (twisted) affine Lie algebra. Conversely, if A is a such matrix, $\mathcal{X}_H$ is a vector field of a Liouville integrable system. It is conjectured that all these integrable systems are a.c.i. Indeed, these systems satisfy the Linearising Criterion [12, Theorem 6.41]. The fact that the system (1.3) is a.c.i. was shown for particular cases of Lie algebras, but not in general. Thus, we can find the proof of the algebraic integrability of all $\mathfrak{a}_l^{(1)}$, for $l \geqslant 1$, in [10]. For the periodic Toda lattice with two degrees of freedom, there exist six cases where the system (1.3) could be a.c.i. for which the Cartan matrix correspond to the (possibly twisted) affine Lie algebras $\mathfrak{a}_2^{(1)}, \mathfrak{d}_3^{(2)}, \mathfrak{c}_2^{(1)}, \mathfrak{a}_4^{(2)}, \mathfrak{g}_2^{(1)}$ and $\mathfrak{d}_4^{(2)}$. As said above, the $\mathfrak{a}_2^{(1)}$ Toda lattice is a.c.i.; see [12] for an alternative proof.

This paper deals with the study of the algebraic integrability and the geometry of the Toda lattice associated to the twisted affine Lie algebra $\mathfrak{d}_3^{(2)}$. We briefly describe the $\mathfrak{d}_3^{(2)}$ Toda lattice. The differential equations of this lattice are given on the five dimensional hyperplane $\mathcal{H} = \left\{(x_0, x_1, x_2, y_0, y_1, y_2) \in \mathbb{C}^6 \mid y_0 + y_1 + y_2 = 0\right\}$ of $\mathbb{C}^6$ by

$$\begin{cases} \dot{x} = x \cdot y, \\ \dot{y} = Ax, \end{cases} \tag{1.4}$$

where $x = (x_0, x_1, x_2)^\top, y = (y_0, y_1, y_2)^\top$, and A is the Cartan matrix of the twisted affine Lie algebra $\mathfrak{d}_3^{(2)}$ given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \tag{1.5}$$

and three independent constants of motion are given by

$$\begin{aligned} F_1 &= y_0^2 + y_2^2 - 4x_0 - 2x_1 - 4x_2, \\ F_2 &= (y_0^2 - 4x_0)(y_2^2 - 4x_2) - x_1(2y_0y_2 - 4x_0 - x_1 - 4x_2), \\ F_3 &= x_0x_1x_2. \end{aligned} \tag{1.6}$$

For $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{C}^3$, let $\mathbb{F}_\mathbf{c}$ be the complex affine variety defined by the intersection of the constants of motion

$$\mathbb{F}_\mathbf{c} = \bigcap_{i=1}^{3} \{m \in \mathcal{H} \mid F_i(m) = c_i\},$$

and

$$\begin{aligned} \Omega := \{\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{C}^3 \mid c_3 \neq 0,\ c_1^2 - 4c_2 \neq 0 \quad \text{and} \\ 6912c_3^2 + 288c_1c_2c_3 + 4c_2^3 - c_1^2c_2^2 - 64c_1^3c_3 \neq 0\}. \end{aligned} \tag{1.7}$$

The main result of this paper (see Section 4) is the following.

Theorem. Let $(\mathcal{H}, \{\cdot,\cdot\}, \mathbb{F})$ denote the integrable system that describes the $\mathfrak{d}_3^{(2)}$ Toda lattice, where $\mathbb{F} = (F_1, F_2, F_3)$ is given by (1.6).

1. $(\mathcal{H}, \{\cdot,\cdot\}, \mathbb{F})$ is an algebraic completely integrable system;
2. For $\mathbf{c} = (c_1, c_2, c_3) \in \Omega$, the invariant surface $\mathbb{F}_{\mathbf{c}}$ is isomorphic to $\mathbb{T}^2_{\mathbf{c}} \backslash \mathcal{D}_{\mathbf{c}}$, where
   (a) $\mathbb{T}^2_{\mathbf{c}}$ is the Jacobian of the hyperelliptic curve (of genus two) $\overline{\Gamma}^{(0)}_{\mathbf{c}}$, defined by
   $$a^4e^2 - (c_1e + 8c_3)a^2e - 4e^3 + c_2e^2 + 4c_1c_3e + 16c_3^2 = 0;$$
   (b) $\mathcal{D}_{\mathbf{c}}$ is a divisor on $\mathbb{T}^2_{\mathbf{c}}$, and consists of three irreducible components $\mathcal{D}^{(0)}_{\mathbf{c}}$, $\mathcal{D}^{(1)}_{\mathbf{c}}$ and $\mathcal{D}^{(2)}_{\mathbf{c}}$ where $\mathcal{D}^{(0)}_{\mathbf{c}}$ and $\mathcal{D}^{(2)}_{\mathbf{c}}$ are both smooth curves, isomorphic to $\overline{\Gamma}^{(0)}_{\mathbf{c}}$, while $\mathcal{D}^{(1)}_{\mathbf{c}}$ is a curve of genus three, with two singular points, defined by
   $$32\alpha\beta^2 + (16\alpha^4 - 8c_1\alpha^2 + c_1^2 - 4c_2)\beta - 32c_3\alpha = 0.$$
   (c) The curves $\mathcal{D}^{(0)}_{\mathbf{c}}$ and $\mathcal{D}^{(2)}_{\mathbf{c}}$ intersect each other transversally in two points, and each of them intersects the curve $\mathcal{D}^{(1)}_{\mathbf{c}}$ at one singular point of the latter.

In this paper, we also establish a link between the $\mathfrak{d}_3^{(2)}$-Toda lattice and the Mumford system [13]. By using a method due to Vanhaecke [14], we construct an explicit morphism between these two systems. Thus, we obtain a new Poisson structure (see Section 6) for the Mumford system and then derive a new Lax equation with spectral parameter for the $\mathfrak{d}_3^{(2)}$ Toda lattice. These results lead to an explicit linearization of the Toda lattice considered.

This paper is organized as follows. In Section 2 we verify that the $\mathfrak{d}_3^{(2)}$ Toda lattice is Liouville integrable. In Section 3 we do the Painlevé analysis of the system. This analysis shows that our integrable system admits three principal balances i.e. three families of Laurent solutions depending on the maximal number of free parameters, to wit four in our case. Thus, by confining each family of Laurent solutions to the invariant manifolds we calculate the Painlevé divisors associated to these principal balances and, for $\mathbf{c} \in \Omega$, we give an explicit embedding of the invariant manifold $\mathbb{F}_{\mathbf{c}}$ in the projective space $\mathbb{P}^{15}$,

$$\begin{aligned} \varphi_{\mathbf{c}} : \qquad \mathbb{F}_{\mathbf{c}} \qquad & \to \mathbb{P}^{15} \\ (x_0, x_1, x_2, y_0, y_1, y_2) & \mapsto (1 : z_1 : \cdots : z_{15}), \end{aligned} \tag{1.8}$$

where the functions z_i behave like $\frac{1}{t}$ at worst when any of the principal balances is substituted in them.

In the Section 4, which is the main part of the paper, we prove the above theorem. Section 5 deals with other elements of the geometry of the system. We determine the positions of the half-periods on the Abelian surface and the tangency locus of the holomorphic vector fields on this surface. Finally in Section 6 we study the connection of our system with the Mumford system and give a morphism between the two. Moreover, we give a new 2 by 2 Lax pair for the system. We finish by giving an explicit linearization of the Toda lattice.

2. LIOUVILLE INTEGRABILITY OF THE $\mathfrak{d}_3^{(2)}$ TODA LATTICE

The equations of motion $\mathfrak{d}_3^{(2)}$ periodic Toda lattice take the following form

$$\begin{aligned} \dot{x}_0 &= x_0 y_0, & \dot{y}_0 &= 2x_0 - x_1, \\ \dot{x}_1 &= x_1 y_1, & \dot{y}_1 &= -2x_0 + 2x_1 - 2x_2, \\ \dot{x}_2 &= x_2 y_2, & \dot{y}_2 &= 2x_2 - x_1. \end{aligned} \tag{2.1}$$

on the five dimension hyperplane $\mathcal{H}$. These differential equations come from (1.3), where A is the Cartan matrix of the twisted affine Lie algebra $\mathfrak{d}_3^{(2)}$ given in (1.5). We denote by $\mathcal{V}_1$ the vector field defined by the above differential equations (2.1). The field $\mathcal{V}_1$ is the Hamiltonian vector field, with Hamiltonian function

$$F_1 = y_0^2 + y_2^2 - 4x_0 - 2x_1 - 4x_2,$$

with respect to the Poisson structure $\{\cdot,\cdot\}$ defined by the following skew-symmetric matrix

$$M := \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & x_0 & -x_0 & 0 \\ 0 & 0 & 0 & -x_1 & 2x_1 & -x_1 \\ 0 & 0 & 0 & 0 & -x_2 & x_2 \\ -x_0 & x_1 & 0 & 0 & 0 & 0 \\ x_0 & -2x_1 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & -x_2 & 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

This Poisson structure is given on $\mathbb{C}^6$; the function $y_0 + y_1 + y_2$ is a Casimir, so that the hyperplane $\mathcal{H}$ is a Poisson subvariety. The rank of the Poisson structure $\{\cdot,\cdot\}$ is 0 on the three-dimensional subspace $\{x_0 = x_1 = x_2 = 0\}$; the rank is 2 on the three four-dimensional subspaces: $\{x_0 = x_1 = 0\}$, $\{x_0 = x_2 = 0\}$ and $\{x_1 = x_2 = 0\}$. Thus, for all points of $\mathcal{H}$ except the four subspaces above the rank is 4. The vector field $\mathcal{V}_1$ admits also the two constants of motion, to wit

$$\begin{aligned} F_2 &= (y_0^2 - 4x_0)(y_2^2 - 4x_2) - x_1(2y_0y_2 - 4x_0 - x_1 - 4x_2), \\ F_3 &= x_0x_1x_2. \end{aligned} \tag{2.3}$$

It is easy to check that F_3 is a Casimir for $\{\cdot,\cdot\}$, and that the function F_2 generates a second Hamiltonian vector field $\mathcal{V}_2$, which commutes with $\mathcal{V}_1$, given by the differential equations

$$\begin{aligned} x_0' &= x_0y_2(y_0y_2 - x_1) - 4x_0x_2y_0, \\ x_1' &= x_1(y_0 + y_2)(x_1 - y_0y_2) + 4x_1(x_2y_0 + x_0y_2), \\ x_2' &= x_2y_0(y_0y_2 - x_1) - 4x_0x_2y_2, \\ y_0' &= 2(x_1x_2 + x_0y_2^2) - 8x_0x_2 + x_1(x_1 - y_0y_2), \\ y_1' &= 2x_1(y_0y_2 - x_0 - x_2) - 2(x_0y_2^2 + x_2y_0^2) - 2(x_1^2 - 8x_0x_2), \\ y_2' &= 2(x_1x_0 + x_2y_0^2) - 8x_0x_2 + x_1(x_1 - y_0y_2), \end{aligned} \tag{2.4}$$

having the same constants of motion. Let us write $\mathbb{F} = (F_1, F_2, F_3) : \mathcal{H} \to \mathbb{C}^3$ for the momentum map. It is clear that $\mathbb{F}$ is involutive. Moreover, $\mathbb{F}$ is independent on a dense open subset of $\mathcal{H}$, hence $(\mathcal{H}, \{\cdot,\cdot\}, \mathbf{F})$ is Liouville integrable.

We finish this section by determining the set of regular values of the map $\mathbb{F}$. To do this, let Π be the set of critical values of the momentum map $\mathbb{F}$

$$\Pi := \{\mathbf{c} \in \mathbb{C}^3 \mid \exists\, m \in \mathbb{F}^{-1}(\mathbf{c}) \text{ with } \mathsf{d}F_1(m) \wedge \mathsf{d}F_2(m) \wedge \mathsf{d}F_3(m) = 0\}.$$

Let us determine Π. The jacobian matrix of $\mathbb{F}$ is given by

$$J_{\mathbb{F}} := \begin{pmatrix} -4 & -2 & -4 & 2y_0 & 2y_2 \\ 4(4x_2 + x_1 - y_2^2) & \star & 4(4x_0 + x_1 - y_0^2) & 2y_0(y_2^2 - 4x_2) - 2x_1 y_2 & \star' \\ x_1 x_2 & x_0 x_2 & x_0 x_1 & 0 & 0 \end{pmatrix},$$

where $\star := 2(2x_0 + x_1 + 2x_2 - y_0 y_2)$ and $\star' := 2y_2(y_0^2 - 4x_0) - 2x_1 y_0$. Let $\mathcal{S}$ be the set of points in $\mathcal{H}$ where the determinants of all 3×3 minors of the matrix $J_{\mathbb{F}}$ cancel. By direct computation, we find that $\mathcal{S}$ is the union of following subvarieties

$$S_1 := \{x_0 = x_1 = 0\}, S_2 := \{x_0 = x_2 = 0\}, S_3 := \{x_1 = x_2 = 0\}, S_4 := \{x_1 = 0, x_0 = x_2 + \frac{1}{4}(y_0^2 - y_2^2)\},$$

$$S_5 := \{x_0 = x_2, y_0 = y_2\}, S_6 := \{y_0 = y_2 = 0, 4x_0 x_2 = x_1(x_0 + x_2)\},$$

$$S_7 := \{4x_2 y_0 = -4x_0 y_2 = (y_2 - y_0)(y_0 y_2 - x_1)\}.$$

We see that the images under $\mathbb{F}$ of S_1, S_2, S_3 and S_4 are contained in the subset $c_3 = 0$; by substituting $x_2 = x_0$ and $y_2 = y_0$ in $F_i = c_i$ $(i = 1, 2, 3)$, one obtains the equality $c_1^2 - 4c_2 = 0$. Also, we verify that a substitution of

$$y_0 = y_2 = 0, \quad x_1 = \frac{4x_0 x_1}{x_0 + x_2}$$

and

$$x_0 = -\frac{1}{4y_2}(y_2 - y_0)(y_0 y_2 - x_1), \quad x_2 = \frac{1}{4y_0}(y_2 - y_0)(y_0 y_2 - x_1),$$

respectively in the equalities $F_i = c_i$ leads, after a direct computation, to

$$6912c_3^2 + 288c_1 c_2 c_3 + 4c_2^3 - c_1^2 c_2^2 - 64c_1^3 c_3 = 0.$$

Thus,

$$\Pi = \{\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{C}^3 \mid c_3 = 0 \quad \text{or } c_1^2 - 4c_2 = 0 \quad \text{or} \\ 6912c_3^2 + 288c_1 c_2 c_3 + 4c_2^3 - c_1^2 c_2^2 - 64c_1^3 c_3 = 0\}.$$

Conversely, it can be shown that each point of Ω is a critical value of the momentum map $\mathbb{F}$. Hence, the set of regular values of the momentum map $\mathbb{F}$ is the Zariski open subset Ω given by

$$\Omega := \{\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{C}^3 \mid c_3 \neq 0,\ c_1^2 - 4c_2 \neq 0 \quad \text{and} \\ 6912c_3^2 + 288c_1 c_2 c_3 + 4c_2^3 - c_1^2 c_2^2 - 64c_1^3 c_3 \neq 0\}. \tag{2.5}$$

For $\mathbf{c} \in \Omega$, we denote by $\mathbb{F}_{\mathbf{c}}$ the fiber over $\mathbf{c}$,

$$\mathbb{F}_{\mathbf{c}} := \mathbb{F}^{-1}(\mathbf{c}) = \bigcap_{i=1}^{3} \{m \in \mathcal{H} : F_i(m) = c_i\}.$$

In conclusion, we have shown

Proposition 1. *For $\mathbf{c} \in \Omega$, the fiber $\mathbb{F}_{\mathbf{c}}$ over $\mathbf{c}$ of the momentum $\mathbb{F}$ is a smooth affine variety of dimension 2 and the rank of the Poisson structure de Poisson (2.2) is maximal, equal to 4 at each point of $\mathbb{F}_{\mathbf{c}}$; moreover the vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ are independent at each point of the fiber $\mathbb{F}_{\mathbf{c}}$.*

3. PAINLEVÉ ANALYSIS OF THE $\mathfrak{d}_3^{(2)}$ TODA LATTICE

On $\mathbb{C}^6$ there are two involutions π and τ which preserve the constants of motion F_1, F_2 and F_3, hence leave the fibers of the momentum map $\mathbb{F}$ invariant. These involutions restrict to the hyperplane $\mathcal{H}$ and are given by

$$\begin{aligned} \pi(x_0, x_1, x_2, y_0, y_1, y_2) &= (x_2, x_1, x_0, y_2, y_1, y_0), \\ \tau(x_0, x_1, x_2, y_0, y_1, y_2) &= (x_0, x_1, x_2, -y_0, -y_1, -y_2). \end{aligned} \tag{3.1}$$

The involution π preserves the vectors fields $\mathcal{V}_1$ et $\mathcal{V}_2$ while the involution τ changes their sign. Both involutions will have strong implications on the geometry of our integrable system. Notice that if we assign the weight 2 to the variables x_0, x_1 and x_2 and the weight 1 to the variables y_0, y_1 and y_2 then the constants of motion F_1, F_2 and F_3 are all weight homogeneous, with weights 2, 4 and 8 respectively. If we give time weight -1 then the vector field $\mathcal{V}_1$ also becomes weight homogeneous. Such a vector field is called *weight homogeneous vector field*. It is shown in [12] that for a such vector field it is easy to find the Laurent solutions and to do the Painlevé analysis of the system.

3.1. Laurent Solutions

The system of differential equations (2.1) has Laurent solutions of the form

$$x(t) = \frac{1}{t^2}\sum_{k=0}^{\infty} x^{(k)} t^k, \qquad y(t) = \frac{1}{t}\sum_{k=0}^{\infty} y^{(k)} t^k, \tag{3.2}$$

i.e. the variables $x = (x_0, x_1, x_2)$ have at most a double pole and the variables $y = (y_0, y_1, y_2)$ have at most a simple pole. In fact, it is shown that all Laurent solutions have this form. By substituting (3.2) in the differential equations (2.1), at the 0-th step, the coefficients of t^{-2} for $x(t)$ and t^{-1} for $y(t)$ lead to a non-linear system, to wit

$$\begin{aligned} 0 &= x_0^{(0)}(2 + y_0^{(0)}), \\ 0 &= x_1^{(0)}(2 + y_1^{(0)}), \\ 0 &= x_2^{(0)}(2 + y_2^{(0)}), \\ 0 &= y_0^{(0)} + 2x_0^{(0)} - x_1^{(0)}, \\ 0 &= y_1^{(0)} - 2x_0^{(0)} + 2x_1^{(0)} - 2x_2^{(0)}, \\ 0 &= y_2^{(0)} + 2x_2^{(0)} - x_1^{(0)}, \end{aligned} \tag{3.3}$$

and at the k-th step ($k \geqslant 1$), the coefficients lead to system of linear equations in $x^{(k)} = (x_0^{(k)}, x_1^{(k)}, x_2^{(k)})$ and $y^{(k)} = (y_0^{(k)}, y_1^{(k)}, y_2^{(k)})$. The solution of the system (3.3) consists of six points

$$(x_0^{(0)}, x_1^{(0)}, x_2^{(0)}, y_0^{(0)}, y_1^{(0)}, y_2^{(0)}) = \begin{cases} (0,0,1,0,2,-2), \\ (0,1,0,1,-2,1), \\ (1,0,0,-2,2,0), \\ (0,4,3,4,-2,-2), \\ (1,0,1,-2,4,-2), \\ (3,4,0,-2,-2,4). \end{cases}$$

We show that the points $m_0 := (0,0,1,0,2,-2)$ leads to a Laurent solution depending on four free parameters, whose the five leading terms (going with steps 1, 2, 2, 4 respectively, are denoted by

a, c, d, e) are given by

$$
\begin{aligned}
x_0(t;m_0) &= c + act + \frac{1}{2}c(2c+a^2)t^2 + O(t^3),\\
x_1(t;m_0) &= et^2 + O(t^3),\\
x_2(t;m_0) &= \frac{1}{t^2} + d + \frac{1}{10}(6d^2-e)t^2 + O(t^3),\\
y_0(t;m_0) &= a + 2ct + act^2 + \frac{1}{3}(2c^2+a^2c-e)t^3 + O(t^4),\\
y_1(t;m_0) &= \frac{2}{t} - a - 2(c+d)t - act^2 + \frac{1}{15}(11e-10c^2-6d^2-5a^2c)t^3 + O(t^4),\\
y_2(t;m_0) &= -\frac{2}{t} + 2dt - \frac{2}{5}(e-d^2)t^3 + O(t^4).
\end{aligned}
\tag{3.4}
$$

The point $m_2 := (1,0,0,-2,2,0) = \pi(m_0)$, so that the Laurent solution $x(t;m_2)$ is obtained from the above formulas for $x(t;m_0)$ by applying the involution π. The point $m_1 := (0,1,0,1,-2,1)$, which is a fixed point of the involution π, leads to the following Laurent solution (with four leading terms)

$$
\begin{aligned}
x_0(t;m_1) &= \beta t + O(t^2),\\
x_1(t;m_1) &= \frac{1}{t^2} + \gamma - \frac{1}{2}(\beta+\delta)t + O(t^2),\\
x_2(t;m_1) &= \delta t + O(t^2),\\
y_0(t;m_1) &= \frac{1}{t} + \alpha - \gamma t + \frac{1}{4}(5\beta+\delta)t^2 + O(t^3),\\
y_1(t;m_1) &= -\frac{2}{t} + 2\gamma t - \frac{3}{2}(\beta+\delta)t^2 + O(t^3),\\
y_2(t;m_1) &= \frac{1}{t} - \alpha - \gamma t + \frac{1}{4}(5\delta+\beta)t^2 + O(t^3),
\end{aligned}
\tag{3.5}
$$

where the four free parameters are denoted by $\alpha, \beta, \gamma, \delta$. The involution τ acts, according to (3.1), on these parameters in the following way

$$\tau : (t, \alpha, \beta, \gamma, \delta) \mapsto (-t, -\alpha, -\beta, \gamma, -\delta).$$

Notice that, the above Laurent solutions are weight homogeneous hence by using majoration method, one shows that they are actually convergent. The other points, solutions of the system (3.3), lead to Laurent solutions that depend on only three parameters; they are not used in what follows. Consequently, we have

Lemma 1. *The system of differential equations (2.1) possesses three Laurent solutions depending on 4 ($= \dim \mathcal{H} - 1$) free parameters, with leading terms given by (3.4), π applied to (3.4) and (3.5).*

3.2. Painlevé Divisors

We now search the formal Painlevé divisors i.e. the algebraic curves defined by the three different principal balances $x(t) = (x_0(t), \ldots, y_2(t))$, confined to a fixed affine invariant surface $\mathbb{F}_{\mathbf{c}}$, $\mathbf{c} \in \Omega$. We have

Proposition 2. *For $\mathbf{c} \in \Omega$, the Painlevé divisors $\Gamma_{\mathbf{c}}^{(0)}$ and $\Gamma_{\mathbf{c}}^{(2)}$ of the balances $x(t;m_0)$ and $x(t;m_2)$ respectively, restricted to the surface $\mathbb{F}_{\mathbf{c}}$, are smooth affine curves. Their equations are given by*

$$a^4e^2 - (c_1e + 8c_3)a^2e - 4e^3 + c_2e^2 + 4c_1c_3e + 16c_3^2 = 0. \tag{3.6}$$

They can be completed into Riemann surfaces of genus 2, which are double covers of $\mathbb{P}^1$, ramified at six points.

Proof. By substituting (3.4) in the equations $F_i(x_0(t),\ldots,y_2(t)) = c_i$ $(i=1,2,3)$ where the F_i are the constants of motion and $(c_1,c_2,c_3)\in\mathbb{C}^3$, we find three algebraic equations in terms of the four free parameters a,c,d,e defining an affine curve in $\mathbb{C}^4$, to wit

$$c_1 = a^2 - 4c - 12d, \quad c_2 = -12a^2d + 48cd + 4e, \quad c_3 = ce.$$

For $\mathbf{c}\in\Omega$, the parameters c,e are different from zero since $c_3\neq 0$. The first and the last equations are linear in c and d; one expresses these latters in terms of regular values of the constants of motion, giving

$$c = \frac{c_3}{e}, \qquad d = \frac{1}{12}\left(a^2 - \frac{4c_3}{e} - c_1\right). \tag{3.7}$$

Then the second equation is reduced, for c_1,c_2,c_3, to the following equation of an affine curve $\Gamma_{\mathbf{c}}^{(0)}$ in $\mathbb{C}^2$, to wit

$$\Gamma_{\mathbf{c}}^{(0)} : a^4e^2 - (c_1e + 8c_3)a^2e - 4e^3 + c_2e^2 + 4c_1c_3e + 16c_3^2 = 0. \tag{3.8}$$

We claim that this affine curve is smooth. Indeed, let us pose

$$f(a,e) := a^4e^2 - (c_1e + 8c_3)a^2e - 4e^3 + c_2e^2 + 4c_1c_3e + 16c_3^2,$$

we easily have

$$\frac{\partial f}{\partial a}(a,e) = 2ae(2ea^2 - c_1e - 8c_3),$$
$$\frac{\partial f}{\partial e}(a,e) = 2a^4e - 2(c_1e + 4c_3)a^2 - 12e^2 + 2c_2e + 4c_1c_3.$$

A point (a,e) is a singular point of the affine curve $\Gamma_{\mathbf{c}}^{(0)}$ if

$$f(a,e) = \frac{\partial f}{\partial a}(a,e) = \frac{\partial f}{\partial e}(a,e) = 0.$$

Since $e\neq 0$, this leads to

$$a = 0, \quad \text{or} \quad a^2 = \frac{8c_3 + c_1e}{2e}.$$

If $a^2 = \dfrac{8c_3 + c_1e}{2e}$, after substitution in the equations

$$f(a,e) = 0 \quad \text{and} \quad \frac{\partial f}{\partial e}(a,e) = 0,$$

we respectively obtain

$$\frac{1}{4}e^2(c_1^2 - 4c_2 + 16e) = 0, \quad \text{and} \quad -\frac{1}{2}e(c_1^2 - 4c_2 + 24e) = 0.$$

But $e\neq 0$, so that $c_1^2 - 4c_2 = -16e = -24e$. This implies that $e = 0$; contradiction!

If $a = 0$, this leads to the following system

$$\begin{cases} -12e^2 + 2c_2e + 4c_1c_3 = 0 \\ -4e^3 + c_2e^2 + 4c_1c_3e + 16c_3^2 = 0. \end{cases}$$

By computing the resultant of the two polynomials forming this system, we obtain

$$-64c_3(6912c_3^2 + 288c_1c_2c_3 + 4c_2^3 - c_1^2c_2^2 - 64c_1^3c_3).$$

This expression is different from zero for $\mathbf{c}\in\Omega$; so that $\Gamma_{\mathbf{c}}^{(0)}$ is a smooth affine curve if $\mathbf{c}\in\Omega$. In order to compactify $\Gamma_{\mathbf{c}}^{(0)}$, for $\mathbf{c}\in\Omega$, into a compact Riemann surface, denoted by $\overline{\Gamma}_{\mathbf{c}}^{(0)}$, we need to add three points at infinity to it, denoted by ∞_ϵ, ∞ where $\epsilon = \pm 1$. A local parameter ς at each of these points is given as follows

$$\infty_\epsilon: \qquad a = \varsigma^{-1}, \qquad e = 4c_3\varsigma^2 + 2(c_1 + \epsilon\eta)c_3\varsigma^4 + O(\varsigma^6), \tag{3.9}$$

$$\infty: \qquad a = \varsigma^{-1}, \qquad e = \frac{1}{4}(\varsigma^{-4} - c_1\varsigma^{-2} + c_2 - 32c_3\varsigma^2 - 16c_1c_3\varsigma^4 + O(\varsigma^6)), \tag{3.10}$$

where η is a fixed square root of $c_1^2 - 4c_2$, which is non-zero for $\mathbf{c} \in \Omega$.
By rewriting (3.8) as follows

$$\left(a^2 - \frac{4c_3}{e}\right)\left(a^2 - \frac{4c_3}{e} - c_1\right) = 4e - c_2,$$

we see that $\Gamma_{\mathbf{c}}^{(0)}$ is a double cover of a rational affine curve defined by

$$\mathcal{E}_{\mathbf{c}} : u(u - c_1) - 4e + c_2 = 0.$$

The covering map is given by

$$\begin{aligned} \psi : \quad \Gamma'_{\mathbf{c}} \quad &\to \mathcal{E}_{\mathbf{c}} \\ (a, e) &\mapsto (u, e) = \left(a^2 - \frac{4c_3}{e}, e\right). \end{aligned}$$

A parametrization of $\mathcal{E}_{\mathbf{c}}$ is given by

$$\mathcal{E}_{\mathbf{c}} = \left\{(u, e) = \left(-t, \frac{t^2 + c_1t + c_2}{4}\right), t \in \mathbb{C}\right\}.$$

Thus, if $\psi(a, e) = (u, e)$, we have

$$a = \pm\sqrt{-\frac{t^3 + c_1t^2 + c_2t - 16c_3}{t^2 + c_1t + c_2}}, \qquad e = \frac{1}{4}(t^2 + c_1t + c_2). \tag{3.11}$$

The cover $\psi : \Gamma_{\mathbf{c}}^{(0)} \to \mathcal{E}_{\mathbf{c}}$ has three branch points : the points (u, e) on the curve $\mathcal{E}_{\mathbf{c}}$ for which $a = 0$ i.e. where u is a root of the polynomial $P(t) = t^3 + c_1t^2 + c_2t - 16c_3$. This leads to three ramification points on $\Gamma_{\mathbf{c}}^{(0)}$. This map is also ramified at infinity. Indeed, writing t in terms of a local parameter ς, we find from (3.11) that

$$(a, e) = \left(\varsigma^{-1} + 8c_3\varsigma^5 + O(\varsigma^7), \frac{\varsigma^{-4}}{4}(1 - c_1\varsigma^2 + c_2\varsigma^4)\right),$$

where $t = -1/\varsigma^2$. This shows that the map is ramified at this point. Let $t_\pm = \frac{1}{2}(c_1 \pm \sqrt{c_1^2 - 4c_2})$ be the two roots of $t^2 + c_1t + c_2$. By letting $t = t_+ + \varsigma^2$ be a local parametrization around t_+, we have

$$a = \frac{1}{\varsigma}\left(\sqrt{\frac{16c_3}{\eta}} + O(\varsigma^2)\right), \qquad e = \varsigma^2\left(\frac{1}{4}\eta + O(\varsigma^2)\right) \qquad \text{where} \qquad \eta = \sqrt{c_1^2 - 4c_2};$$

thus we find a point (a, e) on $\Gamma_c^{(0)}$ which is a ramification point. Also, if we write $t = t_- + \varsigma^2$, we have

$$a = \frac{1}{\varsigma}\left(\sqrt{\frac{-16c_3}{\eta}} + O(\varsigma^2)\right), \qquad e = -\varsigma^2\left(\frac{1}{4}\eta + O(\varsigma^2)\right);$$

which leads to an another ramification point, showing that the Riemann surface $\overline{\Gamma}_{\mathbf{c}}^{(0)}$ is a double cover of a rational curve $\overline{\mathcal{E}}_{\mathbf{c}}^{(0)}$ closure of the curve $\mathcal{E}_{\mathbf{c}}^{(0)}$ with six ramification points, so that its genus is 2 by the Riemann–Hurwitz formula. These six points are Weierstrass points of the Riemann surface $\overline{\Gamma}_{\mathbf{c}}^{(0)}$. Upon computing the Painlevé divisor which corresponds to the principal balance $x(t; m_2)$, we find the same equation (3.8), since the involution π preserves the constants of motion.

Proposition 3. *For $\mathbf{c} \in \Omega$, the Painlevé divisor $\Gamma_{\mathbf{c}}^{(1)}$ which corresponds to the principal balance $x(t; m_1)$ is a smooth hyperelliptic curve of genus 3. It may be completed into a Riemann surface $\overline{\Gamma}_{\mathbf{c}}^{(1)}$ by adding four points at infinity.*

Proof. By substituting the principal balance $x(t; m_1)$ in the equations $F_i = c_i$, $i = 1, \ldots, 3$, where $\mathbf{c} = (c_1, c_2, c_3) \in \Omega$, the resulting expressions are independent of t. This leads to three algebraic equations in terms of the four free parameters $\alpha, \beta, \gamma, \delta$, to wit

$$c_1 = 2\alpha^2 - 6\gamma, \quad c_2 = \alpha^4 + 6\alpha^2\gamma + 8\alpha(\beta - \delta) + 9\gamma^2, \quad c_3 = \beta\delta. \tag{3.12}$$

It is clear that, for $\mathbf{c} \in \Omega$, the parameters β and δ are non-zero since $c_3 \neq 0$. If we eliminate the two parameters γ, δ in (3.12), we obtain an algebraic relation between α and β, which is the equation of an affine curve in $\mathbb{C}^2$, defined by

$$\Gamma_{\mathbf{c}}^{(1)} : 32\alpha\beta^2 + (16\alpha^4 - 8c_1\alpha^2 + c_1^2 - 4c_2)\beta - 32c_3\alpha = 0. \tag{3.13}$$

Let us set

$$g(\alpha, \beta) := 32\alpha\beta^2 + (16\alpha^4 - 8c_1\alpha^2 + c_1^2 - 4c_2)\beta - 32c_3\alpha. \tag{3.14}$$

We have

$$\frac{\partial g}{\partial \alpha}(\alpha, \beta) = 32\beta^2 + (64\alpha^3 - 16c_1\alpha)\beta - 32c_3, \tag{3.15}$$

$$\frac{\partial g}{\partial \beta}(\alpha, \beta) = 64\alpha\beta + 16\alpha^4 - 8c_1\alpha^2 + c_1^2 - 4c_2. \tag{3.16}$$

Let us suppose that

$$g(\alpha, \beta) = \frac{\partial g}{\partial \alpha}(\alpha, \beta) = \frac{\partial g}{\partial \beta}(\alpha, \beta) = 0,$$

i.e. (α, β) is a singular point of the curve $\Gamma_{\mathbf{c}}^{(1)}$. As α et β are non-zero, (3.16) implies

$$\beta = -\frac{16\alpha^4 - 8c_1\alpha^2 + c_1^2 - 4c_2}{64\alpha}.$$

A substitution of β in terms of α in the equations

$$(3.14) - \alpha(3.15) = 0 \quad \text{and} \quad (3.14) + \alpha(3.15) = 0,$$

leads respectively to

$$H_1(\alpha)H_2(\alpha) = 0 \qquad \text{and} \qquad \frac{1}{4}\alpha H_3(\alpha) = 0,$$

where

$$\begin{aligned}
H_1(\alpha) &= 16\alpha^4 - 8c_1\alpha^2 + c_1^2 - 4c_2, \\
H_2(\alpha) &= 48\alpha^4 - 8c_1\alpha^2 - c_1^2 + 4c_2, \\
H_3(\alpha) &= 64\alpha^6 - 48c_1\alpha^4 + 4(3c_1^2 - 4c_2)\alpha^2 - c_1^3 + 4c_1c_2 + 256c_3.
\end{aligned}$$

The resultant of polynomials H_1 et H_3 give the square of $2^{28}c_3^2$ and the resultant of H_2 et H_3 the square of

$$2^{20}(6912c_3^2 + 288c_1c_2c_3 + 4c_2^3 - c_1^2c_2^2 - 64c_1^3c_3),$$

thus for $\mathbf{c} \in \Omega$, the affine curve $\Gamma_{\mathbf{c}}^{(1)}$ is smooth, hence it can be compactified into a Riemann surface, denoted by $\overline{\Gamma}_{\mathbf{c}}^{(1)}$, by adding four points at infinity $\infty_1, \ldots, \infty_4$ which are given in terms of a local parameter ς by

$$\infty_1 : \qquad \beta = \varsigma, \qquad \alpha = \frac{1}{32c_3}(c_1^2 - 4c_2)\varsigma + O(\varsigma^3), \tag{3.17}$$

$$\infty_2: \qquad \alpha = \varsigma^{-1}, \qquad \beta = 2c_3\varsigma^3 + c_1c_3\varsigma^5 + \left(\frac{3}{8}c_3c_1^2 + \frac{1}{2}c_2c_3\right)\varsigma^7 + O(\varsigma^9), \tag{3.18}$$

$$\infty_3: \qquad \alpha = \varsigma^{-1}, \qquad \beta = \frac{1}{32}(-16\varsigma^{-3} + 8c_1\varsigma^{-1} + (4c_2 - c_1^2)\varsigma - 64c_1\varsigma^3 + O(\varsigma^5), \tag{3.19}$$

$$\infty_4: \qquad \beta = \varsigma^{-1}, \qquad \alpha = -\frac{1}{32}(c_1^2 - 4c_2)\varsigma + O(\varsigma^3). \tag{3.20}$$

By the change of variable $\xi = 64\alpha\beta + 16\alpha^4 - 8c_1\alpha^2 + c_1^2 - 4c_2$, the curve $\Gamma_{\mathbf{c}}^{(1)}$ is birational to the hyperelliptic curve

$$\mathcal{E}_{\mathbf{c}}^{(1)} : \xi^2 = (16\alpha^4 - 8c_1\alpha^2 + c_1^2 - 4c_2)^2 + 4096c_3\alpha^2,$$

so that $\Gamma_{\mathbf{c}}^{(1)}$ is a genus three curve.

The invariant manifold $\mathbb{F}_{\mathbf{c}}$ can be embedded explicitly in a projective space i.e. we can construct an embedding $\varphi : \mathbb{F}_{\mathbf{c}} \to \mathbb{P}^N$. The idea of the construction is based on the following fact. If the $\mathfrak{d}_3^{(2)}$ Toda lattice is an irreducible a.c.i. system, then as we have seen above a divisor $\mathcal{D}$ can be added to a Zariski open subset of the $\mathcal{H}$, having the effect of compactifying all fibers $\mathbb{F}_{\mathbf{c}}$, where $\mathbf{c} \in \Omega$. The divisor that is added to $\mathbb{F}_{\mathbf{c}}$ will be denoted by $\mathcal{D}_{\mathbf{c}}$ and the resulting torus by $\mathbb{T}_{\mathbf{c}}^2$. The vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ extend to linear (hence holomorphic) vector fields on this partial compactification of $\mathcal{H}$, hence we may consider the integral curves of $\mathcal{V}_1$, starting from any component $\mathcal{D}_{\mathbf{c}}^{(i)}$. This gives a Laurent solution depending on four parameters, hence it must coincide with one of the Laurent solution $x(t; m_i)$. Let f be a polynomial function on $\mathcal{H}$. Since $\overline{\mathcal{V}}_1$ is transversal to the divisor $\mathcal{D}$, the pole order of the Laurent series $f(t; m_i)$, obtained by substituting the series $x(t; m_i)$ in the function f, equals the order of $f_{|\mathbb{T}_{\mathbf{c}}^r}$ along the divisor $\mathcal{D}_{\mathbf{c}}^{(i)}$, where $f_{|\mathbb{T}_{\mathbf{c}}^2}$ is by definition $f_{|\mathbb{F}_{\mathbf{c}}}$, viewed as a meromorphic function on $\mathbb{T}_{\mathbf{c}}^2$. Since the third power of an ample divisor on an Abelian variety is very ample, we look for all polynomials which have a simple pole at most when any of the three principal balances are substituted in them. Precisely, we look for a maximal independent set of functions which are independent when restricted to $\mathbb{F}_{\mathbf{c}}$. By direct computation, one finds the sixteen weight homogeneous polynomials of weight at most 8.

$$\begin{aligned}
&z_0 = 1, && z_7 = x_1x_0,\\
&z_1 = y_0, && z_8 = x_1x_2,\\
&z_2 = y_0 + y_2, && z_9 = y_0y_2z_4 + x_1(y_0 - y_2)z_2,\\
&z_3 = x_1 - y_0y_2, && z_{10} = x_1x_0(y_0 - y_2),\\
&z_4 = 4(x_0 - x_2) - (y_0 - y_2)z_2, && z_{11} = x_1x_2(y_2 - y_0),\\
&z_5 = x_1y_0 + y_2(4x_0 - y_0^2), && z_{12} = x_1x_0z_3,\\
&z_6 = x_1y_2 + y_0(4x_2 - y_2^2), && z_{13} = x_1x_2z_3,
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
z_{14} &= x_1x_0((y_2 - y_0)z_3 - 4x_0y_2),\\
z_{15} &= x_1x_2((y_0 - y_2)z_3 - 4x_2y_0).
\end{aligned}$$

The involution π acts, according to (3.1), on the independent polynomials z_i as follows

$$\pi(z_0, z_1, \ldots, z_{15}) = (z_0, z_2 - z_1, z_2, z_3, -z_4, z_6, z_5, z_8, z_7, -z_9, z_{11}, z_{10}, z_{13}, z_{12}, z_{15}, z_{14}). \tag{3.22}$$

We consider the regular map,

$$\begin{aligned}
\varphi_{\mathbf{c}} : \qquad \mathbb{F}_{\mathbf{c}} &\to \mathbb{P}^{15}\\
(x_0, x_1, x_2, y_0, y_1, y_2) &\mapsto (1 : z_1 : \cdots : z_{15}),
\end{aligned} \tag{3.23}$$

For $\mathbf{c} \in \Omega$, the map $\varphi_{\mathbf{c}}$ is an embedding of $\mathbb{F}_{\mathbf{c}}$ in the projective space $\mathbb{P}^{15}$. We will see that this embedding extends to $\Gamma_{\mathbf{c}}^{(i)}$, $i = 1, 2, 3$.

4. ALGEBRAIC INTEGRABILITY OF THE $\mathfrak{d}_3^{(2)}$ TODA LATTICE

In order to show that the $\mathfrak{d}_3^{(2)}$ Toda lattice is algebraic completely integrable (a.c.i.), we show that, for $\mathbf{c} \in \Omega$, the fiber

$$\mathbb{F}_{\mathbf{c}} = \mathbb{F}^{-1}(\mathbf{c}) = \bigcap_{i=1}^{3} \{m \in \mathcal{H} : F_i(m) = c_i\}$$

is an affine part of Abelian surface, on which the vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ restrict to linear vector fields. To do this, we must check that $\mathbb{F}_{\mathbf{c}}$ satisfies the conditions of the following theorem.

