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Equational type characterization for σ-complete MV-algebras

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Abstract

In the framework of algebras with infinitary operations, an equational base for the category of σ-complete MV-algebras is given. In this way, we study some particular objects as simple algebras, directly irreducible algebras, injectives, etc. A completeness theorem respect to the standard MV-algebra, considered as σ-complete MV-algebra, is obtained. Finally we apply this result to the study of σ-complete Boolean algebras and σ-complete product MV-algebras.

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Introduction

The theory of σ-complete MV-algebras was studied by several authors in an attempt to extend classical results related to σ-complete Boolean algebras to MV-algebras [1, 10, 23]. Another motivation is rooted in the study of new algebraic and topological representation of MV-algebras [8, 6, 16]. The aim of this paper is to investigate, the category of σ-complete MV-algebras as a class of algebras endowed with infinitary operations.

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Słomiński [27] showed that many results on classical universal algebra can be generalized to the case of infinitary operations. In this framework we develop an equational base that characterizes the class of σ-complete MV-algebras. This equational system has a rigorous motivation when we consider Łukasiewicz tribes as algebras endowed with a denumerable operation given by a truncated series, pointwise defined, in a power of the real interval $[0,1]$.

The paper is organized as follows. In Section 1, we recall some basic notion of abstract algebras with infinitary operations. In Section 2, we review some basic properties of MV-algebras. Moreover we study certain properties of the distance function in MV-algebras. In Section 3, we introduce a class of algebras with infinitary operations called MV_ω-algebras. This class is defined by equations and captures basic properties of Łukasiewicz tribes. In Section 4, we investigate the relationships between σ-complete MV-algebras and MV_ω-algebras. More precisely we prove that the category of MV_ω-algebras and the category of σ-complete MV-algebras is the same. In Section 5, we study MV_ω-algebras as a particular case of monadic MV-algebras. In Section 6, we study sub-structures in the category of MV_ω-algebras. In Section 7, the theory of filters and congruences in MV_ω-algebras is developed. Section 8 is dedicated to the study of directly irreducible and simple $\mathcal{MV}_\omega$-algebras. In Section 9, an standard completeness theorem for MV_ω-algebras is obtained. In Section 10 and Section 11, we apply the results of the previous sections to the study of σ-complete Boolean algebras and σ-complete product MV-algebras respectively. Finally in Section 12, injective objects in σ-complete MV-algebras and σ-complete product MV-algebras are characterized.

1 Basic notions

In what follows we adapt the terminology and some results present in [27] to the class of algebras admitting at most numerable operations. Let us denote by $\mathbb{N}$ the set of natural numbers starting with 1. Let A be a non-empty set and α be an ordinal where $\alpha \leq \omega$. If f is a function with domain A^α where $\vec{a} = (a_i)_{i\in\alpha} \in A^\alpha$ then $f(\vec{a})$ means the value $f(a_0, a_1, \ldots a_i, \ldots)$ where $i \in \alpha$.

An *ω-type* is a set τ of operation symbols having ordinal numbers $\alpha \leq \omega$ for arities. Let τ be an ω-type. An *ω-algebra* of type τ is a pair $\langle A, F\rangle$ where A is a non-empty set and F is a family of operations on A indexed by the type τ such that, corresponding to each α-ary function symbol $\varphi \in \tau$, there is an α-ary operation $\varphi^A : A^\alpha \to A$ in F. An ω-algebra A is *trivial* iff it has only one element.

Let A and B be two ω-algebras of type τ. Then B is a *sub-algebra* of A iff $B \subseteq A$ and for every $\varphi \in \tau$, φ^B is φ^A restricted to B. A function $f : A \to B$ is said to be a *τ-homomorphism* iff for each operation symbol $\varphi \in \tau$ with arity $\alpha \leq \omega$ and for each family $(x_i)_{i\in\alpha}$ on A, $f(\varphi^A(x_0, x_1, \ldots)) = \varphi^B(f(x_0), f(x_1), \ldots)$. A is said to be *rigid* iff the identity τ-homomorphism is the only automorphism.

An equivalence relation θ on A is an *ω-congruence* iff θ satisfies the following *compatibility property*: for each operation symbol $\varphi \in \tau$ of arity $\alpha \leq \omega$, and elements $a_i, b_i \in A$, if $(a_i, b_i) \in \theta$ holds for $i \in \alpha$ then $(\varphi(\vec{a}), \varphi(\vec{b})) \in \theta$ holds, where $\vec{a} = (a_i)_{i\in\alpha}$ and $\vec{b} = (b_i)_{i\in\alpha}$. It is clear that the *diagonal relation* Δ on A and the *all relations* A^2, denoted by ∇, are ω-congruences. The set of all ω-congruences on A is denoted by $Con_\omega(A)$ and $\langle Con_\omega(A), \subseteq\rangle$ is a complete lattice. A is *simple* iff $Con_\omega(A) = \{\Delta, \nabla\}$. A has the *congruence extension property* (CEP) iff for each sub-algebra B and $\theta \in Con_\omega(B)$ there is a $\phi \in Con_\omega(A)$ such that $\theta = \phi \cap A^2$. If $\theta \in Con_\omega(A)$ then the *quotient algebra* of A by θ is the algebra whose universe is the set $A/_\theta$ and whose operations satisfy $\varphi^{A/\theta}(x_0/_\theta, x_0/_\theta, \ldots) = \varphi^A(x_0, x_1, \ldots)/_\theta$ where $\varphi \in \tau$ has arity $\alpha \leq \omega$ and $(x_i)_{i\in\alpha}$ is sequence on A. Note that $A/_\theta$ is an ω-algebra of type τ and the natural map $p_\theta : A \to A/_\theta$ is a surjective τ-homomorphism. If $f : A \to B$ is a τ-homomorphism then $Ker(f) = \{(a, b) \in A^2 : f(a) = f(b)\}$ is an ω-congruence.

The *direct product* of a family $(A_i)_{i\in I}$ of ω-algebras of type τ, denoted by $\prod_{i\in I} A_i$, is the ω-algebra of type τ obtained by endowing the set-theoretical Cartesian product of the family with the operation of type τ, defined pointwise. For each $j \in I$ the j^{th}-projection π_j is a τ-homomorphism onto A_j. A is *directly indecomposable* iff A is not τ-isomorphic to a direct product of two non trivial algebras of type τ.

A class $\mathcal{A}$ of ω-algebras is called *ω-variety* iff it is closed with respect to direct products, sub-algebras and homomorphic images. Let τ be an ω-type and X be a set of variables. The set $Term_\tau(X)$ of terms over X is the smallest set such that:

i $X \cup \tau_0 \subseteq Term_\tau(X)$ where τ_0 is the set of constant operation symbols.

ii If $\varphi \in \tau$ is α-ary ($\alpha \leq \omega$) and $\vec{p} = (p_i)_{i\in\alpha} \subseteq Term_\tau(X)$ then, $\varphi(\vec{p}) \in Term_\tau(X)$.

For $t \in Term_\tau(X)$ we often write $t(\vec{x})$ to indicate that the variables $\vec{x} = (x_i)_{i\in\alpha\leq\omega}$ occurring in t are among $(x_i)_{i\in\alpha\leq\omega}$.

Let A be an ω-algebra of type τ. For $t(\vec{x}) \in Term_\tau(X)$ where $\vec{x} = (x_i)_{i\in\alpha\leq\omega}$, we define the *term function* $t^A : A^\alpha \to A$ as follows:

i if t is a variable x_i, then $t^A(\vec{a}) = a_i$ for $\vec{a} = (a_j)_{j\in\alpha} \in A^\alpha$, i.e. t^A is the i^{th}-projection map.

ii if t is of the form $\varphi(\vec{p}(\vec{x}))$ where $\varphi \in \tau$ is α-ary ($\alpha \leq \omega$) and $\vec{p}(\vec{x}) = (p_k(\vec{x}))_{k\in\alpha} \subseteq Term_\tau(X)$ then $\varphi^A(\vec{a}) = \varphi^A(\vec{p}^A(\vec{a}))$ where $\vec{p}^A = (p_k^A)_{k\in\beta}$.

An *equation* of type τ over X is an expression of the form $p = q$ where $p, q \in Term_\tau(X)$. An ω-algebra A of type τ satisfies an equation $p = q$ (abbreviated by $A \models p = q$) iff $p^A = q^A$. Let $\mathcal{A}$ be a class of ω-algebras of type τ. Then the equation $p = q$ is satisfied in $\mathcal{A}$ (abbreviated by $\mathcal{A} \models p = q$) iff for each $A \in \mathcal{A}$, $A \models p = q$. Let E be a set of equations of type τ over X. We denote by $Alg_\tau(E)$ the class of ω-algebras of type τ that satisfies the equations in E. $\mathcal{A}$ is said to be *equationally definable* iff there exists a set E of equations of type τ over X such that $\mathcal{A} = Alg_\tau(E)$.

Theorem 1.1 [27, § 7] *Let $\mathcal{K}$ be an equationally definable class of ω-algebras. Then $\mathcal{K}$ is an ω-variety.* □

Let $\mathcal{A}$ be a category of ω-algebras. An algebra A in $\mathcal{A}$ is *injective* iff for every $\mathcal{A}$-monomorphism $f : B \to C$ and every $\mathcal{A}$-homomorphism $g : B \to A$ there exists an $\mathcal{A}$-homomorphism $h : C \to A$ such that $g = h \circ f$ ($\circ$ denote the composition of $\mathcal{A}$-homomorphisms).

2 MV-algebras

Introduced by Chang in [3, 4], this structure represents the algebraic counterpart of infinite-valued propositional calculus of Łukasiewicz. In this section we first recall from [5] some basic facts about MV-algebras. Subsequently we study certain properties of the distance function in MV-algebras that play an important role in the following section.

An *MV-algebra* is an algebra $\langle A, \oplus, \neg, 0\rangle$ of type $\langle 2, 1, 0\rangle$ satisfying the following equations:

MV1 $\langle A, \oplus, 0\rangle$ is an abelian monoid, MV2 $\neg\neg x = x$,

MV3 $x \oplus \neg 0 = \neg 0$,

MV4 $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

We denote by $\mathcal{MV}$ the variety of MV-algebras. In agreement with the usual MV-algebraic operations we define

$$x \odot y = \neg(\neg x \oplus \neg y), \qquad x \vee y = (x \to y) \to y = (x \odot \neg y) \oplus y,$$

$$x \to y = \neg x \oplus y, \qquad x \wedge y = x \odot (x \to y),$$

$$1 = \neg 0.$$

Moreover, we use the following notation: $\bigoplus_{i=1}^{n} x_i = x_1 \oplus x_2 \oplus \ldots \oplus x_n$ and $\bigodot_{i=1}^{n} x_i = x_1 \odot x_2 \odot \ldots \odot x_n$. On each MV-algebra A we can define an order $x \leq y$ iff $x \to y = 1$. This order turns $\langle A, \wedge, \vee, 0, 1\rangle$ in a bounded distributive lattice with 1 the greatest element and 0 the smallest element.

Lemma 2.1 *Let A be an MV-algebra. Then:*

1. $(x \to y) \vee (y \to x) = 1$. (*prelinearity condition*)
2. *For each* $n \in \mathbb{N}$, $(\bigodot_{i=1}^{2n} x) \wedge (\bigodot_{i=1}^{2n} y) \leq (\bigodot_{i=1}^{n} x) \odot (\bigodot_{i=1}^{n} y)$.
3. *For each* $n \in \mathbb{N}$, $\bigodot_{i=1}^{2^n}(x \vee y) \leq (\bigodot_{i=1}^{2^n} x) \vee (\bigodot_{i=1}^{2^n} y)$.

Proof: 1) See [16, Proposition 1.1.7]

2) It is shown in [16, Lemma 2.4.1] that $(x\odot x)\wedge(y\odot y) \leq (x\odot y)$. Therefore $(\bigodot_{i=1}^{2n} x)\wedge(\bigodot_{i=1}^{2n} y) = ((\bigodot_{i=1}^{n} x)\odot(\bigodot_{i=1}^{n} x))\wedge((\bigodot_{i=1}^{n} y)\odot(\bigodot_{i=1}^{n} y)) \leq (\bigodot_{i=1}^{n} x) \odot (\bigodot_{i=1}^{n} y)$.

3) We use induction on n. It is shown in [15, Lemma 2.2.24] the following inequality $(x \vee y) \odot (x \vee y) \leq (x \odot x) \vee (y \odot y)$. That constitutes the base of the induction. Suppose that $\bigodot_{i=1}^{2^k}(x \vee y) \leq (\bigodot_{i=1}^{2^k} x) \vee (\bigodot_{i=1}^{2^k} y)$ whenever $k < n$. Then $\bigodot_{i=1}^{2^n}(x \vee y) = (\bigodot_{i=1}^{2^{n-1}}(x \vee y)) \odot (\bigodot_{i=1}^{2^{n-1}}(x \vee y)) \leq ((\bigodot_{i=1}^{2^{n-1}} x) \vee (\bigodot_{i=1}^{2^{n-1}} y)) \odot ((\bigodot_{i=1}^{2^{n-1}} x) \vee (\bigodot_{i=1}^{2^{n-1}} y)) \leq (\bigodot_{i=1}^{2^n} x) \vee (\bigodot_{i=1}^{2^n} y)$.
$\square$

Let A be an MV-algebra and x be an element in A. x is called *nilpotent* iff there exists a natural number n such that $\bigodot_{i=1}^{n} x = 0$ and it is called a *unity* iff $x \neq 1$ and $1 = \neg x \to \bigodot_{i=1}^{n} x$ for each $n \in \mathbb{N}$. We say that x is

Boolean iff $x \oplus x = x$. The set of Boolean elements of A will be denoted by $B(A)$. We can prove that: $x \in B(A)$ iff $x \odot x = x$ iff $x \vee \neg x = 1$ iff $x \wedge \neg x = 0$ iff $\forall y \in A : x \oplus y = x \vee y$ iff $\forall y \in A : x \odot y = x \wedge y$ [5, Theorem 1.5.3]. Equipped with the operations of A, $B(A)$ is a sub-MV-algebra of A which is a Boolean algebra.

A very important example of MV-algebra is $[0,1]_{MV} = \{[0,1], \oplus, \neg, 0\}$ where $[0,1]$ is the real unit segment and $\oplus$ and $\neg$ are defined as follows:

$$x \oplus y = \min(1, x+y) \qquad \neg x = 1 - x$$

The derived operations in $[0,1]_{MV}$ are given by $x \odot y = \max(0, x+y-1)$ (called Łukasiewicz t-norm) and $x \to y = \min(1, 1-x+y)$. The MV-lattice structure is the natural order in $[0,1]$. For each integer $n \geq 2$ the n-element set $Ł_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ yields an example of sub-algebra of $[0,1]_{MV}$. Moreover $[0,1]_{MV}$ and all the sub-algebras are *rigid* algebras [5, Corollary 7.2.6].

Let A be an MV-algebra. A subset $F \subseteq A$ is called *filter* iff it satisfies the following conditions:

1. $1 \in F$,
2. if $x \in F$ and $x \to y \in F$ then $y \in F$.

It is easy to verify that a non-empty subset F is a filter iff F is an increasing set (i.e. if $a \in F$ and $a \leq b$ then $b \in F$) and if $a, b \in F$ then $a \odot b \in F$. F is said to be *proper* iff 0 does not belong to F. The intersection of any family of filters of A is again a filter of A. We denote by $\langle X \rangle_{MV}$ the filter generated by $X \subseteq A$, i.e., the intersection of all filters of A containing X. We abbreviate this as $\langle a \rangle_{MV}$ when $X = \{a\}$ and it is easy to verify that $\langle X \rangle_{MV} = \{x \in A : \exists \ w_1 \cdots w_n \in X \text{ such that } x \geq w_1, \odot \cdots, \odot w_n\}$.

For any filter F of A, $\theta_F = \{(x,y) \in A^2 : x \to y, y \to x \in F\} = \{(x,y) \in A^2 : \exists a \in F : x \odot a \leq y \, and \, y \odot a \leq x\}$ is a congruence on A. Moreover $F = \{x \in A : (x,1) \in \theta_F\}$. Conversely, if $\theta \in Con(A)$ then $F_\theta = \{x \in A : (x,1) \in \theta\}$ is a filter and $(x,y) \in \theta$ iff $(x \to y, 1) \in \theta$ and $(y \to x, 1) \in \theta$. Thus the correspondence $F \to \theta_F$ is a bijection from the set of filters of A onto the set $Con(A)$. If F is a filter of A, we shall write $A/_F$ instead of $A/_{\theta_F}$, and for each $x \in A$ we shall write $x/_{\theta_F}$ or $x/_F$ for the equivalence class of x. F is called *prime* iff for each $x, y \in A$ $x \to y \in F$ or $y \to x \in F$. It is well known that F is prime iff $A/_F$ is tottaly ordered. F is said to be *stonean filter* iff for every $x \in F$ there is $z \in F \cap B(A)$ such that $z \leq x$.

Let A be an MV-algebra. A is simple iff it is MV-isomorphic to a subalgebra of $[0,1]_{MV}$ iff for each $x < 1$, x is nilpotent. [5, Theorem 3.5.1]. We call *radical of* A the intersection of all maximal filters of A. The radical of A will be denoted by $Rad(A)$ and we can see that $Rad(A) = \{x \in A : x \text{ is unity}\}$ [5, Proposition 3.6.4]. A is semisimple iff $Rad(A) = \{1\}$ iff it is a subdirect product of subalgebras of $[0,1]_{MV}$ [5, Proposition 3.6.1].

An important characterization of the equations in $\mathcal{MV}$ is given by:

$$\mathcal{MV} \models t = s \iff [0,1]_{MV} \models t = s \quad \text{[5, Theorem 2.5.3]} \tag{1}$$

Now we study some properties about the distance function in MV-algebras that play an important role in the following section. Let A be an MV-algebra. The *distance function* $d : A \times A \to A$ is defined by

$$d(x,y) = (x \odot \neg y) \oplus (y \odot \neg x)$$

In $[0,1]_{MV}$ the distance function is given by $d(x,y) = \mid x - y \mid$. In this case $d(x,y)$ gives the usual distance in the unitary real interval.

Proposition 2.2 *In every MV-algebra we have:*

1. $d(x,y) = 0$ *iff* $x = y$,
2. $d(x,y) = d(y,x) = d(\neg x, \neg y)$,
3. $d(x,z) \leq d(x,y) \oplus d(y,z)$,
4. $d(x \oplus s, y \oplus t) \leq d(x,y) \oplus d(y,t)$,
5. $x \oplus d(x, x \vee y) = x \vee y$,
6. *if* $x \leq y$ *then* $x \oplus d(x,y) = y$.

Proof: 1...4) See [5, Proposition 1.2.4]. 5) By the characterization of the $\mathcal{MV}$-equations, we study this equation in $[0,1]_{MV}$. In fact: $x \oplus d(x, x \vee y) = x \oplus \mid x - x \vee y \mid = x \oplus (x \vee y - x) = \min(x + x \vee y - x, 1) = \min(x \vee y, 1) = x \vee y$.

6) Suppose that $x \leq y$. By item 5, $x \oplus d(x,y) = x \oplus d(x, x \vee y) = x \vee y = y$. $\square$

Let A be an MV-algebra and $\vec{x} = (x_i)_{i \in \mathbb{N}}$ be a sequence in A. Then we define the following sequences:

$$\begin{array}{ll} Sup_0(\vec{x}) = 0 & Sum_0(\vec{x}) = 0 \\ Sup_n(\vec{x}) = \bigvee_{i=1}^{n} x_n & Sum_n(\vec{x}) = \bigoplus_{i=1}^{n} x_i \end{array}$$

Proposition 2.3 *Let A be an MV-algebra and $\vec{x} = (x_i)_{i\in\mathbb{N}}$ be a sequence in A. Then for each $n \in \mathbb{N}$*

1. $\bigoplus_{i=1}^{n} d(Sup_{i-1}(\vec{x}), Sup_i(\vec{x})) = Sup_n(\vec{x})$,
2. $\bigoplus_{i=1}^{n} d(Sum_{i-1}(\vec{x}), Sum_i(\vec{x})) = Sum_n(\vec{x})$ (⊕-telescopic property).

Proof: 1) We use induction on n. If $n = 1$ then $\bigoplus_{i=1}^{1} d(Sup_{i-1}(\vec{x}), Sup_i(\vec{x})) = d(0, Sup_1(\vec{x})) = Sup_1(\vec{x})$. Suppose that the proposition holds for $k < n$. Then $\bigoplus_{i=1}^{n} d(Sup_{i-1}(\vec{x}), Sup_i(\vec{x})) = (\bigoplus_{i=1}^{n-1} d(Sup_{i-1}(\vec{x}), Sup_i(\vec{x}))) \oplus d(Sup_{n-1}(\vec{x}), Sup_n(\vec{x})) = Sup_{n-1}(\vec{x}) \oplus d(Sup_{n-1}(\vec{x}), Sup_n(\vec{x}))$. By Proposition 2.2-6, $Sup_{n-1}(\vec{x}) \oplus d(Sup_{n-1}(\vec{x}), Sup_n(\vec{x})) = Sup_n(\vec{x})$ since, $Sup_{n-1}(\vec{x}) \leq Sup_n(\vec{x})$. Hence $\bigoplus_{i=1}^{n} d(Sup_{i-1}(\vec{x}), Sup_i(\vec{x})) = Sup_n(\vec{x})$ for each $n \in \mathbb{N}$.