Theorem 1 (Complex Liouville Theorem [12]). *Let $\mathcal{A} \subset \mathbb{C}^s$ be non singular affine variety of dimension r with r holomorphic fields $\mathcal{V}_1, \ldots, \mathcal{V}_r$ and let $\varphi : \mathcal{A} \to \mathbb{C}^N \subset \mathbb{P}^N$ an embedding. We define $\Delta := \overline{\varphi(\mathcal{A})} \backslash \varphi(\mathcal{A})$ and denote by $\mathcal{D}$ the union of all irreducible components of Δ of dimension $r-1$. Suppose the following*
(1) $[\mathcal{V}_i, \mathcal{V}_j] = 0$ for $1 \leqslant i, j \leqslant r$;
(2) At every point $m \in \mathcal{A}$ the vector fields $\mathcal{V}_1, \ldots, \mathcal{V}_r$ are independent;
(3) The vector field $\varphi_* \mathcal{V}_1$ extends to a vector field $\overline{\mathcal{V}}_1$ holomorphic on a neighborhood of $\mathcal{D}$ in $\mathbb{P}^N$;
(4) The integral curves of $\overline{\mathcal{V}}_1$ that start at points $m \in \mathcal{D}$ go immediately into $\varphi(\mathcal{A})$.
Then $\overline{\varphi(\mathcal{A})}$ is a Abelian variety $\mathbb{T}^r$ of dimension r and the fields $\varphi_* \mathcal{V}_1, \ldots, \varphi_* \mathcal{V}_r$ extend to holomorphic on $\overline{\varphi(\mathcal{A})}$. Moreover $\overline{\varphi(\mathcal{A})} = \varphi(\mathcal{A}) \cup \Delta$, i.e. $\mathcal{D} = \Delta$.

For our case, we take for $\mathcal{A}$ the surface $\mathbb{F}_{\mathbf{c}}$, $\mathbf{c} \in \Omega$. According to Theorem 1, $\mathbb{F}_{\mathbf{c}}$ is smooth and the conditions (1) and (2) of the theorem are satisfied. To conclude that the fiber $\mathbb{F}_{\mathbf{c}}$ sastifies the conditions (3) and (4), we use the embedding $\varphi_{\mathbf{c}}$ given in (3.23). This embedding induces three maps of each curve $\Gamma_{\mathbf{c}}^{(i)}$ to $\mathbb{P}^{15}$. Indeed, by substituting the three principal balances in this embedding and letting $t \to 0$ we find an embedding of each of the three curves $\Gamma_{\mathbf{c}}^{(0)}, \Gamma_{\mathbf{c}}^{(1)}$ and $\Gamma_{\mathbf{c}}^{(2)}$ in $\mathbb{P}^{15}$, denoted respectively by $\varphi_{\mathbf{c}}^{(0)}, \varphi_{\mathbf{c}}^{(1)}$ and $\varphi_{\mathbf{c}}^{(2)}$. Explicitely, we have

$$\begin{aligned} \varphi_{\mathbf{c}}^{(0)} : (a, e) \mapsto (0 : 0 : -2 : 2a : 0 : 2a^2 - 8c : 0 : 0 : 0 : 2a(a^2 - 4c + 12d) :& \\ 0 : -2e : 0 : 2ae : 0 : 2a^2 e),& \qquad (4.1) \end{aligned}$$

$$\begin{aligned} \varphi_{\mathbf{c}}^{(1)} : (\alpha, \beta) \mapsto (0 : 1 : 2 : 0 : -4\alpha : \alpha^2 + 3\gamma : \alpha^2 + 3\gamma : \beta : \delta : 4(\alpha^3 + 3\alpha\gamma + \beta - \delta) : 2\alpha\beta :& \\ -2\alpha\delta : \beta(\alpha^2 + 3\gamma) : \delta(\alpha^2 + 3\gamma) : -2\beta(\alpha^3 + 3\alpha\gamma + 2\beta) : 2\delta(\alpha^3 + 3\alpha\gamma - 2\delta)),& \qquad (4.2) \end{aligned}$$

$$\begin{aligned} \varphi_{\mathbf{c}}^{(2)} : (a, e) \mapsto (0 : -2 : -2 : 2a : 0 : 0 : 2a^2 - 8c : 0 : 0 : -2a(a^2 - 4c + 12d) :& \\ -2e : 0 : 2ae : 0 : 2a^2 e : 0).& \qquad (4.3) \end{aligned}$$

We recall that in (4.1) and (4.3) $c = c_3/e$, $d = (a^2 e - 4c_3 - c_1 e)/12e$ while in (4.2) $\gamma = (2\alpha^2 - c_1)/6$, $\delta = c_3/\beta$. We see that $\varphi_{\mathbf{c}}^{(0)}, \varphi_{\mathbf{c}}^{(1)}$ and $\varphi_{\mathbf{c}}^{(2)}$ are indeed embeddings of the affine curve since a and e (resp. α and β) appear linearly in (4.1) and (4.3) (resp. (4.2)). It is clear by seeing the three leadings coordinates of each embedding that the image curves are disjoint. However, they are not complete. We denote by $\mathcal{D}_{\mathbf{c}}^{(i)} := \overline{\varphi_{\mathbf{c}}^{(i)}(\Gamma_{\mathbf{c}}^{(i)})}$ the projective closures of the images of these embeddings. These give us three divisors on the (possibly singular) surface $\overline{\varphi_{\mathbf{c}}(\mathbb{F}_{\mathbf{c}})}$. Let $\mathcal{D}_{\mathbf{c}} := \cup_{i=0}^{2} \mathcal{D}_{\mathbf{c}}^{(i)}$. In order to study the singularities of $\mathcal{D}_{\mathbf{c}}$, we use a local parameter ς around each of the points at infinity in the corresponding map and we let $\varsigma \to 0$. Thus, by substituting (3.9) and (3.10) in (4.1), we find the following leading terms, where $\epsilon = \pm 1$.

$$\varphi_{\mathbf{c}}^{(0)}(\infty_\epsilon) \sim (0 : 0 : \varsigma : 1 : 0 : \frac{1}{2}(c_1 + \epsilon\eta)\varsigma : 0 : 0 : 0 : \epsilon\eta : 0 : 0 : 0 : 0 : 0 : 4c_3\varsigma),$$

$$\varphi_{\mathbf{c}}^{(0)}(\infty) \sim (0 : \cdots : 0 : -\varsigma^2 : 0 : \varsigma : 0 : 1 - c_1\varsigma^2),$$

Letting $\varsigma \to 0$, we find the following image points in $\mathbb{P}^{15}$:

$$P_\epsilon := (0:0:0:1:0:0:0:0:0:\epsilon\eta:0:\cdots:0),$$
$$S' := (0:\cdots:0:0:1)\,.$$

We see at once that these three points are different, so that $\varphi_{\mathbf{c}}^{(0)}$ is injective, and since the linear terms in ς are non-vanishing we conclude that the image curve $\mathcal{D}_{\mathbf{c}}^{(0)}$ is non-singular and isomorphic to $\Gamma_{\mathbf{c}}^{(0)}$. Applying the involution π which acts on the functions z_i according to (3.22), we find

$$\varphi_{\mathbf{c}}^{(2)}(\infty_\epsilon) \sim (0:-\varsigma:\varsigma:1:0:0:\frac{1}{2}(c_1+\epsilon\eta)\varsigma:0:0:-\epsilon\eta:0:0:0:0:4c_3\varsigma:0),$$
$$\varphi_{\mathbf{c}}^{(2)}(\infty) \sim (0:\cdots:-\varsigma^2:0:\varsigma:0:1-c_1\varsigma^2:0),$$

leading to the following three different points in $\mathbb{P}^{15}$:

$$P'_\epsilon := (0:0:0:1:0:0:0:0:0:-\epsilon\eta:0:\cdots:0),$$
$$S'' := (0:\cdots:0:1:0)\,.$$

One verifies also that $\mathcal{D}_{\mathbf{c}}^{(2)}$ is a non-singular curve and isomorphic to $\Gamma_{\mathbf{c}}^{(2)}$. We have $P_\epsilon = P'_{-\epsilon}$, which leads to two intersection points of $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$. Comparing the term in ς of $\varphi_{\mathbf{c}}^{(0)}(\infty_\epsilon)$ with the term in ς of $\varphi_{\mathbf{c}}^{(2)}(\infty_{-\epsilon})$, we conclude that the image curves $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$ intersect tranversally at these two points P_ϵ. Finally, by substituting (3.17) to (3.20) in (4.2), we have the following leading terms:

$$\varphi_{\mathbf{c}}^{(1)}(\infty_1) \sim (0:\cdots:-2\varsigma:8\varsigma:0:0:0:c_1\varsigma:0:8c_3),$$
$$\varphi_{\mathbf{c}}^{(1)}(\infty_2) \sim (0:\cdots:0:-\varsigma^2:0:\varsigma:0:1-c_1\varsigma^2),$$
$$\varphi_{\mathbf{c}}^{(1)}(\infty_3) \sim (0:\cdots:-\varsigma^2:0:-\varsigma:0:1-c_1\varsigma^2:0),$$
$$\varphi_{\mathbf{c}}^{(1)}(\infty_4) \sim (0:\cdots:-2\varsigma:0:-8\varsigma:0:0:c_1\varsigma:0:8:0).$$

We find the two image points in $\mathbb{P}^{15}$, to wit

$$\begin{aligned} \lim_{\varsigma\to 0}\varphi_{\mathbf{c}}^{(1)}(\infty_1) &= \lim_{\varsigma\to 0}\varphi_{\mathbf{c}}^{(1)}(\infty_2) = (0:\cdots:0:0:1) = S',\\ \lim_{\varsigma\to 0}\varphi_{\mathbf{c}}^{(1)}(\infty_3) &= \lim_{\varsigma\to 0}\varphi_{\mathbf{c}}^{(1)}(\infty_4) = (0:\cdots:0:1:0) = S''. \end{aligned} \tag{4.4}$$

We see that the curve $\mathcal{D}_{\mathbf{c}}^{(1)}$ is singular at the points S' and S''. Computing an extra term in the series, it follows easily that these points are ordinary double points, i.e., the two branches of $\mathcal{D}_{\mathbf{c}}^{(1)}$ meet transversally at the points, as indicated in Figure 1. Comparing the terms in ς and ς^2 of $\varphi_{\mathbf{c}}^{(0)}(p)$, for p close to ∞ with the terms in ς and ς^2 of $\varphi_{\mathbf{c}}^{(1)}(p)$, for p close to ∞_3 it follows that $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(1)}$ are tangent at S', and the tangency is double. Similary, we show that $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$ are tangent at S'', and the tangency is double.

Summarizing, the Painlevé divisor $\mathcal{D}_{\mathbf{c}}$ consists of three curves $\mathcal{D}_{\mathbf{c}}^{(0)}$, $\mathcal{D}_{\mathbf{c}}^{(1)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$ of genus 2, 3 and 2 respectively. The curves $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$ are non-singular, intersect transversally in the two points P_ϵ. The curve $\mathcal{D}_{\mathbf{c}}^{(1)}$ is singular, admits two double points S' and S'', its intersections with $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$ respectively. The precise correspondence between the seven points at infinity ∞, ∞_+, ∞_-, ∞_i $(i = 1,\ldots,4)$ and the four points P_+, P_-, S' and S'' under the three embeddings $\varphi_{\mathbf{c}}^{(i)}$ is given in Table 1.

We now need to show that the vector field $(\varphi_{\mathbf{c}})_*\mathcal{V}_1$ extends to a holomorphic vector field on a neighborhood of $\mathcal{D}_{\mathbf{c}}$ in $\mathbb{P}^{15}$ (condition (3) of Theorem 1). We will be able to show that $(\varphi_{\mathbf{c}})_*\mathcal{V}_1$ extends to a holomorphic vector field on all $\mathbb{P}^{15}$, which implies in particular holomorphicity in a

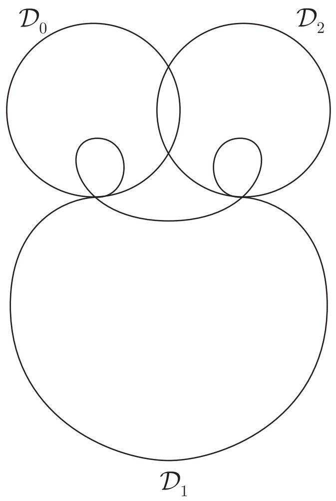

Fig. 1. Curves completing the $\mathfrak{d}_3^{(2)}$-Toda invariant surfaces into Abelian surfaces, where $\mathcal{D}_i := \mathcal{D}_{\mathbf{c}}^{(i)}$.

neighborhood of $\mathcal{D}_{\mathbf{c}}$. The construction of the extension is based on the following theorem, which says that, in appropriate projective coordinates, any holomorphic vector field on an Abelian variety is a quadratic vector field, hence it is globally defined and holomorphic.

Theorem 2 ([12]). *Let $\mathcal{L}$ be an ample line on an irreducible Abelian variety $\mathbb{T}^r$ and let $\mathcal{V}$ be a holomorphic vector field on $\mathbb{T}^r$. Denoting by $\varphi : \mathbb{T}^r \to \mathbb{P}^N$ the Kodaira embedding that corresponds to $\mathcal{L}^{\otimes^3}$ there exists a quadratic vector field $\overline{\mathcal{V}}$ on $\mathbb{P}^N$ such that $\varphi_* \mathcal{V} = \overline{\mathcal{V}}$.*

Thus, in view of Hartog's theorem we must show that the vector field $(\varphi_c)_* \mathcal{V}_1$ can be written as quadratic vector fields in two charts $Z_0 \neq 0$ and $Z_1 \neq 0$. In the chart $Z_0 \neq 0$ we just use the z_i as coordinates. It is easily to check that the quadratic equations of $(\varphi_c)_* \mathcal{V}_1$ can be written in this chart in the following form

$$\begin{aligned}
\dot{z}_1 &= -\frac{1}{4}(c_1 + 6z_3 - z_4 - 2z_1(-4z_1 + 3z_2)), \\
\dot{z}_2 &= -\frac{1}{2}(c_1 + 6z_3 - z_2^2) + 4z_1(z_1 - z_2), \\
\dot{z}_3 &= \frac{1}{2}(z_2 z_3 - z_5 - z_6),
\end{aligned}$$

Table 1. Correspondence between singular points

	∞_-	∞_+	∞
$\mathcal{D}_{\mathbf{c}}^{(0)}$	P_+	P_-	S'
$\mathcal{D}_{\mathbf{c}}^{(2)}$	P_-	P_+	S''

	∞_1	∞_2	∞_3	∞_4
$\mathcal{D}_{\mathbf{c}}^{(1)}$	S'	S'	S''	S''

$$\dot z_4 = -z_1(c_1+z_4) - 2z_6,$$
$$\dot z_5 = -4(z_7-z_8) + \frac{1}{4}(2c_2 + 3z_3(c_1+z_4) - 2(z_2-z_1)(3z_6-z_5)),$$
$$\dot z_6 = 4(z_7-z_8) + \frac{1}{4}(2c_2 + 3z_3(c_1+z_4) - 2z_1(3z_5-z_6)),$$
$$\dot z_7 = z_7(z_1-z_2),$$
$$\dot z_8 = -z_1z_8,$$
$$\dot z_9 = 12z_{10} - 8z_1(z_7-z_8) - z_3(z_5+z_6) + z_1(z_9-2c_2) + \frac{1}{2}(z_4(z_5+3z_6) - c_1z_2z_3),$$
$$\dot z_{10} = \frac{1}{2}(z_4z_7 + z_{10}(-z_2+2z_1)),$$
$$\dot z_{11} = -\frac{1}{2}(z_4z_8 + z_{11}(-z_2+2z_1)),$$
$$\dot z_{12} = -\frac{1}{2}(-z_3z_{10} + z_7(z_5+z_6)),$$
$$\dot z_{13} = -\frac{1}{2}(-z_3z_{11} + z_8(z_5+z_6)),$$
$$\dot z_{14} = c_3(4z_3 - z_4 + c_1) + \frac{1}{2}(z_{14}(3z_1-2z_2) + z_5z_{10} + 4z_7^2),$$
$$\dot z_{15} = c_3(4z_3 + z_4 + c_1) + \frac{1}{2}(z_{15}(z_2-3z_1) + z_6z_{11} + 4z_8^2).$$

This establishes the fact that $(\varphi_{\mathbf{c}})_*\mathcal{V}_1$ extends to a holomorphic vector field in the chart $Z_0 \neq 0$. In the same way, one finds, after some computation, the quadratic differential equations [15] in the chart $Z_1 \neq 0$. According to Hartog's theorem, this shows that the field $(\varphi_{\mathbf{c}})_*\mathcal{V}_1$ extends to a holomorphic vector field on $\mathbb{P}^{15}$, which we denote by $\overline{\mathcal{V}}_1$. Thus, we have verified that $\varphi_{\mathbf{c}}$ satisfies condition (3) in the Theorem 1.

The final thing to be shown is that the integral curves of the holomorphic vector field $\overline{\mathcal{V}}$ that start at any point $m \in \Delta_{\mathbf{c}}$ go immediately into $\varphi_{\mathbf{c}}(\mathbb{F}_{\mathbf{c}})$ i.e.

$$\{\Phi_t(m) \mid 0 < |t| < \epsilon\} \subset \varphi_{\mathbf{c}}(\mathbb{F}_{\mathbf{c}}),$$

where Φ_t is the flow of $\overline{\mathcal{V}}_1$ on $\mathbb{P}^{15}$. This leads to show that $z_j(m)$, $1 \leqslant j \leqslant 15$, are finite. There are three types of points $m \in \Delta_{\mathbf{c}}$ that need to consider.

(1) the points in the images $\varphi_{\mathbf{c}}^{(i)}(\Gamma_{\mathbf{c}}^{(i)})$, for i, $i = 1,2,3$;

(2) the points in $K^{(i)} = \mathcal{D}_{\mathbf{c}}^{(i)} \setminus \varphi_{\mathbf{c}}^{(i)}(\Gamma_{\mathbf{c}}^{(i)})$, for i, $i = 1,2,3$;

(3) the points in $\Delta_{\mathbf{c}} \setminus \cup_{i=1}^3 \mathcal{D}_{\mathbf{c}}^{(i)}$.

For points of (1): let P_0 be any point in $\varphi_{\mathbf{c}}^{(i)}(\Gamma_{\mathbf{c}}^{(i)})$ and $p \in \Gamma_{\mathbf{c}}^{(i)}$ be such that $\varphi_{\mathbf{c}}^{(i)}(p_0) = P_0$. Since the embedding functions z_i are polynomials in the phase variables $x_0, \ldots, y_2$ and that the Laurent series $x_j(t;\Gamma_{\mathbf{c}}^{(i)})$ are convergent, we may pick $\epsilon > 0$ such that $z_j(t;\Gamma_{\mathbf{c}}^{(i)})$ is finite for $j = 1,\ldots,15$ and for t such that $0 < |t| < \epsilon$; hence $\Phi_t(P)$ does not belong to the hyperplane $(z_0 = 0)$ i.e. $\Phi_t(P) \subset \varphi_{\mathbf{c}}(\mathbb{F}_{\mathbf{c}})$. The points for (2) are the points P_ϵ, S' and S''. Let Q_0 be any of these points and $q_0 \in \overline{\Gamma}_{\mathbf{c}}^{(i)}$ such that $\varphi_{\mathbf{c}}^{(i)}(q_0) = Q_0$. If $z_\alpha \neq 0$ is a chart around the point Q_0, we must show that

$$\lim_{q\to q_0} y_0(t;q) = \lim_{q\to q_0} \frac{1}{z_\alpha}(t;q) \neq 0.$$

Indeed, if it is the case, all $z_j(t;q_0)$ are finite for $|t|$ small and non-zero because $z_j = y_j/y_0$ where $y_j = z_j/z_\alpha$ for $j = 1,\ldots,15$. Let us consider the two intersection points P_ϵ of $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$. We read from (3.9) and (4.1) that the leading coefficient of $z_3(t;\Gamma_{\mathbf{c}}^{(0)})$ has a pole for $\varsigma = 0$, that is maximal with the leading coefficient of $z_9(t;\Gamma_{\mathbf{c}}^{(0)})$, and thus the function z_3 defines a chart about these two points. By substituting (3.4) in the function z_3, we have

$$z_3(t;\Gamma_{\mathbf{c}}^{(0)}) = \frac{2a}{t} + 4c + 2a(c-d)t + \frac{1}{3}(2a^2c + e + 4c(c-3d))t^2 + O(t^3).$$

The first few terms of the inverse of this series are given by

$$\frac{1}{z_3(t;\Gamma_{\mathbf{c}}^{(0)})} = \frac{t}{2a} - \frac{c}{a^2}t^2 + O(t^3), \tag{4.5}$$

by substituting (3.7) in the second term of (4.5) and by rewriting the coefficients in terms of local parameter ς around $\infty_\epsilon \in \overline{\Gamma}_{\mathbf{c}}^{(0)}$, by using (3.9), one finds :

$$\lim_{\varsigma\to 0}\frac{1}{z_3}(t;\varsigma) = -\frac{1}{4}t^2 + O(t^3) \neq 0.$$

which shows that the flow of $\overline{\mathcal{V}}_1$, that starts from the two points P_ϵ, goes immediately into affine.

For the point S', one considers the series $1/z_{15}(t;\Gamma_{\mathbf{c}}^{(1)})$; the choice of z_{15} is based on the fact that $z_{15} \neq 0$ defines a chart in a neighborhood of ∞_1, as can be read off from (4.4). The calculation needs six others terms in the principal balance (3.5). By rewriting the coefficients of the series $1/z_{15}(t;\Gamma_{\mathbf{c}}^{(1)})$ in terms of a local parameter ς around S' by using (3.17), the result is that

$$\lim_{p\to\infty_1}\frac{1}{z_{15}}(t;\Gamma_{\mathbf{c}}^{(1)}) = \frac{1}{288}t^7 + O(t^8).$$

Also, by considering the principal balance (3.4) with six others terms and by using (3.9), we find

$$\lim_{p\to\infty_1}\frac{1}{z_{15}}(t;\Gamma_{\mathbf{c}}^{(0)}) = \frac{1}{288}t^7 + O(t^8),$$

and so the series are not identically zero and it follows that the integral curves of $\overline{\mathcal{V}}_1$ starting at S' go immediately into $\varphi_{\mathbf{c}}(\mathbb{F}_{\mathbf{c}})$. For the points S'', the only non-zero entry corresponds to the function z_{14}; so the function z_{14} defines a chart around S''. We show that the flow of $\overline{\mathcal{V}}_1$, that start from the point S'', goes into affine immediately by checking that the following limits are different from zero.

$$\lim_{p\to\infty}\frac{1}{z_{14}}(t;\Gamma_{\mathbf{c}}^{(0)}) = \lim_{p\to\infty_3}\frac{1}{z_{14}}(t;\Gamma_{\mathbf{c}}^{(1)}) = \frac{1}{288}t^7 + O(t^8).$$

Thus, the integral curves of the holomorphic vector field $\overline{\mathcal{V}}_1$ starting from all points of $\mathcal{D}_{\mathbf{c}}$ go immediately into the affine $\varphi_{\mathbf{c}}(\mathbb{F}_{\mathbf{c}})$.

Finally, we turn to the points (3). We must show that there are no such points. Since $\mathbb{F}_{\mathbf{c}}$ is irreducible and $\varphi_{\mathbf{c}}$ is regular the divisor $\mathcal{D}_{\mathbf{c}}$ is connected. Thus, if $\mathcal{D}_{\mathbf{c}}$ contains irreducible components that are different from $\mathcal{D}_{\mathbf{c}}^{(0)}$, $\mathcal{D}_{\mathbf{c}}^{(1)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$ then at least one of the former ones must intersect one of the latter ones; moreover this must happen at the points (2). Therefore, we must verify that no other divisor passes through each of these points. To do this, we compute the degree of $\mathcal{D}_{\mathbf{c}}$ at the four points P_ϵ, S' and S'' and we show that it coincides with the sum of the multiplicities of each divisor passing through these points.

For the two points P_ϵ, we consider the function $1/z_3$, which is a defining function for the divisor $\mathcal{D}_{\mathbf{c}}$ around P_ϵ. Thereby we know that the vector field $\overline{\mathcal{V}}_1$ is tranversal to $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$ at these points. By rewriting the coefficients of the second term of (4.5) we have, in terms of a local parameter ς in a neighborhood of the points $\infty_\epsilon \in \overline{\Gamma}_{\mathbf{c}}^{(0)}$, that

$$\frac{1}{z_3(t;\Gamma_{\mathbf{c}}^{(0)})} = \frac{1}{4}(2\varsigma t - t^2) + O(t^3, \varsigma t^2),$$

it follows that the multiplicity of $\mathcal{D}_{\mathbf{c}}$ at each of the two points P_ϵ is 2, which coincides with the sum of the order of $1/z_3$ on each of the two intersecting branches.

For the points S', we use the series $1/z_{15}(t;\Gamma_{\mathbf{c}}^{(1)})$. We need the seven leading terms of this series. We rewrite the free parameters in terms of a local parameter ς around ∞_1 (3.9). The resulting series in t and ς should start at degree 3 since the point S' has multiplicity 2 and 1 on the divisor $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(1)}$ respectively and the function z_{15} has a simple pole over each of these divisors. Indeed, we have

$$\frac{1}{z_{15}(t;\Gamma_{\mathbf{c}}^{(1)})} = -\frac{1}{4c_3^2}\varsigma^2 t + O(t^3).$$

For the point S'', we have in terms of a local parameter ς in a neighborhood of $\infty_4 \in \overline{\Gamma}_{\mathbf{c}}^{(1)}$ that

$$\frac{1}{z_{14}(t;\Gamma_{\mathbf{c}}^{(1)})} = -\frac{1}{4}\varsigma^2 t + O(t^3),$$

which shows that there are no others divisors passing through S''. Notice that, since the vector field $\overline{\mathcal{V}}_1$ is only tangent to one of the branches of $\mathcal{D}_{\mathbf{c}}^{(1)}$ which cross at the points S' and S'', we have taken the series expansion along the non-tangent branch.

The conditions of the Complex Liouville Theorem being satisfied, it follows that, for $\mathbf{c} \in \Omega$ the projective variety $\overline{\varphi_{\mathbf{c}}(\mathbb{F}_{\mathbf{c}})} = \varphi_{\mathbf{c}}(\mathbb{F}_{\mathbf{c}}) \cup \mathcal{D}_{\mathbf{c}}$ is an Abelian surface and that the restriction of the vector fields $\overline{\mathcal{V}}_1$ and $\overline{\mathcal{V}}_2$ to these Abelian surfaces is linear. Since $\overline{\varphi_{\mathbf{c}}(\mathbb{F}_{\mathbf{c}})}$ contains a smooth curve of genus 2, it is the Jacobian of this curve. We have therefore proved the following theorem.

Theorem 3. *Let $(\mathcal{H}, \{\cdot,\cdot\}, \mathbb{F})$ denote the integrable system that describes the $\mathfrak{d}_3^{(2)}$-Toda lattice, where $\mathbb{F} = (F_1, F_2, F_3)$ and $\{\cdot,\cdot\}$ are given respectively by (2.3) and (2.2), with commuting vector fields (2.1) and (2.4). The weights of the space variables are given by $\varpi(x_0, x_1, x_2, y_0, y_1, y_2) = (2, 2, 2, 1, 1, 1)$.*

1. $(\mathcal{H}, \{\cdot,\cdot\}, \mathbb{F})$ *is a weight homogeneous algebraic completely integrable system;*

2. *For $\mathbf{c} = (c_1, c_2, c_3) \in \Omega$, the invariant surface $\mathbb{F}_{\mathbf{c}}$ is isomorphic to $\mathbb{T}_{\mathbf{c}}^2 \backslash \mathcal{D}_{\mathbf{c}}$, where*

   (a) $\mathbb{T}_{\mathbf{c}}^2$ is the Jacobian of the hyperelliptic curve (of genus two) $\overline{\Gamma}_{\mathbf{c}}^{(0)}$, defined by

   $$a^4e^2 - (c_1e + 8c_3)a^2e - 4e^3 + c_2e^2 + 4c_1c_3e + 16c_3^2 = 0;$$

   (b) $\mathcal{D}_{\mathbf{c}}$ is a divisor on $\mathbb{T}_{\mathbf{c}}^2$, and consists of three irreducible components $\mathcal{D}_{\mathbf{c}}^{(0)}$, $\mathcal{D}_{\mathbf{c}}^{(1)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$ where $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$ are both smooth curves, isomorphic to $\overline{\Gamma}_{\mathbf{c}}^{(0)}$, while $\mathcal{D}_{\mathbf{c}}^{(1)}$ is a curve of genus three, with two singular points, defined by

   $$32\alpha\beta^2 + (16\alpha^4 - 8c_1\alpha^2 + c_1^2 - 4c_2)\beta - 32c_3\alpha = 0.$$

3. *For $\mathbf{c} = (c_1, c_2, c_3) \in \Omega$, the curves $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$ intersect each other transversally in two points, and each of them intersects the curve $\mathcal{D}_{\mathbf{c}}^{(1)}$ at one singular point of the latter.*

5. GEOMETRY OF THE $\mathfrak{d}_3^{(2)}$ TODA LATTICE

Several elements of the geometry of our integrable system have been established in Theorem 3. In this section, we give other of them.

5.1. *Half-periods on* $\mathbb{T}^2_{\mathbf{c}}$

Proposition 4. *For* $\mathbf{c} \in \Omega$*, the Abelian surface* $\mathbb{T}^2_{\mathbf{c}}$ *admits ten half-periods at infinity : the Weierstrass points on* $\mathcal{D}^{(0)}_{\mathbf{c}}$ *and* $\mathcal{D}^{(2)}_{\mathbf{c}}$ *(in particular the four singular points of the divisor* $\mathcal{D}_{\mathbf{c}}$*) and six half-periods on the affine part* $\mathbb{F}_{\mathbf{c}}$*.*

Proof. Recall that the involution τ flips the sign of the vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$, so that it is the (-1)-involution on the Abelian surface $\mathbb{T}^2_{\mathbf{c}}$. The half-periods on $\mathbb{T}^2_{\mathbf{c}}$ are given by the fixed points of this involution. It acts on the parameters $a, b\,\text{et}\,t$ of the principal balance $x(t; m_0)$ and $x(t; m_2)$ as follows

$$\tau(t, a, e) = (-t, -a, e),$$

therefore the half-periods of $\mathbb{T}^2_{\mathbf{c}}$ that lie on $\mathcal{D}^{(0)}_{\mathbf{c}}$ and $\mathcal{D}^{(2)}_{\mathbf{c}}$ are given by the 10 points corresponding to $a = 0$ and $a = \infty$, six points on on each curve with two common points : the points P_+ et P_- (See Figure 1). The involution τ fixing 16 points on $\mathbb{T}^2_{\mathbf{c}}$, there remain 6 others. By substituting $y_0 = y_1 = y_2 = 0$ in the equations

$$F_i = c_i, \qquad (i = 1, 2, 3),$$

where the F_i are the constants of motion, we obtain the system

$$\begin{cases} x_0x_1x_2 = c_3, \\ -4(x_0 + x_2) - 2x_1 = c_1, \\ 16x_0x_2 + x_1(4x_0 + x_1 + 4x_2) = c_2. \end{cases} \tag{5.1}$$

If we solve the second equation in terms of x_1 and by substituting in the others equations, this leads to

$$\begin{cases} (x_0 - x_2)^2 = \dfrac{c_1^2 - 4c_2}{16}, \\ x_0x_2(4x_0 + 4x_2 + c_1) + c_3 = 0. \end{cases} \tag{5.2}$$

Let us pose $\Delta^2 = c_1^2 - 4c_2$; in this case, we have $x_2 = x_0 \pm \dfrac{\Delta}{4}$. Then the system (5.2) leads to the two equations

$$x_0(4x_0 \pm \Delta)(8x_0 \pm \Delta + c_1) + 8c_3 = 0. \tag{5.3}$$

In both cases, the discriminant of the left member is equal to

$$6912c_3^2 + 288c_1c_2c_3 + 4c_2^3 - c_1^2c_2^2 - 64c_1^3c_3,$$

so that there are no double roots for $\mathbf{c} \in \Omega$. For $\mathbf{c} \in \Omega$, if there exists a common root x_0 for both equations (5.3), the corresponding values of x_2 would be different since $c_1^2 - 4c_2 \neq 0$. Hence, we have verified by direct computation that for $\mathbf{c} \in \Omega$, the (-1)-involution admits precisely six fixed points (half-periods) on the affine part $\mathbb{F}_{\mathbf{c}}$ of Abelian surface $\mathbb{T}^2_{\mathbf{c}}$.

5.2. *The Holomorphic Differentials on* $\mathcal{D}_{\mathbf{c}}$ *and Tangency Locus of* $\mathcal{V}_1$

In order to determine the tangency locus of the vector field $\mathcal{V}_1$ on the divisor $\mathcal{D}_{\mathbf{c}}$, we compute the holomorphic differentials ω_1 and ω_2 on the three irreducible components of $\mathcal{D}_{\mathbf{c}}$ that come from the differentials $\mathsf{d}t_1$ and $\mathsf{d}t_2$ on the Abelian surface $\mathbb{T}^2_{\mathbf{c}}$. We know that all irreducible components have multiplicity 1. Let $\mathcal{D}'$ be one of these components. Let us choose the functions y_0 and y among the functions $z_0, \ldots, z_{15}$ defined in (3.21) such that y_0 has a simple pole on $\mathcal{D}'$ and such that $1/y_0$ and y/y_0 define a holomorphic chart around a generic point of $\mathcal{D}'$. We write the Laurent series of the functions y_0 and y with respect to the component $\mathcal{D}'$ as follows

$$y_0(t; \mathcal{D}') = \frac{y_0^{(0)}}{t} + y_0^{(1)} + O(t), \qquad y(t; \mathcal{D}') = \frac{y^{(0)}}{t} + y^{(1)} + O(t).$$

Because of the above condition on y_0, we have that $y_0^{(0)} \neq 0$. We have (see [16])

$$\begin{pmatrix} \mathsf{d}\left(\frac{1}{y_0}\right) \\ \\ \mathsf{d}\left(\frac{y}{y_0}\right) \end{pmatrix} = \begin{pmatrix} \mathcal{V}_1\left[\frac{1}{y_0}\right] & \mathcal{V}_2\left[\frac{1}{y_0}\right] \\ \\ \mathcal{V}_1\left[\frac{y}{y_0}\right] & \mathcal{V}_2\left[\frac{y}{y_0}\right] \end{pmatrix} \begin{pmatrix} \mathsf{d}t_1 \\ \\ \mathsf{d}t_2 \end{pmatrix} \tag{5.4}$$

which we solve for $\mathsf{d}t_1$ and $\mathsf{d}t_2$, and which we restrict to $\mathcal{D}'$, to find

$$\begin{pmatrix} \omega_2 \\ \\ \omega_1 \end{pmatrix} = \frac{1}{\delta} \begin{pmatrix} \mathcal{V}_2\left[y/y_0\right]_{|\mathcal{D}'} & -\mathcal{V}_2\left[1/y_0\right]_{|\mathcal{D}'} \\ \\ -\mathcal{V}_1\left[y/y_0\right]_{|\mathcal{D}'} & 1/y_0^{(0)} \end{pmatrix} \begin{pmatrix} 0 \\ \\ \mathsf{d}\left(y^{(0)}/y_0^{(0)}\right) \end{pmatrix},$$

where δ is the determinant of the matrix in (5.4), restricted to $\mathcal{D}'$,

$$\delta = \frac{1}{(y_0^{(0)})^2} \begin{vmatrix} y_0^{(0)} & \mathcal{V}_2\left[1/y_0\right]_{|\mathcal{D}'} \\ y_0^{(0)}y^{(1)} - y^{(0)}y_0^{(1)} & \mathcal{V}_2\left[y/y_0\right]_{|\mathcal{D}'} \end{vmatrix},$$

which is non-zero by the above assumptions on $y^{(0)}$ and y. It follows that the holomorphic differentials $\omega_1 = \mathsf{d}t_{2|\mathcal{D}'}$ and $\omega_2 = \mathsf{d}t_{1|\mathcal{D}'}$ are given by

$$\begin{aligned} \omega_1 &= \frac{1}{\delta y_0^{(0)}}\mathsf{d}\left(\frac{y^{(0)}}{y_0^{(0)}}\right), \\ \omega_2 &= -\frac{1}{\delta}\mathcal{V}_2\left[\frac{1}{y_0}\right]_{|\mathcal{D}'}\mathsf{d}\left(\frac{y^{(0)}}{y_0^{(0)}}\right). \end{aligned} \tag{5.5}$$

The zeros of ω_1 and ω_2 provide the points of tangency of the vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively. Notice that since the degree on the canonical bundle on a divisor of genus g is $2g-2$, the tangency locus of $\mathcal{V}_1$ (resp. $\mathcal{V}_2$) consists of $2g-2$ points, including multiplicities.

Having explained the method, which is an idea due to Luc Haine (see [16]), we now turn to the holomorphic differentials on the divisor $\mathcal{D}_{\mathbf{c}}$. In the above notations, we choose $y_0 := z_2$ and $y := z_4$ so that, restricted to $\mathcal{D}_{\mathbf{c}}^{(1)}$, we have

$$\begin{aligned} y_0^{(0)} &= 2, & y_0^{(1)} &= 0, \\ y^{(0)} &= -4\alpha, & y^{(1)} &= 0, \end{aligned}$$

and using (2.4) and (3.5),

$$\mathcal{V}_2\left[\frac{1}{z_2}\right]_{|\mathcal{D}_{\mathbf{c}}^{(1)}} = -\frac{\alpha^2+3\gamma}{2}, \qquad \mathcal{V}_2\left[\frac{z_4}{z_2}\right]_{|\mathcal{D}_{\mathbf{c}}^{(1)}} = 4\alpha(\beta+\frac{c_3}{\beta}).$$

It follows that

$$\delta = \frac{1}{4}\begin{vmatrix} 2 & -\dfrac{\alpha^2+3\gamma}{2} \\ 0 & 4\alpha(\beta+\dfrac{c_3}{\beta}) \end{vmatrix} = 2\alpha(\beta+\frac{c_3}{\beta}).$$

The holomorphic differentials $\mathsf{d}t_1$ and $\mathsf{d}t_2$, restricted to $\mathcal{D}_{\mathbf{c}}^{(1)}$, are therefore given by

$$\omega_1 = -\frac{\beta\mathsf{d}\alpha}{2\alpha(\beta^2+c_3)}, \qquad \omega_2 = -\frac{\beta(4\alpha^2+c_1)}{4\alpha(\beta^2+c_3)}\mathsf{d}\alpha. \tag{5.6}$$

For the computation of the holomorphic differentials on the divisors $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$, let us consider the functions $y_0 = z_2$ and $y = z_3$; these functions restricted to $\mathcal{D}_{\mathbf{c}}^{(0)}$ and $\mathcal{D}_{\mathbf{c}}^{(2)}$, lead to

$$y_0^{(0)} = -2, \qquad y_0^{(1)} = a,$$
$$y^{(0)} = 2a, \qquad y^{(1)} = 4c,$$

and using (2.4) and (3.4) we have

$$\mathcal{V}_2\left[\frac{1}{z_2}\right]_{|\mathcal{D}_{\mathbf{c}}^{(0)}} = -\frac{a^2-4c}{2}, \qquad \mathcal{V}_2\left[\frac{z_3}{z_2}\right]_{|\mathcal{D}_{\mathbf{c}}^{(0)}} = -\frac{a^4}{2} - 2(e-2a^2c-12cd).$$

We find that

$$\begin{aligned}\delta &= \frac{1}{4}\begin{vmatrix} 2 & \dfrac{a^2-4c}{2} \\ 2a^2+8c & \dfrac{a^4}{2}+2(e-2a^2c-12cd)\end{vmatrix} \\ &= e - c(a^2-4c+12d).\end{aligned}$$

Thus, the holomorphic differentials ω_1 and ω_2 on $\mathcal{D}_{\mathbf{c}}^{(0)}$ are given by

$$\omega_1 = \frac{2\mathsf{d}a}{\delta}, \qquad \omega_2 = -\frac{a^2+3c}{2\delta}\mathsf{d}a.$$

Proposition 5. *The vector field* $\overline{\mathcal{V}}_1$ *is*

(i) doubly tangent to one of branches of $\mathcal{D}_{\mathbf{c}}^{(1)}$ *passing through the point* S' *(resp.* S''*) and transversal to the other branch;*

(ii) transversal to $\mathcal{D}_{\mathbf{c}}^{(0)}$ *and* $\mathcal{D}_{\mathbf{c}}^{(2)}$ *at their intersection points* P_ϵ*;*

(iii) doubly tangent to $\mathcal{D}_{\mathbf{c}}^{(0)}$ *(resp.* $\mathcal{D}_{\mathbf{c}}^{(2)}$*) at the point* S' *(resp.* S''*).*

Proof. (i) In a neighborhood of the point S' we have, in terms of a local parameter ς, that $\omega_1 = (-\varsigma^2 + O(\varsigma^3))\mathsf{d}\varsigma$, as follows by substituting (3.18) in the differential ω_1 (5.6), which shows that ω_1 has double zero at the point S'. Also, by rewriting ω_1 in terms of a local parameter ς around the point ∞_1 by using (3.17), we have that $\omega_1 = (-1/(2c_3))\mathsf{d}\varsigma$ is non-zero. It follows that the vector field $\overline{\mathcal{V}}_1$ is only doubly tangent to one of the branches of $\mathcal{D}_{\mathbf{c}}^{(1)}$ which cross at the point S' and tranversal to the other branch. This is done similary at the point S'' by using the involution π which preserves $\mathcal{V}_1$.

(ii) and (iii): If we write ω_1, restricted to $\mathcal{D}_{\mathbf{c}}^{(0)}$, in terms of a local parameter ς in a neighborhood of the two points ∞_ϵ, by using (3.7) and (3.9), then we find

$$\omega_1 = -\frac{2\epsilon}{\eta}\mathsf{d}\varsigma,$$

where $\epsilon = \pm 1$ and η is a fixed square root of $c_1^2 - 4c_2$. This shows that ω_1 does not vanish, so that the vector field $\overline{\mathcal{V}}_1$ is tranversal to $\mathcal{D}_{\mathbf{c}}^{(0)}$ at these two points P_ϵ. Using the involution π, it follows that $\overline{\mathcal{V}}_1$ is also transversal to $\mathcal{D}_{\mathbf{c}}^{(2)}$ at these points. Finally, in a neighborhood of the point ∞, by using (3.10), we have

$$\omega_1 = -4\varsigma^2\mathsf{d}\varsigma,$$

whence the vector field $\overline{\mathcal{V}}_1$ is doubly tangent to $\mathcal{D}_{\mathbf{c}}^{(0)}$ at S'; by the involution π, it is also doubly tangent to $\mathcal{D}_{\mathbf{c}}^{(2)}$ at S''.

6. MORPHISM TO MUMFORD SYSTEM, LAX EQUATION AND LINEARIZATION

In this section, we show the link between the $\mathfrak{d}_3^{(2)}$-Toda lattice and the Mumford system [13] by using a method developed by Vanhaecke [17]. We give an explicit morphism of integrable systems between this systems. We finish the section by given a new Lax equation and a explicit linearization of our integrable system.