2) Note that $\vec{s} = (Sum_i(\vec{x}))_{i\in\mathbb{N}}$ is an increasing sequence. Then $Sup_i(\vec{s}) = Sum_i(\vec{x})$. Hence, by item 1,

$$\begin{aligned}\bigoplus_{i=1}^{n} d(Sum_{i-1}(\vec{x}), Sum_i(\vec{x})) &= \bigoplus_{i=1}^{n} d(Sup_{i-1}(\vec{s}), Sup_i(\vec{s})) \\ &= Sup_n(\vec{s}) = Sum_n(\vec{x})\end{aligned}$$

□

In each MV-algebra the following forms of distributive laws are known: if $\bigvee_{i\in I} x_i$ exists in A then:

$$x \odot \bigvee_{i\in I} x_i = \bigvee_{i\in I}(x \odot x_i) \quad \textit{and} \quad x \wedge \bigvee_{i\in I} x_i = \bigvee_{i\in I}(x \wedge x_i) \tag{2}$$

An interesting consequence of (2) is the following:

Proposition 2.4 *Let A be an MV-algebra and suppose that $\bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i$ exists in A. Then:*

$$\bigvee_{n\in\mathbb{N}} (x \oplus \bigoplus_{i=1}^{n} x_i) = x \oplus \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i$$

Proof: Observe that, for each $n \in \mathbb{N}$, $x \oplus \bigoplus_{i=1}^{n} x_i \leq x \oplus \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i$. Let k be an upper bound of the sequence $(x \oplus \bigoplus_{i=1}^{n} x_i)_{n\in\mathbb{N}}$. From definition of $\wedge$, it follows that:

$$\begin{aligned}\neg x \odot (x \oplus \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i) &= \neg x \odot (\neg x \to \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i) = \neg x \wedge \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i \\ &= \bigvee_{n\in\mathbb{N}} \neg x \wedge \bigoplus_{i=1}^{n} x_i = \bigvee_{n\in\mathbb{N}} \neg x \odot (\neg x \to \bigoplus_{i=1}^{n} x_i) \\ &= \bigvee_{n\in\mathbb{N}} \neg x \odot (x \oplus \bigoplus_{i=1}^{n} x_i) \leq \bigvee_{n\in\mathbb{N}} \neg x \odot k = \neg x \odot k\end{aligned}$$

Therefore, by residuation and definition of $\vee$, we have:

$$x \oplus \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i \leq \neg x \to (\neg x \odot k) = x \oplus (\neg x \odot k) = x \vee k = k$$

since $x \leq k$. Hence, $\bigvee_{n\in\mathbb{N}}(x \oplus \bigoplus_{i=1}^{n} x_i) = x \oplus \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i$.

□

We say that an MV-algebra A is *σ-complete* iff suprema and infima exist for all denumerable subsets in A. Every σ-complete MV-algebra is semisimple.

Proposition 2.5 [1, Proposition 1] *The only σ-complete and simple MV-algebras (up to isomorphisms) are* $[0,1]_{MV}$ *and the finite chains* $Ł_n$. □

3 An algebraic framework for Łukasiewicz tribes

Łukasiewicz tribes are collections of fuzzy sets closed under the standard Łukasiewicz complementation and Łukasiewicz sum with countably many arguments. Here, we introduce and study an equational class of ω-algebras, called MV_ω-algebras, that capture some basic properties of Łukasiewicz tribes when they are viewed as algebras with infinitary operations.

Definition 3.1 Let X be a non-empty set. A collection $M \subseteq [0,1]^X$ is called *Łukasiewicz tribe* iff

1. $\mathbf{0} \in M$ where $\mathbf{0}(x) = 0$ for each $x \in X$.
2. If $f \in M$, then $\neg f : X \to [0,1]$ defined as $\neg f(x) = 1 - f(x)$ belongs to M.

3. If $\vec{f} = (f_i)_{i\in\mathbb{N}}$ is a sequence in M then $\sum_{\text{Ł}} \vec{f} : X \to [0,1]$ defined as

$$\sum\nolimits_{\text{Ł}} \vec{f}(x) = \begin{cases} \sum_{i=1}^{\infty} f_i(x), & \text{if } \sum_{i=1}^{\infty} f_i(x) \text{ converges in } [0,1] \\ 1, & \text{if } \exists k \in \mathbb{N} \text{ s.t. } \bigoplus_{i=1}^{k} f_i(x) = 1 \end{cases}$$

belongs to M.

Every Łukasiewicz tribe $M \subseteq [0,1]^X$ is a σ-complete MV-algebra. Denumerable suprema, resp. infima, on M coincide with denumerable suprema, resp. infima, in $[0,1]$ applied pointwisely to functions on X with values in $[0,1]$.

Proposition 3.2 *Let $M \subseteq [0,1]^X$ be a Łukasiewicz tribe, $h, g \in M$ and $\vec{f} = (f_i)_{i\in\mathbb{N}}$ be a sequence in M. Then:*

1. *If we define $h \oplus g = \sum_{\text{Ł}}(h, g, \mathbf{0}, \mathbf{0}\ldots)$ then $\langle M, \oplus, \neg, 0\rangle$ is an MV-algebra. Moreover, the lattice order structure associated to $\langle M, \oplus, \neg, 0\rangle$ is defined pointwisely on X.*
2. *$\sum_{\text{Ł}} \vec{f} = \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_i = \sum_{\text{Ł}}(d(Sum_i(\vec{f}), Sum_{i-1}(\vec{f})))_{i\in\mathbb{N}}$,*
3. *$\sum_{\text{Ł}}(d(Sup_i(\vec{f}) \wedge g, Sup_{i-1}(\vec{f}) \wedge g))_{i\in\mathbb{N}} \leq g$*

Proof: 1) Immediate.

2) If $\vec{f} = (f_i)_{i\in\mathbb{N}}$ is a sequence in M and $x \in X$, we define $\vec{f_x}$ as the sequence in the interval $[0,1]$ given by $\vec{f_x} = (f_i(x))_{i\in\mathbb{N}}$. Then we have to prove that $\sum_{\text{Ł}} \vec{f}(x) = \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_i(x) = \sum_{\text{Ł}}(d(Sum_i(\vec{f_x}), Sum_{i-1}(\vec{f_x})))_{i\in\mathbb{N}}$ for each $x \in X$.

We first suppose that for each $n \in \mathbb{N}$, $\bigoplus_{i=1}^{n} f_i(x) \leq 1$. Then for each $n \in \mathbb{N}$, $\bigoplus_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} f_i(x) = Sum_n(\vec{f_x})$. Since $(Sum_n(\vec{f_x}))_{n\in\mathbb{N}}$ is an increasing bounded sequence, by the monotone convergence principle, $(Sum_n(\vec{f_x}))_{n\in\mathbb{N}}$ is a convergent sequence in $[0,1]$ and

$$\bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_i(x) = \bigvee_{n\in\mathbb{N}} Sum_n(\vec{f_x}) = \lim_{n\to\infty} Sum_n(\vec{f_x}) = \sum_{i=1}^{\infty} f_i(x) = \sum\nolimits_{\text{Ł}} \vec{f}(x)$$

Note that $\sum_{i=1}^{\infty} f_i(x) = (f_1(x) - 0) + (f_1(x) + f_2(x) - f_1(x)) + (f_1(x) + f_2(x) + f_3(x) - f_1(x) - f_2(x)) \ldots = \sum_{i=1}^{\infty} d(Sum_i(\vec{f_x}), Sum_{i-1}(\vec{f_x}))$. Hence, $\sum_{\text{Ł}} \vec{f} = \sum_{\text{Ł}} d(Sum_i(\vec{f}), Sum_{i-1}(\vec{f}))_{i\in\mathbb{N}}$.

Now we suppose that there exists $n \in \mathbb{N}$ such that $\bigoplus_{i=1}^{n} f_i(x) = 1$. Then $\sum_{Ł} \vec{f}(x) = 1 = \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_i(x)$. By Proposition 2.3-2, we have that $\bigoplus_{i=1}^{n} d(Sum_i(\vec{f_x}), Sum_{i-1}(\vec{f_x})) = Sum_n(\vec{f_x}) = \bigoplus_{i=1}^{n} f_i(x) = 1$. Consequently, $\sum_{Ł} d(Sum_i(\vec{f}), Sum_{i-1}(\vec{f}))_{i\in\mathbb{N}} = 1 = \sum_{Ł} \vec{f}(x)$. Hence $\sum_{Ł} \vec{f}(x) = \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_i(x) = \sum_{Ł}(d(Sum_i(\vec{f_x}), Sum_{i-1}(\vec{f_x})))_{i\in\mathbb{N}}$.

3) Consider the sequence $\vec{y} = (f_i \wedge g)_{i\in\mathbb{N}}$ in M. Note that

$$Sup_n(\vec{y}) = \bigvee_{i=1}^{n} (f_i \wedge g) = g \wedge \bigvee_{i=1}^{n} f_i = g \wedge Sup_n(\vec{f}) \leq g$$

By Proposition 2.3-1, for each $n \in \mathbb{N}$

$$\bigoplus_{i=1}^{n} d(Sup_i(\vec{y}), Sup_{i-1}(\vec{y})) = Sup_n(\vec{y}) \leq g$$

Hence, by item 2,

$$\begin{aligned}
\sum\nolimits_{Ł} d(Sup_i(\vec{f}) \wedge g, Sup_{i-1}(\vec{f}) \wedge g)_{i\geq 1} &= \sum\nolimits_{Ł} d(Sup_i(\vec{y}), Sup_{i-1}(\vec{y}))_{i\geq 1} \\
&= \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} d(Sup_i(\vec{y}), Sup_{i-1}(\vec{y})) \\
&= \bigvee_{n\in\mathbb{N}} Sup_n(\vec{y}) \leq g
\end{aligned}$$

□

Remark 3.3 Let M be a Łukasiewicz tribe. The set of operations $\langle \sum_{Ł}, \neg, \mathbf{0} \rangle$ suggest that M can be seen as an ω-algebra of type $\langle \omega, 1, 0 \rangle$ equipped with an underlying MV-structure definable form $\sum_{Ł}$. This motivates the following abstract framework for Łukasiewicz tribes based on ω-algebras.

Let A be a non-empty set. Let $\vec{x} = (x_i)_{i\in\mathbb{N}}$ be a sequence in A. If $s : \mathbb{N} \to \mathbb{N}$ is a bijective function then we define the *$\vec{x}$-permutation* $s(\vec{x})$ as $s(\vec{x}) = (x_{s(i)})_{i\in\mathbb{N}}$. Let $\sum$ be an operation of type ω in A (i.e. $\sum : A^{\mathbb{N}} \to A$). For the value $\sum(\vec{x})$ we use the following notations:

$$\sum(\vec{x}) = \sum \vec{x} = \sum_{i\in\mathbb{N}} x_i$$

Let $n \geq 1$ and consider the subsequence $\vec{x}_{\geq n} = (x_n, x_{n+1} \ldots)$ of $\vec{x}$. Then we define the expression $\sum_{i \geq n} x_i$ by

$$\sum_{i \geq n} x_i = \sum \vec{x}_{\geq n}$$

$\sum$ is said to be *commutative* iff $\sum \vec{x} = \sum s(\vec{x})$ for each $\vec{x} \in A^{\mathbb{N}}$ and for each $\vec{x}$-permutation s. Suppose that $\sum$ is commutative. An element $0 \in A$ is said to be *neutral element* for $\sum$ iff for each $x \in A$, $\sum(x, 0, 0 \ldots) = x$.

Definition 3.4 Consider the structure $\langle A, \sum, 0 \rangle$ of type $\langle \omega, 0 \rangle$ such that $\sum$ is a commutative operation and 0 is neutral element for $\sum$. Define the operation $\oplus : A^2 \to A$ such that, for each $x, y \in A$,

$$x \oplus y = \sum(x, y, 0, 0 \ldots)$$

Then we say that $\langle A, \sum, 0 \rangle$ is an *Abelian ω-monoid* iff $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

It is clear that if $\langle A, \sum, 0 \rangle$ is an Abelian ω-monoid then $\langle A, \oplus, 0 \rangle$ is an Abelian monoid.

Definition 3.5 An *MV_ω-algebra* is an ω-algebra $\langle A, \sum, \neg, 0 \rangle$ of type $\langle \omega, 1, 0 \rangle$ such that, for each sequence $\vec{x} = (x_i)_{i \in \mathbb{N}}$ in A, satisfies:

Σ 1. $\langle A, \sum, 0 \rangle$ is an Abelian ω-monoid,

Σ 2. $\sum \vec{x} = x_1 \oplus \sum_{i \geq 2} x_i$,

Σ 3. $\langle A, \oplus, \neg, 0 \rangle$ is an MV-algebra,

Σ 4. $\sum \vec{x} = \sum_{i \in \mathbb{N}} d(Sum_i(\vec{x}), Sum_{i-1}(\vec{x}))$,

Σ 5. $(\sum_{i \in \mathbb{N}} d(Sup_i(\vec{x}) \wedge y, Sup_{i-1}(\vec{x}) \wedge y)) \to y = 1$.

Note that axioms Σ 3, Σ 4 and Σ 5 capture the basic properties of Łukasiewicz tribes given in Proposition 3.2.

We denote by $\mathcal{MV}_\omega$ the category whose object are MV_ω-algebra and whose arrows are functions preserving the operations $\sum, \neg, 0$. These arrows are called *MV_ω-homomorphisms*. Since $\mathcal{MV}_\omega$ is equationally definable, by Theorem 1.1, it is an ω-variety. In agreement with the usual MV_ω-algebraic operation we define:

$$\bigodot \vec{x} = \bigodot_{i \in \mathbb{N}} x_i = \neg \sum_{i \in \mathbb{N}} \neg x_i \quad \text{where } \vec{x} = (x_i)_{i \in \mathbb{N}}$$

Example 3.6 By Proposition 3.2, each Łukasiewicz tribe with the signature $\langle \sum_{Ł}, \neg, 0\rangle$ is an MV_ω-algebra. In particular we denote by $[0,1]_{MV_\omega}$ the standard MV_ω-algebra $\langle [0,1], \neg, \sum_{Ł}, 0\rangle$ where $\sum_{Ł}$ is defined as

$$\sum\nolimits_{Ł} \vec{x} = \begin{cases} \sum_{i=1}^{\infty} x_i, & \text{if } \sum_{i=1}^{\infty} x_i \text{ converges in } [0,1] \\ 1, & \text{if } \exists k \in \mathbb{N} \text{ s.t. } \bigoplus_{i=1}^{k} x_i = 1 \end{cases}$$

Clearly, the underlying MV-structure associated to $[0,1]_{MV_\omega}$ coincides with the standard MV-algebra $[0,1]_{MV}$.

Proposition 3.7 *Let A be an MV_ω-algebra and $\vec{x} = (x_i)_{i\in\mathbb{N}}$ be a sequence in A. Then:*

1. *For each $n \in \mathbb{N}$, $\sum \vec{x} = \bigoplus_{i=1}^{n} x_i \oplus \sum_{i>n} x_i = Sum_n(\vec{x}) \oplus \sum_{i>n} x_i$,*
2. *$\sum \vec{x} = \bigvee_{n\in\mathbb{N}} (\bigoplus_{i=1}^{n} x_i) = \bigvee_{n\in\mathbb{N}} Sum_n(\vec{x})$,*
3. *for each $n_0 \in \mathbb{N}$, $\sum_{i\in\mathbb{N}} x = \sum_{i>n_0} x$,*
4. *if $\vec{x} = (x_1, x_2 \ldots x_n, 0, 0, 0 \ldots)$ then $\sum \vec{x} = \bigoplus_{i=1}^{n} x_i$,*
5. *if $\vec{x} = (x_1, x_2 \ldots x_n, 1, 1, 1 \ldots)$ then $\bigodot \vec{x} = \bigodot_{i=1}^{n} x_i$,*
6. *$\bigodot \vec{x} = \bigwedge_{n\in\mathbb{N}} (\bigodot_{i=1}^{n} x_i)$.*

Proof: 1) We use induction on n. By Σ 2, if $n = 1$, $\sum_{i\in\mathbb{N}} x_i = x_1 \oplus \sum_{i>2} x_i$. Suppose that $\sum_{i\in\mathbb{N}} x_i = \bigoplus_{i=1}^{n} x_i \oplus \sum_{i>n} x_i$. Then $\sum_{i\in\mathbb{N}} x_i = \bigoplus_{i=1}^{n} x_i \oplus (x_{n+1} \oplus \sum_{i>n+1} x_i) = (\bigoplus_{i=1}^{n} x_i \oplus x_{n+1}) \oplus \sum_{i>n+1} x_i = \bigoplus_{i=1}^{n+1} x_i \oplus \sum_{i>n+1} x_i$ since $\oplus$ is associative.

2) Let $y = \sum \vec{x} = \sum_{i\in\mathbb{N}} x_i$. Since $y = Sum_n(\vec{x}) \oplus \sum_{i>n} x_i$ then $Sum_n(\vec{x}) \leq y$ for each $n \in \mathbb{N}$. Let k be an upper bound of the sequence $\vec{s} = (Sum_n(\vec{x}))_{n\in\mathbb{N}}$. Since for each $n \in \mathbb{N}$ $Sup_n(\vec{s}) = Sum_n(\vec{x})$, by Σ 4 and Σ 5 we have:

$$\begin{aligned} y &= \sum_{i\in\mathbb{N}} x_i = \sum_{i\in\mathbb{N}} d(Sum_i(\vec{x}), Sum_{i-1}(\vec{x})) \\ &= \sum_{i\in\mathbb{N}} d(Sum_i(\vec{x}) \wedge k, Sum_{i-1}(\vec{x}) \wedge k) \\ &= \sum_{i\in\mathbb{N}} d(Sup_i(\vec{s}) \wedge k, Sup_{i-1}(\vec{s}) \wedge k) \leq k \end{aligned}$$

Hence $y = \sum_{i\in\mathbb{N}} x_i = \bigvee_{n\in\mathbb{N}} Sum_n(\vec{x})$.

3) $\bigoplus_{i=1}^{n} x = \bigoplus_{i=n_0+1}^{n+n_0+1} x$. Then $\bigvee_{i\in\mathbb{N}}(\bigoplus_{i=1}^{n} x) = \bigvee_{i\in\mathbb{N}}(\bigoplus_{i=n_0+1}^{n+n_0+1} x)$ and by item 2, $\sum_{i\in\mathbb{N}} x = \sum_{i>n_0} x$ for each $n_0 \in \mathbb{N}$.

4) Immediate from item 1 and $\Sigma 1$.

5) By item 4, $\bigodot(x_1, x_2 \ldots x_n, 0, 0, 0 \ldots) = \neg \sum(\neg x_1, \neg x_2 \ldots \neg x_n, 0, 0, 0 \ldots) = \neg(\bigoplus_{i=1}^{n} \neg x_i \oplus \sum_{i>n} x_i) = \neg(\bigoplus_{i=1}^{n} \neg x_i) \oplus 0 = \bigodot_{i=1}^{n} x_i$.

6) By item 2, $\bigodot \vec{x} = \neg \sum_{i\in\mathbb{N}} \neg x_i = \neg \bigvee_{n\in\mathbb{N}}(\bigoplus_{i=1}^{n} \neg x_i) = \bigwedge_{n\in\mathbb{N}} \neg(\bigoplus_{i=1}^{n} \neg x_i) = \bigwedge_{n\in\mathbb{N}} \bigodot_{i=1}^{n} x_i$.

$\square$

Note that Proposition 3.7-2 allows to see, in an abstract way, the operation Σ as a kind of "*limit*" of partial sums. Our next aim is to analyze a kind of version of the fact that for convergent positive term series, not only the original series converge to a limit, but also for any reordering it converges to the same limit. We first have to introduce some terminology:

Let A be an MV_ω-algebra and $\vec{x} = (x_i)_{i\in\mathbb{N}}$ be a sequence in A. Let I be a non-empty subset of $\mathbb{N}$ and consider the sequence $\vec{x}_I = (x_i^I)_{i\in\mathbb{N}}$ where

$$x_i^I = \begin{cases} x_i & \text{if } i \in I, \\ 0, & \text{otherwise.} \end{cases}$$

Then we define the expression $\sum_{i\in I} x_i$ as follows:

$$\sum_{i\in I} x_i = \sum \vec{x}_I$$

With these notations we have:

Proposition 3.8 *Let A be an MV_ω-algebra, $\vec{x} = (x_i)_{i\in\mathbb{N}}$ be a sequence in A and I be a non-empty subset of $\mathbb{N}$. Then:*

1. $\sum_{i\in I} x_i \leq \sum \vec{x}$.
2. *If I is a finite set then* $\sum_{i\in I} x_i = \bigoplus_{i\in I} x_i$.