The (-1)-involution sur $\mathbb{T}^2_{\mathbf{c}}$ leads to a singular quotient $\mathbb{T}^2_{\mathbf{c}}/(-1)$ which is called the Kummer surface of $\mathbb{T}^2_{\mathbf{c}}$. In order to use Vanhaecke's method, it is necessary to find an equation of this surface. Since one of the components of $\mathcal{D}_{\mathbf{c}}$ is isomorphic to a smooth curve, say $\overline{\Gamma}_{\mathbf{c}}$, of genus two, the sections of the line bundle $\left[2\mathcal{D}^{(0)}_{\mathbf{c}}\right]$ embeds the Kummer surface of $\mathbb{T}^2_{\mathbf{c}}$ in the projective space $\mathbb{P}^3$. To construct this map ψ, we consider the meromorphic functions on $\mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}})$ which have at worst a double pole along $\mathcal{D}_{\mathbf{c}}$, isomorphic to $\overline{\Gamma}_{\mathbf{c}}$. In others terms, we look for a basis of polynomials on $\mathcal{H}$ which have a double pole in t when the principal balance (3.4) is substituted into them and no poles when the other principal balances are substituted. It is easily shown that the space of such polynomials is spanned by

$$\begin{aligned} &\theta_0 := 1, && \theta_2 := (y_0^2 - x_1 - 4x_0)x_2, \\ &\theta_1 := x_2, && \theta_3 := x_1 x_2^2. \end{aligned} \tag{6.1}$$

Let us consider the Kodaira map corresponding of these functions

$$\begin{aligned} \psi_{\mathbf{c}}: \quad \mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}}) &\to \qquad \mathbb{P}^3 \\ (x_0, \dots, y_2) &\mapsto (\theta_0 : \theta_1 : \theta_2 : \theta_3). \end{aligned}$$

This latter maps the surface $\mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}})$ to its Kummer surface, which is a singular quartic in $\mathbb{P}^3$. The map $\psi_{\mathbf{c}}$ induces on $\Gamma_{\mathbf{c}}$ the map $\psi^{(0)}_{\mathbf{c}} : (a, e) \mapsto (0 : 1 : (a^2 e - 4c_3)/e : e)$ by taking the coefficients of t^{-2} in the Laurent series $\theta_i(t; m_0)$ obtained by substituting the principal balance $x(t; m_0)$ in the funtions θ_i, $i = 0, \dots, 3$. Let ∞ be the Weierstrass point (3.10) on $\overline{\Gamma}_{\mathbf{c}}$, we have

$$\begin{aligned} \psi^{(0)}_{\mathbf{c}}(\infty) &= \lim_{\varsigma \to 0} (0 : 4\varsigma^4 : 4\varsigma^2 + O(\varsigma^6) : 1 - c_1\varsigma^2 + O(\varsigma^4)) \\ &= (0 : 0 : 0 : 1), \end{aligned}$$

so the basis $(\theta_0, \theta_1, \theta_2, \theta_3)$ is suitably selected (see [17, Theorem 9]). An equation for this quartic surface can be computed by eliminating the variables x_0, x_1, x_2, y_0, y_2 from (6.1) and from the equations

$$\begin{aligned} &x_0 x_1 x_2 = c_3, \\ &y_0^2 + y_2^2 - 4x_0 - 2x_1 - 4x_2 = c_1, \\ &(y_0^2 - 4x_0)(y_2^2 - 4x_2) - x_1(2y_0 y_2 - 4x_0 - x_1 - 4x_2) = c_2, \end{aligned} \tag{6.2}$$

where the first members are the constants of motion F_1, F_2 and F_3. From (6.1) we have

$$x_0 = \frac{c_3 \theta_1}{\theta_3}, \quad x_1 = \frac{\theta_3}{\theta_1^2}, \quad x_2 = \theta_1. \tag{6.3}$$

Consequently from the second equation of (6.1), after from the second equation de (6.2) one draws respectively

$$y_0^2 = \frac{1}{\theta_1^2 \theta_3}(\theta_1\theta_2\theta_3 + \theta_3^2 + 4c_3\theta_1^3) \quad \text{et} \quad y_2^2 = \frac{1}{\theta_1^2}(\theta_3 + c_1\theta_1^2 - \theta_1\theta_2 + \theta_1^3). \tag{6.4}$$

Let us rewrite the last equation of (6.2) as follows

$$2x_1 y_0 y_2 = \left((y_0^2 - 4x_0)(y_2^2 - 4x_2) - c_2\right) + x_1(4x_0 + x_1 + 4x_2).$$

Upon taking the square of each member, and upon subtituting (6.3), (6.4) into it, we obtain the equation of the Kummer surface of Jac($\overline{\Gamma}_{\mathbf{c}}$); it can put in the form

$$((4\theta_1 + c_1)^2 - 4(4\theta_2 + c_2))\theta_3^2 + 2f_3(\theta_1, \theta_2)\theta_3 + f_4(\theta_1, \theta_2) = 0, \tag{6.5}$$

where f_3 (respectively f_4) is a polynomial of degree three (respectively four) in θ_1 and θ_2, given by

$$f_3(\theta_1, \theta_2) = (4\theta_1 + c_1)(\theta_2(c_1\theta_1 - \theta_2) - \theta_1(c_2\theta_1 + 4c_3)) + 8c_3\theta_2,$$
$$f_4(\theta_1, \theta_2) = (c_2\theta_1^2 + \theta_2^2 - 4c_3\theta_1 - c_1\theta_1\theta_2)^2.$$

6.1. Morphism to Mumford System and Lax Equation

Let us consider the coefficient of θ_3^2 in the equation (6.5) in terms of initial variables x_i et y_i, to wit

$$\Delta = (4x_2 + c_2)^2 - 4(4x_2(y_0^2 - 4x_0 - x_1y_0) + c_3).$$

Let $u(\lambda)$ be a monic polynomial in λ whose discriminant is Δ, we have

$$\begin{aligned} u(\lambda) &= \lambda^2 + (4x_2 + c_1)\lambda + 4x_2(y_0^2 - 4x_0 - x_1) + c_2 \\ &= \lambda^2 + (y_0^2 + y_2^2 - 4x_0 - 2x_1)\lambda + (x_1 - y_0y_2)^2 - 4x_0(y_2^2 - x_1). \end{aligned}$$

Let s_1 ant s_2 be the roots of the polynomial $u(\lambda)$, we have

$$s_1 + s_2 = -4x_2 - c_1, \qquad s_1s_2 = 4x_2(y_0^2 - 4x_0 - x_1) + c_2. \tag{6.6}$$

which imply,with respect to the vector field $\mathcal{V}_1$ (2.1) that

$$\dot{s}_1 + \dot{s}_2 = -4x_2y_2, \qquad s_2\dot{s}_1 + s_1\dot{s}_2 = 4x_2(y_2(y_0^2 - 4x_0) - x_1y_0). \tag{6.7}$$

Let $v(\lambda)$ be the polynomial defined (up to a multiplicative constant) as being the derivative of the polynomial $u(\lambda)$ with respect to the vector field $\mathcal{V}_1$, we have

$$v(\lambda) = i[4x_2y_2\lambda + 4x_2(y_2(y_0^2 - 4x_0) - x_1y_0)],$$

one verifies, by a direct computation, that the expression $f(\lambda) - v^2(\lambda)$ is divisible by $u(\lambda)$ where

$$f(\lambda) = (\lambda^3 + c_1\lambda^2 + c_2\lambda - 16c_3)(\lambda^2 + c_1\lambda + c_2).$$

Notice that the affine curve $y^2 = f(\lambda)$ is (birational to) the affine curve $\Gamma_{\mathbf{c}}$, by adding the three Weierstrass points at infinity (see (3.11)).

We now define a morphism $\phi : \mathcal{H} \to \mathbb{C}^7$ from the $\mathfrak{d}_3^{(2)}$-Toda lattice to the genus 2 odd Mumford system, where $\mathbb{C}^7$ is the phase space of the Mumford system. The homomorphism ϕ is explicitly given by the map

$$(x_0, x_1, x_2, y_0, y_2) \mapsto \begin{cases} u(\lambda) = \lambda^2 + u_1\lambda + u_0, \\ v(\lambda) = v_1\lambda + v_0, \\ w(\lambda) = \lambda^3 + w_2\lambda^2 + w_1\lambda + w_0, \end{cases} \tag{6.8}$$

where

$$\begin{aligned} u_1 &= y_0^2 + y_2^2 - 4x_0 - 2x_1, & v_1 &= 4ix_2y_2, \\ u_0 &= (x_1 - y_0y_2)^2 - 4x_0(y_2^2 - x_1), & v_0 &= 4ix_2(y_2(y_0^2 - 4x_0) - x_1y_0), \end{aligned}$$

$$\begin{aligned} w_2 &= y_0^2 + y_2^2 - 4x_0 - 8x_2 - 2x_1 = u_1 - 8x_2, \\ w_1 &= (x_1 - y_0y_2)^2 + 8(4x_0 + x_1 + 2x_2 - y_0^2)x_2 - 4x_0(y_2^2 - x_1), \\ w_0 &= -16x_2(x_0x_1 - x_2(y_0^2 - 4x_0)). \end{aligned}$$

The polynomial $w(\lambda)$ is the polynomial of degree 2 defined by $w(\lambda) = (f(\lambda) - v^2(\lambda))/u(\lambda)$. Of course this map is regular; moreover it is birational to its image. Next it is easy to check that

$$\begin{aligned}
\phi^*(H_0) &= \phi^*(u_0 w_0 + v_0^2) = -16F_2F_3,\\
\phi^*(H_1) &= \phi^*(u_1 w_0 + u_0 w_1 + 2v_1 v_0) = -16F_1F_3 + F_2^2,\\
\phi^*(H_2) &= \phi^*(u_0 w_2 + u_1 w_1 + w_0 + v_1^2) = 2F_1F_2 - 16F_3,\\
\phi^*(H_3) &= \phi^*(u_0 + w_1 + u_1 w_2) = 2F_2 + F_1^2,\\
\phi^*(H_4) &= \phi^*(u_1 + w_2) = 2F_1,
\end{aligned}$$

where $H_0, \ldots, H_4$ are the constants of motion of the Mumford system. We obtain a new Poisson structure on $\mathbb{C}^7$ for the Mumford system, given by skew-symmetric matrix $X := M - M^\top$ where

$$M := \begin{pmatrix}
0 & 0 & \{u_1,v_1\} & \{u_1,v_0\} & \{u_1,w_2\} & \{u_1,w_1\} & \{u_1,w_0\}\\
0 & 0 & \{u_0,v_1\} & \{u_0,v_0\} & \{u_0,w_2\} & \{u_0,w_1\} & \{u_0,w_0\}\\
0 & 0 & 0 & 0 & \{v_1,w_2\} & \{v_1,w_1\} & \{v_1,w_0\}\\
0 & 0 & 0 & 0 & \{v_0,w_2\} & \{v_0,w_1\} & \{v_0,w_0\}\\
0 & 0 & 0 & 0 & 0 & \{w_2,w_1\} & \{w_2,w_0\}\\
0 & 0 & 0 & 0 & 0 & 0 & \{w_1,w_0\}\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The Poisson brackets of this matrix are given by

$$\begin{aligned}
\{u_1,v_1\} &= \frac{i}{8}((u_1+w_2)^2 - 4(u_1^2 - u_0 + w_1)), & \{u_1,w_0\} &= i(u_1 - w_2)v_0,\\
\{u_1,v_0\} &= -\frac{i}{16}[(u_1-w_2)(u_1^2 - w_2^2 + 4u_0 + 4w_1) + 8(v_1^2 + w_0)], & \{u_0,v_1\} &= \{u_1,v_0\},\\
\{u_1,w_2\} &= -2iv_1, & \{u_0,v_0\} &= \star,\\
\{u_1,w_1\} &= i(u_1-w_2)v_1 - 2iv_0, & \{u_0,w_2\} &= -2iv_0,
\end{aligned}$$

$$\begin{aligned}
\{u_0,w_1\} &= -i(u_1+w_2)v_0 - 2iu_0v_1,\\
\{u_0,w_0\} &= \frac{i}{8}\big[(u_1-w_2)^2(v_1(u_1+w_2)+2v_0) + 4(u_0-w_1)(v_1(w_2-u_1)+2v_0))\\
&\qquad + 8v_1(v_1^2+w_0)\big],\\
\{v_1,w_2\} &= \frac{i}{8}((3u_1-w_2)^2 - 4(u_1^2+u_0-w_1)),\\
\{v_1,w_1\} &= -\frac{i}{16}[(u_1-w_2)(3u_1^2+w_2^2+4u_1w_2+4w_1-12u_0) + 8(v_1^2-w_0)],\\
\{v_1,w_0\} &= \frac{i}{16}(u_1-w_2)\left[(u_1+w_2)(u_1^2-w_2^2-4(u_0-w_1)) + 8v_1^2\right],\\
\{v_0,w_2\} &= -\frac{i}{16}[(u_1-w_2)(u_1^2-w_2^2+4w_1-12u_0) + 8(v_1^2-w_0)],\\
\{v_0,w_1\} &= *,\\
\{v_0,w_0\} &= \frac{i}{16}\big((u_1-w_2)^2(u_1+w_2)(u_0-w_1) - 2w_0(u_1^2+w_2^2) - 4(u_1-w_2)(u_0-w_1)^2\\
&\qquad + 8v_1^2(u_0-w_1) + 4u_1w_0w_2\big),\\
\{w_2,w_1\} &= i(u_1-w_2)v_1,\\
\{w_2,w_0\} &= i(u_1-w_2)v_0,
\end{aligned}$$

$$\{w_1, w_0\} = -\frac{i}{8}\big[(u_1-w_2)^2(v_1(u_1+w_2)+2v_0) - 4(u_0-w_1)(v_1(u_1-w_2)+2v_0) \\ - 8v_1(w_0-v_1^2)\big],$$

where

$$\{u_0, v_0\} = -\frac{i}{16}[(u_1-w_2)(((u_1+w_2)(u_1(u_1-w_2)-2u_0)+4u_1w_1) \\ + 8(u_0(u_0-w_1)+u_1(w_0+v_1^2))],$$

$$\{v_0, w_1\} = -\frac{i}{16}[(u_1-w_2)(w_2(u_1^2-w_2^2)-2u_0(3w_2-u_1)+4w_1w_2) \\ + 8(w_2v_1^2-w_0u_1-u_0(u_0+w_1))].$$

One easily verifies that the functions $H_0, \ldots, H_4$ are in involution for this Poisson structure and that the functions H_4 and $H_5 = 4H_3 - H_4^2$ generate the two commutating vector fields of the odd Mumford system on $\mathbb{C}^7$. Since the Poisson brackets are weight homogeneous, it is clear that this new structure is not a linear combinaison of the three linear Poisson structures known [14] ; moreover this Poisson structure is not compatible with them.

Theorem 4. *A Lax representation of the vector field $\mathcal{V}_1 = \mathcal{X}_{F_1}$ is given by*

$$\frac{d}{dt}\begin{pmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda)\end{pmatrix} = -\frac{\sqrt{-1}}{2}\left[\begin{pmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda)\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ b(\lambda) & 0\end{pmatrix}\right]$$

where $b(\lambda) = \lambda - 8x_2$ is the polynomial part of the rational function $w(\lambda)/u(\lambda)$.

6.2. Linearization and Integration

We now proceed to the linearization of the integrable system, by using the method developed by Vanhaecke. Since in a one hand we have $f(x) - v^2(x) = u(x)w(x)$, and in an other hand the variables s_1 and s_2 are roots of the polynomial $u(x)$, we have from (6.6)

$$\begin{aligned} \sqrt{f(s_k)} &= v(s_k), \\ &= 4ix_2y_2s_k + 4ix_2(y_2(y_0^2-4x_0)-x_1y_0), \qquad k = 1,2 \\ &= -i(\dot{s}_1+\dot{s}_2)s_k + i(s_2\dot{s}_1 + s_1\dot{s}_2). \end{aligned}$$

So that

$$\sqrt{f(s_1)} = -i(s_1-s_2)\dot{s}_1, \quad \text{et} \quad \sqrt{f(s_2)} = i(s_1-s_2)\dot{s}_2.$$

It follows that, in terms of the variables s_1 and s_2 the differential equations (2.1) of the vector $\mathcal{V}_1$ can be written in the form

$$\begin{cases} \dfrac{\mathsf{d}s_1}{\sqrt{f(s_1)}} + \dfrac{\mathsf{d}s_2}{\sqrt{f(s_2)}} = 0, \\ \dfrac{s_1\mathsf{d}s_1}{\sqrt{f(s_1)}} + \dfrac{s_2\mathsf{d}s_2}{\sqrt{f(s_2)}} = i\mathsf{d}t, \end{cases} \tag{6.9}$$

where $f(x) = (x^3 + c_1x^2 + c_2x - 16c_3)(x^2 + c_1x + c_2)$. The form (6.9) is equivalent to

$$\frac{\mathsf{d}}{\mathsf{d}t}\left(\sum_{k=1}^{2}\int_{0_k}^{Q_k}\overrightarrow{\omega}\right) = \begin{pmatrix} 0 \\ i \end{pmatrix},$$

where $\overrightarrow{\omega} = \left(\frac{\mathrm{d}x}{\sqrt{f(x)}}, \frac{x\mathrm{d}x}{\sqrt{f(x)}}\right)^{\top}$ is a basis for holomorphic differentials on $\overline{\Gamma}_{\mathbf{c}}$, $Q_1 := \left(s_1, \sqrt{f(s_1)}\right)$, $Q_2 := \left(s_2, \sqrt{f(s_2)}\right)$ two points of $\Gamma_{\mathbf{c}}$ and $Q_1 + Q_2 = \left(s_1, \sqrt{f(s_1)}\right) + \left(s_2, \sqrt{f(s_2)}\right)$ viewed as a divisor on the genus 2 hyperelliptic curve $\Gamma_{\mathbf{c}}$. Thus, by integrating (6.9), we see that the flow of $\overline{\mathcal{V}}_1$ is linear on the Jacobian of this curve $\overline{\Gamma}_{\mathbf{c}}$. By using [13, Theorem 5.3], one shows that the symmetric functions s_1 and s_2, and hence the original phase variables can be written in terms of theta functions.

ACKNOWLEGMENTS

This work is a part of my Ph.D. thesis at the University of Poitiers (France). I would like to sincerely thank my advisor Pol Vanhaecke.

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 356–373. © Pleiades Publishing, Ltd., 2011.

Isometric Embeddings of Infinite-dimensional Grassmannians

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Received May 9, 2010; accepted February 1, 2011

Abstract—We investigate geometric properties of the (Sato–Segal–Wilson) Grassmannian and its submanifolds, with special attention to the orbits of the KP flows. We use a coherent-states model, by which Spera and Wurzbacher gave equations for the image of a product of Grassmannians using the Powers–Størmer purification procedure. We extend to this product Sato's idea of turning equations that define the projective embedding of a homogeneous space into a hierarchy of partial differential equations. We recover the BKP equations from the classical Segre embedding by specializing the equations to finite-dimensional submanifolds.
We revisit the calculation of Calabi's diastasis function given by Spera and Valli again in the context of C^*-algebras, using the τ-function to give an expression of the diastasis on the infinite-dimensional Grassmannian; this expression can be applied to the image of the Krichever map to give a proof of Weil's reciprocity based on the fact that the distance of two points on the Grassmannian is symmetric. Another application is the fact that each (isometric) automorphism of the Grassmannian is induced by a projective transformation in the Plücker embedding.

MSC2010 numbers: 37K20, 22R67, 47L90, 53C24

DOI: 10.1134/S1560354711030099

Keywords: Calabi's diastasis function, Canonical Anticommutation Relations, Universal Grassmann Manifold, Weil reciprocity, τ-function

INTRODUCTION

The connection between infinite-dimensional Grassmannians and KdV- or, more generally, KP-type equations is by now classical (see, e.g., [1, 2] and references therein). Sato's interpretation of the KP hierarchy as the Plücker equations for the Grassmannian motivated the discovery of new links between finite-dimensional representation theory and quantized, C^*-algebra, versions [3, 4].

This note focuses on applications of integrability to the geometric and metric properties of the infinite-dimensional Grassmann manifold. Sato's formal setting is the simplest to describe, and it will be used; but the functional-theoretic setting, particularly the one introduced by Spera et al. [4–7] in which a C^*-algebra acts on the Grassmannian, allows us to obtain new results.

We review the construction [4] of the Determinant and Pfaffian bundles over the Grassmannian, and the "boson-fermion" correspondence, comparing the approach in [7] and the "Sato approach" [1]. Through this correspondence, the defining equations of the embedded manifolds have been related to various integrable hierarchies. In the framework of [1], the (representation-theoretic) description of an orbit translates into PDEs via the τ-function expression of the Plücker coordinates, which turn out to be derivatives of the τ-function. But there is a different approach ("baby-KP"), that uses the embedding algebraically to characterize polynomial tau functions of the KP hierarchy. We derive the BKP hierarchy (see e.g. [8]) in each approach, from the Pfaffian line bundle, via the Segre map introduced in [4].

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One metric property that the Spera–Valli theory introduced is the local invariant defined for a Kähler manifold by Calabi [9], who called it "diastasis" because of its relation with the geodesic distance. Calabi's study of the diastasis was motivated by the problem of complex-isometric embeddings: specifically, he generalized a result of L. Bieberbach (1932) who produced a model of nonsingular, complete surface in a separable Hilbert space, isomorphic to the hyperbolic disc, and proved that the whole group of Möbius isometries can be induced by affine isometries of the ambient Hilbert space. More recently, algebraic invariants of subvarieties of projective space were used to express their (Fubini–Study) curvature [10, 11]. On the other hand, work on geometric quantization [12] reintroduced the diastasis in the context of a "coherent-states map", which embeds a Kähler manifold in the projectivized space of sections of a given holomorphic bundle. This is ideally suited to Sato's Plücker embedding. Starting with the expression of the diastasis in unitary coordinates, Spera and Valli show the rigidity of the embedding [7, Th. 4.1], also observed by Calabi, which we reinterpret algebro-geometrically, for the Grassmannian and Segre manifolds; it seems to us surprising that isomorphisms of these manifolds which were produced by differential algebra are in fact restrictions of projective maps. Our methods thus differ from the loop-group-theoretic ones used by Dorfmeister et al. (cf. [13], e.g.) to study (iso)metric and homogeneous properties of the Segal–Wilson Grassmannian.

We review the construction of Sato's Grassmannian, as well as the coherent-states approach, Determinant and Pfaffian bundles, and boson-fermion correspondence in both settings, in Section 1. Section 2 contains the results on the diastasis, and Section 3 the applications.

1. DETERMINANTS

1.1. Sato's Universal Grassmann Manifold (UGM)

The geometry and representation theory of Sato's infinite-dimensional Grassmannian (UGM) was importantly related to PDEs (Partial Differential Equations) by Sato's result [2]. We recall that setting vis-à-vis the finite-dimensional theory so as to extend it below.

First we define the UGM. The Grassmannian can be studied in two categories: formal, or analytic. Both approaches are used below; the former is appropriate in the model of projective geometry, Plücker and Segre embeddings; the latter is needed to use results that hold for C^* algebras. The type used will be clear from the context. The field of coefficients will be the complex numbers $\mathbb{C}$, unless specified to be the reals $\mathbb{R}$.

Let V be a $\mathbb{C}$-vector space, $\dim V = m + n = N$; we recall the Plücker embedding of the Grassmann manifold $\mathrm{Gr}(m, V)$ of m-dimensional subspaces of V, also abbreviated $\mathrm{Gr}(m, N)$,

$$\mathrm{Gr}(m, V) = \{m\text{−frames in } V\}/GL(m) \hookrightarrow \mathbb{P}(\wedge^m V),$$

$$\pi^{(0)}, \ldots, \pi^{(m-1)} \mapsto \pi^{(0)} \wedge \ldots \wedge \pi^{(m-1)}.$$

If we fix a basis $e_0, \ldots, e_{N-1}$ of V, and write $\pi^{(i)} = \pi_{0,i}e_0 + \ldots + \pi_{N-1,i}e_{N-1}$, then $\pi^{(0)} \wedge \ldots \wedge \pi^{(m-1)} = \sum\limits_{0\leqslant \ell_0<\ldots<\ell_{m-1}<N} \pi_{\ell_0\ldots\ell_{m-1}} e_{\ell_0} \wedge \ldots \wedge e_{\ell_{m-1}}$ with $\pi_{\ell_0\ldots\ell_{m-1}} = \det(\pi_{\ell_i,j})_{i,j=0,\ldots,m-1}$.

Fact I. A point in the ambient $\mathbb{P}(\wedge^m V)$ lies in the embedded $\mathrm{Gr}(m, V) \Leftrightarrow$ its projective coordinates $\pi_{\ell_0\ldots\ell_{m-1}}$ $(0 \leqslant \ell_i < N)$ satisfy the Plücker relations (PR):

$$\sum_{i=0}^{m} (-1)^i \pi_{k_0\ldots k_{m-2}\ell_i} \pi_{\ell_0\ldots\hat{\ell}_i\ldots\ell_m} = 0. \tag{PR}$$

Therefore,

$$\mathrm{Gr}(m, V) = (\widetilde{\mathrm{Gr}}(m, V)\backslash\{0\})/GL(1)$$

where:
$\widetilde{\mathrm{Gr}}(m, V) = \{(\pi_Y)_{Y\subset\Delta_{mN}}$ satisfying the Plücker relations$\}$ is a line bundle over $\mathrm{Gr}(m, V)$, Y is a Young diagram consisting of rows of length $(\ell_{m-1} - (m - 1), \ldots, \ell_1 - 1, \ell_0)$, so it is contained in the rectangle Δ_{mN}.

Fact II. Let $m \leqslant m', n \leqslant n', N' = m' + n'$

(i) If $(\pi'_Y)_{Y \subset \Delta_{m'N'}}$ satisfies (PR), so does its restriction to Y's within Δ_{mN}

(ii) If $(\pi_Y)_{Y \subset \Delta_{mN}}$ satisfies (PR), so does $(\pi'_Y)_{Y \subset \Delta_{m'N'}}$ where $\pi'_Y = 0$ unless $Y \subset \Delta_{mN}$, yielding the commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{Gr}}(m', N') & \overset{\text{project}}{\longrightarrow} & \widetilde{\mathrm{Gr}}(m, N) \\ \downarrow \text{identity} & & \downarrow \text{identity} \\ \widetilde{\mathrm{Gr}}(m', N') & \overset{\text{embed}}{\hookleftarrow} & \widetilde{\mathrm{Gr}}(m, N) \end{array}$$

Define: $\mathrm{Gr} = (\widetilde{\mathrm{Gr}} \backslash \{0\})/GL(1)$ where $\widetilde{\mathrm{Gr}} = \{(\pi_Y)_{Y \text{ all Young diagrams}}$ satisfy all Plücker relations$\}$:

$$\begin{array}{ccc} \widetilde{\mathrm{Gr}} & \overset{\text{project}}{\longrightarrow} & \widetilde{\mathrm{Gr}}(m, N) \\ \uparrow \text{dense} & & \downarrow \text{identity} \\ \widetilde{\mathrm{Gr}}^{\text{fin}} & \overset{\text{embed}}{\hookleftarrow} & \widetilde{\mathrm{Gr}}(m, N) \end{array}$$

where $\widetilde{\mathrm{Gr}}^{\text{fin}} = \{(\xi) \in \widetilde{\mathrm{Gr}} : \xi_Y = 0 \text{ for almost all } Y\} = \bigcup_{m,N} \widetilde{\mathrm{Gr}}(m, N)$.

Time deformations can be defined by:

$\pi_Y(t) := \sum_{\text{all } Y'} \chi_{Y'/Y}(t) \pi_{Y'}$ where $\chi_{Y'/Y}(t) := \det(p_{\ell'_i - j}(t))$

$p_0(t) = 1$, $p_n(t) := \sum_{\nu_1 + 2\nu_2 + 3\nu_3 + \ldots = n} t_1^{\nu_1} t_2^{\nu_2} \ldots / (\nu_1! \nu_2! \ldots)$.

Write $\chi_{Y/\emptyset}$ as χ_Y, $\chi_Y(t) = \det(p_{\ell_i - j}(t))$ are the Schur functions.

To connect with the KP hierarchy, let $w_n(x, t) := (-1)^n \frac{\pi_{\Delta_{n,1}}(x+t)}{\pi_\emptyset(x+t)}$ where $x + t = (x + t_1, t_2, \ldots)$ and $S := 1 + w_1(x, t)\partial^{-1} + \ldots$ a formal pseudodifferential operator.

Note. The Plücker coordinate

$$\pi_\emptyset(t) = \sum_{\text{all } Y} \chi_Y(t) \pi_Y = \tau(\pi, t)$$

is a 'generating function' for Plücker coordinates,

$$\pi_Y(t) = \chi_Y(\partial_t) \pi_\emptyset(t) \qquad \partial_t := (\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots)$$

By reducing to $\mathrm{Gr}(m, N)$ and checking, we see that every $\pi_Y(t)$ satisfies the Plücker relations, so we have a dynamical system on $\widetilde{\mathrm{Gr}}$, which satisfies the KP hierarchy: if

$$\mathcal{L} = S \partial S^{-1}$$

then $\partial_{t_n} S = B_n S - S\partial^n \iff [\partial_{t_n} - B_n, \partial_{t_k} - B_k] = 0 \iff \partial_{t_n} \mathcal{L} = [(\mathcal{L}^n)_+, \mathcal{L}]$, where $B_n := (S \partial^n S^{-1})_+$.

Lastly, introducing Hirota's bilinear operator:

$$\partial_{t_n} F \cdot F = \left(\frac{\partial}{\partial t'_n} - \frac{\partial}{\partial t_n} \right) F(t) F(t') \Big|_{t=t'}, \quad t = (t_1, t_2, \ldots),\ t' = (t'_1, t'_2, \ldots),$$

Conclusion [2]**:**

Although any $f(t) \in \mathbb{C}[[t_1, t_2, \ldots]]$ admits a formal expression of the form $f(t) = \sum_Y c_Y \chi_Y(t)$, where the coefficients are

$$c_Y = \chi_Y(\partial_t) f(t)|_{t=0},$$

it represents the τ- function for some $\pi \in \widetilde{\mathrm{Gr}} \Longleftrightarrow$ its coefficients satisfy the differential Plücker relations (dPR),

$$\sum_{i=0}^{m}(-1)^i \chi_{k_0\ldots k_{m-1}\ell_i}\left(\frac{\partial_t}{2}\right)\chi_{\ell_0\ldots\hat{\ell}_i\ldots\ell_m}\left(-\frac{\partial_t}{2}\right)\tau\cdot\tau = 0 \tag{dPR}$$

which is the KP hierarchy in Hirota bilinear form.

Now we recall the Borel–Weil theorem which classifies the irreducible representations of a (real) compact Lie group (whose complexification we call G for short, though in [14] the notation is $G_{\mathbb{C}}$) in the finite-dimensional setting: we refer to [14, Ch. 2, Ch. 11] for details as well as the infinite-dimensional extension of the theory.

Viewing the (complex) Grassmannian as a homogeneous space, with the [14] notations for the unitary groups, $U(k)$ (complex-entry matrices which equal the inverse of their complex-conjugate transpose) and

$$\mathrm{Gr}(k,n) \cong GL_n(\mathbb{C})/U(k)\times U(n-k),$$

the theorem says that the representation spaces in question are the spaces of holomorphic sections of L_λ, the line bundle associated to λ, the opposite of a dominant weight.

The consequence that most matters to us pertains to the embedding of $\mathrm{Gr}(k,n)$ in projective space given by the dual Det^* of the determinant line bundle: The space of sections of Det^* on $\mathrm{Gr}(k,V)$ is naturally isomorphic to $\wedge^k(V^*)$, where V is an n-dimensional space containing the elements of the Grassmannian.

In this finite-dimensional setting, the Plücker relations give quadrics whose intersection defines the image of the Grassmannian in the embedding by Det^*.

The tau function τ defined above as $\pi_\emptyset(t)$ is the section of the dual of the (tautological) determinant bundle of Sato's infinite-dimensional Grassmann manifold [1], [15]. Over the "big cell", it is possible to normalize the coordinates on the Grassmannian and view τ as a function (of the Plücker coordinates) — this is our tacit assumption throughout the paper — and express it via Schur functions.

1.2. Canonical Anticommutation Relations (CAR)-algebra Model

The Sato Grassmannian Gr will be thought of as embedded into the Hilbert-space (or restricted) Grassmannian of Segal–Wilson (analytic approach). We briefly recall the main points of the C^*-algebraic approach to Hilbert space Grassmannians and Determinant line bundles developed in [3, 4, 16]. As general references on the C^*-algebra theory employed here one may consult, e.g. [17, 18].

We list some basic definitions and properties for the reader's convenience. A *C^*-algebra* A is a Banach algebra equipped with an isometric antilinear involution (or $*$-operation) $x \mapsto x^*$ such that $(xy)^* = y^*x^*$ and $\| x^*x \| = \| x \|^2$ (the latter property characterizes C^*-algebras within involutive Banach algebras). Specifically, a C^*-algebra can always be realized as a norm-closed involutive subalgebra of $B(H)$ (bounded linear operators of a Hilbert space H). A *state* ω on A is a positive linear functional over A of norm one (namely $\omega(x^*x) \geqslant 0\ \forall x \in A$ and $\| \omega \| := \sup_{\|x\|=1}\omega(x) = 1$). A cornerstone of the theory is the Gelfand–Naimark–Segal(GNS)-construction, which associates to any state ω a representation ρ_ω of A by bounded linear operators on a Hilbert space H_ω, such that $\omega(x) = \langle \rho_\omega(x)\xi_\omega, \xi_\omega\rangle$, with $\xi_\omega \in H_\omega$ being a (unit norm) *cyclic vector* (namely, $\rho_\omega(A)\xi_\omega$ is dense in H_ω). As a simple yet fundamental example consider $C^0(\Omega)$, the C^*-algebra of continuous functions on a compact space Ω, acting naturally, via multiplication, on $L^2(\Omega,\mu)$, where μ is any positive normalized Borel measure on Ω (a state on $C^0(\Omega)$). Applying $C^0(\Omega)$ to the function 1 one gets a dense set in $L^2(\Omega,\mu)$ (Riesz-Fisher): thus we have a GNS-triple $(\rho_\mu, H_\mu = L^2(\Omega,\mu), \xi_\mu = 1)$.

We are now going to apply the above to Grassmannians.

For definitess, one can think of the standard Fourier decomposition of the Hilbert space $H = L^2(S^1, d\theta) = H_+ \oplus H_-$. We first recall the definition of CAR-algebra and gauge-invariant quasi-free states upon it.

Given a complex, separable Hilbert space K (called the *one-particle* space), the CAR (Canonical Anticommutation Relations) algebra $A(K)$ is the universal unital C^*-algebra generated by *creation* operators $a^*(f)$ (depending linearly on $f \in K$) and their adjoints $a(f)$, *annihilation* operators (depending antilinearly on $f \in K$), fulfilling the following relations:

$$a^*(f)a(g) + a(g)a^*(f) = \langle f, g\rangle \cdot \mathbf{1} \qquad a(f)a(g) \ + \ a(g)a(f) = 0$$

with $f,\ g \in K$, $\langle , \rangle$ denoting the scalar product in K, depending linearly on the first factor, and $\mathbf{1}$ being the identity element of $A(K)$ (slightly different conventions are often used in the general references given above). The unitary group $U(K)$, which is the natural symmetry group of K, acts on $A(K)$ through C^*-automorphisms α_U defined by $\alpha_U(a(f)) := a(Uf)$ for $U \in U(K)$ and $f \in K$.

Next, we briefly review the definition of gauge-invariant quasi-free state of the CAR algebra. For the C^*-algebra $A(K)$, consider the *state* ω_E associated to a hermitian projection operator E on K (that is, $E = E^* = E^2$, where A^* denotes the adjoint of a (bounded) operator A) via its *n-point functions*

$$\omega_E(a^*(f_1)\dots a^*(f_n)a(g_1)\dots a(g_m)) = \delta_{n,m}\det(\langle f_i, Eg_j\rangle)$$

for $f_i,\ g_j \in K$. In fact, these are determined by the *2-point functions*, the non-trivial ones reading:

$$\omega_E(a^*(f)a(g)) = \langle f, Eg\rangle$$

for $f,\ g \in K$. These states are called *gauge-invariant* since they are invariant under the $U(1)$-action given by the multiplication by unit-norm complex numbers on K and *quasi-free* since they relate to free fermionic quantum fields. The GNS construction now associates to every state ω_E a representation ρ_E of $A(K)$ by bounded operators on a complex Hilbert space $(\mathcal{H}_E, \langle\ ,\ \rangle_{\mathcal{H}_E})$ and a cyclic unit vector ξ_E (unique up to a phase) such that

$$\omega_E(A) = \langle \rho_E(A)\xi_E, \xi_E\rangle_{\mathcal{H}_E}$$

for all A in $A(K)$. Since the state ω_E is *pure*, the GNS-theory tells us in addition that ρ_E is an irreducible representation of $A(K)$. The cases $E = 0$ and $E = I$, respectively, yield representations called Fock and anti-Fock, respectively.

The GNS vector ξ_E is characterized (up to a phase) by the property:

$$\rho_E(a^*(f))\xi_E = 0\,\forall f \in EK, \quad \rho_E(a(g))\xi_E = 0\,\forall g \in (I-E)K.$$

Let K' denote the dual of K; we have $\langle f', g'\rangle_{K'} = \langle g, f\rangle_K$ ($f \mapsto f'$ denotes the natural *antilinear* duality operator). We introduce (in a special case) what is known as the Powers–Størmer *purification map*: the state ω_E on $A(K) = A(K \oplus 0)$ is the restriction of the state $\omega_{E\oplus F'}$ on $A(K \oplus K')$, where $F = I - E$, and F' is the corresponding adjoint operator on K'. The space $(E \oplus F')(K \oplus K')$ is the *isotropic* subspace in $K \oplus K'$ corresponding to EK (with respect to the complex bilinear form induced by the scalar product in $K \oplus K'$, given by evaluation). This will have a natural counterpart in terms of Grassmannians. Recall the crucial *Fock-anti-Fock* (FaF) *correspondence*, i.e. the canonical complex linear C^*-algebra isomorphism $\chi : A(K) \to A(K')$ induced by the map $a(f) \mapsto a^*(f')$. One has $\omega_{F'} \circ \chi = \omega_E$ and χ induces a map (denoted by the same symbol) which intertwines the representations $(\rho_E, \mathcal{H}_E, \xi_E)$ and $(\rho_{F'}, \mathcal{H}_{F'}, \xi_{F'})$ of $A(K)$ and $A(K')$, respectively. In particular, χ intertwines the corresponding Fock and anti-Fock representations.

A special case of a theorem of Powers and Størmer states that two quasi-free representations of $A(K)$, $(\rho_E, \mathcal{H}_E, \xi_E)$ and $(\rho_F, \mathcal{H}_F, \xi_F)$, say (corresponding to hermitian projections E and F), are unitarily equivalent if and only if $E - F$ is a Hilbert–Schmidt operator).

We consider the following "restricted" Grassmannians (see [14, §6.2] for the original definition)

$$\mathrm{Gr} = \mathrm{Gr}(H, H_+), \ \ GR := \mathrm{Gr}(H_{\mathbb{C}}, H_+ \oplus \overline{H}_-)$$

(the second one coming from an obvious purification); as shown in [3], [4], points W in Gr (say, for definiteness), identified first with orthogonal projections E_W onto the respective subspaces, correspond to *gauge-invariant quasi-free states* $\omega_{E_W} \equiv \omega_W$ of the CAR-algebra $A(H)$ yielding (irreducible) GNS-representations (of $A(H)$), ρ_W, unitarily equivalent to the (anti)-Fock one (ρ_{H_+}).

By virtue of the theorem of Powers and Størmer recalled above, one has ultimately the following simple characterization ([3]):

$$\mathrm{Gr} = \mathrm{Gr}(H, H_+) = \{W < H \mid E_W - E_+ \text{ is Hilbert–Schmidt}\}$$

Thus, points in Gr are concretely realized as *rays* $\langle \xi_W \rangle$, ξ_W being the (unique up to phase) GNS cyclic vector realizing the state $\omega_W(\cdot) = \langle \rho_W(\cdot)\xi_W, \xi_W\rangle$ in the (anti)-Fock GNS-representation space $\mathcal{H}_+$, i.e. as points in $P(\mathcal{H}_+)$; these rays yield, in turn, the fibres of the *Determinant line bundle* Det (pull-back of the tautological bundle $\mathcal{O}(-1) \to \mathbb{P}(\mathcal{H}_+)$); this is the standard Plücker embedding

$$Pl : \mathrm{Gr} \hookrightarrow \mathbb{P}(\mathcal{H}_+).$$

The whole setting is $U_{res}(H)$-equivariant. Recall that the restricted unitary group $U_{res}(H)$ consists of the unitaries in H commuting with the polarization operator $E_+ - E_-$ up to a Hilbert–Schmidt operator, and that one has the homogeneous Kähler manifold description

$$\mathrm{Gr} = \mathrm{Gr}(H, H_+) = U_{res}(H)/\ U(H_+) \times U(H_-)$$

By virtue of [3] and [15], we connect two theories:

Fact. The determinant bundle Det coincides with the natural determinant bundle over Sato's UGM, in the analytic context [15] (which will be reviewed in subsection 3.2).

For comparison, we present the objects more specifically. Let us fix the GNS-representation $(\rho_+, \mathcal{H}_+, \xi_+)$. We drop the suffix + for ease of notation. Any other GNS-representation corresponding to $Y \in \mathrm{Gr}(K, K_+)$ is unitarily equivalent to it and can thus be realized on $\mathcal{H}$. The GNS state becomes a vector state and gives rise to a complex line in $\mathcal{H}$, which will be denoted by Det_Y where $Y = UK_+$, $U \in U_{res}(K, K_+)$. In view of $U_{res}(K, K_+)$-equivariance, one has:

$$\mathrm{Det}_Y = \mathrm{Det}_{UK_+} = \tilde{U}\mathrm{Det}_{K_+}$$

where $\tilde{U}$ denotes the (linear) unitary operator that corresponds to U up to a phase on $\mathcal{H}$. There is a natural central extension of $U_{res}(K, K_+)$ by $U(1)$ denoted by $U^{\sim}_{res}(K, K_+)$. By equivariance, this gives rise to a holomorphic hermitian line bundle $\mathrm{Det} \to \mathrm{Gr}(K, K_+)$:

i) $\mathrm{Det} \to \mathrm{Gr}(K, K_+)$ is a holomorphic hermitian line bundle and $U^{\sim}_{res}(K, K_+)$ acts equivariantly on the total space; specifically we have $\mathrm{Det}_{UH_+} = \mathbb{C} \cdot \tilde{U}\xi_+$, where $\tilde{U}$ in $U^{\sim}_{res}(K, K_+)$ lies over U in $U_{res}(K, K_+)$.

ii) $\mathcal{H}^*$ is canonically realized inside $\Gamma_{\mathcal{O}}(\mathrm{Gr}(K, K_+), \mathrm{Det}^*)$.

iii) The bundle Det is isomorphic to the Determinant bundle of [14].

An explicit description is

$$\mathrm{Det} = \{(Y, \xi) \in \mathrm{Gr}(K) \times \mathcal{H} \ \mid \ a^*(y)\xi = 0 \quad \forall y \in Y \text{ and } a(y^\perp)\xi = 0 \quad \forall y^\perp \in Y^\perp\}$$

(operators are meant to act on the fixed representation space). Informally, ξ can be viewed as an infinite wedge product $\xi = y_1 \wedge y_2 \wedge \ldots$ manufactured from an (admissible, in a suitable technical sense) orthonormal basis of Y. In quantum-field theoretic terms Det_Y corresponds to the Dirac vacuum filled with the antiparticles relative to Y (*Dirac sea*). The upshot is the following [4, Corollary 8.2]: The Hilbert-Space Grassmannian $\mathrm{Gr}(H, H_+)$ is diffeomorphic, via the Plücker map and the GNS construction, to the (projective) $U_{res}(H, H_+)$-orbit of the GNS-vector ξ_+ associated to the reference quasi-free state ω_+ in $\mathcal{H}_+$, the latter group acting via the projectively unitarily implemented automorphism group α associated to U.