Proof: 1) By definition of $\vec{x}_I$, for each $n \in \mathbb{N}$, $\bigoplus_{i=1}^{n} x_i^I \leq \bigoplus_{i=1}^{n} x_i$. Then, by Proposition 3.7-2, $\sum_{i\in I} x_i = \sum \vec{x}_I = \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i^I \leq \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i = \sum \vec{x}$. 2) Follows from Proposition 3.7-4.

□

Proposition 3.9 *Let A be an MV_ω-algebra and $\vec{x} = (x_i)_{i\in\mathbb{N}}$ be a sequence in A. Let $(I_n)_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ such that I_n is a non-empty set. Then:*

$$\sum \vec{x} = \sum_{n\in\mathbb{N}} (\sum_{i\in I_n} x_i)$$

Proof: We first prove that $(\sum_{t\in I_a} x_t) \oplus (\sum_{j\in I_b} x_j) = \sum_{i\in I_a\cup I_b} x_i$. By Proposition 2.4 and Proposition 3.7-2 we have:

$$\begin{aligned} (\sum_{t\in I_a} x_t) \oplus (\sum_{j\in I_b} x_j) &= (\bigvee_{r\in\mathbb{N}} \bigoplus_{t=1}^{r} x_t^{I_a}) \oplus (\bigvee_{s\in\mathbb{N}} \bigoplus_{j=1}^{s} x_j^{I_b}) \\ &= \bigvee_{r\in\mathbb{N}} \bigvee_{s\in\mathbb{N}} (\bigoplus_{t=1}^{r} x_t^{I_a} \oplus \bigoplus_{j=1}^{s} x_j^{I_b}) \end{aligned}$$

Observe that, for each $r, s \in \mathbb{N}$, $\bigoplus_{t=1}^{r} x_t^{I_a} \oplus \bigoplus_{j=1}^{s} x_j^{I_b} \leq \sum_{i\in I_a\cup I_b} x_i$ and then $(\sum_{t\in I_a} x_t) \oplus (\sum_{j\in I_b} x_j) \leq \sum_{i\in I_a\cup I_b} x_i$. Conversely, $\bigoplus_{i=1}^{m} x_i^{I_a\cup I_b} \leq \bigvee_{r\in\mathbb{N}} \bigvee_{s\in\mathbb{N}} (\bigoplus_{t=1}^{r} x_t^{I_a} \oplus \bigoplus_{j=1}^{s} x_j^{I_b})$ and then $\sum_{i\in I_a\cup I_b} x_i = \bigvee_{m\in\mathbb{N}} \bigoplus_{i=1}^{m} x_i^{I_a\cup I_b} \leq (\sum_{t\in I_a} x_t) \oplus (\sum_{j\in I_b} x_j)$. It proves that $(\sum_{t\in I_a} x_t) \oplus (\sum_{j\in I_b} x_j) = \sum_{i\in I_a\cup I_b} x_i$. Then, by induction, we obtain

$$\bigoplus_{j=1}^{n} (\sum_{i\in I_j} x_i) = \sum_{i\in\bigcup_{j=1}^{n} I_j} x_i$$

for each $n \in \mathbb{N}$. Consequently, by Proposition 3.8-1,

$$\begin{aligned} \sum_{n\in\mathbb{N}} (\sum_{i\in I_n} x_i) &= \bigvee_{n\in\mathbb{N}} (\bigoplus_{j=1}^{n} \sum_{i\in I_j} x_i) \\ &= \bigvee_{n\in\mathbb{N}} \sum_{i\in\bigcup_{j=1}^{n} I_j} x_i \\ &\leq \sum \vec{x} \end{aligned}$$

Now we prove the other inequality. Note that, if $n \in \mathbb{N}$ then there exists $m \in \mathbb{N}$ such that $\bigoplus_{i=1}^{n} x_i \leq \bigoplus_{j=1}^{m} \sum_{t \in I_j} x_t$. In fact, we can take $m = \min\{k \in \mathbb{N} : \{x_1, \ldots x_n\} \subseteq \bigcup_{j=1}^{k} I_j\}$. Thus, $\bigoplus_{i=1}^{n} x_i \leq \bigvee_{m \in \mathbb{N}} \bigoplus_{j=1}^{m} \sum_{t \in I_j} x_t$ and

$$\sum \vec{x} = \bigvee_{n \in \mathbb{N}} \bigoplus_{i=1}^{n} x_i \leq \bigvee_{m \in \mathbb{N}} \bigoplus_{j=1}^{m} \sum_{t \in I_j} x_t = \sum_{j \in \mathbb{N}} (\sum_{i \in I_j} x_i)$$

Hence the equation $\sum \vec{x} = \sum_{n \in \mathbb{N}} (\sum_{i \in I_n} x_i)$ holds in A.

□

4 MV_ω-algebras and σ-complete MV-algebras

In this section we will show that the class $\mathcal{MV}_\omega$ equationally defines the class of σ-complete MV-algebras. We denote by $\sigma\mathcal{MV}$ the category whose objects are σ-complete MV-algebras and whose arrows (called σMV-homomorphims) are MV-homomorphisms preserving denumerable suprema and consequently, denumerable infima.

Proposition 4.1 *Let A be a σ-complete MV-algebra. If we define*

$$\sum_{i \in \mathbb{N}} x_i = \bigvee_{n \in \mathbb{N}} \bigoplus_{i=1}^{n} x_i$$

for all sequence $\vec{x} = (x_i)_{n \in \mathbb{N}}$ in A then $\langle A, \sum, \neg, 0 \rangle$ is an MV_ω-algebra.

Proof: By the definition of $\sum$, $\langle A, \sum, 0 \rangle$ is an Abelian ω-monoid. This proves $\Sigma 1$). By Proposition 2.4, $x_1 \oplus \sum_{i \geq 2} x_i = x_1 \oplus \bigvee_{n \geq 2} \bigoplus_{i=2}^{n} x_i = \bigvee_{n \geq 2} (x_1 \oplus \bigoplus_{i=2}^{n} x_i) = \bigvee_{n \in \mathbb{N}} \bigoplus_{i=1}^{n} x_i = \sum_{i \in \mathbb{N}} x_i$ and we have proved $\Sigma 2$). It is clear that $x \oplus y = \Sigma(x, y, 0, 0 \ldots)$ and then $\langle A, \sum, \neg, 0 \rangle$ define the MV-structure on A. This proves $\Sigma 3$). It follows from Proposition 2.3-2 that:

$$\begin{aligned} \sum_{i \in \mathbb{N}} d(Sum_i(\vec{x}), Sum_{i-1}(\vec{x})) &= \bigvee_{n \in \mathbb{N}} \bigoplus_{i=1}^{n} d(Sum_i(\vec{x}), Sum_{i-1}(\vec{x})) \\ &= \bigvee_{n \in \mathbb{N}} Sum_n(\vec{x}) = \bigvee_{n \in \mathbb{N}} \bigoplus_{i=1}^{n} x_i = \sum_{i \in \mathbb{N}} x_i \end{aligned}$$

and we have proved $\Sigma 4$). In order to prove Σ 5), consider the sequence $\vec{y} = (k \wedge x_i)_{n \in \mathbb{N}}$. Note that, for each $n \in \mathbb{N}$, $Sup_n(\vec{y}) = \bigvee_{i=1}^{n} (k \wedge x_i) =$

$k \wedge \bigvee_{i=1}^{n} x_i = k \wedge Sup_n(\vec{x})$. Then, by Proposition 2.3-1,

$$\begin{aligned} \sum_{i\in\mathbb{N}} d(Sup_i(\vec{x}) \wedge k, Sup_{i-1}(\vec{x}) \wedge k) &= \sum_{i\in\mathbb{N}} d(Sup_i(\vec{y}), Sup_{i-1}(\vec{y})) \\ &= \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} d(Sup_i(\vec{y}), Sup_{i-1}(\vec{y})) \\ &= \bigvee_{n\in\mathbb{N}} Sup_n(\vec{y}) = \bigvee_{n\in\mathbb{N}} Sup_n(\vec{x}) \wedge k \leq k \end{aligned}$$

Therefore $\langle A, \sum, \neg, 0\rangle$ is an MV_ω-algebra.

□

Proposition 4.2 *Let A be an MV_ω-algebra. Then $\langle A, \oplus, \neg, 0\rangle$ is a σ-complete MV-algebra in which for each sequence $\vec{x} = (x_i)_{i\in\mathbb{N}}$ in A,*

$$\bigvee_{i\in\mathbb{N}} x_i = \sum_{i\in\mathbb{N}} d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x}))$$

Moreover, $\bigodot \vec{x} \leq \bigwedge_{i\in\mathbb{N}} x_i \leq \bigvee_{i\in\mathbb{N}} x_i \leq \sum \vec{x}$.

Proof: Let $y = \sum_{i\in\mathbb{N}} d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x}))$. By Proposition 2.3-1 and Proposition 3.7-2 , for each $n \in \mathbb{N}$,

$$\begin{aligned} x_n &\leq Sup_n(\vec{x}) = \bigoplus_{i=1}^{n} d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x})) \\ &\leq \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x})) \\ &= \sum_{i\in\mathbb{N}} d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x})) = y \end{aligned}$$

Therefore y is an upper bound of the sequence $\vec{x} = (x_i)_{i\in\mathbb{N}}$. Let k be an upper bound of $\vec{x}$. Then for each $i \in \mathbb{N}$, $Sup_i(\vec{x}) \leq k$ and by Σ 5, $y = \sum_{i\in\mathbb{N}} d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x})) = \sum_{i\in\mathbb{N}} d(Sup_i(\vec{x}) \wedge k, Sup_{i-1}(\vec{x}) \wedge k) \leq k$. Hence $y = \bigvee_{i\in\mathbb{N}} x_i$ and $\langle A, \oplus, \neg, 0\rangle$ is a σ-complete MV-algebra.

Since A is σ-complete MV-algebra, $\bigvee_{i\in\mathbb{N}} x_i$ exists in A. Then, by Proposition 3.7-2, $\sum \vec{x} = \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i \geq \bigvee_{n\in\mathbb{N}} \bigvee_{i=1}^{n} x_i = \bigvee_{i\in\mathbb{N}} x_i$. The rest of the inequality follows from duality.