The Plücker equations for the embedding were given in [3] in the concise form

$$a^*(w)\xi_W = 0 \ \ \forall w \in W$$

where $a^*(w) = a(w)^*$ is the *creation operator* pertaining to w (this is, in turn, a manifestation of the *Pauli Exclusion Principle* in quantum physics).

Let $\mathbf{S}$ denote the set of subsets S of the integers for which both $S \setminus \mathbb{N}$ and $\mathbb{N} \setminus S$ are finite (cf. [15] or [14, § 7.1]). Let $\mathbf{S}'$ denote the set of all $S' = \mathbb{Z} \setminus S$ with S in $\mathbf{S}$. Then [3], [4], one finds the following geometric expression for the Plücker coordinates

$$\pi_S(W) = \langle W, H_{\mathbb{N}}\rangle_{\ell^2(\mathbf{S})} = \langle W, H_S\rangle$$

where $H_{\mathbb{N}} = H_+$, and H_S is manufactured in the following way (it is useful to keep the infinite wedge product description in mind): one takes the standard orthonormal basis in $H = L^2(S^1, d\theta)$, and replaces the standard orthonormal basis in H_+ by the basis vectors labelled by S. In fact, what should be taken are scalar products between GNS-vectors.

1.3. The Pfaffian Line Bundle

Let $(H, g = (,))$ be a real Hilbert space and $H_{\mathbb{C}} = H \otimes_{\mathbb{R}} \mathbb{C}$ its complexification, endowed with the $\mathbb{C}$-linear extension $B = g^{\mathbb{C}}$ of the metric and the canonical hermitian structure $\langle u, v\rangle = B(u, \overline{v})$, where $\overline{h \otimes \lambda} = h \otimes \overline{\lambda}$ denotes the canonical conjugation of $H_{\mathbb{C}}$. The *Clifford algebra* is the real (universal) Banach algebra with unit $\mathbf{1}$ generated by operators $\gamma(u)$, $u \in H$ fulfilling the anticommutation relation (u,$v \in H$)

$$\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = g(u, v) \cdot \mathbf{1}.$$

(the present convention adheres to quantum-field theorists' usage). The complex Clifford algebra $C(H_{\mathbb{C}}, B)$ is then by definition

$$C(H_{\mathbb{C}}, B) = C(H, g) \otimes_{\mathbb{R}} \mathbb{C}$$

and turns out to be a unital C^*-algebra isomorphic to $A(W)$, where W is any B-isotropic subspace of $H_{\mathbb{C}}$ such that $\overline{W} = W^{\perp}$. Any such W gives rise to a *complex structure* J on H, that is $J \in \mathcal{J}(H)$, where

$$\mathcal{J}(H) = \{J : H \to H \mid J \text{ is a } g \text{ isometric and } J^2 = -1\}$$

(the *Siegel manifold*, or *isotropic Grassmannian*). Indeed J yields $W = Eig(J^{\mathbb{C}}, +i)$, the eigenspace of eigenvalue $+i$ of the complexification $J^{\mathbb{C}}$ of J, and conversely the operator

$$J^{\mathbb{C}} = i \cdot (E_W - E_{W^{\perp}})$$

preserves H and yields a $J \equiv J_W$:

$$J = i \cdot (E_W - E_{W^{\perp}})|_H, \tag{1.1}$$

denoting the orthoprojector onto a subspace $Y \subset H_{\mathbb{C}}$ by E_Y. In analogy to the unitary group case, O in $O(H_{\mathbb{C}}, B)$, the orthogonal group associated with B, acts on $C(H_{\mathbb{C}}, B)$ by a C^*-automorphism group α_O, extending $\alpha_O(\gamma(v)) = \gamma(Ov)$. The same holds, of course, for $A(W)$. These particular automorphisms are called *Bogolubov automorphisms* (or *transformations*, see, e.g., [17–21]).

Remark. If H is a complex Hilbert space, regarded as vector space over the reals, the relationship between $A(H)$ and $C(H)$ is displayed by the formula

$$\gamma(f) = 2^{-\frac{1}{2}}(a(f) + a^*(f))$$

The assignment $f \mapsto \gamma(f)$ is called *Majorana field* in quantum field theory.

The automorphism-group implementation problem can be posed for the orthogonal group in the Clifford algebra context as well.

Consider the anti-Fock state ω_I on $A(W)$ and its corresponding GNS-representation $(\rho_I, \mathcal{H}_I, \xi_I)$. Each U in $U(W)$ obviously preserves the projector I, so by Powers–Størmer it is implemented, but in fact we have a larger symmetry group $O(H_{\mathbb{C}}, B)$ coming from the Clifford point of view. Since these automorphisms α_O (for O in $O(H_{\mathbb{C}}, B)$) of $A(W)$ are not induced from operators on W in general, we need the Shale–Stinespring theorem to decide when O is implemented, which we can formulate in the following guise:

For O in $O(H_{\mathbb{C}}, B)$ there exists $\tilde{O} \in U(\mathcal{H}_I)$ such that $\rho_I \circ \alpha_O(a) = \tilde{O} \circ \rho_I(a) \circ \tilde{O}^{-1}$ for all $a \in A(W)$ if and only if $O \in O_{res}(H_{\mathbb{C}}, B) := O(H_{\mathbb{C}}, B) \cap U_{res}(H_{\mathbb{C}}, H_+)$.

Now, in close analogy to the previous case, the $O_{res}(H_{\mathbb{C}}, B)$-orbit of the projector onto W in $H_{\mathbb{C}}$ yields by definition the (restricted) isotropic Grassmannian, or restricted Siegel manifold $\mathcal{J}_{res}(H)$.

Equivalently, it is easy to see that $\mathcal{J}_{res}(H)$ describes all isometric complex structures on H differing from J_W by a Hilbert–Schmidt operator. In conclusion,

$$\mathcal{J}_{res}(H) \cong O_{res}(H_{\mathbb{C}}, B)/U(W) \cong \{J \in \mathcal{J}(H) \mid J - J_W \text{ is Hilbert–Schmidt}\}.$$

It follows that there is an $O_{res}(H_{\mathbb{C}}, B)$-equivariant embedding

$$i \ : \ \mathcal{J}_{res}(H) \hookrightarrow \mathrm{Gr}(H_{\mathbb{C}}, W)$$

sending J in $\mathcal{J}_{res}(H)$ to $Y = Eig(J^{\mathbb{C}}, i)$ and having as its diffeomorphic image $\{Y \in \mathrm{Gr}(H_{\mathbb{C}}, W) \mid Y$ is B-isotropic and $Y^{\perp} = \overline{Y}\}$. Also, $\mathcal{J}_{res}(H)$ parametrizes all anti-Fock states $\omega := \omega_I \circ \alpha_O$ yielding GNS representations of the CAR algebra $A(W)$ unitarily equivalent to the reference one ρ_I. Working in the reference GNS space $\mathcal{H}_I$, the complex lines generated by their GNS vectors will realize the fibres of the Pfaffian line bundle, defined below. We also recall how a Bogolubov automorphism α_O associated to O in $O_{res}(H_{\mathbb{C}}, B)$ acts on $A(W)$: an element u of $H_{\mathbb{C}}$ considered as $\gamma(u)$ in $C(H_{\mathbb{C}}, B) \cong A(W)$ is mapped, up to a factor, to $a^*(E_W O u) + a(\overline{E_{\overline{W}} O u})$ (cf. [20]).

The FaF correspondence has a geometrical counterpart which will be also relevant in our discussion of the Segre embedding. Let us take a complex polarized Hilbert space $H = H_+ \oplus H_-$, which we regard as a real one. Let us set $\overline{H}_- := \overline{H_-}$ and, correspondingly, denote its projector by $\overline{E}_-$.

Upon complexification, any subspace $W_1 \in \mathrm{Gr}(H, H_+)$ goes to an isotropic subspace $Y = W_1 \oplus \overline{W_1^{\perp}} \in \mathcal{J}_{res}(H) \subset \mathrm{Gr}(H_{\mathbb{C}}, W)$, where $W = H_+ \oplus \overline{H}_-$: this is a geometric version of Powers–Størmer purification. The above map is equivariant with respect to the natural embedding $U_{res}(H, H_+) \hookrightarrow O_{res}(H_{\mathbb{C}}, B)$. Let us notice that, for $Y \in \mathcal{J}_{res}(H)$, $A(Y)$ and $A(\overline{Y})$ are canonically isomorphic by the FaF correspondence, since $\overline{Y}$ can be identified $\mathbb{C}$-linearly, via B, with the dual Y'. Let $\mathcal{H}_{E_+}$ respectively $\mathcal{H}_{\overline{E}_-}$ denote the GNS-spaces carrying the GNS representations of $A(H)$, respectively $A(\overline{H})$, induced by ω_{E_+}, respectively $\omega_{\overline{E}_-}$. We have the following (notation of [14] and [3]):

i) The following isomorphisms hold: $\mathcal{H}_{E_+} \cong \ell^2(\mathbf{S})$, $\mathcal{H}_{\overline{E}_-} \cong \ell^2(\mathbf{S}')$, and $\mathcal{H}_{E_+ \oplus \overline{E}_-} \cong \ell^2(\mathbf{S}) \hat{\otimes} \ell^2(\mathbf{S}') \cong \mathcal{H}_{E_+} \hat{\otimes} \mathcal{H}_{\overline{E}_-}$.

ii) The FaF correspondence reads as follows, in terms of the natural orthonormal bases in $\mathcal{H}_{E_+}$ and $\mathcal{H}_{\overline{E}_-}$ (with a slight abuse of notation)

$$\chi : H_S \mapsto \overline{H_S^{\perp}} = \overline{H}_{S'}.$$

iii) The correspondence also gives rise to a Kähler isometry

$$\mathrm{Gr}(H, H_+) \cong \mathrm{Gr}(\overline{H}, \overline{H}_-)$$

(which allows identification of the two manifolds).

iv) The embedding $\mathrm{Gr}(H, H_+) \hookrightarrow \mathrm{Gr}(H_{\mathbb{C}}, W)$ reads, in terms of Plücker coordinates:

$$\pi_{(S,T')}(Y) = \langle W_1 \oplus \overline{W_1^{\perp}}, H_S \oplus \overline{H_{T'}^{\perp}} \rangle = \langle W_1, H_S \rangle \langle W_1, H_T \rangle = \pi_S(W_1) \pi_T(W_1).$$

In particular,

$$\pi_{(S,S')}(W_1 \oplus \overline{W_1^{\perp}}) = \pi_S(W_1)^2.$$

We briefly recall the infinite-dimensional version of the spin representation. Let $\mathcal{S}$ be the GNS representation space $\mathcal{H}_{I_W}^{A(W)}$ of the CAR algebra $A(W)$ and $O_{res}^{\sim}(H_{\mathbb{C}}, B)$ the central $U(1)$-extension of $O_{res}(H_{\mathbb{C}}, B)$ implementing the Bogolubov transformations α_O as unitary endomorphisms of $\mathcal{S}$. This linear action of $O_{res}^{\sim}(H_{\mathbb{C}}, B)$ on $\mathcal{S}$ is called the *Spin^c-representation of* $O_{res}(H_{\mathbb{C}}, B)$.

We recall that in contrast with the finite-dimensional case, one cannot render it two-valued only. We shall realize this representation in a Borel–Weil fashion by resorting to the *Pfaffian line bundle*. The latter is given as

$$\mathrm{Pf} = \{(Y, \xi) \in \mathcal{J}_{res}(H) \times \mathcal{H}_I^{A(W)} \mid (a^*(y_W) + a(\overline{y_{W^\perp}}))\xi = 0 \ \ \forall y = y_W + y_{W^\perp} \in Y\},$$

where y_W and $y_{W^\perp}$ are the projections onto W and $W^\perp$respectively; if we denote by ρ_I the GNS-representation of $A(W)$ associated to the anti-Fock state ω_{I_W}, the operators act in the anti-Fock representation $\rho_I := \rho_{I_W}$. More precisely, we have the following result (Theorem 9.2 of [4]):

i) $\mathrm{Pf} \to \mathcal{J}_{res}(H)$ is a holomorphic hermitian line bundle.

ii) $O^{\sim}_{res}(H_{\mathbb{C}}, B)$ acts equivariantly on Pf.

iii) $\mathcal{S}^* = (\mathcal{H}_I^{A(W)})^*$ is canonically realized inside the holomorphic-section module of Pf^*, i.e. we have a Borel–Weil description of the infinite-dimensional spinor module.

iv) Using the embedding $i : \mathcal{J}_{res}(H) \hookrightarrow \mathrm{Gr}(H_{\mathbb{C}}, W)$ given above we have $i^*\mathrm{Det} = \mathrm{Pf}^{\otimes 2}$.

The crucial point of the proof is that, at the level of GNS vectors, one has:

$$\mathrm{Pf}_W \otimes \mathrm{Pf}_W \cong \mathbb{C}\xi_{I_W \oplus 0_{\overline{W}}} = \mathbb{C}\xi_{E_W} \subset \mathcal{H}_{E_W}^{A(\mathcal{H}_{\mathbb{C}})}$$

which is the line Det_W. The whole construction enjoys $O_{res}(H_{\mathbb{C}}, B)$-equivariance.

Let **Pf** denote the projective embedding of $\mathcal{J}_{res}(H)$ into $\mathbb{P}(S)$ by means of the vector space of sections $\mathcal{S}^*$ of the line bundle Pf^*. Then the restricted isotropic Grassmannian $\mathcal{J}_{res}(H)$ is diffeomorphic, via the map **Pf**, to the (projective) $O_{res}(H_{\mathbb{C}}, B)$-orbit in $\mathbb{P}(\mathcal{S})$ of the anti-Fock state vector ξ_{I_W}. Furthermore, for a product of Grassmannians, the upshot is [4, Theorem 10.1]:

i) Upon restriction to the "small" Grassmannian $\mathrm{Gr}(H) = \mathrm{Gr}(H, H_+)$, $\mathrm{Pf}|_{\mathrm{Gr}(H)} = \mathrm{Det}$.

ii) There exists a natural Segre embedding $Seg : \mathrm{Gr}(H) \times \mathrm{Gr}(H) \to \mathrm{Gr}(H_{\mathbb{C}}) = \mathrm{Gr}(H_{\mathbb{C}}, W)$ defined by: $Seg((W_1, W_2)) = W_1 \oplus \overline{(W_2{}^\perp)}$ (projective embeddings understood). The embedding is realized via the bundle $\mathrm{Det}^*_{\mathrm{Gr}(H)} \boxtimes \mathrm{Det}^*_{\mathrm{Gr}(H)}$ ("box" product).

The reader of [14, Proposition (12.3.1)] may note that the second factor of the box product is missing for the map **sq** from their diagram.

In subsection 3.1, we will write the analog of the (PR) for this product. The classical interpretation of the KP hierarchy as Plücker equations for the τ-function [2] uses a vertex operator $\exp(x\Lambda)$ (more generally, $\exp(x\Lambda + y\Lambda^2 + t\Lambda^3)$, e.g.),

$$\Lambda := \begin{bmatrix} 0 & 1 & 0 & \dots & & \\ 0 & 0 & 1 & 0 & \dots & \\ & \ddots & \ddots & & \ddots & \end{bmatrix}, \text{ an } \infty \times \infty \text{ matrix;}$$

evolutions are defined on $\mathrm{Gr}(2, \infty)$ by multiplication $\exp(x\Lambda) \cdot F$, using a frame $F = (\partial_x^{i-1} f_j)_{\infty \times 2}$, where f_1, f_2 is a fundamental solution of the equation $f''(x) + a(x, y, t)f'(x) + b(x, y, t)f(x) = 0$ (a boson-fermion correspondence, cf. subsection 1.4); the (PR) for this frame are the KP equation in Hirota's bilinear form [22].

Instead, we will use directly the geometry of the finite-dimensional space of soliton τ-functions. Again, we review this technique for the KP case, where the equations define a Grassmannian.

The subspaces of the Sato Grassmannian in question have Baker functions

$$\psi_\lambda^{(n)} = \frac{\partial^n}{\partial z^n} e^{xz + yz^2 + tz^3}\bigg|_{z=\lambda},$$

and the Plücker equations hold for the coefficients $c_{i_1 \dots i_k}$ of

$$\tau(x, y, t) = \sum_{1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n} c_{i_1 \dots i_k} \omega_{i_1 \dots i_k}(x, u, t),$$

where $\omega_{i_1\ldots i_k}$ =Wronskian$(\phi_{i_1},\ldots,\phi_{i_k})$ for any basis $\phi_{i_1},\ldots,\phi_{i_k}$ of the Grassmannian subspace.

The statement is: τ satisfies

$$-3\tau_y^2+3\tau_{xx}^2+3\tau\tau_{yy}+4\tau_t\tau_x-4\tau\tau_{xt}-4\tau_x\tau_{xxxx}+\tau\tau_{xxxx}=0$$

if and only if the coefficients satisfy the Plücker relations:

$$\sum_{i=1}^{k+1}(-1)^i c_{a_1\ldots a_{k-1}b_i}c_{b_1\ldots b_{i-1}b_{i+1}\ldots b_{k+1}}=0,$$

for all subsets $A=\{a_1,\ldots,a_{k-1}\}$ and $B=\{b_1,\ldots,b_{k+1}\}$ of $\{1,2,\ldots,n\}$, for any possible choice of k. Using this formulation one can check the strikingly simple:

Remark. Every function τ that satisfies $\tau_{xx}=\tau_y$ and $\tau_{xxx}=\tau_t$ is a solution of the bilinear KP equation.

1.4. Boson-fermion Correspondence

A correspondence between bosons and fermions is an important tool in physics and representation theory; we do not delve on its significance, but rather, identify it in our setting so as to transliterate objects and formulas used below.

In the conventional Sato formalism (cf. e.g. [1]), the correspondence can be used to take the Grassmannian of subspaces of analytic functions on the disc to formal pseudodifferential operators; in two-dimensional quantum-field theory, as explained in [14, introduction to Ch. 9 and § 10.7], it is a correspondence between an exterior algebra and a sum of symmetric algebras.

Let ψ_n, ψ_n^*, $n\in 2\mathbb{Z}+1$, be elements of a Lie algebra that acts on a Hilbert space, and satisfy the relations

$$[\psi_m,\psi_n]_+=[\psi_m^*,\psi_n^*]_+=0,\quad [\psi_m^*,\psi_n]_+=\delta_{m,n},$$

where $[X,Y]_+=XY+YX$. The vacua $\langle m|$, $|m\rangle$, $m\in 2\mathbb{Z}+1$ are defined by the conditions

$$\langle m|\psi_n=0\quad\text{for } n\leqslant m,\qquad \langle m|\psi_n^*=0\quad\text{for } n>m,$$

$$\psi_n|m\rangle=0\quad for\ n>m,\qquad \psi_n^*|m\rangle=0\quad for\ n\leqslant m.$$

They are related by

$$\psi_m^*|m-2\rangle=|m\rangle,\quad \langle m-2|\psi_m=\langle m|.$$

The Fock spaces H_m, H_m^* are constructed from $|m\rangle$ and $\langle m|$ respectively by the action of an equal number of ψ_k and ψ_l^*. The pairing between H_m and H_m^* is defined by normalizing

$$\langle m|m\rangle=1.$$

Let us set

$$h_{-2k}=\sum_{n\in 2\mathbb{Z}+1}\psi_n\psi_{n+2k}^*,$$

$$T=\exp(-\sum_{k=1}^{\infty}\frac{1}{k}J_{2k}h_{-2k}),$$

where J_{2k} are commutative variables. Notice that $\langle m|T=\langle m|$ for any m. The boson-fermion correspondence gives the isomorphism of bosonic and fermionic Fock spaces, $H_{2m-1}^*\simeq\mathbb{C}[J_2,J_4,\ldots]$,

$$\langle 2m-1|a\mapsto\langle 2m-1|aT|2m-1\rangle. \qquad (\mathrm{b-f})_{\mathrm{rep}}$$

The above boson-fermion correspondence may be understood geometrically in the setting of subsection 1.2 as follows [3, 5, 7]: in view of the above algebraic construction of Det, one has [15] a natural holomorphic section τ_W of the dual Det* of the Determinant line bundle, naturally associated to W, hence to the CAR-algebra state ω_W, with cyclic vector $\xi_W \in \mathcal{H}_+$. It can by given explicitly by

$$\tau_W((W', v)) = \langle v, \xi_W \rangle, \ v \in \mathrm{Det}_{W'}.$$

The assignment

$$\mathcal{H}_+ \ni \xi_W \mapsto \tau_W \in \Gamma_{L^2}(\mathrm{Det}^* \to \mathrm{Gr}) \qquad (\mathrm{b-f})_{\mathrm{geo}}$$

is precisely the *boson-fermion correspondence*, in the language of [3], [4]. Note here that we are in the analytic, not formal, category, and what is meant as L^2-holomorphic sections over an infinite-dimensional space, denoted by Γ_{L^2} above, are actually hyperplane sections in [4] (we also note that the relevant measure-theoretic framework is developed in [23]); their use is legitimate thanks to the fact that the sums converge over "admissible bases" [14, Prop. 7.5.2], [15, Section 10]; in [9, Lemma 1], the holomorphicity of the embedding is checked locally. In physics, τ_W is an example of *coherent state* [24]. We shall give a more concrete description of τ in the following sections; here we only observe that in representation-theoretic terms τ_W can be identified with the Sato τ-function:

$$\tau(x, g) = \langle \Omega \mid e^{H(t)} g \, \Omega \rangle = \sum_Y c_Y(g) \chi_Y(t)$$

where Ω is an admissible basis corresponding to the diagram $\emptyset$, $g = e^{\sum t_i z^i}$ acts on Gr (the action is reviewed in some technical detail in subsection 3.2), $H(t) = \sum (1/i)(\partial/\partial t_i)$ and the (projective) *Plücker coordinates* $c_Y(g)$ (Y a Young (or Maya) diagram) [1]) are given by

$$c_Y(g) = \chi_Y(\partial_t) \tau(t, g)|_{t=0}$$

($\{\chi_Y\}$ denoting the corresponding Schur functions), cf. subsection 1.1. This notation shows the analogy between the formulas $(\mathrm{b-f})_{\mathrm{rep}}$ and $(\mathrm{b-f})_{\mathrm{geo}}$; we do not review more extensive definitions of the notation because we do not make specific use of this formula.

2. DIASTASIS

2.1. Determinant Formula

The *Calabi diastasis function* D on a Kähler manifold is manufactured through the choice of a local Kähler potential f; in local complex coordinates,

$$D(z, w) = f(z, z) + f(w, w) - f(z, w) - f(w, z)$$

(where a sesquiholomorphic local extension of the Kähler potential is understood), and is coordinate-independent, yielding a metric invariant. In [7], the Calabi diastasis function of Gr (or GR) was calculated, as the pull-back under the Plücker embedding of the natural projective-space diastasis (induced by the Fubini–Study metric by the polarizing line bundle Det*). The formula reads as follows, where a point of the embedded Grassmannian is identified by its Plücker coordinates (subsection 1.1) and denoted by $[\tau]$, and we modify a notation by $'$ to denote a second object in the same set:

$$D([\tau], [\tau']) = \log \frac{\sum_Y |c_Y|^2 \cdot \sum_Y |c'_Y|^2}{|\sum_Y c_Y \bar{c}'_Y|^2}$$

written compactly as

$$D([\tau], [\tau']) = \log \frac{\| \tau \|^2 \cdot \| \tau' \|^2}{|\langle \tau, \tau' \rangle|^2}.$$

In particular we get the canonical Fubini–Study Kähler potential:

$$D([\tau = 1], [\tau]) = \log \sum_Y |c_Y|^2 = \log \| \tau \|^2 .$$

For the Kähler form of Gr one has the standard formula

$$\omega = \frac{i}{2}\partial\overline{\partial}\log \| \tau \|^2 .$$

From Calabi's global-rigidity theorem [9, Th. 7] we deduce that any isometric automorphism of the Grassmannian is projectively induced.

We already mentioned the fact that τ is a holomorphic section of Det^* ("coherent state", in physical terms [24]). It can also be represented by projectively induced coherent-state functions via

$$\tau \hookrightarrow \psi_{[c_Y]} : [c'_Y] \mapsto \frac{\sum_Y c'_Y \bar{c}_Y}{\sum_Y |c_Y|^2}$$

$$\tau \hookrightarrow \psi_\tau : [\tau'] \mapsto \frac{\langle \tau', \tau \rangle}{\| \tau \|^2}$$

(here the symbol $[\xi]$ denotes the projectivization of the vector ξ). Also notice that the boson-fermion correspondence yields the following equality between "boson" and "fermion" transition probabilities ([3, 7])

$$|\langle \xi_{W_1}, \xi_{W_2} \rangle|^2 = |\langle \tau_{W_1}, \tau_{W_2} \rangle|^2 = \exp(-D(W_1, W_2)).$$

In fact, one has performed geometric quantization on Gr (and GR).

Remark. The formal expression of the Baker function can be interpreted in terms of the general remark that

$$\tau' = \frac{\tau'}{\tau} \cdot \tau$$

as holomorphic sections of the same line bundle: the Baker function arises as a suitable quotient of τ's and is an ordinary function. This will be used below.

2.2. Diastasis and Weil Reciprocity

We make use of a formula, derived in [6, Prop. 4.1] using basic potential theory, that connects the diastasis $\mathcal{D}_P(Q)$ for two points P, Q on a (compact) Riemann surface with the Green's function:

$$G_P(Q) = \log |f(Q)|^2 - \mathcal{D}_P(Q), \qquad (\mathcal{D})$$

where $f(Q)$ is a local meromorphic function having a simple zero at P. Since the Green's function is symmetric, one can use this fact to give a proof of Weil's reciprocity. Stated in the setting of [25, II.3], the reciprocity says:

Theorem 1. *Let f, g be meromorphic functions on the compact Riemann surface S, with disjont divisors of zeros and poles; then,*

$$\prod_{P \in S} f^{\mathrm{ord}_P(g)}(P) = \prod_{P \in S} g^{\mathrm{ord}_P(f)}(P).$$

Proof. The products can be calculated using $G_P(Q) = \log |f(Q)|^2 - \mathcal{D}_P(Q)$, $G_Q(P) = \log |g(P)|^2 - \mathcal{D}_Q(P)$ by $(\mathcal{D})$. Since $\mathcal{D}_P(Q) = \mathcal{D}_Q(P)$ as follows from the definition, and $G_P(Q) = G_Q(P)$, we deduce $|f(Q)|^2 = |g(P)|^2$, i.e. f and g differ by an *a priori* variable phase factor; to obtain the reciprocity formula, observe that in relating the diastasis and the Green's function [6, Prop. 4.1] the local "Bochner coordinate" $f(Q)$ at P ($g(P)$ at Q, resp.) is appropriately normalized and so the phase factor is equal to 1 by holomorphicity.

2.3. *Automorphisms of Projective Subvarieties*

In subsection 3.2 we will give an application of the following result of Calabi's [9, Th. 2]:

Let M be a Kähler manifold, and assume that $(\dots)$ M can be isometrically and complex analytically embedded in a unitary space $\mathbb{C}^N$ of dimension N ($N \leqslant \infty$), so that it does not lie in any proper complex analytic linear subspace of $\mathbb{C}^N$. Then $(\dots)$ the embedding is determined up to within the group of motions in $\mathbb{C}^N$.

The question of comparing the group of automorphisms induced on an algebraic subvariety X from automorphisms of the ambient space, with the 'intrinsic' group $\mathrm{Aut}X$ of the variety itself is significant. By viewing the embedding as a polarization, this area investigates "linearizable group actions"; in the complex-analytic category cf., e.g., [26] and [27]. We restrict the question to projective subvarieties, and automorphisms induced by linear transformations of the projective space where the variety is embedded.

To recall basic examples, the plane cubic, which intrinsically is an elliptic curve once an origin is chosen, has more automorphisms than those induced by linear projectivities (for example, collineations that fix the curve will have to permute the inflexion points, hence are a finite group, whereas the automorphism group of the curve includes all translations on the curve), whereas for any smooth plane quartic X, $\mathrm{Aut}X$ is linearly induced because the embedding is canonical.

A lesser-known example (although deduced from first principles) which has to do with moduli spaces was given by Newstead [28]: The moduli space of (fixed) odd-determinant, rank-2 vector bundles over a hyperelliptic curve of genus two can be identified with Klein's quadratic complex, $Q \subset \mathbb{P}^5$. Newstead's result is that any automorphism of the moduli space is induced by a projective transformation. The first step of the proof consists in the observation, the one relevant to us in terms of Fubini–Study diastatic rigidity, that every automorphism of a non-singular quadratic complex in $\mathbb{P}^5$ is a projective equivalence. This again (as for the example of curves we gave) follows from the fact that Q has a 'canonical' line-bundle, with Chern class equal to that of the restriction H of the hyperplane bundle, since $H^1(Q) = 0$, and from $\dim H^0(Q, H) = 6$.

For current work on the case of hypersurfaces, cf. [29].

3. APPLICATIONS

3.1. *Baby-KP, Giant-KP*

Victor G. Kac in a seminar talk (U.C. Berkeley, 1992) called the defining equations of the finite-dimensional Grassmannian under the Plücker embedding "baby KP".

A key point of the Sato theory was that the KP hierarchy describes the GL_∞-orbit of the highest weight vector in a fundamental representation; here GL_∞ is used to denote the suitable subgroup of the general linear group in infinite dimension that acts on admissible frames [15], but the present section can be read formally, or locally, and we give the action explicitly in coordinates below.

When V is a finite-dimensional vector space and $F := \oplus_{j\geqslant 0} \wedge^j V$, the operators $v(u) = v \wedge u$ and $v^*(u_1 \wedge \dots \wedge u_k) = \sum_{i=1}^k (-1)^{i-1} v^*(u_i) u_2 \wedge \dots \wedge u_{i-1} \wedge u_{i+1} \wedge \dots \wedge u_k$ give a representation of the Clifford algebra $\mathcal{C}(V \oplus V^*)$ on F. Choosing a basis $\{e_1, \dots, e_n\}$ of V, $|j\rangle := e_j \wedge e_{j-1} \wedge \dots \wedge e_1$ is a highest-weight vector for the action of $G := GL(n)$; the orbit $\widetilde{\mathrm{Gr}}_j$ of $|j\rangle$ under G gives a $\mathbb{C}^*$-bundle: $\widetilde{\mathrm{Gr}}_j \to \mathrm{Gr}(k, n)$; the operator $S = \sum e_j \otimes e_j^*$ commutes with G and,

Baby-KP: An element $\tau \in \wedge^j V$, $\tau \neq 0$ is in the orbit $G|j\rangle$ if and only if $S(\tau \otimes \tau) = 0$.

However, the original way of setting up 'Giant-KP', namely a hierarchy of PDEs following from the Hirota bilinear equation for the τ function (cf. subsection 1.1) is based of the boson-fermion correspondence. We recall that process briefly, but in a rather different approach [30], also yielding PDEs of the KP hierarchy. It is the latter that we extend to Pfaffian bundles in this paper. In what follows we use the setting and notation of [15].

The finite-dimensional geometry is recovered by defining Hirota's equations on the subgrassmannian $\mathrm{Gr}_0 = \cup_k z^{-k} H_+ / z^k H_+$.

There are two key ideas [30]. One is the identification $\lambda \otimes \lambda$ with $L := \mathbb{C}[t_i^1] \otimes \mathbb{C}[t_i^2]$, where $\lambda \cong \cup_k V_k,\ V_k := \{z^{-k}, z^{-k+1}, \ldots, z^k\}$; to write the infinite-dimensional analog of the Plücker relations one defines the action of the group $GL^+ \subset GL_{res}$ given on $\mathbb{C}[t_i]$ by differentiation in t_i and multiplication by $2t_i$ (2 is the twist that gives the central extension of the group that acts on the infinite-dimensional frames).

Now the Heisenberg algebra (with basis $\{p_i, q_i, 2\},\ i \geqslant 1$ and relations $[p_i, q_i] = 2$) acts on L_{high}, the submodule generated by $\Omega \otimes \Omega$ where $\Omega \in \mathbb{C}[t_i]$, each of the two factors of the tensor product, is the highest-weight vector and on $L_{\text{low}} = L_{\text{high}}^{\perp}$, with inner product on $\mathbb{C}[t_i]$ under the isomorphism resulting in:

$$\langle P, Q\rangle = P\left(\frac{1}{j}\frac{\partial}{\partial t_j}\right) Q(t_i)|_{\mathbf{t}=\mathbf{0}}$$

(we resort to the indices i and j to signify the two sets of variables, but the functions are intended to depend on the whole sequence of variables, as in the $\tau(t)$ of previous sections; no specific i or j is fixed in these formulas). Lastly, with the change of variables $x_i = t_i^1 + t_i^2,\ y_i = t_i^1 - t_i^2$, the Plücker relations for $v \otimes v \in L_{\text{high}},\ v \otimes v = \tau(t_i^1)\tau(t_i^2) = \tau(x_i + y_i)\tau(x_i - y_i)$ (where τ is a polynomial) become:

$$P\left(\frac{1}{j}\frac{\partial}{\partial x_j}\right) Q\left(\frac{1}{j}\frac{\partial}{\partial y_j}\right) \tau(x_i + y_i)\tau(x_i - y_i)|_{\mathbf{x}=\mathbf{0},\mathbf{y}=\mathbf{0}} = 0,$$

Hirota's bilinear expression of the KP hierarchy.

For example: the Plücker relation for $\mathrm{Gr}(2,4) \subset \mathrm{Gr}_0$, namely $\pi_{12}\pi_{34} - \pi_{13}\pi_{24} + \pi_{14}\pi_{23} = 0$, becomes

$$e_{-2,-1} \otimes e_{0,1} - e_{-2,0} \otimes e_{-1,1} - e_{-2,1} \otimes e_{-1,0} + e_{0,1} \otimes e_{-2,-1} - e_{-1,1} \otimes e_{-2,0} + e_{-1,0} \otimes e_{-2,1} \in L_{\text{low}},$$

then using the isomorphism (via Schur functions), $y_1^4 - 12y_1y_3 + 12y_2^2 \in L_{\text{low}}$, namely

$$\left(\left(\frac{\partial}{\partial y_1}\right)^4 - 4\left(\frac{\partial}{\partial y_1}\right)\left(\frac{\partial}{\partial y_3}\right) + 3\left(\frac{\partial}{\partial y_2}\right)^2\right) \tau(x+y)\tau(x-y)|_{\mathbf{y}=\mathbf{0}} = 0,$$

the KP equation in Hirota form.

Our contribution is to apply this algebraic setting to the Segre embedding identified by Spera and Wurzbacher [4, Th. 10.1]:

Theorem 2. *Under the Segre embedding* $Seg : \mathrm{Gr} \times \mathrm{Gr} \to \mathrm{Gr}(H, W)$ *defined (projectively) by:*

$$Seg((W_1, W_2)) = W_1 \oplus \overline{W_2}^{\perp},$$

the τ *function, which is the normalized section of the embedding line bundle* $\mathrm{Det}^*_{\mathrm{Gr}} \boxtimes \mathrm{Det}^*_{\mathrm{Gr}}$*, satisfies the Segre hierarchy,* $\tau(x_i + y_i)\tau(x_i - y_i) \in L_{\text{high}}$.

The PDEs are obtained by replacing the (PR) by the Segre equations [31, (2.9) c)], namely the Segre quadrics and, separately, the (PR) of $\mathrm{Gr}(n/2, n)$ and $\mathrm{Gr}(m/2, m)$ in the two sets of bi-homogeneous coordinates in $\mathbb{P}^n \times \mathbb{P}^m$; the boson-fermion counterpart of the algebraic equations is applied to any polynomial τ-functions associated to $W_1, W_2 \in \mathrm{Gr}_0$, with $z^{n/2}H_+ \subset W_1 \subset z^{-n/2}H_+, z^{m/2}H_+ \subset W_2 \subset z^{-m/2}H_+$.

We now also derive the equations in general Hirota form using the boson-fermion correspondence.

From subsections 1.2, 1.3 and 1.4 we have, in terms of τ (as sections of the appropriate line bundles)

$$\tau_{\mathrm{Gr}}^2 = \tau_{GR}\,|_{\mathrm{Gr}}\ .$$

We already observed that τ_{GR} is actually Sato's τ related to the KP-hierarchy, after specializing to finite dimensional subspaces, and we write simply $\tau_{GR} \equiv \tau_{KP}$. The BKP-hierarchy, see e.g. [8]), can be defined by the property

$$\tau_{BKP}^2 = \tau_{KP}\,|_{x_2=x_4=\ldots=0}$$

(formula (6.7) in [8]). One has, accordingly,

$$\tau_{\mathrm{Gr}} = \tau_{BKP},$$

using the Pfaffian line bundle construction. Thus one gets, in Sato's setting, a Segre hierarchy:

$$\tau_S := \tau_{Gr} \otimes \tau_{Gr} = \tau_{GR} \mid_{Gr \times Gr} .$$

The BKP-hierarchy issues from the spin representation of the orthogonal affine Lie subalgebra B_∞ of $A_\infty = gl(\infty)$ ([8]), so our derivation gives the geometric counterpart of the representation-theoretic approach.

3.2. Dualities

There are different types of dualities on Sato's Grassmannian. The geometric version of the finite-dimensional duality, in which the dual of a vector space V is the space of linear maps on V, $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, can be viewed as acting on Gr via the identification given by Sato's non-degenerate bilinear map,

$$\langle \psi_W, \psi_{W'} \rangle = \oint \psi_W \psi_{W'} d\zeta.$$

For a quick review of the symbols, letting $H = H_- \oplus H_+$ be the Hilbert space decomposition as in [15], $H_+ = \{(\sum_{i=0}^{\infty} a_i z^i) \text{ s.t. } \sum |a_i|^2 < \infty, a_i \in \mathbb{C}\}$, $H_- = \{(\sum_{i=-\infty}^{-1} a_i z^i) \text{ s.t. } \sum |a_i|^2 < \infty, a_i \in \mathbb{C}\}$, we consider the Grassmannian Gr of closed subspaces W of H whose projection $\pi_+ \colon W \to H/H_- \cong H_+$, is Fredholm of index 0, and whose projection $\pi_- \colon W \to H_-$ is Hilbert–Schmidt.

The fiber of the dual determinant bundle over W is:
$\mathrm{Det}^*|_W = \Lambda^{\mathrm{top}}\mathrm{coker}(\pi_+ \colon W \to H/H_-) \otimes (\Lambda^{\mathrm{top}}\ker(\pi_+ \colon W \to H/H_-))^{\otimes -1}$. The bundle Det^* is equipped with a canonical section $\sigma = \det(\pi_+)$ (for analytic justifications of the statements we refer to proofs in [14, Ch. 7]). If $W + H_- = H$, then $\sigma(W) = 1 \in \Lambda^{\mathrm{top}}0 \otimes (\Lambda^{\mathrm{top}}0)^{\otimes -1} = \mathbb{C}$. If $W + H_- \neq H$, then $\sigma(W) = 0$. The fiber $\mathrm{Det}^*|_W = \Lambda^{\mathrm{top}}(\mathrm{coker}(\pi_+)) \otimes (\Lambda^{\mathrm{top}}\ker(\pi_+))^{\otimes -1} = \Lambda^{\mathrm{top}}(H/(H_- + W)) \otimes (\Lambda^{\mathrm{top}}(H_- \cap W))^{\otimes -1}$.

Consider the subgroup $GL_{\mathrm{res}}(H)$ of $GL(H) = GL(H_- \oplus H_+)$ whose elements have off-diagonal blocks of Hilbert–Schmidt class. Its identity component $GL_{\mathrm{res}}(H)^0$ acts on Gr. This action lifts projectively to the dual determinant bundle Det^* on Gr. This provides us with a central extension G of $GL_{\mathrm{res}}(H)^0$. Thus an element $g \in G$ which covers an $h \in GL_{\mathrm{res}}(H)$ is an ordered pair (h, α), where α is an (holomorphic) isomorphism $\alpha \colon h^*\mathrm{Det}^* \to \mathrm{Det}^*$. We denote the composite map $\alpha \circ h^*$ by $g^* \colon \mathrm{Det}^* \to \mathrm{Det}^*$. Since the only global holomorphic functions on Gr are the constant functions, G is an extension of $GL_{\mathrm{res}}(H)^0$ by $\mathbb{C}^*$.

Given $W \in \mathrm{Gr}$ we choose $0 \neq \delta \in \mathrm{Det}^*|_W$, and define $\tau_W(g) = (((g^{-1})^*\sigma)/\delta)(W)$ [15]. This is well-defined as a function of g only up to a constant. For $W \in U_{0,0}$, where we denote by $U_{M,n}$, for any finite-dimensional subspace M of H, the following subset of Gr: $\{W | W \cap (M + z^{-n}H_-) = 0\}$, we may normalize so that $\tau_W(g) = ((g^{-1})^*\sigma/\sigma)(W)$, $g \in G$.

Denote by $\Gamma_{+,N}$ the subgroup of $GL_{\mathrm{res}}(H)^0$ consisting of holomorphic nonvanishing functions on N which take the value 1 at 0, N a neighborhood of the unit disc D_0 around 0, $N \supset D_0 = \{z \mid |z| \leqslant 1\}$. $\Gamma_{+,N}$ acts on H by multiplication. $\Gamma_+ := \bigcup_N \Gamma_{+,N}$ is contained in a parabolic subgroup $P \subset GL_{\mathrm{res}}(H)^0$, such that $\mathrm{Gr} = GL_{\mathrm{res}}(H)^0/P$. The central extension G of $GL_{\mathrm{res}}(H)^0$ splits over P. We choose such a splitting and henceforth use the notation $\Gamma_{+,N}$ to describe the subgroup of G: $\Gamma_+ \subset P \subset G$. Γ_+ can conveniently be coordinatized: all $g \in \Gamma_+$ can be written as $g = \exp(\sum_{i=1}^{\infty} t_i z^i)$, $t_i \in \mathbb{C}$.

Define $q_\zeta = 1 - (z/\zeta) \in \Gamma_+$, $|\zeta| > 1$, $\zeta \in \mathbb{C} \cup \infty$ by $q_\zeta = 1 - (z/\zeta)$.

Given a $W \in \mathrm{Gr}$, we introduce the Baker–Akhiezer wave function $\psi_W \colon G_W \to W$ defined on the subset $G_W = \{g \in G | g^{-1}W + H_- = H\}$, by requiring that $g^{-1}\psi_W(g) \equiv 1 \otimes (dz)^{1/2} \bmod H_-$. The image of ψ_W is W. By definition, the expression $g^{-1}\psi_W(g) \otimes (dz)^{-1/2}$ extends to a holomorphic function $\tilde{\psi}_W(g)$ on the disc around ∞, $D_\infty := \{z \in \mathbb{P}^1 \mid |z| > 1\}$, whose boundary value on the unit circle S^1 is in $L^2(S^1)$. It is easy to see that $\psi_W(\Gamma_+ \cap G_W)$ generates W as a Hilbert space.

To recall the formula describing ψ_W in terms of τ, we think of $\tilde{\psi}_W(g)$ as a complex valued function of g and z, analytic in z: $\tilde{\psi}_W(g, z)$, $g \in G_W$, $|z| > 1$. Then we have

$$\tilde{\psi}_W(g, z)|_{z=\zeta} = \tau_W(gq_\zeta)/\tau_W(g),$$

where $q_\zeta = 1 - (z/\zeta) \in \Gamma_+$, and also $g \in \Gamma_+ \cap G_W$ [15, (5.14)]. Here, q_ζ is in Γ_+ provided $|\zeta| > 1$. For a $g \in \Gamma_{+,N}$ we may think of $\psi_W(g) \otimes (dz)^{-1/2}$ as holomorphic function on $N \cap D_\infty$ with L^2 boundary value on the unit circle S^1.

The boson-fermion correspondence (cf. subsection 1.4) in this notation associates to an element of the Grassmannian (big cell) via its Baker function, the pseudodifferential operator $\mathcal{L}$ such that $\mathcal{L}\psi = z\psi$, and the duality given by taking orthogonal spaces becomes: $\mathcal{L} = \sum u_k(x)\partial^k \mapsto \mathcal{L}^* = \sum(-1)^k\partial^k u_k(x)$ with the abbreviation $\partial = \partial/\partial t_1$.

Our result, translating the projective rigidity proved by diastasis, is that the duality is induced on the Plücker embedding by a projective transformation. Note first that our Plücker and Segre embeddings are holomorphic, hence preserve the diastasis that we induced from the Fubini–Study metric (using holomorphic bundles to pull it back). Next, we use the following result (Wigner's theorem, see e.g. [32, Th. 51], and Appendix D, *ibid.* for a proof): any transformation ϕ preserving (in absolute value) transition amplitudes, i.e. $|\langle\phi(x), \phi(y)\rangle| = |\langle x, y\rangle|$ when x, y are any two unit vectors in the space, is induced by a unitary or anti-unitary transformation T_ϕ of the space, unique up to a phase.

Lemma. *The Plücker (Segre) embedding is an isometry. Consequently any (isometric) automorphism of the Grassmannian is induced by a linear transformation of the vector space underlying the target projective space, i.e. it is projectively induced.*

Proof. To show that the Plücker embedding is an isometry (the proof is the same for the Segre embedding), we have to compare the metric on the Grassmannian with that on the target projective space. We recalled above that on the Grassmannian the metric is given by the pairing

$$\langle\psi_W, \psi_{W'}\rangle = \oint \psi_W\psi_{W'}d\zeta.$$

If we abbreviate the notation by $\psi_W = e^\alpha\tilde{\psi}_W$ where $e^\alpha = g = \exp(\sum_{i=1}^\infty t_i z^i)$ and $\tilde{\psi}_W$ is analytic and normalized, then the expression for the metric in the target linear space, given in terms of the diastasis, is:

$$D(v, w) = -2\log\frac{||v||^2 \cdot ||w||^2}{|\langle v, w\rangle|^2} = -2\log|\langle\tilde{\psi}_W, \tilde{\psi}_{W'}\rangle|$$

and the map is an isometry (up to a scalar which does not affect projective coordinates or the statement about linearity). Therefore, upon passing to the corresponding coherent states ($W \mapsto \tau_W$), we can resort to Wigner's theorem above, achieving the conclusion (cf. [7])

There is a third involution of great importance in integrable-equation theory (the "bispectral involution"), namely the interchange of the spectral parameter z and the evolution parameter $t_1 = x$. While it can be defined on the formal Grassmannian, on the Hilbert-space Grassmannian the definition has to be restricted to the "adelic" [33] part $\mathrm{Gr}^{\mathrm{ad}}$. We remark that, since its effect on the Baker–Akhiezer function ψ_W defined above is simply to interchance the t_1 and the z variables, it is a holomorphic map, as well as an isometry, again using the expression for the diastasis. By Calabi's result, we can only say that this involution is induced from a projective transformation of the linear span of the adelic Grassmannian under the Plücker embedding; however, one can always extend this projective transformation to the whole projective space, so the result holds for this involution as well. In conclusion, we have the following:

Theorem 3. *An automorphism of the Grassmannian as well as the Segre variety given by the Pfaffian embedding, in particular the dualities induced by Hirota's bilinear form and by the bispectral involution, are induced by projective transformations of the projective space into which they are Plücker-, resp. Segre-, embedded.*

ACKNOWLEDGEMENTS

We are deeply grateful to the anonymous Referee for his or her expert and painstaking reading and cross-checking; in particular, for the reference to the alternative analytic study of [13] (and the papers that ensued), and to the analytic issue treated in [26, 27] (and the papers that ensued). E.P. very gratefully acknowledges the research support of the NSF-DMS-0808708.

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 374–395. © Pleiades Publishing, Ltd., 2011.

Separation of Variables and Explicit Theta-function Solution of the Classical Steklov–Lyapunov Systems: A Geometric and Algebraic Geometric Background

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Received June 30, 2010; accepted August 26, 2010

Abstract—The paper revises the explicit integration of the classical Steklov–Lyapunov systems via separation of variables, which had been first made by F. Kötter in 1900, but was not well understood until recently. We give a geometric interpretation of the separating variables and then, applying the Weierstrass hyperelliptic root functions, obtain explicit theta-function solution to the problem. We also analyze the structure of poles of the solution on the Jacobian on the corresponding hyperelliptic curve. This enables us to obtain a solution for an alternative set of phase variables of the systems that has a specific compact form.
In conclusion we discuss the problem of integration of the Rubanovsky gyroscopic generalizations of the above systems.

MSC2010 numbers: 37J35, 70E40, 70H06, 70G55, 14H40, 14K25, 14H42

DOI: 10.1134/S1560354711030105

Keywords: Steklov–Lyapunov system, explicit solution, separation of variables, algebraic integrability

1. INTRODUCTION

The motion of a rigid body in the ideal incompressible fluid is described by the classical Kirchhoff equations

$$\dot{K} = K \times \frac{\partial H}{\partial K} + p \times \frac{\partial H}{\partial p}, \quad \dot{p} = p \times \frac{\partial H}{\partial K},$$

where $K, p \in \mathbb{R}^3$ are the vectors of the total angular momentum and the momentum, respectively, and $H = H(K, p)$ is the Hamiltonian, which is quadratic in K, p. Note that this system always possesses two trivial integrals (Casimir functions of the coalgebra $e^*(3)$) $\langle K, p\rangle, \langle p, p\rangle$ and the Hamiltonian itself is also a first integral.

Steklov [1] noticed that the classical Kirchhoff equations are integrable under certain conditions i.e., when the Hamiltonian has the form

$$H_1 = \frac{1}{2}\sum_{\alpha=1}^{3}\Big(b_\alpha K_\alpha^2 + 2\nu b_\beta b_\gamma K_\alpha p_\alpha + \nu^2 b_\alpha (b_\beta - b_\gamma)^2 p_\alpha^2\Big), \qquad (\alpha, \beta, \gamma) = (1, 2, 3)\,, \tag{1.1}$$

b_1, b_2, b_3 and ν being arbitrary parameters. Under the Steklov condition, the equations possess fourth additional integral

$$H_2 = \frac{1}{2}\sum_{\alpha=1}^{3}\Big(K_\alpha^2 - 2\nu b_\alpha K_\alpha p_\alpha + \nu^2 (b_\beta - b_\gamma)^2 p_\alpha^2\Big)\,. \tag{1.2}$$

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Later Lyapunov [2] discovered an integrable case of the Kirchhoff equations whose Hamiltonian was a linear combination of the additional integral (1.2) and the two trivial integrals. Thus, the Steklov and Lyapunov integrable systems actually define different trajectories on the same invariant manifolds, two-dimensional tori. This fact was first noticed in [3].

In the sequel, without loss of generality, we assume $\nu = 1$ (this can always be made by an appropriate rescaling $p \to p/\nu$).

The Kirchhoff equations with the Hamiltonians (1.1), (1.2) were first solved explicitly by Kötter [4], who used the change of variables $(K, p) \to (z, p)$:

$$2z_\alpha = K_\alpha - (b_\beta + b_\gamma)p_\alpha\,, \qquad \alpha = 1,2,3\,, \quad (\alpha,\beta,\gamma) = (1,2,3), \tag{1.3}$$

which transforms the Steklov–Lyapunov systems to the form

$$\dot z = z \times Bz - Bp \times Bz\,, \quad \dot p = p \times Bz\,, \qquad B = \operatorname{diag}(b_1, b_2, b_3) \tag{1.4}$$

and, respectively,

$$\dot z = p \times Bz\,, \quad \dot p = p \times (z - Bp)\,. \tag{1.5}$$

In [4], Kötter implicitly showed that the above systems admit the following Lax representation with 3×3 skew-symmetric matrices and a spectral parameter

$$\begin{gathered} \dot L(s) = [\,L(s), A(s)\,]\,, \qquad L(s), A(s) \in so(3), \quad s \in \mathbb{C}\,, \\ L(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\Big(\sqrt{s - b_\gamma}\,(z_\gamma + sp_\gamma)\Big)\,, \end{gathered} \tag{1.6}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor. Equations (1.4) and (1.5) are generated by the operators

$$A(s)_{\alpha\beta} = \frac{\varepsilon_{\alpha\beta\gamma}}{s}\sqrt{(s-b_\alpha)(s-b_\beta)}\,b_\gamma z_\gamma\,, \quad \text{resp.} \quad A(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\sqrt{(s-b_\alpha)(s-b_\beta)}\,p_\gamma. \tag{1.7}$$

The radicals in (1.6)–(1.7) are single-valued functions on the elliptic curve $\widehat{\mathcal{E}}$, the 4-sheeted unramified covering of the plane curve $\mathcal{E} = \{w^2 = (s-b_1)(s-b_2)(s-b_3)\}$. For this reason, the Lax representation has an elliptic spectral parameter.

Writing out the characteristic equation for $L(s)$, we arrive at the following family of quadratic integrals

$$\mathcal{F}(s) = \sum_{\gamma=1}^{3}(s-b_\gamma)(z_\gamma + sp_\gamma)^2 \equiv J_1 s^3 + J_2 s^2 + 2sH_2 - 2H_1\,, \tag{1.8}$$

where

$$H_1 = \frac{1}{2}\langle z, Bz\rangle\,, \quad H_2 = \frac{1}{2}\langle z, z\rangle - \langle Bz, p\rangle\,, \quad J_2 = 2\langle z, p\rangle - \langle Bp, p\rangle\,, \quad J_1 = \langle p, p\rangle\,. \tag{1.9}$$

It is seen that under the Kötter substitution (1.3) the functions J_1, J_2 transform into invariants of the coalgebra $e^*(3)$, whereas the integrals $H_1(z,p)$, $H_2(z,p)$ (up to a linear combination of the invariants) become the Hamiltonians (1.1), (1.2).

An analog of the elliptic Lax pair (1.6) was later rediscovered in [5] and was used to obtain theta-function solution of the systems by using the method of Baker–Akhieser functions (see [6]). However, the resulting formulas appeared to be quite tedious, and it was not clear how to compare or identify them with the theta-function solution of Kötter.

Note that the latter was obtained in the classical manner, i.e., by a separation of variables and reduction of the equations of motion to quadratures, which have the form of the Abel–Jacobi map associated to a genus 2 hyperelliptic curve. The phase variables of the Kirchhoff equations have been expressed in terms of the separating variables in a quite symmetric but complicated way. Until recently, various attempts to check these expressions, as well as the reduction to quadratures made by Kötter, even using packages of modern computer algebra, were not successful. This even led to an opinion among some specialists that the results of [4] are not reliable, hence useless.

One of the first steps in verification of Kötters' calculations was made in [7], where the Steklov–Lyapunov systems on $e^*(3)$, as well as their higher-dimensional generalizations, have been considered as Poisson reductions of certain Hamiltonian systems in a bigger phase space. The latter systems were shown to possess 2×2 matrix Lax representations in a generalized Gaudin form with a rational spectral parameter. This fact easily allowed separating variables to be found, which coincided with those suggested by Kötter, and, as a byproduct, to prove their commutativity with respect to the Lie–Poisson bracket on $e^*(3)$. A similar approach to the separation of variables was made in [8].

The main aim of the present paper is to reconstruct the rest of the results of Kötter's paper [4][1)]. We shall use the original notation of [4], when possible.

2. SEPARATION OF VARIABLES BY F. KÖTTER.

The explicit solution of the Steklov–Lyapunov systems in the generic case was given by Kötter in the brief communication [4], where he presented the following scheme.

Let us fix the constants of motion in (1.9), then the invariant polynomial (1.8) can be written as

$$\mathcal{F}(s) = \sum_{\gamma=1}^{3} (s - b_\gamma)(z_\gamma + s p_\gamma)^2 = c_0(s - c_1)(s - c_2)(s - c_3), \qquad c_0, c_1, c_2, c_3 = \text{const.} \tag{2.1}$$

Assume that all b_α are distinct and that no one of c_α coincides with b_1, b_2, b_3. Then the real level variety of the four first integrals of the problem (given by the coefficients at s^3, s^2, s, s^0) is a union of two-dimensional tori in $\mathbb{R}^6 = (z, p)$. We restrict ourselves to this generic situation, excluding the other cases, which correspond to periodic or asymptotic motions of the body.

Let now λ_1, λ_2 be the roots of the equation

$$f(\lambda) = \sum_{i=1}^{3} \frac{(z_j p_k - z_k p_j)^2}{\lambda - b_i} = 0 , \qquad (i, j, k) = (1, 2, 3) . \tag{2.2}$$

Then for fixed c_0, c_1, c_2, c_3 the variables z, p can be expressed in terms of λ_1, λ_2 in such a way that for any $s \in \mathbb{C}$ the following relation holds (see formula (7) in [4])

$$z_i + s p_i = \sqrt{c_0}\, \frac{x_i \sum\limits_{\alpha=1}^{3} (s - c_\alpha) \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \left(\frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{(\lambda_1 - b_i)(\lambda_2 - c_\alpha)} - \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{(\lambda_2 - b_i)(\lambda_1 - c_\alpha)} \right)}{(\lambda_1 - \lambda_2) \sum\limits_{\alpha=1}^{3} \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)}} , \tag{2.3}$$

where

$$\Phi(\lambda) = (\lambda - b_1)(\lambda - b_2)(\lambda - b_3) , \quad \psi(\lambda) = (\lambda - c_1)(\lambda - c_2)(\lambda - c_3) , \tag{2.4}$$

$$x_i = \frac{z_j p_k - z_k p_j}{|z \times p|} = \frac{\sqrt{(\lambda_1 - b_i)(\lambda_2 - b_i)}}{\sqrt{(b_i - b_j)(b_i - b_k)}} , \tag{2.5}$$
$$(i, j, k) = (1, 2, 3) , \quad (\alpha, \beta, \gamma) = (1, 2, 3) .$$

Setting in the above expression $s \to \infty$ and $s = 0$, one obtains the corresponding formulas for p_i, z_i.

As was mentioned in [4], for any $\alpha = 1, 2, 3$, the branches of $\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}$ in the numerator and the denominator of (2.3) must be the same.

Next, the evolution of λ_1, λ_2 is described by the quadratures

$$\begin{aligned} \frac{d\lambda_1}{\sqrt{R(\lambda_1)}} + \frac{d\lambda_2}{\sqrt{R(\lambda_2)}} &= \delta_1\, dt , \\ \frac{\lambda_1\, d\lambda_1}{\sqrt{R(\lambda_1)}} + \frac{\lambda_2\, d\lambda_2}{\sqrt{R(\lambda_2)}} &= \delta_2\, dt, \end{aligned} \tag{2.6}$$

1)Note that apart from the solutions of the Kirchhoff equations, Kötter also provided (although in an extremely brief form) the theta-solutions describing the motion of the group $E(3)$, that is, the components of the rotation matrix of the body and the trajectory of its center in space. We were unable to reconstruct these solutions.

$$R(\lambda) = -\Phi(\lambda)\psi(\lambda)$$

with certain constants δ_1, δ_2 depending on the choice of the Hamiltonian only. In other words, in the variables λ_1, λ_2 the systems separate.

Note that the paper [4] does not describe explicitly how to find δ_1, δ_2. They were calculated in [7–9].

The above quadratures rewritten in the integral form

$$\begin{aligned} &\int_{\lambda_0}^{\lambda_1} \frac{d\lambda}{2\sqrt{R(\lambda)}} + \int_{\lambda_0}^{\lambda_2} \frac{d\lambda}{2\sqrt{R(\lambda)}} = u_1, \\ &\int_{\lambda_0}^{\lambda_1} \frac{\lambda\, d\lambda}{2\sqrt{R(\lambda)}} + \int_{\lambda_0}^{\lambda_1} \frac{\lambda\, d\lambda}{2\sqrt{R(\lambda)}} = u_2\,, \end{aligned} \tag{2.7}$$

$$u_1 = \delta_1 t + u_{10}, \quad u_2 = \delta_2 t + u_{20}, \tag{2.8}$$

contain two holomorphic differentials on the genus 2 hyperelliptic curve $\mu^2 = -\Phi(\lambda)\psi(\lambda)$ and represent the Abel–Jacobi map from the symmetric product $\Gamma \times \Gamma$ to the Jacobian variety of Γ.

Inverting the map (2.7) and substituting symmetric functions of $\lambda_1, \lambda_2, \mu_1, \mu_2$ into (2.3), one finally finds z, p as functions of u_1, u_2 and, therefore, of time t (see Theorem 2 below).

Everyone who had read paper [4] might be surprised by how Kötter managed to invent the intricate substitution $(z, p) \to (\lambda_1, \lambda_2, c_0, c_1, c_2, c_3)$ and to represent the result in the symmetric form (2.3). Unfortunately, the author of the paper gave no explanations of his computations. Nevertheless, it is clear that behind the striking formulas there must be a certain geometric idea, which we try to reconstruct in the next section.

The real case. Assume, without loss of generality, that $b_1 < b_2 < b_3$ and that z_i, p_i are all real and correspond to generic constants of motion c_α in (2.1). Then one has

Proposition 1.

1) *The constants* c_1, c_2, c_3 *are either all real or 2 of them are complex conjugated. If any* c_α *is real, then it belongs to the segment* $[b_1, b_3]$.

2) *The separating variables* λ_1, λ_2 *are also real and, if* $\lambda_1 \leqslant \lambda_2$*, they vary in subsets of* $[b_1, b_2]$ *and* $[b_2, b_3]$ *respectively.*

Proof. 1) The polynomial $\mathcal{F}(s)$ in (2.1) has real coefficients, hence its roots are either real or complex conjugated. Next, setting in (2.1) $s = c_\alpha$ (real), we obtain

$$\sum_{\gamma=1}^{3} (c_\alpha - b_\gamma)(z_\gamma + c_\alpha p_\gamma)^2 = 0.$$

Since z_i, p_i are real and generic, then $(z_\gamma + c_\alpha p_\gamma)^2$ are all non-negative. Moreover, since $z_\gamma + c_\alpha p_\gamma$ are not integrals of the motion, at certain time their squares are all positive. Hence, the above sum can be zero iff $b_1 \leqslant c_\alpha \leqslant b_3$. This holds for any real c_α.

2) In view of (2.5), x_1, x_2, x_3 are all real and $x_1^2 + x_2^2 + x_3^2 = 1$. This is possible only when the coordinates λ_1, λ_2 are real (and not complex conjugated), and $\lambda_1 \in [b_1, b_2]$, $\lambda_2 \in [b_2, b_3]$. More precisely, in view of the quadratures (2.7), the coordinates can vary only in subsets of these segments, for which $R(\lambda_1) \geqslant 0, R(\lambda_2) \geqslant 0$. □

Notice that if z_i, p_i are not generic as was assumed above, it is possible that $z + c_\alpha p \equiv 0$ and item (1) of the Proposition does not hold. This corresponds to the case of particular periodic solutions, which we do not treat here.

Remark 1. There is a natural conjecture that the fibers of the momentum map $\mathcal{M} : \mathbb{R}^6 = (z, p) \longrightarrow \mathbb{R}^4 = (J_1, J_2, h_1, h_2)$ are singular if and only if the corresponding curve $\mu^2 = -\Phi(\lambda)\psi(\lambda)$ is singular. This happens when either one of the roots c_α (or more) coincides with b_i or two roots c_α, c_β collide inside the segment $[b_1, b_3]$.

Then Proposition 1 provides sufficient tools to construct the bifurcation diagram of the momentum map $\mathcal{M}$, which was first presented in [10] by using different techniques.

Alternative variables. For our purposes we shall also use another set of phase variables which depend linearly on z, p. Namely, putting in the family (1.8) successively $s = b_1$, $s = b_2$, $s = b_3$ we obtain three independent quadratic integrals defining rank 3 quadrics in $\mathbb{P}^6$:

$$\begin{aligned} (b_1 - b_2)(z_2 + b_1 p_2)^2 + (b_1 - b_3)(z_3 + b_1 p_3)^2 &= \mathcal{F}(b_1)\,, \\ (b_2 - b_1)(z_1 + b_2 p_1)^2 + (b_2 - b_3)(z_3 + b_2 p_3)^2 &= \mathcal{F}(b_2)\,, \\ (b_3 - b_1)(z_1 + b_3 p_1)^2 + (b_3 - b_2)(z_2 + b_3 p_2)^2 &= \mathcal{F}(b_3)\,. \end{aligned} \tag{2.9}$$

Then it is natural to introduce new variables

$$\begin{aligned} v_1 &= \sqrt{(b_2 - b_3)(b_1 - b_2)}\,(z_2 + b_1 p_2)\,, \\ v_2 &= \sqrt{(b_2 - b_3)(b_3 - b_1)}\,(z_3 + b_1 p_3)\,, \\ v_3 &= \sqrt{(b_3 - b_1)(b_1 - b_2)}\,(z_1 + b_2 p_1)\,, \\ v_4 &= \sqrt{(b_2 - b_3)(b_3 - b_1)}\,(z_3 + b_2 p_3)\,, \\ v_5 &= \sqrt{(b_3 - b_1)(b_1 - b_2)}\,(z_1 + b_3 p_1)\,, \\ v_6 &= \sqrt{(b_2 - b_3)(b_1 - b_2)}\,(z_2 + b_3 p_2)\,, \end{aligned} \tag{2.10}$$

which imply

$$p_1 = \frac{v_3 - v_5}{\sqrt{\mathcal{S}}\sqrt{b_2 - b_3}}, \quad p_2 = \frac{v_1 - v_6}{\sqrt{\mathcal{S}}\sqrt{b_3 - b_1}}, \quad p_3 = \frac{v_2 - v_4}{\sqrt{\mathcal{S}}\sqrt{b_1 - b_2}}, \tag{2.11}$$

$$z_1 = \frac{b_3 v_3 - b_2 v_5}{\sqrt{\mathcal{S}}\sqrt{b_2 - b_3}}, \quad z_2 = \frac{b_3 v_1 - b_1 v_6}{\sqrt{\mathcal{S}}\sqrt{b_3 - b_1}}, \quad z_3 = \frac{b_2 v_2 - b_1 v_4}{\sqrt{\mathcal{S}}\sqrt{b_1 - b_2}}, \tag{2.12}$$

$$\mathcal{S} = (b_1 - b_2)(b_2 - b_3)(b_3 - b_1).$$

Then the integrals (2.9) and $(p, p) = J_1$ take the following compact form

$$\begin{aligned} v_1^2 - v_2^2 &= \psi(b_1)\,/\,(b_2 - b_3)\,, \\ v_3^2 - v_4^2 &= \psi(b_2)\,/\,(b_3 - b_1)\,, \\ v_5^2 - v_6^2 &= \psi(b_3)\,/\,(b_1 - b_2)\,, \end{aligned} \tag{2.13}$$

$$\frac{(v_3 - v_5)^2}{b_2 - b_3} + \frac{(v_1 - v_6)^2}{b_3 - b_1} + \frac{(v_2 - v_4)^2}{b_1 - b_2} = J_1(b_1 - b_2)(b_2 - b_3)(b_3 - b_1)\,.$$

The Steklov–Lyapunov systems written in terms of $v_1, \dots, v_6$, as well as the integrals (2.13), are quite similar to those describing the reduction of the integrable geodesic flow on the group $SO(4)$ with the diagonal metric II to the algebra $so(4)$, which was considered in detail in [11, 12]. In fact, as was shown by several authors (see e.g., [5]), there is a linear isomorphism connecting the above systems[2)]. In Section 5 we shall use this property and the results of [12] to obtain theta function expressions for the sums and differences of v_i, which have an especially simple form.

3. A GEOMETRIC BACKGROUND OF KÖTTER'S SOLUTION

Let $(x_1 : x_2 : x_3)$ be homogeneous coordinates in $\mathbb{P}^2$ defined up to multiplication by the same non-zero factor. Consider a line l in $\mathbb{P}^2 = (x_1 : x_2 : x_3)$ defined by equation

$$y_1 x_1 + y_2 x_2 + y_3 x_3 = 0.$$

Following Plücker (see e.g., [13]), the coefficients y_1, y_2, y_3 can be regarded as homogeneous coordinates of a point in the dual projective space $\left(\mathbb{P}^2\right)^*$. Now let l_1, l_2 be two lines in $\mathbb{P}^2$ with the Plücker coordinates $(y_1^{(1)}, y_2^{(1)}, y_3^{(1)})$, $(y_1^{(2)}, y_2^{(2)}, y_3^{(2)})$.

[2)] On the other hand, one of the Steklov–Lyapunov systems on $e^*(3)$ can also be regarded as a limit of the system on $so(4)$.

Then, for any constants $\lambda, \mu \in \mathbb{C}$ not vanishing simultaneously, the linear combination $\lambda y_\alpha^{(1)} + \mu y_\alpha^{(2)}$ also gives Plücker coordinates of a line $l_{\lambda,\mu} \in \mathbb{P}^2$. Hence, we arrive at an important geometric object, *a pencil of lines* in $\mathbb{P}^2$, i.e., the one-parameter family $l_{\lambda,\mu}$. It is remarkable that all the lines of a pencil intersect at the same point $\mathcal{P} \in \mathbb{P}^2$. The point $\mathcal{P}$ is called *the focus* of the pencil.

Theorem 1. ([13]) *Let $l_{\lambda,\mu}$ be a pencil of lines in $\mathbb{P}^2$ defined by Plücker coordinates $\lambda y_\alpha^{(1)} + \mu y_\alpha^{(2)}, (\lambda : \mu) \in \mathbb{P}$. Then the homogeneous coordinates of the focus are*

$$\mathbf{P} = \left(y_2^{(1)} y_3^{(2)} - y_3^{(1)} y_2^{(2)} : y_1^{(1)} y_3^{(2)} - y_3^{(1)} y_1^{(2)} : y_1^{(1)} y_2^{(2)} - y_2^{(1)} y_1^{(2)} \right) .$$

Next, consider the family of confocal quadrics in $\mathbb{P}^2$

$$Q(s) = \left\{ \frac{x_1^2}{s-b_1} + \frac{x_2^2}{s-b_2} + \frac{x_3^2}{s-b_3} = 0 \right\} \tag{3.1}$$

and a fixed point $P = (X_1 : X_2 : X_3)$. Then one defines the spheroconical coordinates λ_1, λ_2 of this point (with respect to $Q(s)$) as the roots of the equation

$$\frac{X_1^2}{\lambda - b_1} + \frac{X_2^2}{\lambda - b_2} + \frac{X_3^2}{\lambda - b_3} = 0.$$

Now, going back to the Steklov–Lyapunov systems, we make the following observation.

Proposition 2. *Let $\mathbf{P}$ be the focus of the pencil of lines in $\mathbb{P}^2$ with the Plücker coordinates*

$$z + sp = (z_1 + sp_1 : z_2 + sp_2 : z_3 + sp_3), \qquad s \in \mathbb{P}.$$

Then the separating variables λ_1, λ_2 defined by formula (2.2) are spheroconical coordinates of $\mathbf{P}$ *with respect to the family of quadrics (3.1).*

Proof. According to Theorem 1, the homogeneous coordinates of the focus $\mathbf{P}$ are

$$(z_2 p_3 - z_3 p_2 : z_3 p_1 - z_1 p_3 : z_1 p_2 - z_2 p_1) , \tag{3.2}$$

hence, the spheroconical coordinates of $\mathbf{P}$ with respect to the family (3.1) are precisely the roots of equation (2.2), i.e., λ_1, λ_2. □

Let a solution $z(t), p(t)$ correspond to constants of motion c_1, c_2, c_3. Then there is also the following property: for $\alpha = 1, 2, 3$, the line $\ell_\alpha(t)$ with the Plücker coordinates $z(t) + c_\alpha p(t)$ remains tangent to the quadric $Q_\alpha = Q(c_\alpha)$. Indeed, setting in the right-hand side of (1.8) $s = c_\alpha$, we obtain

$$\sum_{i=1}^{3} (c_\alpha - b_i)(z_i + c_\alpha p_i)^2 = 0 ,$$

which represents the condition of tangency of the line ℓ_α and the quadric Q_α.

As a result, the following configuration holds: *the three (moving) lines ℓ_1, ℓ_2, ℓ_3 in $\mathbb{P}^2$ intersect at the same (moving) point* $\mathbf{P}$ *and are tangent to the fixed quadrics Q_1, Q_2, Q_3 respectively.* An example of such a configuration is shown in Fig. 1.

This geometric property is reminiscent of the famous Chasles theorem in the Jacobi problem on geodesics on an ellipsoid Q in $\mathbb{R}^3$: the tangent line to a geodesic remains also tangent to a fixed quadric confocal to Q.

It also follows from the above that a solution $z(t)$, $p(t)$ defines a trajectory of the focus $\mathbf{P}$ on $\mathbb{P}^2$ or on $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$, and it is natural to suppose that the Steklov–Lyapunov systems define certain dynamical systems on the sphere. Indeed, some of these systems were studied in [8] and were shown to be related to a generalization of the classical Neumann system with an additional quartic potential.

In the sequel our main goal will be to recover the variables z and p as functions of the spheroconical coordinates of the focus $\mathbf{P}$, that is, to reconstruct the Kötter formula (2.3).

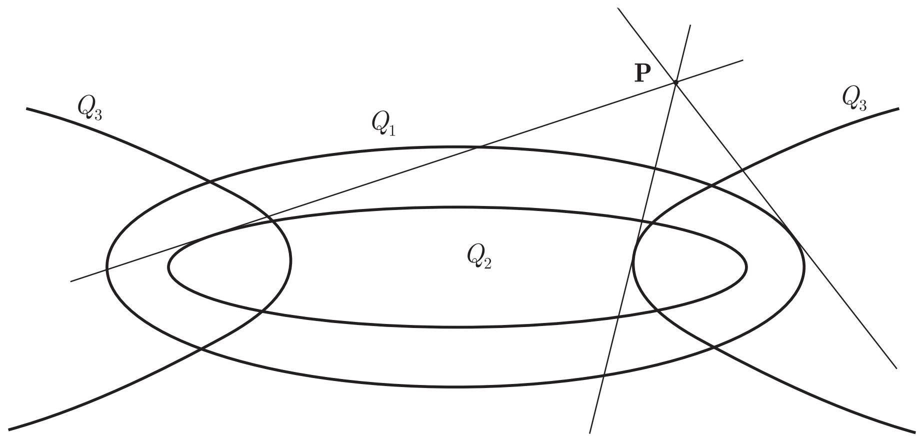

Fig. 1. A configuration of tangent lines in $\mathbb{R}^2 = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)$ for the case $b_1 < c_1 < b_2 < c_2 < c_3 < b_3$, when the quadrics Q_α are two ellipses and a hyperbola.

Obviously, the solution is not unique: due to the presence of square roots in (2.5), each pair $(\lambda_1, \lambda_2), \lambda_k \neq b_1, b_2, b_3$ gives 4 points on $\mathbb{P}^2$, and for each point $\mathbf{P}$ that does not lie on any of the quadrics $Q(c_\alpha)$, there are $2^3 = 8$ different possible configurations of tangent lines ℓ_1, ℓ_2, ℓ_3 (Fig. 1 shows just one of them). Thus, under the above generality conditions, a pair (λ_1, λ_2) gives 32 different tangent configurations.

Reconstruction of z, p in terms of the separating variables. Let $(\mathbb{P}^2)^* = (G_1 : G_2 : G_3)$ be the dual of $\mathbb{P}^2 = (x_1 : x_2 : x_3)$ (G_i being the Plücker coordinates of lines in $\mathbb{P}^2$). It is convenient to regard G_i also as Cartesian coordinates in $(\mathbb{C}^3)^* = (G_1, G_2, G_3)$. The pencil $\sigma(\mathbf{P})$ of lines in $\mathbb{P}^2$ with the focus (3.2) is represented by a line in $(\mathbb{P}^2)^*$ or by plane

$$\pi = \{(z_2p_3 - z_3p_2)G_1 + (z_3p_1 - z_1p_3)G_2 + (z_1p_2 - z_2p_1)G_3 = 0\} \subset (\mathbb{C}^3)^*.$$

Consider the line $\bar{\sigma}(\mathbf{P}) = \{z + sp \,|\, s \in \mathbb{R}\} \subset (\mathbb{C}^3)^*$. Obviously, $\{z + sp\} \subset \pi$. Now let us use the condition for the three lines ℓ_1, ℓ_2, ℓ_3 defined by the points $z + c_1p$, $z + c_2p$, $z + c_3p$ in $(\mathbb{P}^2)^*$ to be tangent to the quadrics $Q(c_1)$, $Q(c_2)$, $Q(c_3)$ respectively. Let $\mathbf{V}_\alpha = (V_{\alpha 1}, V_{\alpha 2}, V_{\alpha 3}) \subset \pi$, $\alpha = 1, 2, 3$ be some vectors in $(\mathbb{C}^3)^*$ representing these points, so that $\ell_\alpha = \{V_{\alpha 1}x_1 + V_{\alpha 2}x_2 + V_{\alpha 3}x_3 = 0\}$. Then we have

$$z + c_1p - \mu_1\mathbf{V}_1 = 0\,, \quad z + c_2p - \mu_2\mathbf{V}_2 = 0\,, \quad z + c_3p - \mu_3\mathbf{V}_3 = 0 \tag{3.3}$$

for some indefinite factors μ_α. This system is equivalent to a homogeneous system of 9 scalar equations for 9 variables $z_\alpha, p_\alpha, \mu_\alpha$, $\alpha = 1, 2, 3$. Thus the latter can be found up to multiplication by a common factor. Namely, we obtain

$$p = \frac{\mu}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)}(c_1\Sigma_1\mathbf{V}_1 + c_2\Sigma_2\mathbf{V}_2 + c_3\Sigma_3\mathbf{V}_3)\,, \tag{3.4}$$

$$z = \frac{\mu}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)}(c_2c_3\Sigma_1\mathbf{V}_1 + c_1c_3\Sigma_2\mathbf{V}_2 + c_1c_2\Sigma_3\mathbf{V}_3)\,, \tag{3.5}$$

$$\Sigma_1 = V_{22}V_{33} - V_{32}V_{23}\,, \quad \Sigma_2 = V_{32}V_{13} - V_{33}V_{12}\,, \quad \Sigma_3 = V_{12}V_{23} - V_{13}V_{22}\,, \tag{3.6}$$

that is,

$$z + sp = \frac{\mu}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} \sum_{\alpha=1}^{3}(c_\alpha s + c_\beta c_\gamma)\Sigma_\alpha\mathbf{V}_\alpha\,, \tag{3.7}$$

$\mu \neq 0$ being an arbitrary factor.

Now we express the components of $\mathbf{V}_\alpha$ in terms of λ_1, λ_2. Up to an arbitrary nonzero factor, they can be found from the system of equations

$$V_{\alpha 1}x_1 + V_{\alpha 2}x_2 + V_{\alpha 3}x_3 = 0\,, \quad \sum_{i=1}^{3}(c_\alpha - b_i)V_{\alpha i}^2 = 0\,, \qquad \alpha = 1,2,3, \tag{3.8}$$

which represent the conditions for the lines ℓ_α to pass through the focus $\mathbf{P} = (x_1 : x_2 : x_3)$ and to touch the quadric $Q(c_\alpha)$. In the sequel we apply the normalization $x_1^2 + x_2^2 + x_3^2 = 1$, which gives rise to the expressions (2.5).

For $\mathbf{P} \notin Q(c_\alpha)$, this system possesses two different solutions, and for $\mathbf{P} \in Q(c_\alpha)$ a single one (the line touches $Q(c_\alpha)$ at the point $\mathbf{P}$). In the latter case we can just put

$$V_{\alpha i} = x_i \,/\, (c_\alpha - b_i)\,. \tag{3.9}$$

Next, it is obvious that under reflection $(x_1 : x_2 : x_3) \to (-x_1 : x_2 : x_3)$, a solution $(V_{\alpha 1} : V_{\alpha 2} : V_{\alpha 3})$ transforms to $(-V_{\alpha 1} : V_{\alpha 2} : V_{\alpha 3})$ (similarly, for the two other reflections). Let us seek solutions of equations (3.8) in the form of symmetric functions of the complex coordinates λ_1, λ_2 such that

1) for $\lambda_1 = c_\alpha$ or $\lambda_2 = c_\alpha$ (i.e., when $\mathbf{P} \in Q(c_\alpha)$) there is a unique solution proportional to (3.9);

2) if λ_1 or λ_2 circles around the point $\lambda = c_\alpha$ on the complex plane λ, the two solutions transform into each other;

3) for $\lambda_1 = b_i$ or $\lambda_2 = b_i$ (i.e., when $x_i = 0$), $V_{\alpha i}$ does not vanish.

Using the Jacobi identities

$$\sum_{i=1}^{n} \frac{a_i^k}{\prod (a_i - a_j)} = \begin{cases} 0, & k < n-1 \\ 1, & k = n-1 \\ \sum\limits_{i=1}^{n} a_i, & k = n, \end{cases} \tag{3.10}$$

one can check that the following expressions satisfy equations (3.8) and the above three conditions:

$$V_{\alpha i} = x_i \left(\frac{\sqrt{\Phi(\lambda_1)(\lambda_2 - c_\alpha)}}{\lambda_1 - b_i} + \frac{\sqrt{\Phi(\lambda_2)(\lambda_1 - c_\alpha)}}{\lambda_2 - b_i} \right), \quad x_i = \frac{\sqrt{(\lambda_1 - b_i)(\lambda_2 - b_i)}}{\sqrt{(b_i - b_j)(b_i - b_k)}}. \tag{3.11}$$

Then, using again the identities (3.10), we calculate the scalar products

$$\langle \mathbf{V}_\alpha, \mathbf{V}_\beta \rangle \equiv (\lambda_2 - \lambda_1)\left(\sqrt{(\lambda_2 - c_\alpha)(\lambda_2 - c_\beta)} - \sqrt{(\lambda_1 - c_\alpha)(\lambda_1 - c_\beta)} \right) \tag{3.12}$$

and, in particular, $\langle \mathbf{V}_\alpha, \mathbf{V}_\alpha \rangle = (\lambda_1 - \lambda_2)^2$ for $\alpha = 1,2,3$.

Next, substituting (3.11) into (3.6) and applying the symbolic multiplication rule $\sqrt{ab}\sqrt{ac} = a\sqrt{bc}$, we find the above factors Σ_α in the form

$$\Sigma_\alpha = (\lambda_1 - \lambda_2)x_1 \left(\sqrt{-(\lambda_1 - c_\gamma)(\lambda_2 - c_\beta)} - \sqrt{-(\lambda_1 - c_\beta)(\lambda_2 - c_\gamma)} \right), \qquad (\alpha, \beta, \gamma) = (1,2,3)\,. \tag{3.13}$$

Further, putting (3.11), (3.13) into (3.7), we obtain

$$z_i + s p_i = \frac{\mu(\lambda_1 - \lambda_2)x_1}{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)} x_i \cdot \sum_{\alpha=1}^{3} (c_\alpha s + c_\beta c_\gamma)$$
$$\times \left[\frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{\lambda_1 - b_i} \left(\sqrt{\frac{\lambda_1 - c_\gamma}{\lambda_2 - c_\gamma}} - \sqrt{\frac{\lambda_1 - c_\beta}{\lambda_2 - c_\beta}} \right) + \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{\lambda_2 - b_i} \left(\sqrt{\frac{\lambda_2 - c_\gamma}{\lambda_1 - c_\gamma}} - \sqrt{\frac{\lambda_1 - c_\beta}{\lambda_2 - c_\beta}} \right) \right]$$

$$\equiv \mu(\lambda_1-\lambda_2)x_1x_i\sum_{\alpha=1}^{3}(s-c_\alpha)\frac{\sqrt{-(\lambda_1-c_\alpha)(\lambda_2-c_\alpha)}}{(c_\alpha-c_\beta)(c_\alpha-c_\gamma)}\left(\frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{(\lambda_1-b_i)(\lambda_2-c_\alpha)}-\frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{(\lambda_2-b_i)(\lambda_1-c_\alpha)}\right), \quad (3.14)$$

which, up to multiplication by a common factor, coincides with the numerator in Kötter's formula (2.3).

To determine the factor μ in (3.7) and in (3.14), we apply the condition $\langle p,p\rangle = c_0$ which follows from (2.1). Then, from (3.4) we get

$$\frac{c_0}{\mu^2}=\frac{|c_1\varSigma_1\mathbf{V}_1+c_2\varSigma_2\mathbf{V}_2+c_3\varSigma_3\mathbf{V}_3|^2}{(c_1-c_2)^2(c_2-c_3)^2(c_3-c_1)^2}. \quad (3.15)$$

Using the expressions (3.12), (3.13), we obtain

$$\begin{aligned}\left|\sum_{\alpha=1}^{3}c_\alpha\Sigma_\alpha\mathbf{V}_\alpha\right|^2 &\equiv \sum_{\alpha=1}^{3}\left[c_\alpha^2\varSigma_\alpha^2\langle\mathbf{V}_\alpha,\mathbf{V}_\alpha\rangle+2c_\beta c_\gamma\varSigma_\beta\varSigma_\gamma\langle\mathbf{V}_\beta,\mathbf{V}_\gamma\rangle\right]\\ &=(\lambda_1-\lambda_2)^3x_1^2\sum_{\alpha=1}^{3}\Bigg[c_\alpha^2(\lambda_1-\lambda_2)\left(\sqrt{-(\lambda_1-c_\gamma)(\lambda_2-c_\beta)}-\sqrt{-(\lambda_1-c_\beta)(\lambda_2-c_\gamma)}\right)^2\\ &\qquad+2c_\beta c_\gamma\left(\sqrt{-(\lambda_1-c_\gamma)(\lambda_2-c_\beta)}-\sqrt{-(\lambda_1-c_\beta)(\lambda_2-c_\gamma)}\right)\\ &\qquad\times\left(\sqrt{-(\lambda_1-c_\alpha)(\lambda_2-c_\gamma)}-\sqrt{-(\lambda_1-c_\gamma)(\lambda_2-c_\alpha)}\right)\\ &\qquad\times\left(\sqrt{-(\lambda_2-c_\beta)(\lambda_2-c_\gamma)}-\sqrt{-(\lambda_1-c_\beta)(\lambda_1-c_\gamma)}\right)\Bigg].\end{aligned}$$

Simplifying the above expression and again using symbolic multiplication of square roots, one can verify that it is a full square of a scalar expression:

$$\left|\sum_{\alpha=1}^{3}c_\alpha\Sigma_\alpha\mathbf{V}_\alpha\right|^2=x_1^2(\lambda_1-\lambda_2)^4\left(\sum_{\alpha=1}^{3}(c_\beta-c_\gamma)\sqrt{-(\lambda_1-c_\alpha)(\lambda_2-c_\alpha)}\right)^2.$$

Hence, from (3.15) we find

$$\frac{\sqrt{c_0}}{\mu}=x_1(\lambda_1-\lambda_2)^2\sum_{\alpha=1}^{3}\frac{\sqrt{-(\lambda_1-c_\alpha)(\lambda_2-c_\alpha)}}{(c_\alpha-c_\beta)(c_\alpha-c_\gamma)}.$$

Combining the latter with (3.14), we finally arrive at (2.3).