□

Proposition 4.3 *$f : A \to B$ is an MV_ω-homomorphism iff f is a σMV-homomorphism.*

Proof: Suppose that f is an MV_ω-homomorphism. By definition of $\oplus$, f is an MV-homomorphism. Let $\vec{x} = (x_i)_{i\in\mathbb{N}}$ be a sequence in A and we define $f(\vec{x}) = (f(x_i))_{i\in\mathbb{N}}$. By Proposition 4.2,

$$\begin{aligned} f(\bigvee_{i\in\mathbb{N}} x_i) &= f(\sum_{i\in\mathbb{N}} d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x}))) \\ &= \sum_{i\in\mathbb{N}} d(Sup_i(f(\vec{x})), Sup_{i-1}(f(\vec{x}))) = \bigvee_{i\in\mathbb{N}} f(x_i) \end{aligned}$$

Thus f is a σMV-homomorphism. The converse is immediate from Proposition 4.1.

□

By Proposition 4.1, each σ-complete MV-algebra $\langle A, \oplus, \neg, 0\rangle$ becomes an MV_ω-algebra $\langle A, \sum, \neg, 0\rangle$ by defining $\sum_{i\in\mathbb{N}} x_i = \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i$. Conversely, if $\langle A, \sum, \neg, 0\rangle$ is an MV_ω-algebra and we consider the operation $x \oplus y = \sum(x, y, 0, 0, \ldots)$, by Proposition 4.2, $\langle A, \oplus, \neg, 0\rangle$ is a σ-complete MV-algebra in which $\bigvee_{i\in\mathbb{N}} x_i = \sum_{i\in\mathbb{N}} d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x}))$. Consequently, we shall use the two terms (MV_ω-algebra and σ-complete MV-algebra) almost as if they were synonymous, selecting on each occasion the one that seems intuitively more appropriate. Therefore, taking into account Proposition 4.3, we can establish the following result:

Theorem 4.4 *$\sigma\mathcal{MV} = \mathcal{MV}_\omega$ as categories, i.e. they have the same objects and the same arrows, resulting $\sigma\mathcal{MV}$ an equationally definable class of ω-algebras.*

□

5 MV_ω-algebras as monadic MV-algebras

Monadic MV-algebras (monadic Chang algebras by Rutledge's terminology) were introduced by Rutledge in [25] and studied by several authors [2, 7]. They provide an algebraic model for the predicate calculus of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. In this section we study the MV_ω-algebra structure as a particular case of monadic MV-algebras.

A *monadic MV-algebra* is an algebra $\langle A, \oplus, \neg, \forall, 0\rangle$ of type $\langle 2, 1, 1, 0\rangle$ such that $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra and in addition $\forall$ satisfies the following equations:

$\forall 1$. $\forall x \leq x$,

$\forall 2$. $\forall(x \wedge y) = \forall x \wedge \forall y$,

$\forall 3$. $\forall(\neg \forall x) = \neg \forall x$,

$\forall 4$. $\forall(\forall x \odot \forall y) = \forall x \odot \forall y$,

$\forall 5$. $\forall(x \odot x) = \forall x \odot \forall x$,

$\forall 6$. $\forall(x \oplus x) = \forall x \oplus \forall x$.

Let A be an MV_ω-algebra. On A we introduce the unary operation $\Box$ as follows: for each $x \in A$ consider the constant sequence $\vec{x} = (x, x, x \ldots)$ then

$$\Box x = \bigodot \vec{x}$$

The unary operation $\Box$, defined in any MV_ω-algebra, plays a crucial role in the rest of the paper.

Proposition 5.1 *Let A be an MV_ω-algebra and $x \in A$. Then*

1. $\Box x \leq x$.
2. $\Box x \in B(A)$.
3. *If* $z \in B(A)$ *then* $\Box z = z$.
4. $\Box\Box x = \Box x$, $\Box(\neg \Box x) = \neg \Box x$ *and* $\Box(\Box x \odot \Box y) = \Box x \odot \Box y$.
5. $\Box x = \max\{z \in B(A) : z \leq x\}$.
6. $\Box x = \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{kn} x) = \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{k^n} x)$ *for each* $k \in \mathbb{N}$.
7. $\Box(x \wedge y) = \Box x \wedge \Box y$.
8. $\Box(x \vee y) = \Box x \vee \Box y$.
9. $\Box(x \to y) \vee \Box(y \to x) = 1$.
10. $\Box(x \odot x) = \Box x \odot \Box x$ *and* $\Box(x \oplus x) = \Box x \oplus \Box x$.

Proof: 1) Immediate.

2) We prove that $\sum_{i\in\mathbb{N}} x \in B(A)$. By Proposition 3.7-(1 and 3), for each $n \in \mathbb{N}$, $\sum_{i\in\mathbb{N}} x = \bigoplus_{i=1}^{n} x \oplus \sum_{i>n} x = \bigoplus_{i=1}^{n} x \oplus \sum_{i\in\mathbb{N}} x$. Hence, by Proposition 2.4, $\sum_{i\in\mathbb{N}} x = \bigvee_{n\in\mathbb{N}}(\bigoplus_{i=1}^{n} x \oplus \sum_{i>n} x) = (\bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x) \oplus$

$\sum_{i\in\mathbb{N}} x = \sum_{i\in\mathbb{N}} x \oplus \sum_{i\in\mathbb{N}} x$ and $\sum_{i\in\mathbb{N}} x \in B(A)$. Hence by definition of $\bigodot$, $\Box x \in B(A)$.

3) If $z \in B(A)$ then, for each $n \in \mathbb{N}$, $z = \bigodot_{i=1}^{n} z$ and $\Box z = z$.

4) Follows from item 3.

5) Let $z \in Z(A)$ such that $z \leq x$. For each $n \in \mathbb{N}$ $z = \bigodot_{i=1}^{n} z \leq \bigodot_{i=1}^{n} x$. Then $z = \Box z \leq \Box x \leq x$ and hence $\Box x = \max\{z \in B(A) : z \leq x\}$.

6) Let $k \in \mathbb{N}$. Note that $\bigodot_{i=1}^{kn} x = \bigodot_{i=1}^{n}(\bigodot_{i=1}^{k} x)$. Consider the sequence $\vec{k_x} = (\bigodot_{i=1}^{k} x, \bigodot_{i=1}^{k} x, \ldots)$. Then, by Proposition 3.7-6, we have that $\bigodot \vec{k}_x = \bigwedge_{n\in\mathbb{N}} \bigodot_{i=1}^{n}(\bigodot_{i=1}^{k} x) = \bigwedge_{n\in\mathbb{N}} \bigodot_{i=1}^{kn} x$. Note that $\bigodot \vec{k}_x$ is a lower bound of the family $(\bigodot_{i=1}^{n} x)_{n\in\mathbb{N}}$. Let m be another lower bound of the family $(\bigodot_{i=1}^{n} x)_{n\in\mathbb{N}}$. Then, for each $n \in \mathbb{N}$, $m \leq \bigodot_{i=1}^{kn} x$ and $m \leq \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{kn} x) = \bigodot \vec{k}_x$. Thus $\bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{kn} x) = \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{n} x) = \bigodot_{i\in\mathbb{N}} x = \Box x$. Notice that since $\Box x \leq \bigodot_{i=1}^{k^n} x \leq \bigodot_{i=1}^{kn} x$, this implies that $\Box x \leq \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{k^n} x) \leq \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{kn} x) = \Box x$. Hence $\Box x = \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{kn} x) = \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{k^n} x)$ holds in A for each $k \in \mathbb{N}$.

7) Since $x \wedge y \leq x, y$ then $\Box(x \wedge y) \leq \Box x, \Box y$ and $\Box(x \wedge y) \leq \Box x \wedge \Box y$. Now we prove that $\Box x \wedge \Box y \leq \Box(x \wedge y)$. By Lemma 2.1-2, for each $n \in \mathbb{N}$, $(\bigodot_{i=1}^{2n} x) \wedge (\bigodot_{i=1}^{2n} y) \leq (\bigodot_{i=1}^{n} x) \odot (\bigodot_{i=1}^{n} y) = \bigodot_{i=1}^{n}(x \odot y) \leq \bigodot_{i=1}^{n}(x \wedge y)$. Then, by item 6, $\Box x \wedge \Box y = \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{2n} x) \wedge \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{2n} y) \leq \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{n}(x \wedge y) = \Box(x \wedge y)$. Hence $\Box(x \wedge y) = \Box x \wedge \Box y$.

8) Let $x_n = \bigodot_{i=1}^{2^n} x$ and $y_n = \bigodot_{i=1}^{2^n} y$. First we shall prove that $\bigwedge_{n\in\mathbb{N}}(x_n \vee y_n) = (\bigwedge_{n\in\mathbb{N}} x_n) \vee (\bigwedge_{n\in\mathbb{N}} y_n)$. Note that $(\bigwedge_{n\in\mathbb{N}} x_n) \vee (\bigwedge_{n\in\mathbb{N}} y_n)$ is a lower bound of the sequence $(x_n \vee y_n)_{n\in\mathbb{N}}$. Let k be another lower bound of $(x_n \vee y_n)_{n\in\mathbb{N}}$. Since $(x_n)_{n\in\mathbb{N}}$ is a decreasing sequence, for each $n_0 \in \mathbb{N}$, $k \leq x_n \vee y_n \leq x_{n_0} \vee y_n$ whenever $n \geq n_0$. Therefore $k \leq \bigwedge_{n\geq n_0}(x_{n_0} \vee y_n) = x_{n_0} \vee \bigwedge_{n\geq n_0} y_n = x_{n_0} \vee \bigwedge_{n\in\mathbb{N}} y_n$ since $(y_n)_{n\in\mathbb{N}}$ is a decreasing sequence. With the same argument, $k \leq \bigwedge_{n_0\in\mathbb{N}}(x_{n_0} \vee \bigwedge_{n\in\mathbb{N}} y_n) = (\bigwedge_{n\in\mathbb{N}} x_n) \vee (\bigwedge_{n\in\mathbb{N}} y_n)$. Thus $\bigwedge_{n\in\mathbb{N}}(x_n \vee y_n) = (\bigwedge_{n\in\mathbb{N}} x_n) \vee (\bigwedge_{n\in\mathbb{N}} y_n)$ and

$$\bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{2^n} x \vee \bigodot_{i=1}^{2^n} y) = (\bigwedge_{n\in\mathbb{N}} \bigodot_{i=1}^{2^n} x) \vee (\bigwedge_{n\in\mathbb{N}} \bigodot_{i=1}^{2^n} y)$$

It is now easy to see that $\Box(x \vee y) = \Box x \vee \Box y$. Taking into account item 6 and Lemma 2.1-3 we have:

$$\Box(x \vee y) \;=\; \bigwedge_{n\in\mathbb{N}} \bigodot_{i=1}^{2^n}(x \vee y) \leq \bigwedge_{n\in\mathbb{N}}(\bigodot_{i=1}^{2^n} x \vee \bigodot_{i=1}^{2^n} y)$$

$$= \ (\bigwedge_{n\in\mathbb{N}} \bigodot_{i=1}^{2^n} x) \vee (\bigwedge_{n\in\mathbb{N}} \bigodot_{i=1}^{2^n} y) = \Box x \vee \Box y$$

The inequality $\Box x \vee \Box y \leq \Box(x \vee y)$ is immediate.