Thus, we derived the remarkable Kötter formula by making use of the geometric interpretation of the variables λ_1,λ_2. We also note that the expressions (2.3) are symmetric in λ_1,λ_2.

Remark 2. As noticed above, an unordered generic pair (λ_1,λ_2) gives 32 different configurations of tangent lines to the quadrics $Q(c_1)$, $Q(c_2)$, $Q(c_3)$. Since the common factor μ in (3.7) is defined up to sign flip, we conclude that, according to the formula (2.3), *to each generic pair* (λ_1,λ_2) *there correspond* 64 *different points* (z,p) *on the invariant manifold (a union of 2-dimensional tori) defined by the constants* c_0,c_1,c_2,c_3. This ambiguity corresponds to different signs of the square roots in the Kötter formula.

In the next section we shall use the expressions (2.3) and the quadratures (2.7) to find explicit theta-functional solutions for the Steklov–Lyapunov systems.

4. EXPLICIT THETA-FUNCTION SOLUTION OF THE STEKLOV-LYAPUNOV SYSTEMS

We first recall some basic formulas describing inversion of the quadratures (2.6), mostly following the description given in [6, 14, 15]. Consider an even order hyperelliptic Riemann surface of genus g represented in the standard form

$$\Gamma = \left\{\mu^2 = (\lambda - E_1)\cdots(\lambda - E_{2g+2})\right\} \in \mathbb{C}^2(\lambda, \mu).$$

It can be regarded as a 2-fold covering of the complex plane $\{\lambda\}$ ramified at $E_1, \dots, E_{2g+2}$. In the sequel we shall think of Γ as of its compactification obtained by adding two infinite points ∞_-, ∞_+, at which the coordinate λ equals infinity.

Choose a canonical basis of cycles $\mathfrak{a}_1, \dots, \mathfrak{a}_g, \mathfrak{b}_1, \dots, \mathfrak{b}_g$ on the Γ such that their intersections are

$$\mathfrak{a}_i \circ \mathfrak{a}_j = \mathfrak{b}_i \circ \mathfrak{b}_j = 0, \quad \mathfrak{a}_i \circ \mathfrak{b}_j = \delta_{ij}, \qquad i, j = 1, \dots, g,$$

$\gamma_1 \circ \gamma_2$ denotes the intersection index of the cycles γ_1, γ_2.

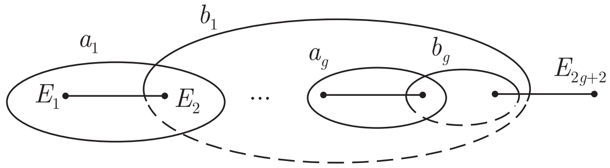

Fig. 2

In the sequel we choose the basis of cycles on Γ as indicated on Fig. 2. The parts of the cycles on the lower λ-sheet are shown by dashed lines.

Next, let $\bar{\omega}_1, \dots, \bar{\omega}_g$ be the conjugated basis of *normalized* holomorphic differentials on Γ such that

$$\oint_{\mathfrak{a}_j} \bar{\omega}_i = 2\pi \jmath\, \delta_{ij}, \qquad \jmath = \sqrt{-1}.$$

The $g \times g$ matrix of b-periods $B_{ij} = \oint_{\mathfrak{b}_j} \bar{\omega}_i$ is symmetric and has a negative definite real part. Consider the period lattice $\Lambda^0 = \{2\pi\jmath\mathbb{Z}^g + B\mathbb{Z}^g\}$ of rank $2g$ in $\mathbb{C}^g = (z_1, \dots, z_g)$. The complex torus $\mathrm{Jac}(\Gamma) = \mathbb{C}^g/\Lambda^0$ is called the Jacobi variety (*Jacobian*) of the curve Γ.

Now consider a generic divisor of points $P_1 = (\lambda_1, \mu_1), \dots, P_g = (\lambda_g, \mu_g)$ on it, and the Abel–Jacobi mapping with a basepoint P_0

$$\int_{P_0}^{P_1} \bar{\omega} + \cdots + \int_{P_0}^{P_g} \bar{\omega} = z, \tag{4.1}$$

$$\bar{\omega} = (\bar{\omega}_1, \dots, \bar{\omega}_g)^T, \quad z = (z_1, \dots, z_g)^T \in \mathbb{C}^g.$$

Under the mapping, functions on $S^g\Gamma$, i.e., symmetric functions of the coordinates of the points $P_1, \dots, P_g$ are $2g$-fold periodic functions of the complex variables $z_1, \dots, z_g$ with the above period lattice Λ^0.

Explicit expressions of such functions can be obtained by means of theta-functions on the universal covering $\mathbb{C}^g = (z_1, \dots, z_g)$ of the complex torus. Recall that customary Riemann's theta-function $\theta(z|B)$ associated with the Riemann matrix B is defined by the series[3)]

$$\theta(z|B) = \sum_{M \in \mathbb{Z}^g} \exp(\langle BM, M\rangle + \langle M, z\rangle), \tag{4.2}$$

[3)] The expression for $\theta(z)$ we use here is different from that chosen in several of books on theta-functions by multiplication of z by a constant factor.

$$\langle M, z\rangle = \sum_{i=1}^{g} M_i z_i, \quad \langle BM, M\rangle = \sum_{i,j=1}^{g} B_{ij} M_i M_j.$$

Equation $\theta(z|B) = 0$ defines a codimension one subvariety $\Theta \in \mathrm{Jac}(\Gamma)$ (for $g > 2$ with singularities) called *theta-divisor*.

We shall also use theta-functions with characteristics $\alpha = (\alpha_1, \ldots, \alpha_g)$, $\beta = (\beta_1, \ldots, \beta_g)$, $\alpha_j, \beta_j \in \mathbb{R}$, which are obtained from $\theta(z|B)$ by shifting the argument z and multiplying by an exponent[4)]:

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z) \equiv \theta\begin{bmatrix}\alpha_1 & \cdots & \alpha_g\\ \beta_1 & \cdots & \beta_g\end{bmatrix}(z) = \exp\{\langle B\alpha, \alpha\rangle/2 + \langle z + 2\pi\jmath\beta, \alpha\rangle\}\, \theta(z + 2\pi\jmath\beta + B\alpha).$$

Then for a pair of characteristics one has the following useful relations

$$\theta\begin{bmatrix}\alpha+\alpha'\\ \beta+\beta'\end{bmatrix}(z) = \exp\{(B\alpha', \alpha')/2 + (z + 2\pi\jmath\beta + 2\pi\jmath\beta', \alpha')\}\, \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z + 2\pi\jmath\beta' + B\alpha'). \tag{4.3}$$

All these functions possess the following property of quasi-periodicity:

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z + 2\pi\jmath K + BM) = \exp(2\pi\jmath\epsilon)\exp\{-\langle BM, M\rangle/2 - \langle M, z\rangle\}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z), \tag{4.4}$$
$$\epsilon = \langle\alpha, K\rangle - \langle\beta, M\rangle,$$

An important particular case is represented by theta-functions with half-integer characteristics

$$\Delta = \begin{pmatrix}\Delta'\\ \Delta''\end{pmatrix}, \quad \eta_i = \begin{pmatrix}\eta_i'\\ \eta_i''\end{pmatrix}, \quad \text{and} \quad \eta_{ij} = \eta_i + \eta_j \pmod{\mathbb{Z}^{2g}/\mathbb{Z}^{2g}}, \quad \Delta', \Delta'', \eta_i', \eta_i'' \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$$

such that

$$2\pi\jmath\,\eta_i'' + B\eta_i' = \int_{E_{2g+2}}^{E_i} \bar{\omega} \pmod{\Lambda}, \tag{4.5}$$
$$2\pi\jmath\Delta'' + B\Delta' = \mathcal{K} \pmod{\Lambda},$$

$\mathcal{K} \in \mathbb{C}^g$ being the vector of the Riemann constants, and E_i briefly denotes the branch point $(E_i, 0)$ on Γ.

The half-integer characteristic $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is odd (even) if $\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z)$ is odd (respectively, even).

For the case $g = 2$ and for the chosen canonical basis of cycles $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2$ on Γ the above characteristics Δ, η_i are

$$\Delta = \begin{pmatrix}1/2 & 1/2\\ 0 & 1/2\end{pmatrix}, \quad \eta_1 = \begin{pmatrix}1/2 & 0\\ 0 & 0\end{pmatrix}, \quad \eta_2 = \begin{pmatrix}1/2 & 0\\ 1/2 & 0\end{pmatrix},$$
$$\eta_3 = \begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}, \quad \eta_4 = \begin{pmatrix}0 & 1/2\\ 1/2 & 1/2\end{pmatrix}, \quad \eta_5 = \begin{pmatrix}0 & 0\\ 1/2 & 1/2\end{pmatrix}, \tag{4.6}$$

and, by convention, η_6 is the zero theta-characteristic. Note also the property

$$\eta_1 + \eta_3 + \eta_5 = \eta_2 + \eta_4 = \Delta \mod \mathbb{Z}^{2g}/\mathbb{Z}^{2g}. \tag{4.7}$$

The six functions $\theta[\Delta + \eta_i](z)$, $i = 1, \ldots, 6$ are odd, that is, $\theta[\Delta + \eta_i](0) = 0$, whereas the other 10 functions $\theta[\Delta + \eta_{ij}](z)$, $i, j \neq 6$ are even. In the case $g = 2$ no one of even theta-functions vanishes at zero.

[4)]Here and below we omit B in the theta-functional notation.

The root functions. To obtain theta-functions solutions to many problems linearized on Jacobians of hyperelliptic curves, one can apply some remarkable relations between roots of certain functions on symmetric products of such curves and quotients of theta-functions with half-integer characteristics, which are historically referred to as *root functions.* For the case of odd order hyperelliptic curves such functions were obtained by Weierstrass and Rosenheim [16, 17], see also [6, 14].

For our purposes it is sufficient to quote only several root functions for the particular case $g = 2$ and the even-order hyperelliptic curve

$$\Gamma = \{\mu^2 = R(\lambda)\}, \quad R(\lambda) = (\lambda - E_1) \cdots (\lambda - E_6).$$

Let us introduce the polynomial $U(\lambda, s) = (s - \lambda_1)(s - \lambda_2)$.

Proposition 3. *Under the Abel–Jacobi mapping (4.1) with $g = 2$ and the basepoint $P_0 = E_6$ the following relations hold*

$$U(\lambda, E_i) \equiv (\lambda_1 - E_i)(\lambda_2 - E_i) = \kappa_i \frac{\theta^2[\Delta + \eta_i](z)}{\theta[\Delta](z - q/2)\,\theta[\Delta](z + q/2)}, \tag{4.8}$$

$$q = \int_{\infty_-}^{\infty_+} \bar{\omega} = 2\int_{E_6}^{\infty_+} \bar{\omega}, \qquad \kappa_i = \text{const}, \qquad i = 1, \dots, 6,$$

$$\frac{1}{\lambda_1 - \lambda_2}\left(\frac{\sqrt{R(\lambda_1)}}{(E_i - \lambda_1)(E_j - \lambda_1)(E_s - \lambda_1)} - \frac{\sqrt{R(\lambda_2)}}{(E_i - \lambda_2)(E_j - \lambda_2)(E_s - \lambda_2)}\right)$$
$$= \kappa_{ijs}\,\frac{\theta[\Delta + \eta_i + \eta_j + \eta_s](z)\,\theta[\Delta](z - q/2)\,\theta[\Delta](z + q/2)}{\theta[\Delta + \eta_i](z)\,\theta[\Delta + \eta_j](z)\,\theta[\Delta + \eta_s](z)}, \tag{4.9}$$

$$\frac{\sqrt{U(\lambda, E_i)}\sqrt{U(\lambda, E_j)}}{\lambda_1 - \lambda_2}\left(\frac{\sqrt{R(\lambda_1)}}{(E_i - \lambda_1)(E_j - \lambda_1)(E_s - \lambda_1)} - \frac{\sqrt{R(\lambda_2)}}{(E_i - \lambda_2)(E_j - \lambda_2)(E_s - \lambda_2)}\right)$$
$$= \kappa'_{ijs}\,\frac{\theta[\Delta + \eta_i + \eta_j + \eta_s](z)}{\theta[\Delta + \eta_s](z)}, \tag{4.10}$$

$$\kappa_{ijs}, \kappa'_{ijs} = \text{const}, \quad i, j, s = 1, \dots, 6, \quad i \neq j \neq s \neq i,$$

where, as above, η_6 is the zero theta-characteristic and ∞_+, ∞_- are the infinite points of the compactified curve Γ. The constant factors $\kappa_i, \kappa_{ijs}, \kappa'_{ijs}$ depend on the moduli of Γ only.

Note that various expressions of symmetric functions of the λ, μ-coordinates on an even hyperelliptic curve were obtained in [18] on the basis of the Klein–Weierstrass realization of Abelian functions outlined in [14] and [19].

Sketch of proof of Proposition 3. The left- and right-hand sides of (4.8) are meromorphic functions on Jac(Γ), which have the same zeros and poles with the same multiplicity. This implies that their quotient is an analytic function on a compact complex manifold without poles and therefore a constant.

The root functions (4.9), (4.10) can be deduced from the corresponding root functions for the case of odd-order hyperelliptic curve, by making a fractionally-linear transformation of λ that sends the Weierstrass point E_{2g+2} on Γ to infinity. □

The constants $\kappa_i, \kappa_{ijs}, \kappa'_{ijs}$ can be calculated explicitly in terms of the coordinates $E_1, \dots, E_6$ and theta-constants by equating λ_1, λ_2 to certain E_i and the argument z to the corresponding half-period in Jac(Γ) (see, e.g., [14]).

Explicit solution. Now we are able to write explicit solution for the Steklov–Lyapunov systems by comparing the root functions (4.8), (4.10) with the Kötter expression (2.3).

Namely, let $\Gamma = \left\{\mu^2 = \Phi(\lambda)\,\psi(\lambda)\right\}$, where the polynomials Φ and ψ are defined in (2.4), and identify (without ordering) the sets

$$\{E_1, E_2, E_3, E_4, E_5, E_6\} = \{b_1, b_2, b_3, c_1, c_2, c_3\}.$$

By $\eta_{b_i}, \eta_{c_\alpha}$ we denote the half-integer characteristics corresponding to the branch points $(b_i, 0), (c_\alpha, 0)$ respectively, according to (4.5).

Theorem 2. *For fixed constants of motion c_1, c_2, c_3 the variables z, p can be expressed in terms of theta-functions of the curve Γ in a such a way that for any $s \in \mathbb{C}$*

$$z_i + sp_i = \frac{\sum_{\alpha=1}^{3} k_{i\alpha}\,(s - c_\alpha)\,\theta\left[\Delta + \eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}\right](z)}{\sum_{\alpha=1}^{3} k_{0\alpha}\,\theta[\Delta + \eta_{c_\alpha}](z)}, \qquad (\alpha, \beta, \gamma) = (1, 2, 3), \tag{4.11}$$

where $k_{i\alpha}, k_{0\alpha}$ are certain constants depending on the moduli of Γ only. The components of the argument z are the following linear functions of t:

$$z_1 = C_{11}\delta_1 t + C_{12}\delta_2 t + z_{10}, \quad z_2 = C_{21}\delta_1 t + C_{22}\delta_2 t + z_{20}, \quad z_{10}, z_{20} = \text{const}, \quad C = A^{-1}, \tag{4.12}$$

A being the matrix of $\mathfrak{a}$-periods of the differentials $d\lambda/\mu, \lambda\, d\lambda/\mu$ on Γ.

Thus, we have recovered the theta-function solution of the systems obtained by Kötter in [4]. The proof of the theorem will be given in the end of the section.

Remark 3. In view of the definition of the theta-function with characteristics, under the argument shift $z \to z - \mathcal{K}$ the special characteristic Δ is killed and the solutions (4.11) are simplified to

$$z_i + sp_i = \frac{\sum_{\alpha=1}^{3} \bar{k}_{i\alpha}(s - c_\alpha)\theta[\eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}](z)}{\sum_{\alpha=1}^{3} \bar{k}_{0\alpha}\,\theta[\eta_{c_\alpha}](z)}, \qquad (\alpha, \beta, \gamma) = (1, 2, 3), \tag{4.13}$$

where the constants $\bar{k}_{i\alpha}, \bar{k}_{0\alpha}$ coincide with $k_{i\alpha}, k_{0\alpha}$ in (4.11) up to multiplication by a quartic root of unity. In each specific case of relative position of b_i, c_α, one can also simplify the sums of characteristics in the numerator of (4.13) by using the relations (4.7).

Remark 4 (Where the solutions are meromorphic). In view of the quasi-periodicity property (4.4), when the complex argument z changes by a period vector in $\operatorname{Jac}(\Gamma)$, the theta-functions in (4.11), (4.13) are multiplied by generally different factors. Hence, the variables z_i, p_i cannot be single valued on the Jacobian variety Γ, and a simple accounting shows that they are meromorphic on $\widetilde{\operatorname{Jac}}(\Gamma)$, the 16-fold unramified covering of it, obtained by doubling of all the four period vectors in $\operatorname{Jac}(\Gamma)$. This implies that $\widetilde{\operatorname{Jac}}(\Gamma)$ is also a principally polarized Abelian variety isomorphic to $\operatorname{Jac}(\Gamma)$. As it follows from the structure of (4.11), all z_i, p_i have a common set of simple poles (the pole divisor), which we denote $\mathcal{D} \subset \widetilde{\operatorname{Jac}}(\Gamma)$.

The degree of the covering $\widetilde{\operatorname{Jac}}(\Gamma) \to \operatorname{Jac}(\Gamma)$ can also be found in another way: According to Remark 2, each generic pair (λ_1, λ_2) corresponds to 64 different points (z, p) on the invariant manifold $\widetilde{\operatorname{Jac}}(\Gamma)$. On the other hand, the same pair gives rise to 4 different points in $\operatorname{Jac}(\Gamma)$ defined by the divisors $\{(\lambda_1, \pm\sqrt{R_6(\lambda_1)}\,), (\lambda_2, \pm\sqrt{R_6(\lambda_2)}\,)\,\}$. Hence a generic point of $\operatorname{Jac}(\Gamma)$ corresponds to 64/4=16 points in $\widetilde{\operatorname{Jac}}(\Gamma)$.

Proof of Theorem 2. The summands in the numerator of the Kötter solution (2.3), when divided by $\lambda_1 - \lambda_2$, can be written as

$$\frac{s - c_\alpha}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{\lambda_1 - \lambda_2} \cdot \left(\frac{\sqrt{\Phi(\lambda_1)\psi(\lambda_2)}}{(\lambda_1 - b_i)(\lambda_2 - c_\alpha)} - \frac{\sqrt{\Phi(\lambda_2)\psi(\lambda_1)}}{(\lambda_2 - b_i)(\lambda_1 - c_\alpha)} \right)$$
$$= \frac{s - c_\alpha}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} \frac{\sqrt{-(\lambda_1 - c_\beta)(\lambda_2 - c_\beta)}\sqrt{-(\lambda_1 - c_\gamma)(\lambda_2 - c_\gamma)}}{\lambda_1 - \lambda_2}$$

$$\times \left(\frac{\mu_1}{(\lambda_1 - b_i)(\lambda_1 - c_\beta)(\lambda_1 - c_\gamma)} - \frac{\mu_2}{(\lambda_2 - b_i)(\lambda_2 - c_\beta)(\lambda_2 - c_\gamma)} \right),$$
$$\mu_1 = \sqrt{\Phi(\lambda_1)\psi(\lambda_1)}, \quad \mu_2 = \sqrt{\Phi(\lambda_2)\psi(\lambda_2)}.$$

The right-hand sides have the form of the root function (4.10). Hence, up to a constant factor, they are equal to

$$(s - c_\alpha)\frac{\theta\left[\Delta + \eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}\right](z)}{\theta[\Delta + \eta_{b_i}](z)}.$$

Next, in view of (4.8), we obtain

$$x_i = \varkappa_i \frac{\theta[\Delta + \eta_{b_i}](z)}{\sqrt{\theta[\Delta](z - q/2)\,\theta[\Delta](z + q/2)}},$$
$$\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)} = \varkappa_\alpha \frac{\theta[\Delta + \eta_{c_\alpha}](z)}{\sqrt{\theta[\Delta](z - q/2)\,\theta[\Delta](z + q/2)}}, \tag{4.14}$$
$$\varkappa_i, \varkappa_\alpha = \text{const},$$
$$\sum_{\alpha=1}^{3} \frac{\sqrt{-(\lambda_1 - c_\alpha)(\lambda_2 - c_\alpha)}}{(c_\alpha - c_\beta)(c_\alpha - c_\gamma)} = \sum_{\alpha=1}^{3} \frac{k_{0\alpha}\theta[\Delta + \eta_{c_\alpha}](z)}{\sqrt{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}}. \tag{4.15}$$

Combining the above expressions, we rewrite the right-hand side of (2.3) in the form

$$\sqrt{c_0}\,\frac{\dfrac{\theta[\Delta + \eta_{b_i}](z)}{\sqrt{\theta\,[\Delta]\,(z - q/2)\theta[\Delta](z + q/2)}} \sum_{\alpha=1}^{3} \dfrac{k_{i\alpha}\,(s - c_\alpha)\,\theta\left[\Delta + \eta_{c_\beta} + \eta_{c_\gamma} + \eta_{b_i}\right](z)}{\theta\left[\Delta + \eta_{b_i}\right](z)}}{\sum_{\alpha=1}^{3} \dfrac{k_{0\alpha}\theta[\Delta + \eta_{c_\alpha}](z)}{\sqrt{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}}},$$

which, after simplifications, gives (4.11).

Expressions (4.12) follow from the relation $(\bar{\omega}_1, \bar{\omega}_2)^T = C(d\lambda/\mu, \lambda\, d\lambda/\mu)$, where, as above, $\bar{\omega}_j$ are the normalized holomorphic differentials on Γ, which implies $(\bar{\omega}_1, \bar{\omega}_2)^T = C(u_1, u_2)^T$, where u_1, u_2 are the right-hand sides of the quadratures (2.8). □

5. THE DIVISOR OF POLES AND THE ALTERNATIVE FORM OF THE THETA-FUNCTION SOLUTION

The nice form of the Kötter solution (4.11) itself tells us a little about the structure of zeros and poles of z_i, p_i on the 2-dimensional Abelian variety $\widetilde{\text{Jac}}(\Gamma)$. It appears however that recent studies allow to give a quite detailed description of the set of common poles of these variables, called the divisor of poles $\mathcal{D}$. Obviously, $\mathcal{D} = \{\sum_{\alpha=1}^{3} k_{0\alpha}\theta[\Delta + \eta_{c_\alpha}](z) = 0\} \subset \widetilde{\text{Jac}}(\Gamma)$.

For each $\alpha = 1, 2, 3$, the zeros of $\theta[\Delta + \eta_{c_\alpha}](z)$ in $\text{Jac}(\Gamma)$ form a translate Θ_α of the theta-divisor Θ by the half-period $2\pi\jmath\eta''_{c_\alpha} + 2B\eta'_{c_\alpha}$.

Each translate passes via six half-periods, and $\Theta_1, \Theta_2, \Theta_3$ have a unique common intersection in the origin (neutral point) $\mathcal{O} \in \text{Jac}(\Gamma)$. This is depicted in Fig. 3a, where Θ_α are shown as circles and the half-periods in $\text{Jac}(\Gamma)$ as black dots. Hence, at $z = \mathcal{O}$ the denominator of (4.11) vanishes. Then, under the covering $\pi : \widetilde{\text{Jac}}(\Gamma) \to \text{Jac}(\Gamma)$, the preimage of $\mathcal{O}$ consists of all the 16 half-periods in $\widetilde{\text{Jac}}(\Gamma)$, which therefore belong to the divisor $\mathcal{D}$.

Note that, according to Remark 4, translations in $\text{Jac}(\Gamma)$ by a complete periods $\mathcal{V}$ correspond to translation in $\widetilde{\text{Jac}}(\Gamma)$ by the half-periods $\mathcal{V}/2$.

Now assume, as above, that $b_1 < b_2 < b_3$ and that $(b_3, 0) = E_6 \in \Gamma$ is the basepoint of the Abel map (4.1) with $g = 2$. Further information about $\mathcal{D}$ is given by

Proposition 4. *The divisor $\mathcal{D} \subset \widetilde{\mathrm{Jac}}(\Gamma)$ is invariant under translations by the half-periods generated by*

$$\mathcal{V}_1/2 = 2\pi \jmath \eta''_{b_1} + 2B\eta'_{b_1}, \quad \mathcal{V}_2/2 = 2\pi \jmath \eta''_{b_2} + 2B\eta'_{b_2}, \qquad \begin{pmatrix} \eta'_{b_i} \\ \eta''_{b_i} \end{pmatrix} = \eta_{b_i}. \tag{5.1}$$

Proof. Choose a generic point $q \in \mathcal{D}$ and let z^* be its projection onto $\mathrm{Jac}(\Gamma)$, which gives

$$f(z^*) = \sum_{\alpha=1}^{3} k_{0\alpha}\, \theta[\Delta + \eta_{c_\alpha}](z^*) = 0.$$

In view of the quasi-periodic property (4.4) and the half-integer characteristics (4.6), under the translations $z^* \to z^* + M\mathcal{V}_1 + N\mathcal{V}_2$, $M, N \in \mathbb{Z}$ all the functions $\theta[\Delta + \eta_{c_\alpha}](z^*)$ are multiplied by the same factor and therefore $f(z^* + M\mathcal{V}_1 + N\mathcal{V}_2) = 0$. Hence the points $z^*/2 + M\mathcal{V}_1/2 + N\mathcal{V}_2/2$ in $\widetilde{\mathrm{Jac}}(\Gamma)$ also belong to $\mathcal{D}$.

One can also show that this does not hold for the translations by the other half-periods. □

Next, analytically, we have

Theorem 3. *The denominator of the solution* (4.11) *admits the factorization*

$$\sum_{\alpha=1}^{3} k_{0\alpha}\theta[\Delta + \eta_{c_\alpha}](z) = \exp(\chi z + \zeta) \cdot \theta[\Delta](z/2)\, \theta[\Delta + \eta_{b_1}]\left(\frac{z}{2}\right)\, \theta[\Delta + \eta_{b_2}]\left(\frac{z}{2}\right)\, \theta[\Delta + \eta_{b_1} + \eta_{b_2}]\left(\frac{z}{2}\right) \tag{5.2}$$

with certain constants χ, ζ.

The proof of the theorem is based on the fourth Riemann identity (see, e.g., [14, 15]) and the theta-formulas of Frobenius and Thomae (see, e.g., [20, 21]). Technically, it is quite tedious and for this reason we move it into Appendix.

Now note that each of the four sets

$$\mathcal{D}_0 = \{\theta[\Delta](z/2\,|B) = 0\}, \quad \mathcal{D}_1 = \{\theta[\Delta + \eta_{b_1}](z/2\,|B) = 0\},$$
$$\mathcal{D}_2 = \{\theta[\Delta + \eta_{b_2}](z/2\,|B) = 0\}, \quad \mathcal{D}_3 = \{\theta[\Delta + \eta_{b_1} + \eta_{b_2}](z/2\,|B) = 0\}$$

describes a translate of the theta-divisor, the genus 2 curve Γ embedded into $\widetilde{\mathrm{Jac}}(\Gamma)$. Then, Theorem 3 says that the pole divisor $\mathcal{D}$ is a union of these translates, which are obtained from each other by shifts by the half-periods $\mathcal{V}_1/2, \mathcal{V}_2/2$, and $\mathcal{V}_3/2 = -\mathcal{V}_1/2 - \mathcal{V}_2/2$ in $\widetilde{\mathrm{Jac}}(\Gamma)$. The union passes through all the 16 half-periods in $\widetilde{\mathrm{Jac}}(\Gamma)$. The action of the translations by $\mathcal{V}_1/2, \mathcal{V}_2/2, \mathcal{V}_3/2$ in $\widetilde{\mathrm{Jac}}(\Gamma)$ on the components $(\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ gives respectively

$$(\mathcal{D}_1, \mathcal{D}_0, \mathcal{D}_3, \mathcal{D}_2), \quad (\mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_0, \mathcal{D}_1), \quad (\mathcal{D}_3, \mathcal{D}_2, \mathcal{D}_1, \mathcal{D}_0). \tag{5.3}$$

All these properties are in complete correspondence with our previous observations about the divisor $\mathcal{D}$.

Also, as was shown in [12] by applying the Kovalevskaya–Painlevé analysis, the pole divisor with the same structure appears in the integrable flow on the algebra $so(4)$ with the diagonal metric II, already mentioned in Introduction. This result of [12] about $\mathcal{D}$ equally holds for our Steklov–Lyapunov systems due to a linear isomorphism between them and the integrable flow on $so(4)$.

The intersection pattern for $\mathcal{D}$ is shown in Fig. 3b, which we borrowed from [12]. Here the circles represent the translates $\mathcal{D}_j$ and the 16 black dots depict the half-periods. Under the projection $\pi : \widetilde{\mathrm{Jac}}(\Gamma) \to \mathrm{Jac}(\Gamma)$ all the above half-periods are mapped onto $\mathcal{O} \in \mathrm{Jac}(\Gamma)$.

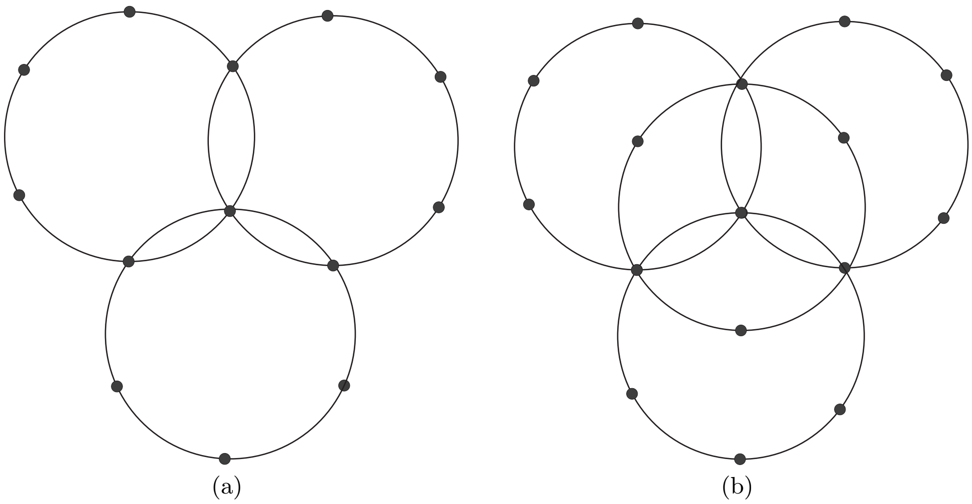

Fig. 3. (a) Configuration of the translates Θ_α in $\mathrm{Jac}(\Gamma)$. (b) The 4 translates of Γ in $\widetilde{\mathrm{Jac}}(\Gamma)$ forming the pole divisor $\mathcal{D}$.

Solutions for the variables v_k. Let us choose the origin of $\widetilde{\mathrm{Jac}}(\Gamma)$ at one of the four triple intersections of $\mathcal{D}_j$ and denote for brevity the four theta-functions in (5.2) as $\theta_0(z/2), \theta_1(z/2), \theta_2(z/2), \theta_3(z/2)$ respectively.

Now we show that the theta-function solutions for the new phase variables $v_1, \dots, v_6$ introduced in (2.10) have a rather specific and compact form. Namely, as it follows from expressions (4.11) and (2.10), the functions $v_1 + v_2$ and $v_1 - v_2$ may have only *simple* poles *at most* along the components of the divisor $\mathcal{D}$. On the other hand, the form of the integrals (2.13) imply the following remarkable property: the poles (the zeros) of $v_1 + v_2$ are the zeros (resp. the poles) of $v_1 - v_2$. Since both functions are meromorphic on $\widetilde{\mathrm{Jac}}(\Gamma)$, none of then can have simple poles along only one component $\mathcal{D}_j$. This necessarily implies that $v_1 + v_2$ has poles along two certain components $\mathcal{D}_{j_1}, \mathcal{D}_{j_2}$ and zeros along the other two components $\mathcal{D}_{j_3}, \mathcal{D}_{j_4}$, and vice versa for $v_1 - v_2$.

The same observations hold for the pairs $(v_3 + v_4,\, v_3 - v_4)$ and $(v_5 + v_6,\, v_5 - v_6)$. Note also that functions from different pairs cannot have the same poles, since in that case they would also have the same zeros and their quotient would be constant, which is not true.

Now let us fix the origin of $\widetilde{\mathrm{Jac}}(\Gamma)$ at one specific triple intersection of $\mathcal{D}_j$ such that the 3 functions $v_1 + v_2, v_3 + v_4, v_5 + v_6$ have a common pole along the component $\mathcal{D}_0$. In this case the following proposition holds.

Proposition 5. *The theta-function solutions for the phase variables v_k have the form*

$$
\begin{aligned}
v_1 + v_2 &= \chi_1 \frac{\theta_1(z/2)\,\theta_2(z/2)}{\theta_0(z/2)\,\theta_3(z/2)}, \quad & v_1 - v_2 &= \chi_2 \frac{\theta_0(z/2)\,\theta_3(z/2)}{\theta_1(z/2)\,\theta_2(z/2)}, \\
v_3 + v_4 &= \chi_3 \frac{\theta_2(z/2)\,\theta_3(z/2)}{\theta_0(z/2)\,\theta_1(z/2)}, \quad & v_3 - v_4 &= \chi_4 \frac{\theta_0(z/2)\,\theta_1(z/2)}{\theta_2(z/2)\,\theta_3(z/2)}, \\
v_5 + v_6 &= \chi_5 \frac{\theta_1(z/2)\,\theta_3(z/2)}{\theta_0(z/2)\,\theta_2(z/2)}, \quad & v_5 - v_6 &= \chi_6 \frac{\theta_0(z/2)\,\theta_2(z/2)}{\theta_1(z/2)\,\theta_3(z/2)},
\end{aligned}
\tag{5.4}
$$

$$
\chi_1, \chi_3, \chi_5 = \text{const}, \qquad \chi_2 = \frac{\psi(b_1)}{(b_2 - b_3)\chi_1}, \quad \chi_4 = \frac{\psi(b_2)}{(b_3 - b_1)\chi_3}, \quad \chi_6 = \frac{\psi(b_3)}{(b_1 - b_2)\chi_5}, \tag{5.5}
$$

where $z = (z_1, z_2)$ depend on t according to (4.12).

Given the solutions (5.4), one can then reconstruct solutions for the original variables z, p by using the relations (2.11).

Proof of Proposition 5. First, note that the functions (5.4) have the same structure of zeros and poles, as prescribed above. Next, as follows from the Kötter formula (2.3) and theta-solutions (4.11), the translations by the period vectors $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_1 + \mathcal{V}_2$ in $\mathrm{Jac}(\Gamma)$ generate the involutions

$$\begin{aligned} \sigma_1 \ &: \ (z_1, p_1, z_2, p_2, z_3, p_3) \mapsto (z_1, p_1, -z_2, -p_2, z_3, p_3), \\ \sigma_2 \ &: \ (z_1, p_1, z_2, p_2, z_3, p_3) \mapsto (-z_1, -p_1, -z_2, -p_2, z_3, p_3), \\ \sigma_3 = \sigma_2 \circ \sigma_1 \ &: \ (z_1, p_1, z_2, p_2, z_3, p_3) \mapsto (-z_1, -p_1, z_2, p_2, z_3, p_3), \end{aligned}$$

which, in view of (2.10), gives rise to the transformations

$$\begin{aligned} \sigma_1 \ &: \ v_2 + v_1 \longleftrightarrow v_2 - v_1, \quad v_4 \pm v_3 \longleftrightarrow v_4 \pm v_3, \quad v_5 + v_6 \longleftrightarrow v_5 - v_6, \\ \sigma_2 \ &: \ v_2 + v_1 \longleftrightarrow v_2 - v_1, \quad v_4 + v_3 \longleftrightarrow v_4 - v_3, \quad v_5 \pm v_6 \longleftrightarrow -(v_5 \pm v_6) \\ \sigma_3 \ &: \ v_2 \pm v_1 \longleftrightarrow v_2 \pm v_1, \quad v_4 + v_3 \longleftrightarrow v_4 - v_3, \quad v_6 + v_5 \longleftrightarrow v_6 - v_5. \end{aligned}$$

Now observe that the relations (5.4) are invariant under the action of σ_i on the left-hand sides and the corresponding transformation of $\theta_0(z/2), \ldots, \theta_3(z/2)$ under the action (5.3). Moreover, one can check that the left- and right-hand sides of (5.4) are multiplied by the same factors under the shift of z by any period vector of $\mathrm{Jac}(\Gamma)$. This proves (5.4).

The relations (5.5) between the constants χ_i follow from the first 3 integrals in (2.13). □

The constants χ_1, χ_2, χ_3 can be calculated explicitly in terms of b_i, c_α and theta-constants of Γ.

As follows from the solutions (5.4), the product $(v_1 + v_2)(v_3 + v_4)$ and the other two similar products have double poles along $\mathcal{D}_0$ only:

$$(v_1 + v_2)(v_3 + v_4) = \varsigma_2 \frac{\theta_2^2(z/2)}{\theta_0^2(z/2)},$$

$$(v_3 + v_4)(v_5 + v_6) = \varsigma_3 \frac{\theta_3^2(z/2)}{\theta_0^2(z/2)}, \quad (v_1 + v_2)(v_5 + v_6) = \varsigma_1 \frac{\theta_1^2(z/2)}{\theta_0^2(z/2)},$$

$$\varsigma_1, \varsigma_2, \varsigma_3 = \text{const}.$$

Analogs of some of these expressions were obtained in paper [22] in relation with separation of variables for the integrable system on $so(4)$ with the diagonal metric II. Due to the linear isomorphism between this system and the Steklov–Lyapunov systems, the separating variables of [22] can also be regarded as new separating variables for (1.4), (1.5).

6. CONCLUSIVE REMARKS AND OUTLOOK

In this paper we gave a justification of the separation of variables and the theta-function solution of the Steklov–Lyapunov systems obtained by F. Kötter [4]. Using some of the results of [11, 12, 22], we also presented such solutions for an alternative set of variables, which have a much simpler form.

On the other hand, there exist several nontrivial integrable generalizations of the systems: one of them was discovered by V. Rubanovsky [23] and describes a motion of a gyrostat in an ideal fluid under the action of the Archimedes torque, which arises when the barycenter of the gyrostat does not coincide with its volume center. In this generalization the Hamiltonian of the Kirchhoff equations, apart form quadratic terms, contains linear (gyroscopic) terms in K, p. Under the change of variables (1.3), the gyroscopic generalizations of the systems (1.4), (1.5) take the form

$$\dot{z} = z \times (Bz - g) - Bp \times (Bz - g), \qquad \dot{p} = p \times (Bz - g)$$

and, respectively,

$$\dot{z} = p \times (Bz - g), \qquad \dot{p} = p \times (z - Bp),$$

where $g = (g_1, g_2, g_3)^T$ is an arbitrary constant vector related to the angular momentum of the rotor inside the gyrostat.

These systems has the same trivial integrals J_1, J_2 given in (1.9), and the following non-homogeneous extensions of the Hamiltonians H_1, H_2:

$$\tilde{H}_1 = \frac{1}{2}\langle z, Bz\rangle - \langle z, g\rangle, \quad \tilde{H}_2 = \frac{1}{2}\langle z, z\rangle - \langle Bz, p\rangle + \langle p, g\rangle .$$

As was shown in [24], these systems admit the following generalizations of the Kötter Lax pair (1.6) with an elliptic spectral parameter:

$$\dot{L}(s) = [\,L(s), A(s)\,], \qquad L(s), A(s) \in so(3), \quad s \in \mathbb{C}, \tag{6.1}$$

$$L(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\Big(\sqrt{s-b_\gamma}\,(z_\gamma + sp_\gamma) + g_\gamma/\sqrt{s-b_\gamma}\Big), \quad (\alpha,\beta,\gamma) = (1,2,3),$$

$$A(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\frac{1}{s}\sqrt{(s-b_\alpha)(s-b_\beta)}\,(b_\gamma z_\gamma - g_\gamma), \quad \text{resp.} \quad A(s)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\sqrt{(s-b_\alpha)(s-b_\beta)}\,p_\gamma .$$

One can observe that the above L-matrix can be written as follows

$$L(s) = \sqrt{\Phi(s)}\,S^{-1}\mathbf{P}(s)S^{-1}, \quad \Phi(s) = (s-b_1)(s-b_2)(s-b_3),$$

$$S = \operatorname{diag}(\sqrt{s-b_1}, \sqrt{s-b_2}, \sqrt{s-b_3}),$$

$$\mathbf{P}(s) = \begin{pmatrix} 0 & -(z_3+sp_3) - \dfrac{g_3}{s-a_3} & z_2 + sp_2 + \dfrac{g_2}{s-a_2} \\ z_3 + sp_3 + \dfrac{g_3}{s-a_3} & 0 & -(z_1+sp_1) - \dfrac{g_1}{s-a_1} \\ -(z_2+sp_2) - \dfrac{g_2}{s-a_2} & z_1 + sp_1 + \dfrac{g_1}{s-a_1} & 0 \end{pmatrix}.$$

Then, multiplying the Lax equation (6.1) from the left by S^{-1} and from the right by S, we get

$$\frac{d}{dt}\left(S^{-2}P(s)\right) = \left[S^{-2}P(s), \mathcal{M}(s)\right] \tag{6.2}$$

with an appropriate matrix $\mathcal{M}(s)$ depending on the choice of the Hamiltonian. Equation (6.2) gives a Lax representation of the Rubanovsky systems with a *rational* spectral parameter and the 3×3 (non-skew-symmetric) Lax matrix

$$\mathcal{L}(s) = S^{-2}\mathbf{P}(s) = \begin{pmatrix} 0 & -\dfrac{(z_3+sp_3)(s-a_3)+g_3}{(s-a_1)(s-a_3)} & \dfrac{(z_2+sp_2)(s-a_2)+g_2}{(s-a_1)(s-a_2)} \\ \dfrac{(z_3+sp_3)(s-a_3)+g_3}{(s-a_2)(s-a_3)} & 0 & -\dfrac{(z_1+sp_1)(s-a_1)+g_1}{(s-a_2)(s-a_1)} \\ -\dfrac{(z_2+sp_2)(s-a_2)+g_2}{(s-a_2)(s-a_3)} & \dfrac{(z_1+sp_1)(s-a_1)+g_1}{(s-a_1)(s-a_3)} & 0 \end{pmatrix}. \tag{6.3}$$

The characteristic equation $|\mathcal{L}(s)\Phi(s) - \mu\mathbf{I}| = 0$ has the form

$$\mu(\mu^2 - F_6(s)) = 0, \quad F_6(s) = \Phi(s)\left(J_1 s^3 + J_2 s^2 + 2\tilde{H}_2 s - 2\tilde{H}_1 + \sum_{\alpha=1}^{3}\frac{g_\alpha}{s-b_\alpha}\right).$$

Since $F_6(s)$ is a polynomial of degree 6, the spectral curve is reduced to hyperelliptic one, $\tilde{\Gamma} = \{\mu^2 = F_6(s)\}$, which, for $g = 0$, coincides with the Kötter curve $\mu^2 = \Phi(s)\psi(s)$ in (2.7), as expected. Note that for $g \neq 0$, the numbers $s = b_i$ are no more s-coordinates of the Weierstrass (branch) points on $\tilde{\Gamma}$. Nevertheless, like in the case of the Steklov–Lyapunov systems, generic complex invariant manifolds of the Rubanovsky systems are affine parts of two-dimensional Jacobian varieties of $\tilde{\Gamma}$ or covering of these varieties.