9) Since $(x \to y) \vee (y \to x) = 1$, it follows by item 8.

10) $x \odot x \leq x$ and then $\Box(x \odot x) \leq \Box x$. Since $\Box x \in B(A)$, $\Box(x \odot x) = \Box x \wedge \Box(x \odot x) = \Box x \odot \Box(x \odot x) \leq \Box x \odot \Box x$. For the converse, $\Box x \leq x$ and then $\Box x \odot \Box x \leq x \odot x$. Thus, by item 4, $\Box x \odot \Box x = \Box(\Box x \odot \Box x) \leq \Box(x \odot x)$. Similarly we prove that $\Box(x \oplus x) = \Box x \oplus \Box x$.

$\Box$

An immediate consequence of Proposition 5.1 is the following:

Theorem 5.2 *Let A be an MV_ω algebra. Then $\langle A, \oplus, \neg, \Box, 0\rangle$ is a monadic MV-algebra.* $\Box$

Remark 5.3 The monadic structure associated to an MV_ω-algebra is a particular case of a more general structure called *MV-algebra with storage* [22] i.e., an MV-algebra equipped with a unitary operation I satisfying, $I(1) = 1$, $I(x) = x \odot I(x)$ and $x \odot I(x \to (x \odot x \odot y)) \leq I(y)$. We can prove that $I(x)$ is the greatest Boolean element $\leq x$. Thus, if A is an MV_ω-algebra, by Proposition 5.1-5, $\langle A, \oplus, \neg, \Box, 0\rangle$ is an MV-algebra with storage.

6 Sub MV_ω-algebras

Let A and B be two σ-complete MV-algebras. If A is a sub-MV-algebra of B, the supremum (infimum) in A of a sequence $(x_i)_{i\in\mathbb{N}}$ of A, will be denoted by $\bigvee^A_{i\in\mathbb{N}} x_i$ ($\bigwedge^A_{i\in\mathbb{N}} x_i$) to distinguish it from the supremum $\bigvee^B_{i\in\mathbb{N}} x_i$ (infimum $\bigwedge^B_{i\in\mathbb{N}} x_i$) in B, which need not belong to A.

Proposition 6.1 *Let A and B be two MV_ω-algebras. The following conditions are equivalent:*

1. *A is a sub-MV_ω-algebra of B.*
2. *A is a sub MV-algebra of $\langle B, \oplus, \neg, 0\rangle$ in which $\bigvee^A_{i\in\mathbb{N}} x_i = \bigvee^B_{i\in\mathbb{N}} x_i$ for each sequence $(x_i)_{i\in\mathbb{N}}$ in A.*

Proof: 1 $\Longrightarrow$ 2) Suppose that A is a sub-MV_ω-algebra of B. Since $\sum$ is closed in A, $\oplus$ and Sup_i are closed operations in A. Thus A is a sub-MV-algebra of B and $\bigvee^A_{i\in\mathbb{N}} x_i = \sum_{i\in\mathbb{N}} d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x}))) = \bigvee^B_{i\in\mathbb{N}} x_i$.

$2 \Longrightarrow 1$) We prove that $\sum$ is closed in A. Let $\vec{x} = (x_i)_{i\in\mathbb{N}}$ be a sequence in A. By Proposition 4.1, $\sum\vec{x} = \bigvee^B_{n\in\mathbb{N}} \bigoplus_{i=1}^n x_i = \bigvee^A_{n\in\mathbb{N}} \bigoplus_{i=1}^n x_i \in A$. Hence $\sum$ is closed in A and A is a sub MV_ω-algebra of B.

□

Proposition 6.2 *Let A be an MV_ω-algebra. Then $B(A)$ equipped with the MV_ω-operations of A is a sub-MV_ω-algebra of A which is a σ-complete Boolean algebra.*

Proof: By [5, Corollary 6.6.5], $B(A)$ is a σ-complete Boolean algebra and the countable operations of $B(A)$ agree with the restriction of the corresponding operations of A. Hence, by Proposition 6.1, $B(A)$ is a sub MV_ω-algebra of A.

□

7 MV_ω-congruences and MV_ω-filters

The aim of this section is to construct a theory of filters and congruences in MV_ω-algebras. Let A be an MV_ω-algebra. An MV_ω-congruence on A is an ω-congruence on A i.e., an equivalence relation $\theta \subseteq A^2$ compatible respect to the signature $\langle\neg, 0\rangle$, satisfying the following condition:

$$\text{if for each } i \in \mathbb{N},\ (x_i, y_i) \in \theta \text{ then } \left(\textstyle\sum_{i\in\mathbb{N}} x_i, \sum_{i\in\mathbb{N}} y_i\right) \in \theta.$$

We shall denote by $Con_{MV_\omega}(A)$ the set of all MV_ω-congruences and by $Con_{MV}(A)$ the set of all MV-congruences of $\langle A, \oplus, \neg, 0\rangle$. It is clear that $Con_{MV_\omega}(A) \subseteq Con_{MV}(A)$.

Proposition 7.1 *Let A be an MV_ω-algebra and $\theta \subseteq A^2$. Then the following assertions are equivalent:*

1. *$\theta \in Con_{MV_\omega}(A)$,*
2. *$\theta \in Con_{MV}(A)$ and the following condition is satisfied: if $(x_i, y_i) \in \theta$ for each $i \in \mathbb{N}$ then, $(\bigvee_{i\in\mathbb{N}} x_i, \bigvee_{i\in\mathbb{N}} y_i) \in \theta$.*

Proof: Assume that $\theta \in Con_{MV_\omega}(A)$. By definition of $\oplus$, θ is an MV-congruence of $\langle A, \oplus, \neg, 0\rangle$. Suppose that for each $i \in \mathbb{N}$, $(x_i, y_i) \in \theta$. Then, for each $i \in \mathbb{N}$, $(d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x})), d(Sup_i(\vec{y}), Sup_{i-1}(\vec{y}))) \in \theta$ where $\vec{x} = (x_i)_{i\in\mathbb{N}}$ and $\vec{y} = (y_i)_{i\in\mathbb{N}}$. By hypothesis,

$$(\sum_{i\in\mathbb{N}} d(Sup_i(\vec{x}), Sup_{i-1}(\vec{x})), \sum_{i\in\mathbb{N}} d(Sup_i(\vec{y}), Sup_{i-1}(\vec{y}))) \in \theta$$

hence by Proposition 4.2, $(\bigvee_{i\in\mathbb{N}} x_i, \bigvee_{i\in\mathbb{N}} y_i) \in \theta$.

For the converse suppose that θ is a MV-congruence on $\langle A, \oplus, \neg, 0\rangle$ that satisfies: if for each $i \in \mathbb{N}$, $(x_i, y_i) \in \theta$ then $(\bigvee_{i\in\mathbb{N}} x_i, \bigvee_{i\in\mathbb{N}} y_i) \in \theta$. Notice that for each $n \in \mathbb{N}$, $(\bigoplus_{i=1}^n x_i, \bigoplus_{i=1}^n y_i) \in \theta$. By hypothesis

$$(\bigvee_{i\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i, \bigvee_{i\in\mathbb{N}} \bigoplus_{i=1}^{n} y_i) \in \theta$$

Hence, by Proposition, 4.1 $(\sum_{i\in\mathbb{N}} x_i, \sum_{i\in\mathbb{N}} y_i) \in \theta$ and $\theta \in Con_{MV_\omega}(A)$.

□

Let A be an MV_ω-algebra and $\theta \in Con_{MV_\omega}(A)$. Then, by Theorem 1.1, the quotient algebra $A/_\theta$ is an MV_ω-algebra and the natural application $p_\theta : A \to A/_\theta$ is an MV_ω-homomorphism. Consequently, for each sequence $(x_i)_{i\in\mathbb{N}}$ in A,

$$\bigvee_{i\in\mathbb{N}} (x_i/_\theta) = (\bigvee_{i\in\mathbb{N}} x_i)/_\theta. \tag{3}$$

In [1] an equivalent result was obtained without an equational theory for σ-complete MV-algebras.

Definition 7.2 Let A be an MV_ω-algebra. A non-empty subset $F \subseteq A$ is an *MV_ω-filter* iff F is an increasing set and, if $(x_i)_{i\in\mathbb{N}}$ is a sequence in F, then $\bigodot_{i\in\mathbb{N}} x_i \in F$.

We shall denote by $Filt_{MV_\omega}(A)$ the set of all MV_ω-filters in A and by $Filt_{MV}(A)$ the set of all MV-filters of $\langle A, \oplus, \neg, 0\rangle$. Clearly $Filt_{MV_\omega}(A) \subseteq Filt_{MV}(A)$.

Proposition 7.3 *Let A be an MV_ω-algebra and F be a non-empty subset of A. Then the following assertions are equivalent:*

1. *$F \in Filt_{MV_\omega}(A)$.*
2. *$F \in Filt_{MV}(A)$ and it is closed by denumerable infima.*

Proof: Suppose that $F \in Filt_{MV_\omega}(A)$. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in F. By Proposition 3.8-1 $\bigodot_{i\in\mathbb{N}} x_i \leq \bigwedge_{i\in\mathbb{N}} x_i$. Since $\bigodot_{i\in\mathbb{N}} x_i \in F$ and F is an increasing set, $\bigwedge_{i\in\mathbb{N}} x_i \in F$. Hence F is closed by denumerable infima.

For the converse, let $(x_i)_{i\in\mathbb{N}}$ be a sequence in F. Since $F \in Filt_{MV}(A)$, for each $i \in \mathbb{N}$, $\bigodot_{i=1}^{n} x_i \in F$. Since F is closed by denumerable infima, by Proposition 3.7-6, $\bigodot_{i\in\mathbb{N}} x_i = \bigwedge_{i\in\mathbb{N}} \bigodot_{i=1}^{n} x_i \in F$. Thus $F \in Filt_{MV_\omega}(A)$.

□

Let A be an MV_ω-algebra. Observe that, the intersection of any family of MV_ω-filters of A is a filter of A. Thus $\langle Filt_{MV_\omega}(A), \subseteq\rangle$ is a complete lattice. We denote by $\langle X\rangle_{MV_\omega}$ the MV_ω-filter generated by $X \subseteq A$, i.e., the intersection of all MV_ω-filters of A containing X. We abbreviate $\langle a\rangle_{MV_\omega}$ when $X = \{a\}$ and we say that $\langle a\rangle_{MV_\omega}$ is the *principal MV_ω-filter* associated to a.