Next, following the methods developed in many publications (see, e.g., [25, 26]), the polynomial Lax matrix $\Phi(s)\mathcal{L}(s)$ allows to construct Darboux coordinates s_1, s_2 (separating variables), and the conjugated momenta μ_1, μ_2 on each 4-dimensional phase space obtained by fixing values of J_1, J_2. Namely let $\mathcal{K}(s,\mu)$ be the adjoint matrix of $\Phi(s)\mathcal{L}(s) - \mu\mathbf{I}$ and $\mathbf{v}$ be an eigenvector of the leading matrix coefficient of $\Phi(s)\mathcal{L}(s)$. Then the conditions

$$\mathcal{K}(s,\mu)\,\mathbf{v} = 0 \tag{6.4}$$

define precicely 2 finite points $(s_1, \mu_1), (s_2, \mu_2)$ on $\tilde{\Gamma}$ that give the separating variables.

Note that, in view of (6.3), one can just take $\mathbf{v} = p$. Then in the Steklov–Lyapunov case ($g = 0$), from the set of equations (6.4) one can eliminate the coordinate μ and obtain the following equation for s

$$(s - b_2)(s - b_3)(z_2 p_3 - z_3 p_2)^2 + (s - b_1)(s - b_3)(z_3 p_1 - z_1 p_3)^2 + (s - b_1)(s - b_2)(z_1 p_2 - z_2 p_1)^2 = 0,$$

which is equivalent to equation (2.2) for the separating variables λ_1, λ_2 of Kötter.

A detailed description of separation of variables and an explicit theta-function solution for the Rubanovsky gyroscopic generalizations will be given in a forthcoming publication.

APPENDIX. PROOF OF THEOREM 3

The proof is based on the fourth Riemann identity (see, e.g., [14, 15])

$$\theta(y_1)\,\theta(y_2)\,\theta(y_3)\,\theta(y_4) = \frac{1}{4}\sum \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(w_1)\,\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(w_2)\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(w_3)\,\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(w_4), \tag{6.5}$$

where the summation is over all the half-period characteristics $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ and the arguments $y_j, w_j \in \mathbb{C}^g$ (in our case $g = 2$) are related as follows

$$(w_1\, w_2\, w_3\, w_4) = (y_1\, y_2\, y_3\, y_4)\, T, \quad T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Now note that, up to multiplication by a simple exponent of z, the theta-product in (5.2) can be written as

$$\theta(z'/2)\,\theta(z'/2 + \mathcal{V}_1)\,\theta(z'/2 + \mathcal{V}_2)\,\theta(z'/2 + \mathcal{V}_1 + \mathcal{V}_2), \tag{6.6}$$

where $z' = z + 2\mathcal{K}$, i.e., the translation by the complete period in $\mathrm{Jac}(\Gamma)$, and $\mathcal{V}_{1,2}$ are defined by (5.1). In view of the identity (6.5), the product (6.6) gives the sum of 16 theta-products, and in each product the variable z' enters only once:

$$\frac{1}{4}\sum_{2(\alpha,\beta)\in(\mathbb{Z}_2)^4} \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(z' + \frac{V_1 + V_2}{2}\right)\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\frac{V_1}{2}\right)\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\frac{V_2}{2}\right)\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0),$$

Next, in view of the property (4.3), this sum can be written as product of an exponent of z and the sum

$$\frac{1}{4}\sum_{2(\alpha,\beta)\in(\mathbb{Z}_2)^4} \theta\begin{bmatrix}\alpha+\alpha'\\ \beta+\beta'\end{bmatrix}(z')\,\theta\begin{bmatrix}\alpha+\alpha'\\ \beta+\beta'\end{bmatrix}\left(-\frac{V_1}{2}\right)\theta\begin{bmatrix}\alpha+\alpha'\\ \beta+\beta'\end{bmatrix}\left(-\frac{V_2}{2}\right)\theta\begin{bmatrix}\alpha+\alpha'\\ \beta+\beta'\end{bmatrix}\left(-\frac{V_1 + V_2}{2}\right),$$

$$2\pi\jmath\,\beta' + B\alpha' = \frac{V_1 + V_2}{2},$$

which, after the corresponding re-indexation of α, β, reads

$$\begin{aligned} &\frac{1}{4}\sum_{2(\alpha,\beta)\in(\mathbb{Z}_2)^4} \epsilon_{\alpha,\beta}\,\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z')\,\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\frac{V_1}{2}\right)\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\frac{V_2}{2}\right)\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\frac{V_1 + V_2}{2}\right)\\ &= -\frac{1}{4}\sum_{i=1}^{6}\theta[\Delta+\eta_i](z')\,\theta[\Delta+\eta_i]\left(\frac{V_1}{2}\right)\theta[\Delta+\eta_i]\left(\frac{V_2}{2}\right)\theta[\Delta+\eta_i]\left(\frac{V_1 + V_2}{2}\right)\\ &+\frac{1}{4}\sum_{1\leqslant i<j\leqslant 5}\theta[\Delta+\eta_{ij}](z')\,\theta[\Delta+\eta_{ij}]\left(\frac{V_1}{2}\right)\theta[\Delta+\eta_{ij}]\left(\frac{V_2}{2}\right)\theta[\Delta+\eta_{ij}]\left(\frac{V_1 + V_2}{2}\right), \end{aligned} \tag{6.7}$$

where $\epsilon_{\alpha,\beta} = -1$ if $\theta[\alpha\,\beta](z)$ is odd and $+1$ otherwise, and, as above, $\eta_{ij} = \eta_j + \eta_j \mod \mathbb{Z}^2/\mathbb{Z}^2$.

In fact, most of the theta-constants in (6.7) are proportional to $\theta[\Delta + \eta_i](0)$, $i = 1, \dots, 6$ and therefore vanish. Namely, in the first sum in the right-hand side of (6.7) all the theta-constants are non-zero if and only if η_i is different from η_{b_1}, η_{b_2}, and $\eta_6 = 0$. In the second sum, if η_i or η_j coincides with η_{b_1} or η_{b_2}, then either the first or the second theta-constant is zero. Otherwise, if $\{\eta_i, \eta_j\} \cap \{\eta_{b_1}, \eta_{b_2}\} = \emptyset$, then, in view of the relations (4.7), the third theta-constant is proportional to $\theta[\Delta + \eta_k](0)$, for a certain $k \in \{1, \dots, 6\}$ and, therefore, equals zero.

Since for the case of genus 2 there are no even theta-functions which vanish for zero value of the argument (see [14]), one concludes that the above sum contains only 3 non-zero theta-products:

$$\sum_{\eta_i \neq \eta_{b_1}, \eta_{b_2}, 0} \theta[\Delta + \eta_i](z')\, \theta[\Delta + \eta_i]\left(\frac{V_1}{2}\right)\, \theta[\Delta + \eta_i]\left(\frac{V_2}{2}\right)\, \theta[\Delta + \eta_i]\left(\frac{V_1 + V_2}{2}\right). \tag{6.8}$$

Now, assume (for the moment) the following ordering of the Weierstrass points:

$$E_1 = b_1, \quad E_2 = c_1, \quad E_3 = b_2, \quad E_4 = c_2, \quad E_5 = c_3, \quad E_6 = b_3, \tag{6.9}$$

Then, in view of (4.3) and the identities (4.7), the sum (6.8), up to a constant common factor, can be written as

$$\begin{aligned} &\varepsilon_1 \theta[\Delta + \eta_{c_1}](z')\, \theta[\eta_{b_1} + \eta_{c_2}](0)\, \theta[\eta_{b_2} + \eta_{c_2}](0)\, \theta[\eta_{c_1} + \eta_{c_3}](0) \\ &\quad + \varepsilon_2 \theta[\Delta + \eta_{c_2}](z')\, \theta[\eta_{b_1} + \eta_{c_1}](0)\, \theta[\eta_{b_2} + \eta_{c_1}](0)\, \theta[\eta_{c_2} + \eta_{c_3}](0) \\ &\qquad + \varepsilon_3 \theta[\Delta + \eta_{c_3}](z')\, \theta[\eta_{b_2}](0)\, \theta[\eta_{b_1}](0)\, \theta[0](0), \end{aligned} \tag{6.10}$$

where now ε_α are certain quartic roots of 1.

Now we are going to show that the denominator in the theta-function solution (4.11) coincides with (6.10) up to multiplication by an exponent of z'.

Namely, in view of (4.15), the sum $\sum_{\alpha=1}^{3} k_{0\alpha} \theta[\Delta + \eta_{c_\alpha}](z)$ can be written as a product of

$$\text{const} \cdot \sqrt{\theta[\Delta](z - q/2)\theta[\Delta](z + q/2)}$$

and the expression

$$\mathcal{G} = \frac{\sqrt{-(\lambda_1 - c_1)(\lambda_2 - c_1)}}{\sqrt{(c_1 - c_2)(c_1 - c_2)}} \sqrt{\frac{c_2 - c_3}{c_1 - c_2}} + \frac{\sqrt{-(\lambda_1 - c_2)(\lambda_2 - c_2)}}{\sqrt{(c_2 - c_1)(c_2 - c_3)}} \sqrt{\frac{c_3 - c_1}{c_1 - c_2}} + \frac{\sqrt{-(\lambda_1 - c_3)(\lambda_2 - c_3)}}{\sqrt{(c_3 - c_1)(c_3 - c_2)}}$$

(*there is no second radical in the third summand!*). Now, make the projective transformation $\lambda \to \nu = \lambda/(\lambda - E_6) = \lambda/(\lambda - b_3)$, which sends the Weierstrass points c_α, b_1, b_2, b_3 on Γ to $\bar{c}_\alpha, \bar{b}_1, \bar{b}_2$, and ∞. The two infinite points over $\lambda = \infty$ are mapped to 2 points over ν_1. This change leaves the sum $\mathcal{G}$ almost invariant: it becomes the product of $\text{const}/\sqrt{(\nu_1 - 1)(\nu_2 - 1)}$ and the sum

$$\bar{\mathcal{G}} = \frac{\sqrt{-(\nu_1 - \bar{c}_1)(\nu_2 - \bar{c}_1)}}{\sqrt{(\bar{c}_1 - \bar{c}_2)(\bar{c}_1 - \bar{c}_2)}} \sqrt{\frac{\bar{c}_2 - \bar{c}_3}{\bar{c}_1 - \bar{c}_2}} + \frac{\sqrt{-(\nu_1 - \bar{c}_2)(\nu_2 - \bar{c}_2)}}{\sqrt{(\bar{c}_2 - \bar{c}_1)(\bar{c}_2 - \bar{c}_3)}} \sqrt{\frac{\bar{c}_3 - \bar{c}_1}{\bar{c}_1 - \bar{c}_2}} + \frac{\sqrt{-(\nu_1 - \bar{c}_3)(\nu_2 - \bar{c}_3)}}{\sqrt{(\bar{c}_3 - \bar{c}_1)(\bar{c}_3 - \bar{c}_2)}}.$$

Under the Abel map (4.1), the radicals in $\bar{\mathcal{G}}$ can be expressed completely in terms of the theta-functions and theta-constants of Γ: Applying the theta-formulae of Frobenius and Thomae for the case when one of the Weierstrass points of the curve lies at infinity (see, e.g., [20, 21]) and keeping the ordering (6.9), we have

$$\begin{aligned} \frac{\sqrt{(\nu_1 - \bar{c}_\alpha)(\nu_2 - \bar{c}_\alpha)}}{\sqrt{(\bar{c}_\alpha - \bar{c}_\beta)(\bar{c}_\alpha - \bar{c}_\gamma)}} &= \pm \varrho_1 \frac{\theta[\Delta + \eta_{c_\beta} + \eta_{\bar{c}_\gamma}](0)}{\theta[\Delta + \eta_{c_1} + \eta_{c_2} + \eta_{c_3}](0)} \frac{\theta[\Delta + \eta_{c_\alpha}](z)}{\theta[\Delta](z)}, \quad (\alpha, \beta, \gamma) = (1, 2, 3), \\ \sqrt{\frac{\bar{c}_2 - \bar{c}_3}{\bar{c}_1 - \bar{c}_2}} &= \varrho_2 \frac{\theta[\eta_{c_2} + \eta_{b_2}](0)\, \theta[\eta_{c_2} + \eta_{b_1}](0)}{\theta[\eta_{c_1} + \eta_{c_2} + \eta_{c_3} + \eta_{b_2}](0)\, \theta[\eta_{c_1} + \eta_{c_2} + \eta_{c_3} + \eta_{b_1}](0)}, \\ \sqrt{\frac{\bar{c}_3 - \bar{c}_1}{\bar{c}_1 - \bar{c}_2}} &= \varrho_3 \frac{\theta[\eta_{c_1} + \eta_{b_1}](0)\, \theta[\eta_{c_1} + \eta_{b_2}](0)}{\theta[\eta_{c_1} + \eta_{c_2} + \eta_{c_3} + \eta_{b_2}](0)\, \theta[\eta_{c_1} + \eta_{c_2} + \eta_{c_3} + \eta_{b_1}](0)}, \end{aligned} \tag{6.11}$$

where $\eta_{c_\alpha}, \eta_{b_1}, \eta_{b_2}$ are the same as above and ϱ_i the appropriate quartic roots of 1. Lastly, we have

$$\sqrt{(\nu_1-1)(\nu_2-1)} = \text{const}\, \frac{\sqrt{\theta\,[\Delta]\,(z-q/2)\,\theta[\Delta](z+q/2)}}{\theta[\Delta](z)}, \tag{6.12}$$

where q is the same as in (4.8), or, in terms of the new coordinate ν on Γ, $q/2 = \int_\infty^{(1,0)} \bar{\omega}$.

Combining the above expressions, we see that in the quotient $\mathcal{G} = \bar{\mathcal{G}}/\sqrt{(\nu_1-1)(\nu_2-1)}$ the term $\theta[\Delta](z)$ is canceled and in the product $\mathcal{G}\sqrt{\theta[\Delta](z-q/2)\theta[\Delta](z+q/2)}$ the square root (6.12) is canceled. Now, simplifying the theta-characteristics in (6.11) by using (4.7) and ignoring common constant factors, we eventually find

$$\begin{aligned}
&\text{const} \sum_{\alpha=1}^{3} k_{0\alpha}\theta[\Delta+\eta_{c_\alpha}](z) \\
&= \bar{\varepsilon}_1\theta[\Delta+\eta_{c_1}](z)\,\theta[\eta_{c_1}+\eta_{c_3}](0)\,\theta[\eta_{c_2}+\eta_{b_1}](0)\,\theta[\eta_{c_2}+\eta_{b_2}](0) \\
&\quad + \bar{\varepsilon}_2\theta[\Delta+\eta_{c_2}](z)\,\theta[\eta_{c_2}+\eta_{c_3}](0)\,\theta[\eta_{c_1}+\eta_{b_1}](0)\,\theta[\eta_{c_1}+\eta_{b_2}](0) \\
&\quad + \bar{\varepsilon}_3\theta[\Delta+\eta_{c_3}](z)\,\theta[\eta_{b_2}](0)\,\theta[\eta_{b_1}](0)\,\theta(0),
\end{aligned} \tag{6.13}$$

$\bar{\varepsilon}_i$ also being certain quartic roots of 1. The latter expression has the same structure as the sum (6.10). Lastly, note that under the shift of z by an appropriate complete period in $\text{Jac}(\Gamma)$ the roots $\bar{\varepsilon}_i$ can be made proportional to any combination of roots ε_α in (6.10). (This corresponds to choosing an appropriate origin in $\widetilde{\text{Jac}}(\Gamma)$.) Hence, we proved the theorem for the chosen ordering (6.9).

To complete the proof for the other possible orderings of b_i, c_α it remains to modify the theta-characteristics in (6.10), (6.13). □

ACKNOWLEDGMENTS

Both authors acknowledge the support of MICINN-FEDER grant MTM2009-06973 and CUR-DIUE grant 2009SGR859. The work of the first author (Yu.F.) was also partially supported by MICINN-FEDER grant MTM2009-12670.

We are grateful to the referee for remarks.

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2011, Vol. 16, Nos. 3–4, pp. 396–414. © Pleiades Publishing, Ltd., 2011.

Integrable Systems on the Sphere Associated with Genus Three Algebraic Curves

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Received March 14, 2011; accepted April 20, 2011

Abstract—New variables of separation for few integrable systems on the two-dimensional sphere with higher order integrals of motion are considered in detail. We explicitly describe canonical transformations of initial physical variables to the variables of separation and vice versa, calculate the corresponding quadratures and discuss some possible integrable deformations of initial systems.

MSC2010 numbers: 70H20; 70H06; 37K10

DOI: 10.1134/S1560354711030117

Keywords: integrable systems, separation of variables, Abel equations

1. INTRODUCTION

A fundamental requirement for new developments in mechanics is to unravel the geometry that underlies different dynamical systems, especially mechanical systems. There are several reasons why this geometrical understanding is fundamental. First, it is a key tool for reduction by symmetries and for the geometric characterization of the integrability and stability theories. Second, the effective use of numerical techniques is often based on the comprehension of the fundamental structures appearing in the dynamics of mechanical and control systems. In fact, geometric analysis of such systems reveals what they have in common and indicates the most suitable strategy to obtain and to analyze their solutions.

Already in 19th century Euler and Lagrange established a mathematically satisfactory foundation of Newtonian mechanics. In [1] Jacobi combined their ideas with the Hamilton optic theory and with the Abel geometric methods into a new Hamilton–Jacobi formalism. The Hamilton–Jacobi formalism was a crucial step towards Liouville's classical definition of the notion of integrability [2] based on the notion of first integrals of motion.

The Liouville definition of integrable Hamiltonian systems naturally covered many classical examples. Among them are the Kepler motion solved by Newton, harmonic oscillators solvable by trigonometric functions, the Euler and Lagrange spinning tops and the Jacobi example of geodesic motion on an ellipsoid solvable by elliptic functions [3], the Neumann system on the sphere [4] and Kowalevski top [5] solved terms of hyperelliptic functions, etc. Recently there has been a great interest in integrable systems prompted by the discovery of a vast class of integrable soliton nonlinear partial differential equations, that admits this type of integrability when dynamics is restricted to finite dimensional Liouville tori and the system appears to be completely integrable in the Liouville–Arnold sense. All of them are more or less connected with the hyperelliptic curves and with the hyperelliptic functions [1, 6, 7]. Below we show that foregoing development of the theory detected a number of cases when associated algebraic curve is non hyperelliptic and and its genus exceeds the number of degrees of freedom [8–10].

Bi-Hamiltonian structures can be seen as a dual formulation of integrability and separability, in the sense that they replace the hierarchy of compatible Poisson structures with the hierarchy of functions in involutions, which may be treated either as integrals of motion or as variables

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of separation for some dynamical system [11]. The Eisenhart–Benenti theory was embedded into the bi-Hamiltonian set-up using the lifting of the conformal Killing tensor that lies at the heart of Benenti's construction, which may be realized as a computer algorithm [12]. The concept of natural Poisson bivectors allows us to generalize this construction and to study systems with quadratic and higher order integrals of motion in the framework of a single theory [8, 11, 13].

The aim of this note is to discuss separation of variables for integrable natural systems on the two-dimensional unit sphere $\mathbb{S}^2$ from [8, 9, 13–15]. In the above mentioned previous papers we focused our attention on the bi-Hamiltonian calculations of the variables of separation starting from the given integrals of motion. This note is devoted to construction of the initial physical variables in terms of variables of separation, to calculation of the corresponding quadratures and to discussion of the possible integrable "gyroscopic" deformations of these systems associated with genus three algebraic curves.

In order to describe integrable systems on the sphere we will use the angular momentum vector $J = (J_1, J_2, J_3)$ and the Poisson vector $x = (x_1, x_2, x_3)$ in a moving coordinate frame attached to the principal axes of inertia. The Poisson brackets between these variables

$$\{J_i\,, J_j\,\} = \varepsilon_{ijk} J_k\,, \qquad \{J_i\,, x_j\,\} = \varepsilon_{ijk} x_k\,, \qquad \{x_i\,, x_j\,\} = 0\,, \tag{1.1}$$

may be associated to the Lie–Poisson brackets on the algebra $e^*(3)$. Using the Hamilton function H and the Lie–Poisson bracket $\{.,.\}$ (1.1) on the Euclidean algebra $e^*(3)$ the customary Euler–Poisson equations may be rewritten in Hamiltonian form

$$\dot{J}_i = \{J_i, H\}\,, \qquad \dot{x}_i = \{x_i, H\}\,. \tag{1.2}$$

Remind that the Lie–Poisson dynamics on $e^*(3)$ can be interpreted as resulting from reduction by the symmetry Euclidean group $E(3)$ of the full dynamics on the twelve-dimensional phase space $T^*E(3)$ [3]. There are two Casimir elements

$$C_1 = |x|^2 \equiv \sum_{k=1}^{3} x_k^2, \qquad C_2 = \langle x, J\rangle \equiv \sum_{k=1}^{3} x_k J_k, \tag{1.3}$$

where $\langle .,. \rangle$ means the inner product. Using canonical transformations $x \to \alpha x$ we will always put $C_1 = 1$ without loss of generality.

If the square integral of motion $C_2 = \langle x, J\rangle$ is equal to zero, rigid body dynamics may be restricted to the unit sphere $\mathbb{S}^2$ and we can use the standard spherical coordinate system on its cotangent bundle $T^*\mathbb{S}^2$

$$\begin{aligned} &x_1 = \sin\phi\sin\theta, \qquad x_2 = \cos\phi\sin\theta, \qquad x_3 = \cos\theta\,,\\ &J_1 = \frac{\sin\phi\cos\theta}{\sin\theta}\,p_\phi - \cos\phi\,p_\theta\,,\ J_2 = \frac{\cos\phi\cos\theta}{\sin\theta}\,p_\phi + \sin\phi\,p_\theta\,,\ J_3 = -p_\phi\,. \end{aligned} \tag{1.4}$$

We use these variables in order to determine canonical variables of separation on $T^*\mathbb{S}^2$.

As usual all the results are presented up to the linear canonical transformations, which consist of rotations

$$x \to \alpha\, U\, x\,, \qquad J \to U\, J\,, \tag{1.5}$$

where α is an arbitrary parameter and U is an orthogonal constant matrix, and shifts

$$x \to x\,, \qquad J \to J + S\, x\,, \tag{1.6}$$

where S is an arbitrary 3×3 skew-symmetric constant matrix.

Of course, any canonical transformation of the spherical variables (1.4) yields automorphism of $e^*(3)$ too. For instance, trivial canonical transformation

$$p_\theta \to p_\theta + f(\theta) \tag{1.7}$$

gives rise to "generalized" shift depending on arbitrary function $f(x_3)$:

$$J_1 \to J_1 - \frac{x_2 f(x_3)}{\sqrt{x_1^2 + x_2^2}}\,, \qquad J_2 \to J_2 + \frac{x_1 f(x_3)}{\sqrt{x_1^2 + x_2^2}}\,, \tag{1.8}$$

This and more complicated canonical transformations of $e^*(3)$ are discussed in [3, 16].

2. KOWALEVSKI TOP AND CHAPLYGIN SYSTEM

Following to [8, 14, 15], we determine canonical coordinates $q_{1,2}$ on $T^*\mathbb{S}^2$ as roots of the following polynomial

$$\begin{aligned} B(\lambda) &= (\lambda - q_1)(\lambda - q_2) = \lambda^2 - \frac{p_\theta^2 \sin^2\theta + p_\phi^2 \cos^2\theta}{\sin^\alpha\theta \cos^2\theta}\,\lambda - a^2 - b^2 \\ &- \frac{(a\cos\alpha\phi - b\sin\alpha\phi)(p_\theta^2\sin^2\theta + p_\phi^2\cos^2\theta)}{\sin^\alpha\theta\cos^2\theta} - \frac{2\sin\theta(a\sin\alpha\phi + b\cos\alpha\phi)p_\phi\, p_\theta}{\sin^\alpha\theta\cos^2\theta}\,. \end{aligned} \tag{2.1}$$

Then we can introduce the auxiliary polynomial

$$A(\lambda) = \frac{\sin\theta p_\theta}{\alpha\cos\theta}\,\lambda + \frac{a\sin\alpha\phi + b\cos\alpha\phi}{\alpha}\,p_\phi - \frac{\sin\theta(a\cos\alpha\phi - b\sin\alpha\phi)}{\alpha\cos\theta}\,p_\theta\,,$$

such that

$$\{B(\lambda), A(\mu)\} = \frac{1}{\lambda - \mu}\Big((\mu^2 - a^2 - b^2)B(\lambda) - (\lambda^2 - a^2 - b^2)B(\mu)\Big)\,, \qquad \{A(\lambda), A(\mu)\} = 0\,.$$

It entails that

$$p_j = -\frac{1}{u_j^2 - a^2 - b^2}\,A(\lambda = q_j)\,, \qquad j = 1, 2, \tag{2.2}$$

are canonically conjugated momenta on $T^*\mathbb{S}^2$ with the standard Poisson brackets

$$\{q_i, p_j\} = \delta_{ij}\,, \quad \{q_1, q_2\} = \{p_1, p_2\} = 0\,.$$

Below we prove that at $\alpha = 1, 2$ these variables are variables of separation for the Kowalevski top and Chaplygin system, respectively.

2.1. Kowalevski Top

Let us consider the Kowalevski top defined by the following integrals of motion

$$\begin{aligned} H_1 &= J_1^2 + J_2^2 + 2J_3^2 + 2bx_1 \\ H_2 &= (J_1^2 + J_2^2)^2 - 4b\Big(x_1(J_1^2 - J_2^2) + 2x_2J_1J_2\Big) - 4b^2x_3^2\,. \end{aligned} \tag{2.3}$$

In the original work of Kowalevski [5] the first step in the separation of variables method is the complexification: she introduces

$$\mathrm{z}_1 = J_1 + iJ_2, \qquad \mathrm{z}_2 = J_1 - iJ_2$$

as independent complex variables. Next she makes her famous change of variables

$$s_{1,2} = \frac{R(\mathrm{z}_1, \mathrm{z}_2) \pm \sqrt{R(\mathrm{z}_1, \mathrm{z}_1)R(\mathrm{z}_2, \mathrm{z}_2)}}{2(\mathrm{z}_1 - \mathrm{z}_2)^2}.$$

The fourth degree polynomials $R(\mathrm{z}_i, \mathrm{z}_k)$ may be found in [3, 5]. It brings the system (1.2) to the form

$$(-1)^k\,(s_1 - s_2)\dot{s}_k = \sqrt{P(s_k)}\,, \qquad k = 1, 2,$$

where

$$P(s) = 4\left((s-H_1)^2 - \frac{H_2+4b^2C_1}{4}\right)\left[s\left((s-H_1)^2 + b^2C_1 - \frac{H_2+4a^2C_1}{4}\right) + b^2C_2\right]. \tag{2.4}$$

Consequently, the initial equations of motion can be written as hyperelliptic quadratures

$$\frac{\dot s_1}{\sqrt{P(s_1)}} + \frac{\dot s_2}{\sqrt{P(s_2)}} = 0\,,$$

$$\frac{s_1\dot s_1}{\sqrt{P(s_1)}} + \frac{s_2\dot s_2}{\sqrt{P(s_2)}} = \mathrm{i}\,,$$

where we can substitute the conjugated momenta p_{s_k} for $\sqrt{P(s_k)}$ in order to get standard Abel–Jacobi form. So, the problem can be integrated in terms of genus two hyperelliptic functions of time. Finally, we have to substitute these functions of time $s_k(t)$ and $p_{s_k}(t)$ into the initial variables x, J, the corresponding expressions may be found in [5, 17].

Discussion of the other variables of separation for some particular subcases in the Kowalevski dynamic may be found in [3]. As usual different variables of separation are related to the distinct integrable deformations of the initial integrals of motion.

2.1.1. New Real Variables of Separation at $C_2 = 0$

According to [8], at $\alpha = 1$ coordinates $q_{1,2}$ (2.1) are variables of separation associated with the Hamilton function

$$H = J_1^2 + J_2^2 + 2J_3^2 + +2ax_2 + 2bx_1\,,$$

which may be reduced to the initial Hamiltonian H_1 using rotations (1.5) around the third axis [16], so we can put $a = 0$ in (2.1) without loss of generality.

Coordinates $q_{1,2}$ (2.1) at $\alpha = 1$ and $a = 0$ are defined by

$$B(\lambda) = (\lambda-q_1)(\lambda-q_2) = \lambda^2 + \frac{\sqrt{x_1^2+x_2^2}\,(J_1^2+J_2^2)}{x_3^2} - \frac{b\Big(x_1(J_1^2-J_2^2) + 2x_2J_1J_2\Big)}{x_3^2} - b^2\,.$$

The conjugated momenta $p_{1,2}$ are equal to

$$p_k = -\frac{A(\lambda = q_k)}{q_k^2 - b^2}\,, \qquad A(\lambda) = \frac{x_1J_2 - x_2J_1}{x_3}\,\lambda + \frac{b\sqrt{x_1^2+x_2^2}J_2}{x_3}\,.$$

These variables $q_{1,2}$ differ by constant terms $\pm b$ from variables introduced in [15]. Inverse transformation reads as

$$\begin{aligned}
x_1 &= \frac{b^2-q_1q_2}{b(q_1-q_2)^2}\Big((b^2-q_1^2)p_1^2 + (b^2-q_2^2)p_2^2\Big) - \frac{2(b^2-q_1^2)(b^2-q_2^2)}{b(q_1-q_2)^2}\,p_1p_2\,,\\
x_2 &= -\frac{\sqrt{(q_1^2-b^2)(b^2-q_2^2)}}{b(q_1-q_2)^2}\Big(b^2(p_1-p_2)^2 - (p_1q_1-p_2q_2)^2\Big)\,,\\
x_3 &= \sqrt{1 - \frac{(b^2-q_1^2)^2p_1^4 + (b^2-q_2^2)^2p_2^4}{(q_1-q_2)^2} + \frac{2(b^2-q_1^2)(b^2-q_2^2)p_1^2p_2^2}{(q_1-q_2)^2}}\,,\\
J_1 &= \frac{\sqrt{(q_1^2-b^2)(b^2-q_2^2)}(p_1q_1-p_2q_2)}{(b^2-q_1^2)p_1^2-(b^2-q_2^2)p_2^2}\,\frac{x_3}{b}\,,\\
J_2 &= -\frac{q_2(b^2-q_1^2)p_1 - q_1(b^2-q_2^2)p_2}{(b^2-q_1^2)p_1^2-(b^2-q_2^2)p_2^2}\,\frac{x_3}{b}\,,
\end{aligned} \tag{2.5}$$

$$J_3 = -\sqrt{(q_1^2-b^2)(b^2-q_2^2)}\,\frac{p_1-p_2}{q_1-q_2}\,.$$

Coordinates of separation take values only in the following intervals

$$q_1 > b > q_2\,,$$

similar to the standard elliptic coordinates on the sphere [3].

In these variables integrals of motion $H_{1,2}$ (2.3) look like

$$H_1 = \frac{(b^2-q_1^2)^2p_1^4-(b^2-q_2^2)^2p_2^4-(q_1^2-q_2^2)}{(b^2-q_1^2)p_1^2-(b^2-q_2^2)p_2^2}$$

$$H_2 = \frac{\Big((b^2-q_1^2)p_1^2-(b^2-q_2^2)p_2^2+q_1+q_2\Big)\Big((b^2-q_1^2)p_1^2-(b^2-q_2^2)p_2^2-q_1-q_2\Big)}{(b^2-q_1^2)p_1^2-(b^2-q_2^2)p_2^2}$$
$$\times\Big((b^2-q_1^2)p_1^2-(b^2-q_2^2)p_2^2+q_1-q_2\Big)\Big((b^2-q_1^2)p_1^2-(b^2-q_2^2)p_2^2-q_1+q_2\Big)\,.$$

It is easy to see that integrals of motion and variables of separation are related via the following separated relations

$$\Phi = \Big(2(q^2-b^2)p^2+H_1+\sqrt{H_2}\Big)\Big(2(q^2-b^2)p^2+H_1-\sqrt{H_2}\Big)-4q^2=0, \tag{2.6}$$

at $q=q_{1,2}$ and $p=p_{1,2}$. Equation $\Phi(q,p)=0$ defines the genus three hyperelliptic curve with the following base of the holomorphic differentials

$$\Omega_1 = \frac{dq}{p(b^2-q^2)\Big(H_1-2(b^2-q^2)p^2\Big)}\,,\qquad \Omega_2 = \frac{qdq}{p(b^2-q^2)\Big(H_1-2(b^2-q^2)p^2\Big)}$$

$$\Omega_3 = \frac{pdq}{H_1-2(b^2-q^2)p^2},.$$

In fact equation (2.6) is invariant with respect to involution $(q,p)\to(-q,p)$. Factorization with respect to this involution gives rise to an elliptic curve.

In variables of separation the equations of motion (1.2) have the following form

$$\frac{\dot q_1}{p_1(b^2-q_1^2)\Big(H_1-2(b^2-q_1^2)p_1^2\Big)}+\frac{\dot q_2}{p_2(b^2-q_2^2)\Big(H_1-2(b^2-q_2^2)p_2^2\Big)}=0\,,$$

$$\frac{\dot q_1}{H_1-2(b^2-q_1^2)p_1^2}+\frac{\dot q_2}{H_1-2(b^2-q_2^2)p_2^2}=-2\,.$$

The above quadratures in the integral form

$$\int_{q_0}^{q_1}\Omega_1+\int_{q_0}^{q_2}\Omega_1=\beta_1\,,\qquad \int_{q_0}^{q_1}\Omega_3+\int_{q_0}^{q_2}\Omega_3=-2t+\beta_2\,, \tag{2.7}$$

represent the Abel–Jacobi map associated with the genus three hyperelliptic curve defined by $\Phi(q,p)=0$. In particular it means that instead of p in $\Omega_{1,3}$ (2.7) we have to substitute the function on q obtained from the separated relation (2.6).

In order to give an explicit theta-function solution one can apply some remarkable relations between roots of certain functions on symmetric products of such curves and quotients of theta-functions with half-integer characteristics, which are historically referred to as root function and are generalized so-called *Wurzelfunktionen* that were used by Jacobi for the case of ordinary hyperelliptic Jacobians [18, 19]. For the case of odd order hyperelliptic curves such functions were obtained by Weierstrass [20]. Inverting the map (2.7) and substituting symmetric functions of q_1, q_2, p_1, p_2 into (2.5), one finally finds x, J as functions of time.

2.1.2. Deformations of the Kowalevski Top

According to [8, 15], using separated relations

$$\Phi_1 = \Big(2(q^2-a^2)p^2 + \widehat{H}_1 + \sqrt{\widehat{H}_2}\Big)\Big(2(q^2-a^2)p^2 + \widehat{H}_1 - \sqrt{\widehat{H}_2}\Big) - 4cu^2 + 4du + e(q^2-a^2)p = 0\,, \tag{2.8}$$

one gets the Hamilton function of the generalized Kowalevski top

$$\widehat{H}_1 = \left(1 - \frac{c-1}{x_3^2}\right)(J_1^2+J_2^2) + 2J_3^2 + 2bx_1 + \frac{d}{\sqrt{x_1^2+x_2^2}} + \frac{e(x_2J_1 - x_1J_2)}{4\sqrt{(x_1^2+x_2^2x_3}}\,. \tag{2.9}$$

Second integral of motion is equal to

$$\begin{aligned}\widehat{H}_2 &= \frac{(x_3^2+c-1)^2}{x_3^4}(J_1^2+J_2^2)^2 - \left(\frac{4(x_1J_1^2 - x_1J_2^2 + 2x_2J_1J_2)(x_3^2+c-1)}{x_3^2} + \frac{J_2\sqrt{x_1^2+x_2^2}e}{x_3}\right)a\\ &- 4(x_3^2+c-1)a^2 + \left(\frac{(2(J_1^2+J_2^2)(c-x_1^2-x_2^2)}{\sqrt{x_1^2+x_2^2}\,x_3^2} + \frac{(x_2J_1-x_1J_2)e}{2(x_1^2+x_2^2)x_3}\right)d + \frac{d^2}{x_1^2+x_2^2}\\ &+ \frac{(J_1^2+J_2^2)(x_2J_1-x_1J_2)(c-x_1^2-x_2^2)e}{2\sqrt{x_1^2+x_2^2}x3^3} + \frac{(x_2J_1-x_1J_2)^2e^2}{16(x_1^2+x_2^2)x_3^2}\end{aligned} \tag{2.10}$$

According to [8, 16] canonical transformation (1.8) reduces the Hamilton function (2.9) to the natural form

$$\widehat{H}_1 = \left(1 - \frac{c-1}{x_3^2}\right)(J_1^2+J_2^2) + 2J_3^2 + 2ax_1 + \frac{d}{\sqrt{x_1^2+x_2^2}} - \frac{e^2}{64(x_3^2+c-1)}\,, \tag{2.11}$$

at

$$f(x_3) = \frac{ex_3}{8(x_3^2+c-1)}\,.$$

At $c = 1$ this system coincides with that of the deformations discussed in [21]. Below we will show only the final form (2.11) of the deformed Hamiltonians and will omit the intermediate form (2.9) for the brevity.

It is easy to calculate the corresponding equations of motion

$$\frac{\dot q_1}{(b^2-q_1^2)\Big(8\widehat{H}_1p_1 + e - 16p_1^3(b^2-q_1^2)\Big)} + \frac{\dot q_2}{p_2(b^2-q_2^2)\Big(8\widehat{H}_1p_2 + e - 16p_2^3(b^2-q_2^2)\Big)} = 0\,,$$

$$\frac{\dot q_1}{8\widehat{H}_1p_1 + e - 16p_1^3(b^2-q_1^2)} + \frac{\dot q_2}{8\widehat{H}_1p_2 + e - 16p_2^3(b^2-q_2^2)} = -\frac{1}{4}\,,$$

and prove that the Abel–Jacobi map on the genus three hyperelliptic curve has the same form (2.7)

$$\int_{q_0}^{q_1}\Omega_1 + \int_{q_0}^{q_2}\Omega_1 = \beta_1\,, \qquad \int_{q_0}^{q_1}\Omega_3 + \int_{q_0}^{q_2}\Omega_3 = -2t + \beta_2\,,$$

where p have to be solution of the separated relation (2.8) and

$$\Omega_1 = \frac{dq}{(b^2-q^2)(8\widehat{H}_1p + e - 16p^3(b^2-q^2))}\,, \qquad \Omega_3 = \frac{dq}{8\widehat{H}_1p + e - 16p^3(b^2-q^2)}\,.$$

2.2. Chaplygin System

Let us consider the Chaplygin system defined by the following Hamilton function

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 - 2a(x_1^2 - x_2^2) - 2bx_1x_2 - \frac{c}{x_3^2}\,. \tag{2.12}$$

At $c = 0$ this system and the corresponding variables of separation have been investigated by Chaplygin [22]. The singular term has been added by Goryachev in [23].

Using rotations (1.5) around the third axis [16] we can put $b = 0$ without loss of generality. In this case the second integral of motion is equal to

$$H_2 = \left(J_1^2 + J_2^2 - \frac{c}{x_3^2}\right)^2 - 4ax_3^2(J_1^2 - J_2^2) + 4a^2x_3^4.$$

According to [8, 14], coordinates $q_{1,2}$ (2.1) are variables of separation for this integrable system at $\alpha = 2$ and $b = 0$. In this case $q_{1,2}$ are roots of the following polynomial (2.1)

$$B(\lambda) = (\lambda - q_1)(\lambda - q_2) = \lambda^2 - \frac{J_1^2 + J_2^2}{x_3^2}\,\lambda - \frac{2aJ_2^2}{x_3^2} + \frac{a(J_1^2 + J_2^2)}{x_3^2} - a^2\,,$$

whereas momenta $p_{1,2}$ are values of the other auxiliary polynomial

$$A(\lambda) = -\frac{x_2J_1 - x_1J_2}{2x_3}\,\lambda - \frac{ax_1x_2J_3}{x_1^2 + x_2^2} - \frac{a(x_1^2 - x_2^2)(x_2J_1 - x_1J_2)}{2(x_1^2 + x_2^2)x_3}\,.$$

at $\lambda = q_{1,2}$ (2.2). Inverse transformation reads as

$$\begin{aligned}
x_1 &= \frac{\sqrt{2(q_1 - a)(a - q_2)}\,(p_1(q_1 + a) - p_2(q_2 + a))}{\sqrt{a}(q_1 - q_2)}\,,\\
x_2 &= \frac{\sqrt{2(q_1 + a)(q_2 + a)}\,(p_1(q_1 - a) - p_2(q_2 - a))}{\sqrt{a}(q_1 - q_2)}\,,\\
x_3 &= \sqrt{1 - 4\frac{(q_1^2 - a^2)p_1^2 - (q_2^2 - a^2)p_2^2}{q_1 - q_2}}\,,\\
J_1 &= \sqrt{\frac{(a + q_1)(a + q_2)}{2a}}\,x_3\,, \qquad J_2 = -\sqrt{\frac{(q_1 - a)(a - q_2)}{2a}}\,x_3\\
J_3 &= -2\sqrt{(q_1^2 - a^2)(a^2 - q_2^2)}\;\frac{p_1 - p_2}{q_1 - q_2}
\end{aligned} \tag{2.13}$$

As usual coordinates of separation take values only in the following intervals

$$q_1 > a > q_2\,.$$

These variables $q_{1,2}$ are related to variables of separation from [14] by the rule $q_k \to q_k + a$.

In variables of separation the integrals of motion read as

$$\begin{aligned}
H_1 &= 4(a^2 - q_1^2)p_1^2 + 4(a^2 - q_2^2)p_2^2 + q_1 + q_2 - \frac{c}{4(a^2 - q_1^2)p_1^2 - 4(a^2 - q_2^2)p_2^2 + q_1 - q_2}\,,\\
H_2 &= \Big(4(a^2 - q_1^2)p_1^2 - 4(a^2 - q_2^2)p_2^2 + q_1 - q_2\Big)^2 + \frac{c^2(q_1 - q_2)^2}{\Big(4(a^2 - q_1^2)p_1^2 - 4(a^2 - q_2^2)p_2^2 + q_1 - q_2\Big)^2}\\
&\quad - 2c(q_1 + q_2)\,.
\end{aligned}$$

It is easy to see that integrals of motion and variables of separation are related via the following separated relations

$$\Phi = \Big(8(q^2-a^2)p^2 - 2q + H_1 - \sqrt{H_2}\Big)\Big(8(q^2-a^2)p^2 - 2q + H_1 + \sqrt{H_2}\Big) - 4cq = 0, \qquad (2.14)$$

at $q = q_{1,2}$ and $p = p_{1,2}$. Equation $\Phi(q,p) = 0$ defines a genus two algebraic curve with the following holomorphic differentials

$$\Omega_1 = \frac{dq}{p(a^2-q^2)\Big(H_1 - 8(a^2-q^2)p^2 - 2q\Big)}, \qquad \Omega_2 = \frac{\Big(4(a^2-q^2)p^2 + q\Big)dq}{p(a^2-q^2)\Big(H_1 - 8(a^2-q^2)p^2 - 2q\Big)}. \qquad (2.15)$$

The corresponding quadratures look like

$$\frac{\dot q_1}{p_1(a^2-q_1^2)\Big(H_1 - 8(a^2-q_1^2)p_1^2 - 2q_1\Big)} + \frac{\dot q_2}{p_2(a^2-q_2^2)\Big(H_1 - 8(a^2-q_2^2)p_2^2 - 2q_2\Big)} = 0$$

$$\frac{\Big(4(a^2-q_1^2)p_1^2 + q_1\Big)\dot q_1}{p_1(a^2-q_1^2)\Big(H_1 - 8(a^2-q_1^2)p_1^2 - 2q_1\Big)} + \frac{\Big(4(a^2-q_2^2)p_2^2 + q_2\Big)\dot q_2}{p_2(a^2-q_2^2)\Big(H_1 - 8(a^2-q_2^2)p_2^2 - 2q_2\Big)} = 8\,.$$

The Abel–Jacobi map on a genus two hyperelliptic curve has the standard form

$$\int_{q_0}^{q_1}\Omega_1 + \int_{q_0}^{q_2}\Omega_1 = \beta_1\,, \qquad \int_{q_0}^{q_1}\Omega_2 + \int_{q_0}^{q_2}\Omega_2 = 8t + \beta_2\,,$$

where p into $\Omega_{1,2}$ have to be a solution of the separated relation (2.14).

2.2.1. Deformations of the Chaplygin System

According to [8, 14], if we substitute these variables of separation onto the following separated relations

$$\Phi_1 = \Big(8(q^2-a^2)p^2 - 2dq + \widehat{H}_1 - \sqrt{\widehat{H}_2}\Big)\Big(8(q^2-a^2)p^2 - 2dq + \widehat{H}_1 + \sqrt{\widehat{H}_2}\Big) - 4cq + e(q^2-a^2)p = 0\,, \qquad (2.16)$$

one gets the Hamilton function of the generalized Chaplygin system

$$\widehat{H}_1 = \left(1 - \frac{1-d}{x_3^2}\right)(J_1^2 + J_2^2) + 2J_3^2 - 2a(x_1^2 - x_2^2) - 2bx_1x_2 - \frac{c}{d-1+x_3^2} + \frac{(x_2J_1 - x_1J_2)e}{8(d - x_1^2 - x_2^2)x_3}\,. \qquad (2.17)$$

As for the Kowalevski top, using canonical transformation (1.8) at

$$f(x_3) = \frac{ex_3\sqrt{1-x_3^2}}{16(d-1+x_3^2)^2}$$

we can reduce the Hamilton function (2.17) to the natural Hamiltonian

$$\widehat{H}_1 = \left(1 - \frac{1-d}{x_3^2}\right)(J_1^2 + J_2^2) + 2J_3^2 - 2a(x_1^2 - x_2^2) - 2bx_1x_2 - \frac{c}{d-1+x_3^2} + \frac{e(x_3^2-1)}{256(d-1+x_3^2)^3}\,.$$

At $d = 1$ the additional term is equal to $e(x_3^{-4} - x_3^{-6})$ and this system coincides with that of the deformations considered in [21].

In this case we have a genus three hyperelliptic curve with holomorphic differentials

$$\Omega_1 = \frac{dq}{(a^2-q^2)\Big(e + 32p\big(\widehat{H}_1 - 8(a^2-q^2)p^2 - 2dq\big)\Big)}\,,$$

$$\Omega_2 = \frac{qdq}{(a^2-q^2)\Big(e+32p\big(\widehat{H}_1 - 8(a^2-q^2)p^2 - 2dq\big)\Big)},$$

$$\Omega_3 = \frac{p^2dq}{e+32p\big(\widehat{H}_1 - 8(a^2-q^2)p^2 - 2dq\big)},$$

and the corresponding quadratures involve all these differentials

$$\int_{q_0}^{q_1}\Omega_1 + \int_{q_0}^{q_2}\Omega_1 = \beta_1\,, \qquad \int_{q_0}^{q_1}(4\Omega_2 + d\Omega_3) + \int_{q_0}^{q_2}(4\Omega_2 + d\Omega_3) = -\frac{t}{4} + \beta_2\,,$$

in contrast to other integrable systems on genus three algebraic curves considered in this note.

3. INTEGRABLE SYSTEMS ASSOCIATED WITH TRIGONAL CURVES

According to [8–10], we introduce other coordinates $q_{1,2}$ on $T^8\mathbb{S}^2$ defined as roots of the following polynomial

$$B(\lambda) = (\lambda - q_1)(\lambda - q_2) = \lambda^2 - \mathrm{i}\sqrt{F}\lambda + \Lambda\,, \quad \mathrm{i} = \sqrt{-1}\,, \tag{3.1}$$

with coefficients

$$F = \Big(g(\theta)p_\theta - \mathrm{i}h(\theta)p_\phi\Big)^2, \qquad \Lambda = \alpha\,\exp\left(\mathrm{i}\phi - \int\frac{h(\theta)}{g(\theta)}\,d\theta\right), \tag{3.2}$$

depending on arbitrary functions $g(\theta)$ and $h(\theta)$. As usual conjugated momenta $p_{1,2}$ are equal to

$$p_k = A(\lambda = q_k)\,, \qquad A(\lambda) = \mathrm{i}\int\frac{d\theta}{g(\theta)} - \frac{\mathrm{i}p_\phi}{\lambda}\,. \tag{3.3}$$

It is easy to prove that these polynomials satisfy the following relations

$$\{B(\lambda), A(\mu)\} = \frac{\lambda}{\mu-\lambda}\left(\frac{B(\lambda)}{\lambda} - \frac{B(\mu)}{\mu}\right), \qquad \{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = 0\,, \tag{3.4}$$

which give rise to canonical Poisson brackets

$$\{q_i, p_j\} = \delta_{ij}\,, \quad \{q_1, q_2\} = \{p_1, p_2\} = 0\,.$$

Substituting variables

$$x = a\,q_k^{-1}\,, \qquad z = a_0p_k\,, \qquad k = 1,2, \qquad a, a_0 \in \mathbb{R}\,, \tag{3.5}$$

into the generic equation of the (3,4) algebraic curve

$$\Phi(z,x) = z^3 + (a_1x + a_2)z^2 + (H_1x^2 + b_1x + b_2)z + x^4 + H_2x^3 + c_1x^2 + c_2x + c_3 = 0\,, \tag{3.6}$$

and solving the resulting equations with respect to $H_{1,2}$, one gets the following Hamilton function

$$H_1 = T + V + \left(\frac{c_2 + \mathrm{i}a_0b_1w_2 - a_0^2a_1w_2^2}{a_0aw_2}\,h + \frac{2a_0a_1w_2 - \mathrm{i}b_1}{aw_2}\right)\mathrm{i}p_\phi - \frac{gw_2(c_2 + \mathrm{i}a_0b_1w_2 - a_0^2a_1w_2^2)}{aa_0}p_\theta\,, \tag{3.7}$$

where geodesic Hamiltonian T and potential V are equal to

$$T = \left(\frac{a_0^2(h^2w_2^2 - 3hw_2 + 3)}{a^2} - \frac{\mathrm{i}a_0a_2(hw_2-1)^2}{a^2w_2} - \frac{b_2h(hw_2-1)}{a^2w_2} + \frac{\mathrm{i}c_3h^2}{a_0a^2w_2}\right)p_\phi^2$$

$$+\ \frac{\mathrm{i}g}{a^2w_2}\left(\Big(2a_0^2w_2^3 - 2\mathrm{i}a_0a_2w_2^2 - 2b_2w2 + \frac{2\mathrm{i}c_3}{a_0}\Big)h - 3a_0^2w_2^2 + 2\mathrm{i}a_0a_2w_2 + b_2\right)p_\phi p_\theta$$

$$+\ \frac{g^2(a_0b_2w_2 + \mathrm{i}a_0^2a_2w_2^2 - a_0^3w_2^3 - \mathrm{i}c_3)}{a^2a_0w_2}\,p_\theta^2$$

$$V = -\frac{\mathrm{i}a^2 \mathrm{e}^{-\mathrm{i}\phi}}{\alpha a_0 w_1 w_2} + \frac{(a_0 b_2 w_2 + \mathrm{i}a_0^2 a_2 w_2^2 - a_0^3 w_2^3 - \mathrm{i}c_3)\alpha w_1 \mathrm{e}^{\mathrm{i}\phi}}{a_0 a^2 w_2} + \frac{\mathrm{i}c_1}{a_0 w_2}\,.$$

Here

$$w_1 = \exp\left(-\int \frac{h(\theta)}{g(\theta)}\,d\theta\right), \qquad w_2 = \int \frac{d\theta}{g(\theta)}\,.$$

The second integral of motion H_2 is a cubic polynomial in momenta p_ϕ and p_θ.

The resulting Hamiltonian H_1 (3.7) has a natural form if and only if

$$2a_0 a_1 w_2 - \mathrm{i}b_1 = 0\,, \quad c_2 + \mathrm{i}a_0 b_1 w_2 - a_0^2 a_1 w_2^2 = 0\,.$$

So, because $w_2 \neq 0$, we have to put

$$a_1 = b_1 = c_2 = 0\,.$$

If we want to obtain diagonal metric, then we have to solve the integral equation

$$2h(a_0^3 w_2^3 - \mathrm{i}a_0^2 a_2 w_2^2 - a_0 b_2 w_2 + \mathrm{i}c_3) - 3a_0^3 w_2^2 + 2\mathrm{i}a_0^2 a_2 w_2 + a_0 b_2 = 0\,, \tag{3.8}$$

with respect to functions $h(\theta), w_2(\theta)$ and parameters a_0, a_2, b_2, c_3. If we want to get real potential

$$V = f_1(\theta)\cos(\phi) + f_2(\theta)$$

in (3.7), we have to add one more equation to (3.8)

$$\mathrm{i}\alpha^2(a_0^3 w_2^3 - \mathrm{i}a_0^2 a_2 w_2^2 - a_0 b_2 w_2 + \mathrm{i}c_3)w_1^2 + a^4 = 0 \tag{3.9}$$

depending also on function w_1 and parameters a (3.5) and α (3.2).

Some particular solutions of these equations have been studied in [8, 9, 24] including integrable systems due to Goryachev, Chaplygin, Dullin, Matveev, etc. For all these systems, we collect a_0 and the zero-valued coefficients in (3.8) in the following table.

Goryachev–Chaplygin top	$a_0 = 2\mathrm{i}a$	$b_2 = c_3 = 0$
Goryachev system	$a_0 = 2\mathrm{i}a/3$	$a_2 = b_2 = 0$
Case 3 from [24]	$a_0 = \mathrm{i}a/3$	$a_2 = b_2 = 0$
Dullin–Matveev system	$a_0 = \mathrm{i}a$	$c_3 = 0$
Case 5 from [24]	$a_0 = \mathrm{i}a/2$	$a_2 = c_3 = 0$

Integrable systems with the same coefficients in the separated relations (3.6) and with different a_0 and a_0' in (3.5) are related by non-canonical transformation of the momenta

$$z = a_0 p_k \to z = a_0' p_k\,, \qquad k = 1, 2. \tag{3.10}$$

3.1. Goryachev–Chaplygin top

Let us consider the Gorychev–Chaplygin top with the following integrals of motion

$$H_1 = J_1^2 + J_2^2 + 4J_3^2 + ax_1 + \frac{b}{x_3^2}\,, \qquad H_2 = 2J_3\left(J_1^2 + J_2^2 + \frac{b}{x_3^2}\right) + ax_3 J_1 \tag{3.11}$$

In this case the variables of separation (3.1,3.3) are determined by

$$q_1 + q_2 = -\frac{2J_3}{x_3^2} - \frac{J_1 + \mathrm{i}J_2}{x_3(x_1 + \mathrm{i}x_2)}\,, \qquad q_1 q_2 = \frac{a}{2x_3^2(x_1 + \mathrm{i}x_2)}\,, \qquad p_{1,2} = \frac{\mathrm{i}x_3^2}{2} + \frac{\mathrm{i}J_3}{q_{1,2}}\,.$$

They are related with initial variables by the rule

$$x_1 + \mathrm{i}x_2 = -\frac{\mathrm{i}a(q_1 - q_2)}{4q_1q_1(p_1q_1 - q_2p_2)}\,, \qquad x_3 = \sqrt{-\frac{2\mathrm{i}(p_1q_1 - q_2p_2)}{q_1 - q_2}}\,,$$

$$J_1 + \mathrm{i}J_2 = \frac{a(q_1^2p_1 - q_2^2p_2)}{2q_1q_2(p_1q_1 - q_2p_2)\sqrt{-\dfrac{2\mathrm{i}(p_1q_1 - q_2p_2)}{q_1 - q_2}}}\,, \qquad J_3 = \frac{\mathrm{i}q_1q_2(p_1 - p_2)}{q_1 - q_2}\,,$$

$$x_1 - \mathrm{i}x_2 = \frac{4q_1q_2}{a(q_1 - q_2)^2}\,,\Big((\mathrm{i} - 2p_1)q_1^2p_1 + (4p_1p_2 - \mathrm{i}p_1 - \mathrm{i}p_2)q_1q_2 + (\mathrm{i} - 2p_2)q_2^2p_2\Big)\,,$$

$$J_1 - \mathrm{i}J_2 = -\frac{8\mathrm{i}q_1q_2}{a(q_1 - q_2)^2\sqrt{-\dfrac{2\mathrm{i}(p_1q_1 - q_2p_2)}{q_1 - q_2}}}\Big(q_1p_1 - q_2p_2)(\mathrm{i} - 2p_1)p_1q_1^2 + $$
$$+ (q_1p_1 + q_2p_2)(\mathrm{i} - 2p_2)p_2q_2^2\Big)\,.$$

Separated relation is given by an equation with real coefficients

$$\Phi(q, \mu) = (\mu^2 - b)q^2 + (\mu^3 - H_1\mu + H_2)q + \frac{a^2}{4} = 0\,, \qquad q = q_{1,2}, \quad \mu = 2\mathrm{i}\, q_{1,2}p_{1,2}\,. \tag{3.12}$$

Equations of motion in variables of separation look like

$$\frac{\dot{q_1}}{q_1(3\mu_1^2 + 2q_1\mu_1 - H_1)} + \frac{\dot{q_2}}{q_2(3\mu_2^2 + 2q_2\mu_2 - H_1)} = 0\,,$$
$$\frac{\mu_1\dot{q_1}}{q_1(3\mu_1^2 + 2q_1\mu_1 - H_1)} + \frac{\mu_2\dot{q_2}}{q_2(3\mu_2^2 + 2q_2\mu_2 - H_1)} = 2\mathrm{i}\,.$$

By making the birational change

$$q = \frac{a^2}{4x}, \qquad \mu = \frac{z}{x} \tag{3.13}$$

the curve (3.12) can be transformed to the canonical trigonal form (3.6) at

$$a_1 = b_1 = b_2 = c_2 = c_3 = 0\,,$$

whereas other parameters are functions on a, b.

3.1.1. Deformation of the Goryachev–Chaplygin Top

Substituting $q = q_{1,2}$ and $\mu = 2\mathrm{i}\, q_{1,2}p_{1,2}$ into the non-hyperelliptic algebraic curve of genus three defined by the following equation

$$\Phi_1(q, \mu) = cq^3 + (\mu^2 + d\mu - b)q^2 + (\mu^3 + e\mu^2 - \widehat{H}_1\mu + \widehat{H}_2)q + \frac{a^2}{4} = 0\,, \tag{3.14}$$

and solving a pair of the resulting equations with respect to $\widehat{H}_{1,2}$ one gets deformation of the initial Hamilton function

$$\widehat{H}_1 = J_1^2 + J_2^2 + 4J_3^2 + ax_1 + \frac{b}{x_3^2} - \left(e - \frac{c - d + e}{x_1^2 + x_2^2} + \frac{c}{x_3^2} - \frac{2c}{x_3^4}\right) J_3 + \frac{(c - dx_3^2 + ex_3^4)^2}{4x_3^6(x_1^2 + x_2^2)}\,,$$

using the generalized shift (1.8) at

$$f = -\frac{\mathrm{i}(ex_3^4 - dx_3^2 + c)}{2\sqrt{1 - x_3^2}\, x_3^3}\,.$$

In this case quadratures are defined by the following differential equations

$$\sum_{k=1}^{2} \frac{\dot q_k}{q_k(3\mu_k^2 + 2\mu_k q_k + 2e\mu_k + dq_k - \widehat{H}_1)} = 0\,, \qquad \mu_k = 2\mathrm{i}\, q_k p_k$$

$$\sum_{k=1}^{2} \frac{\mu_k \dot q_k}{q_k(3\mu_k^2 + 2\mu_k q_k + 2e\mu_k + dq_k - \widehat{H}_1)} = 2\mathrm{i}\,.$$

If $c = 0$ and $d = e$, one gets the usual Goryachev–Chaplygin gyrostat with the Hamiltonian

$$\widehat{H}_1 = J_1^2 + J_2^2 + 4J_3^2 - eJ_3 + ax_1 + \frac{b}{x_3^2}\,.$$

In this case equation (3.14) defines a genus two hyperelliptic curve instead of trigonal one.

3.2. Goryachev System

Let us consider the Gorychev system on the sphere defined by the following integrals of motion

$$H_1 = J_1^2 + J_2^2 + \frac{4}{3}J_3^2 + \frac{ax_1}{x_3^{2/3}} + \frac{b}{x_3^{2/3}}\,,$$
$$H_2 = -\frac{2J_3}{3}\left(J_1^2 + J_2^2 + \frac{8}{9}J_3^2 + \frac{b}{x_3^{2/3}}\right) + \frac{a(3x_3J_1 - 2x_1J_3)}{3x_3^{2/3}}\,. \tag{3.15}$$

The corresponding variables of separation $q_{1,2}$ and $p_{1,2}$ (3.1,3.3) are obtained from

$$q_1 + q_2 = \frac{x_3^{4/3}J_3}{1 - x_3^2} + \frac{\mathrm{i}(J_1x_2 - x_1J_2)x_3^{1/3}}{1 - x_3^2}\,, \quad q_1q_2 = \frac{a}{2(x_1 + \mathrm{i}x_2)}\,, \quad p_{1,2} = \frac{3\mathrm{i}x_3^{2/3}}{2} + \frac{\mathrm{i}J_3}{q_{1,2}}\,.$$

Inverse transformation looks like

$$x_1 + \mathrm{i}x_2 = \frac{a}{2q_1q_2}\,, \qquad J_3 = \frac{\mathrm{i}q_1q_2(p_1 - p_2)}{q_1 - q_2}\,,$$
$$x_1 - \mathrm{i}x_2 = 2\frac{q_1q_2(1 - x_3^2)}{a}\,, \qquad J_1 + \mathrm{i}J_2 = -\frac{a(q_1 + q_2)}{2q_1q_2x_3^{1/3}}\,,$$
$$J_1 - \mathrm{i}J_2 = -\frac{4\mathrm{i}q_1^2q_2^2(p_1 - p_2)}{a(q_1 - q_2)}\,x_3 + \frac{2q_1q_2(1 - x_3^2)(q_1 + q_2)}{ax_3^{1/3}}\,, \tag{3.16}$$

where

$$x_3 = \left(-\frac{2\mathrm{i}(p_1q_1 - p_2q_2)}{3(q_1 - q_2)}\right)^{3/2}.$$

Separated relation is given by an equation with real coefficients

$$\Phi(q,\mu) = q^4 - bq^2 + (\mu^3 - H_1\mu + H_2)q + \frac{a^2}{4} = 0\,, \qquad \text{at} \quad q = q_{1,2}, \quad \mu = \frac{2\mathrm{i}}{3}\, q_{1,2}p_{1,2}\,. \tag{3.17}$$

In this case the quadratures read

$$\int_{q_0}^{q_1} \frac{dq}{q(3\mu^2 - H_1)} + \int_{q_0}^{q_2} \frac{dq}{q(3\mu^2 - H_1)} = \beta_1\,,$$
$$\int_{q_0}^{q_1} \frac{\mu dq}{q(3\mu^2 - H_1)} + \int_{q_0}^{q_2} \frac{\mu dq}{q(3\mu^2 - H_1)} = \frac{2\mathrm{i}}{3}\,t + \beta_2\,. \tag{3.18}$$

As usual, here μ is a function on q obtained from the separated relation (3.17).

3.2.1. Deformation of the Goryachev System

Using a trigonal curve of genus three defined by the following equation

$$\Phi_1 = q^4 + cq^3 - bq^2 + (\mu^3 + d\mu^2 - \widehat{H}_1\mu + \widehat{H}_2)q + \frac{a^2}{4} = 0\,, \tag{3.19}$$

instead of (3.17) one gets deformation of the initial Hamilton function (3.15)

$$\widehat{H}_1 = H_1 - \left(\frac{d}{3} + \frac{d + cx_3^{2/3}}{x_1^2 + x_2^2}\right) J_3 + \frac{(c + dx_3^{4/3})^2}{4(x_1^2 + x_2^2)x_3^{2/3}} \tag{3.20}$$

after the generalized shift (1.8) at

$$f = -\frac{\mathrm{i}(c + dx_3^{4/3})}{2\sqrt{1 - x_3^2 x_3^{1/3}}}\,.$$

The corresponding equations of motion look like

$$\frac{\dot q_1}{q_1(3\mu_1^2 + 2d\mu_1 - \widehat{H}_1)} + \frac{\dot q_2}{q_2(3\mu_2^2 + 2d\mu_2 - \widehat{H}_1)} = 0\,, \qquad \mu_k = \frac{2\mathrm{i}}{3}\, q_k p_k$$

$$\frac{\mu_1 \dot q_1}{q_1(3\mu_1^2 + 2d\mu_1 - \widehat{H}_1)} + \frac{\mu_1 \dot q_2}{q_2(3\mu_2^2 + 2d\mu_2 - \widehat{H}_1)} = \frac{2\mathrm{i}}{3}\,.$$

Birational transformation (3.13) maps the curve (3.19) to the canonical trigonal form (3.6) at $a_2 = b_1 = b_2 = 0$.

3.3. Case 3 from [24]

Let us consider one more integrable system from [24] defined by the following integrals of motion

$$H_1 = J_1^2 + J_2^2 + \left(\frac{1}{12} + \frac{(2x_3+1)}{2(x_3+1)}\right) J_3^2 + \frac{ax_1}{(x_3+1)^{5/6}} + \frac{b}{(x_3+1)^{1/3}}\,,$$

$$H_2 = \frac{1}{27}\, J_3^3 - \frac{1}{3}\, J_3 H_1 - a(x_3+1)^{1/6}\, J_1 + \frac{ax_1 J_3}{2(x_3+1)^{5/6}}\,. \tag{3.21}$$

The corresponding variables of separation $q_{1,2}$ and $p_{1,2}$ (3.1,3.3) are obtained from

$$q_1 + q_2 = -\frac{(1+x_3)^{2/3} J_3}{2(x_3 - 1)} - \frac{\mathrm{i}(x_2 J_1 - x_1 J_2)}{(1+x_3)^{1/3}(x_3 - 1)}\,, \quad q_1 q_2 = \frac{a\sqrt{1+x_3}}{2(x_1 + \mathrm{i}x_2)}\,, \quad p_{1,2} = 3\mathrm{i}(1+x_3)^{1/3} + \frac{\mathrm{i}J_3}{q_{1,2}}\,.$$

Inverse transformation looks like

$$x_1 + \mathrm{i}x_2 = \frac{a\sqrt{1+x_3}}{2q_1q_2}\,, \qquad J_3 = \frac{\mathrm{i}q_1q_2(p_1 - p_2)}{q_1 - q_2}\,,$$

$$x_1 - \mathrm{i}x_2 = -\frac{2q_1q_2(x_3^2 - 1)}{a\sqrt{1+x_3}}\,, \qquad J_1 + \mathrm{i}J_2 = \frac{\mathrm{i}a(p_1 - p_2)}{4(q_1 - q_2)\sqrt{1+x_3}} - \frac{a(q_1 + q_2)}{2q_1q_2(1+x_3)^{1/6}}\,,$$

$$J_1 - \mathrm{i}J_2 = -\frac{\mathrm{i}q_1^2q_2^2(3x_3+1)(p_1 - p_2)}{a(q_1 - q_2)\sqrt{1+x_3}} - \frac{2q_1q_2(q_1+q_2)(x_3-1)}{a(1+x_3)^{1/6}}\,, \tag{3.22}$$

where

$$x_3 = \frac{\mathrm{i}(p_1q_1 - p_2q_2)^3}{27(q_1 - q_2)^3} - 1\,.$$

Separated relations are defined by an equation with the real coefficients

$$\Phi(q,\mu)=2q^4-bq^2+(\mu^3\,q-H_1\mu+H_2)q+\frac{a^2}{4}=0\,,\qquad q=q_{1,2},\quad \mu=\frac{\mathrm{i}\,q_{1,2}p_{1,2}}{3} \tag{3.23}$$

The corresponding quadratures are given by

$$\begin{aligned}&\int_{q_0}^{q_1}\frac{\dot q}{q(3\mu^2-H_1)}+\int_{q_0}^{q_2}\frac{\dot q}{q(3\mu^2-H_1)}=\beta_1\,,\\ &\int_{q_0}^{q_1}\frac{\mu\dot q}{q(3\mu^2-H_1)}+\int_{q_0}^{q_2}\frac{\mu\dot q}{q(3\mu^2-H_1)}=\frac{\mathrm{i}}{3}\,t+\beta_2\,.\end{aligned} \tag{3.24}$$

As for the Goryachev system, the birational change (3.13) transforms the equation (3.23) to the canonical trigonal form (3.6) at

$$a_1=a_2=b_1=b_2=c_2=0\,,$$

It allows us to prove that integrals of motion for this system (3.21) are related to integrals of motion (3.15) for the Goryachev system by the non-canonical transformation (3.10).

It may seem that quadratures (3.18) and (3.24) are trivially related by change of time

$$t\to 2t\,,$$

but we have to keep firmly in mind that μ in (3.18) is a function on q obtained from (3.17), whereas μ in (3.24) is another function on q obtained from (3.23).

3.3.1. Deformation of the System (3.21)

Similar to the Goryachev system, we can add two terms to the initial trigonal curve of genus three (3.23)

$$\Phi_1=2q^4+cq^3-bq^2+(\mu^3+d\mu^2-\widehat{H}_1\mu+\widehat{H}_2)q+\frac{a^2}{4}=0\,. \tag{3.25}$$

Deformation of the initial Hamilton function (3.21) looks like

$$\widehat{H}_1\,=H_1-\left(\frac{d}{6}-\frac{d}{x_3-1}-\frac{c(1+x_3)^{1/3}}{2(x_3-1)}\right)J_3+\frac{\left(d\sqrt{1+x_3}+c(1+x_3)^{-1/6}\right)^2}{4(1-x_3)} \tag{3.26}$$

after canonical transformation (1.8) at

$$f=-\frac{\mathrm{i}\Big(d(1+x_3)+c(1+x_3)^{1/3}\Big)}{2\sqrt{1-x_3^2}}\,.$$

In this case equations of motion are equal to

$$\begin{aligned}&\frac{\dot q_1}{q_1(3\mu_1^2+2d\mu_1-\widehat{H}_1)}+\frac{\dot q_2}{q_2(3\mu_2^2+2d\mu_2-\widehat{H}_1)}=0\,,\qquad \mu_k=\frac{\mathrm{i}}{3}\,q_kp_k\,,\\ &\frac{\mu_1\dot q_1}{q_1(3\mu_1^2+2d\mu_1-\widehat{H}_1)}+\frac{\mu_2\dot q_2}{q_2(3\mu_2^2+2d\mu_2-\widehat{H}_1)}=\frac{\mathrm{i}}{3}\,.\end{aligned} \tag{3.27}$$

3.4. Dullin–Matveev System

Let us consider the Dullin–Matveev system [25] defined by the following integrals of motion

$$
\begin{aligned}
H_1 &= J_1^2 + J_2^2 + \left(1 + \frac{x_3}{x_3+c} - \frac{x_3^2 - |x|^2}{4(x_3+c)^2}\right) J_3^2 + \frac{ax_1}{(x_3+c)^{1/2}} + \frac{b}{x_3+c}\,,\\
H_2 &= -\left(J_1^2 + J_2^2 - \frac{J_3^2}{4} + \frac{(4x_3^2 + 6x_3c + c^2 + |x|^2)J_3^2}{4(x_3+c)^2} + \frac{b}{x_3+c}\right) J_3\\
&\quad + a\sqrt{x_3+c}J_1 - \frac{ax_1J_3}{2\sqrt{x_3+c}}\,.
\end{aligned}
\tag{3.28}
$$

According to [8] variables of separation $q_{1,2}$ and $p_{1,2}$ are defined by (3.1,3.3)

$$
q_1+q_2 = -\frac{J_3}{2(c+x_3)} - \frac{J_1+\mathrm{i}J_2}{x_1+\mathrm{i}x_2}\,,\qquad q_1q_2 = \frac{a}{2(x_1+\mathrm{i}x_2)\sqrt{c+x_3}}\,,\qquad p_{1,2} = \mathrm{i}(c+x_3) + \frac{\mathrm{i}J_3}{q_{1,2}}
$$

or by inverse transformation

$$
\begin{aligned}
x_1+\mathrm{i}x_2 &= \frac{a}{2\sqrt{-\dfrac{\mathrm{i}(p_1q_1-p_2q_2)}{q_1-q_2}q_1q_2}}\,,\quad x_3 = -\frac{\mathrm{i}(p_1q_1-q_2p_2)}{q_1-q_2} - c\,,\quad J_3 = \frac{\mathrm{i}q_1q_2(p_1-p_2)}{q_1-q_2}\,,\\
x_1-\mathrm{i}x_2 &= -\frac{2\sqrt{-\dfrac{\mathrm{i}(p_1q_1-p_2q_2)}{q_1-q_2}q_1q_2}}{a(q_1-q_2)^2}\Big((c+1+\mathrm{i}p_1)q_1 - (c+1+\mathrm{i}p_2)q_2\Big)\\
&\quad\times\Big((c-1+\mathrm{i}p_1)q_1 - (c-1+\mathrm{i}p_2)q_2\Big)\,,\\
J_1+\mathrm{i}J_2 &= -\frac{a(q_1(2q_1+q_2)p_1 - q_2(2q_2+q_1)p_2)}{4\sqrt{-\dfrac{\mathrm{i}(p_1q_1-p_2q_2)}{q_1-q_2}}\,,q_1q_2(p_1q_1-p_2q_2)}\,,\\
J_1-\mathrm{i}J_2 &= \frac{\mathrm{i}q_1q_2}{a\sqrt{-\dfrac{\mathrm{i}(p_1q_1-p_2q_2)}{q_1-q_2}}(q_1-q_2)}\Big((c+1+\mathrm{i}p_1)(c-1+\mathrm{i}p_1)(2p_1q_1-3q_2p_1-p_2q_2)q_1^3\\
&\quad + \big(2\mathrm{i}(p_1+p_2)c - 4p_1p_2\big)(p_1-p_2)q_1^2q_2^2 - (c+1+\mathrm{i}p_2)(c-1+\mathrm{i}p_2)(2p_2q_2-3q_1p_2-p_1q_1)q_2^3\,.
\end{aligned}
\tag{3.29}
$$

The corresponding separated relations are defined by an equation with the real coefficients

$$
\Phi(q,\mu) = \mu(c^2-1)q^3 + (2c\mu^2 - b)q^2 + (\mu^3 - H_1\mu + H_2)\,q + \frac{a^2}{4} = 0\,,\qquad q = q_{1,2},\quad \mu = \mathrm{i}\,q_{1,2}p_{1,2}\,,
\tag{3.30}
$$

and the quadratures in differential form look like

$$
\begin{aligned}
&\frac{\dot q_1}{q_1\Big((c^2-1)q_1^2 + 4cq_1\mu_1 + 3\mu_1^2 - H_1\Big)} + \frac{\dot q_2}{q_2\Big((c^2-1)q_2^2 + 4cq_2\mu_2 + 3\mu_2^2 - H_1\Big)} = 0\,,\\
&\frac{\mu_1\dot q_1}{q_1\Big((c^2-1)q_1^2 + 4cq_1\mu_1 + 3\mu_1^2 - H_1\Big)} + \frac{\mu_2\dot q_2}{q_2\Big((c^2-1)q_2^2 + 4cq_2\mu_2 + 3\mu_2^2 - H_1\Big)} = \mathrm{i}\,.
\end{aligned}
$$

3.4.1. Deformation of the Dullin–Matveev System

Substituting $q = q_{1,2}$ and $\mu = \mathrm{i}\, q_{1,2}p_{1,2}$ into the non-hyperelliptic algebraic curve of genus three defined by the following equation

$$\Phi_1 = \mu(c^2-1)q^3 + (2c\mu^2 + d\mu - b)q^2 + (\mu^3 + e\mu^2 - \widehat{H}_1\mu + \widehat{H}_2)q + \frac{a^2}{4} = 0\,, \tag{3.31}$$

and solving a pair of the resulting equations with respect to $H_{1,2}$ one gets deformation of the initial Hamilton function (3.28)

$$\widehat{H}_1 = H_1 - \frac{1}{2}\left(e - \frac{d}{c+x_3} + \frac{(ce-d)x_3 + e}{x_1^2+x_2^2}\right) J_3 + - \frac{(ce-d+x_3e)^2}{4(x_1^2+x_2^2)} \tag{3.32}$$

after the generalized shift (1.8) at

$$f = -\frac{\mathrm{i}(ce-d+x_3e)}{2\sqrt{1-x_3^2}}\,.$$

Using the same birational change (3.13) the curve (3.19) can be transformed to the canonical trigonal form (3.6) at $c_2 = c_3 = 0$.

In this case the equations of motion read as

$$\sum_{k=1}^{2} \frac{\dot q_k}{q_k(3\mu_k^2 + 4cq_k\mu_k + 2e\mu_k + q_k^2(c^2-1) + dq_k - \widehat{H}_1)} = 0\,, \qquad \mu_k = \mathrm{i}q_kp_k\,,$$

$$\sum_{k=1}^{2} \frac{\mu_k\dot q_k}{q_k(3\mu_k^2 + 4cq_k\mu_k + 2e\mu_k + q_k^2(c^2-1) + dq_k - \widehat{H}_1)} = \mathrm{i}\,. \tag{3.33}$$

3.5. Case 5 from [24]

Let us consider the last integrable system from [24] with integrals of motion

$$H_1 = J_1^2 + J_2^2 + \left(\frac{3}{16} + \frac{8x_3+5}{8(x_3+1)}\right) J_3^2 + \frac{ax_1}{(x_3+1)^{3/4}} + \frac{b}{\sqrt{x_3+1}}\,,$$

$$H_2 = \frac{1}{8}J_3^3 - \frac{1}{2}H_1J_3 + a(x_3+1)^{1/4}J_1 - -\frac{ax_1J_3}{4(x_3+1)^{3/4}}\,. \tag{3.34}$$

The corresponding variables of separation $q_{1,2}$ and $p_{1,2}$ (3.1,3.3) are obtained from

$$q_1+q_2 = \frac{(3x_3+1)J_3}{4\sqrt{x_3+1}(1-x_3)} + \frac{\mathrm{i}(x_2J_1-x_1J_2)}{\sqrt{x_3+1}(1-x_3)}\,, \quad q_1q_2 = \frac{a(x_3+1)^{1/4}}{2(x_1+\mathrm{i}x_2)}\,, \quad p_{1,2} = 2\mathrm{i}\sqrt{x_3+1} + \frac{\mathrm{i}J_3}{q_{1,2}}\,.$$

Inverse transformation looks like

$$x_1 + \mathrm{i}x_2 = \frac{a(x_3+1)^{1/4}}{2q_1q_2}\,, \qquad J_3 = \frac{\mathrm{i}q_1q_2(p_1-p_2)}{q_1-q_2}\,,$$

$$x_1 - \mathrm{i}x_2 = -\frac{2q_1q_2(x_3^2-1)}{a(x_3+1)^{1/4}}\,, \qquad J_1 + \mathrm{i}J_2 = \frac{\mathrm{i}a(p_1-p_2)}{8(q_1-q_2)(x_3+1)^{3/4}} - \frac{a(q_1+q_2)}{2q_1q_2(x_3+1)^{1/4}}\,,$$

$$J_1 - \mathrm{i}J_2 = -\frac{\mathrm{i}q_1^2q_2^2(7x_3+1)(p_1-p_2)}{2a(q_1-q_2((x_3+1)^{1/4}} - \frac{2q_1q_2(q_1+q_2)(x_3-1)(x_3+1)^{1/4}}{a}\,, \tag{3.35}$$

where

$$x_3 = -\frac{(p_1q_1-p_2q_2)^2}{4(q_1-q_2)^2} - 1\,.$$

Separated relations are defined by

$$\Phi(q,\mu) = -2\mu q^3 - bq^2 + (\mu^3 - H_1\mu + H_2)q + \frac{a^2}{4} = 0\,, \qquad q = q_{1,2}, \quad \mu = \frac{\mathrm{i}}{2}\, q_{1,2}p_{1,2} \tag{3.36}$$

and we have the following quadratures in differential form

$$\begin{aligned} &\frac{\dot q_1}{q_1(3\mu_1^2 - H_1 - 2q_1^2)} + \frac{\dot q_2}{q_2(3\mu_2^2 - H_1 - 2q_2^2)} = 0\,,\\ &\frac{\mu_1\dot q_1}{q_1(3\mu_1^2 - H_1 - 2q_1^2)} + \frac{\mu_2\dot q_2}{q_2(3\mu_2^2 - H_1 - 2q_2^2)} = \frac{\mathrm{i}}{2}\,. \end{aligned} \tag{3.37}$$

3.5.1. Deformation of the System (3.34)

Let us add three terms to the initial trigonal curve of genus three (3.36)

$$\Phi_1 = (c - 2\mu)q^3 - (d\mu + b)q^2 + (\mu^3 + e\mu^2 - \widehat{H}_1\mu + \widehat{H}_2)q + \frac{a^2}{4} = 0\,. \tag{3.38}$$

The corresponding deformation of the initial Hamilton function (3.34) has the form

$$\begin{aligned} \widehat{H}_1 = H_1 &- \left(\frac{e}{4} + \frac{c+2e}{2(1-x_3)} + \frac{c}{4(1+x_3)} + \frac{d(x_3^2 + 4x_3 + 3)}{4(1-x_3)(1+x_3)^{3/2}}\right) J_3\\ &+ \frac{1}{4(1-x_3)}\left(e\sqrt{1+x_3} + d + \frac{c}{\sqrt{1+x_3}}\right)^2, \end{aligned} \tag{3.39}$$

after canonical transformation (1.8) at

$$f = -\frac{\mathrm{i}c + \mathrm{i}e(1+x_3)}{2\sqrt{1-x_3^2}} - \frac{\mathrm{i}d}{2\sqrt{1-x_3}}\,.$$

The corresponding quadratures are defined by

$$\begin{aligned} &\sum_{k=1}^{2} \frac{\dot q_k}{q_k(3\mu_k^2 + 2e\mu_k - 2q_k^2 - dq_k - \widehat{H}_1)} = 0\,, \qquad \mu_k = \frac{\mathrm{i}}{2}\, q_k p_k\,,\\ &\sum_{k=1}^{2} \frac{\mu_k\dot q_k}{q_k(3\mu_k^2 + 2e\mu_k - 2q_k^2 - dq_k - \widehat{H}_1)} = \frac{\mathrm{i}}{2}\,. \end{aligned} \tag{3.40}$$

Non canonical transformations (3.10) relate equations (3.40) with similar equations (3.33) for the deformed Dullin–Matveev system.

4. CONCLUSION

In [9, 14, 15] some new variables of separation for various integrable systems on the sphere with higher order integrals of motion have been obtained by the brute force method. In [8, 13] we introduce a concept of natural Poisson tensors, which allows us to understand the geometric origin of this method and to find some common attributes of the variables of separation for the Kowalevski top, Chaplygin system, Goryachev–Chaplygin gyrostat, Goryachev and Dullin–Matveev systems, etc.

In this more technical paper we continue our investigations in order to describe explicitly canonical transformations of initial physical variables to variables of separation and vice versa, to calculate the corresponding quadratures and to discuss possible integrable deformations of these systems associated with genus three hyperelliptic and non-hyperelliptic algebraic curves.

In Section 2 we consider real variables of separation for which the separation relations have the real coefficients only. In Section 3 we discuss complex variables of separation and the separation relations with the real coefficients as above. Similar complex variables satisfying to the real separated equations for the Kowalevski top and Goryachev–Chaplygin gyrostat have been found in [26], for the Kowalevski–Goryachev–Chaplygin gyrostat in [27] and for the Steklov–Lyapunov system in [28]. Real variables for the Steklov–Lyapunov system have been discussed in [29]; in contrast to these variables, complex variables [28] can be generalized to the Rubanovsky case [30]. These and other known complex variables lying on the real algebraic curves are discussed in [3].

Further inquiry is related to numerical, algebro-geometric and topological analysis of the obtained quadratures. For dynamical systems associated with the (3,4) trigonal curve (3.6) we also want to discuss an application of the Kowalevski–Painlevé criteria to these systems, because in the generic case solutions of the corresponding quadratures are non-meromorphic functions of time.

We would like to thank A. V. Borisov and Yu. N. Fedorov for helpful discussions. The work is supported by the Grant of The Government of the Russian Federation for the support of the scientific research project implemented under the supervision of leading scientists at Russian institutions of higher education (11.G34.31.0039).

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