Proposition 7.4 *Let A be an MV_ω-algebra and X be a non-empty subset of A. Then*

$$\langle X\rangle_{MV_\omega} = \{a \in A : \exists (x_i)_{i\in\mathbb{N}} \subseteq X \;\; s.t. \; \bigodot_{i\in\mathbb{N}} x_i \leq a\}$$

In particular, for each $x \in A$, $\langle x\rangle_{MV_\omega} = \langle \Box x\rangle_{MV_\omega} = [\Box x)$.

Proof: We first prove that the set

$$F_X = \{a \in A : \exists (x_i)_{i\in\mathbb{N}} \subseteq X \; s.t. \; \bigodot_{i\in\mathbb{N}} x_i \leq a\}$$

is an MV_ω-filter. It is obvious that F_X is an increasing set. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence in F_X. Then, for each a_i in the sequence, there exists a sequence $(x_{j_i})_{j_i\in\mathbb{N}_i}$ in X such that $\bigodot_{j_i\in\mathbb{N}_i} x_{j_i} \leq a_i$ where $\mathbb{N}_i = \mathbb{N} \times \{i\}$. Since $\bigcup_{i\in\mathbb{N}} \mathbb{N}_i$ is a denumerable set, we can consider $\bigcup_{i\in\mathbb{N}} \mathbb{N}_i$ endowed with an order isomorphic to $\mathbb{N}$ given by a bijective function $\gamma : \bigcup_{i\in\mathbb{N}} \mathbb{N}_i \to \mathbb{N}$. Thus we can assume that $\mathbb{N} = \bigcup_{i\in\mathbb{N}} \mathbb{N}_i$ in which $(\mathbb{N}_i)_{i\in\mathbb{N}}$ is a denumerable partition of $\mathbb{N}$. Moreover we consider the sequence $\vec{x} = (x_k)_{k\in\mathbb{N}}$ such that $x_k = x_{j_i}$ iff $\gamma(j, i) = k$. By Proposition 3.9 $\bigodot \vec{x} = \bigodot_{i\in\mathbb{N}}(\bigodot_{j_i\in\mathbb{N}_i} x_{j_i}) \leq \bigodot_{i\in\mathbb{N}} a_i$. Since $\vec{x}$ is a sequence in X, $\bigodot_{i\in\mathbb{N}} a_i \in F_X$. Hence F_X is a MV_ω-filter.

Since $X \subseteq F_X$ then $\langle X\rangle_{MV_\omega} \subseteq F_X$. Conversely, let $a \in F_X$. Then there exists a sequence $(x_i)_{i\in\mathbb{N}}$ in X such that $x = \bigodot_{i\in\mathbb{N}} x_i \leq a$. If F is an MV_ω-filter containing X then, $x \in F$ and $a \in F$ since F is an increasing set. Thus $a \in \langle X\rangle_{MV_\omega}$ and $F_X \subseteq \langle X\rangle_{MV_\omega}$. Consequently $F_X = \langle X\rangle_{MV_\omega}$.

□

Proposition 7.5 *Let A be an MV_ω-algebra, $F \in Filt_{MV_\omega}(A)$ and B be a sub MV_ω-algebra of A. Then:*

1. *F is a Stonean MV-filter.*
2. *$F \cap B \in Filt_{MV_\omega}(B)$.*
3. *If $G \in Filt_{MV_\omega(B)}$ and G_A is the MV_ω-filter of A generated by G then $G = G_A \cap B$.*
4. *$F \cap B(A) = \{\Box x : x \in F\}$.*
5. *$F = \langle F \cap B(A)\rangle_{MV_\omega}$.*

Proof: 1) Since F is closed by $\bigodot$, for each $x \in F$, $x \geq \Box x \in F$. Hence F is a Stonean MV-filter. 2) Straightforward. 3) Clearly $G \subseteq G_A \cap B$. To see the converse, let $a \in G_A \cap B$. Then, by Proposition 7.4, there exists a sequence $(x_i)_{i\in\mathbb{N}} \subseteq G$ such that $\bigodot_{i\in\mathbb{N}} x_i \leq a$. Since G is an MV_ω-filter and $a \in B$, it follows that $a \in G$. 4) If $x \in F \cap B(A)$ then $x = \Box x$ and $x \in \{\Box x : x \in F\}$. Thus $F \cap B(A) \subseteq \{\Box x : x \in F\}$. The other inclusion is trivial. 5) We prove that $F \subseteq \langle F \cap B(A)\rangle_{MV_\omega}$. By item 4, if $x \in F$ then $\Box x \in F \cap B(A)$ and $\Box x \leq x$. Hence $x \in \langle F \cap B(A)\rangle_{MV_\omega}$ and $F \subseteq \langle F \cap B(A)\rangle_{MV_\omega}$. The other inclusion is trivial.

□

Let A be an MV_ω-algebra. Given $\theta \in Con_{MV_\omega}(A)$ we define:

$$F_\theta = \{x \in A : (x, 1) \in \theta\}$$

Conversely, given $F \in Filt_{MV_\omega}(A)$ we define:

$$\theta_F = \{(x, y) \in A^2 : \exists a \in F : x \odot a \leq y \textit{ and } y \odot a \leq x\}$$

Theorem 7.6 *Let A be an MV_ω-algebra. The maps $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$ are mutually inverse lattice-isomorphisms between $Con_{MV_\omega}(A)$ and $Filt_{MV_\omega}(A)$.*

Proof: We first prove that if $F \in Filt_{MV_\omega(A)}$ then $\theta_F \in Con_{MV_\omega}(A)$. First, observe that θ_F is an MV-congruence of $\langle A, \oplus, \neg, 0\rangle$. By Proposition 7.1, we have to prove that if $(x_i, y_i)_{i\in\mathbb{N}}$ is a sequence in θ_F then $(\bigwedge_{i\in\mathbb{N}} x_i, \bigwedge_{i\in\mathbb{N}} y_i) \in \theta_F$. For each $i \in \mathbb{N}$, there exists $z_i \in F$ such that $x_i \odot z_i \leq y_i$ and $y_i \odot z_i \leq x_i$. By Proposition 7.3, $z = \bigwedge_{i\in\mathbb{N}} z_i \in F$. Since $x_i \odot z \leq y_i$ we have that $x_i \leq z \to y_i$ and then $\bigwedge_{i\in\mathbb{N}} x_i \leq z \to y_i$. Thus for each $i \in \mathbb{N}$, $z \odot \bigwedge_{i\in\mathbb{N}} x_i \leq y_i$

and then $z \odot \bigwedge_{i\in\mathbb{N}} x_i \leq \bigwedge_{i\in\mathbb{N}} y_i$. By the same argument we can prove that $z \odot \bigwedge_{i\in\mathbb{N}} y_i \leq \bigwedge_{i\in\mathbb{N}} x_i$. Hence $(\bigwedge_{i\in\mathbb{N}} x_i, \bigwedge_{i\in\mathbb{N}} y_i) \in \theta_F$ and $\theta_F \in Con_{MV_\omega}(A)$.

To complete the proof, suppose now that $\theta \in_{MV_\omega} (A)$. Since F_θ is an MV-filter of $\langle A, \oplus, \neg, 0\rangle$, by Proposition 7.3, we have to prove that F_θ is closed by denumerable infima. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in F_θ i.e., $(x_i, 1) \in \theta$ for each $i \in \mathbb{N}$. By Proposition 7.1 we can prove that $(\bigwedge_{i\in\mathbb{N}} x_i, 1) \in \theta$. Hence $\bigwedge_{i\in\mathbb{N}} x_i \in F_\theta$ and $F_\theta \in Filt_{MV_\omega(A)}$.

Since $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$ are mutually inverse lattice-isomorphisms between $Con_{MV}(A)$ and $Filt_{MV}(A)$, by the precedent argument, $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$ are mutually inverse lattice-isomorphisms between $Con_{MV_\omega(A)}$ and $Filt_{MV_\omega(A)}$.

□

The latter theorem together with Proposition 7.5-3, allows to establish the following result:

Proposition 7.7 *If A is an MV_ω-algebra then A satisfies CEP.*

□

Remark 7.8 In the literature, MV-filters closed by denumerable infima have also been termed σ-filters [1, 8]. They are a natural generalization of σ-filters for σ-complete Boolean algebras [14, 18, 26]. Thus, according to the Proposition 7.3 and Theorem 7.6, σ-filters determine the congruences theory for MV_ω-algebras.

8 Direct products and simple MV_ω-algebras

The aim of this section is to describe directly irreducible and simple algebras in $\mathcal{MV}_\omega$. Our results depend of the fact that Boolean elements of MV-algebras determine a direct decomposition of the algebra. We begin by briefly recalling some basic notions about direct products decompositions of MV-algebras.

Let A be an MV-algebra, $z \in B(A)$ and consider the segment

$$[0, z] = \{x \in A : 0 \leq x \leq z\}$$

Note that $\oplus$ is a closed operation in $[0, z]$. If we define the unary operation $\neg_z x$ in $[0, z]$ by the formula $\neg_z x = z \wedge \neg x$ then $[0, z]_{MV} = \langle [0, z], \oplus, \neg_z, 0, z\rangle$ is an MV-algebra. The map $B(A) \ni z \mapsto \theta_z = \{(a, b) \in A^2 : a \wedge z = b \wedge z\}$ is a Boolean isomorphism between $B(A)$ and the Boolean sublattice of

$Con_{MV}(A)$ of factor congruences. The correspondence $x/_{\theta_z} \mapsto x \wedge z$ defines an MV-isomorphism from A/θ_z onto $[0,z]_{MV}$ and $x \mapsto (x \wedge z, x \wedge \neg z)$ defines an MV-isomorphism from A onto $[0,z]_{MV} \times [0,\neg z]_{MV}$. Conversely, if $f : A \to A_1 \times A_2$ is a MV-isomorphism, the element $z \in A$ such that $f(z) = (1,0)$ is the unique element in $B(A)$ such that A_1 is MV-isomorphic to $[0,z]_{MV}$ and A_2 is MV-isomorphic to $[0,\neg z]_{MV}$.

In what follows we shall establish analogous results for MV_ω-algebras.

Proposition 8.1 *Let A be an MV_ω-algebra and $z \in B(A)$. Then:*

1. *The operation $\sum$ of A is closed in $[0,z]$ and the structure $[0,z]_\omega = \langle [0,z], \sum, \neg_z, 0\rangle$ is an MV_ω-algebra.*
2. *$\theta_z = \{(a,b) \in A^2 : a \wedge z = b \wedge z\} \in Con_\omega(A)$ and the correspondence $x/\theta_z \mapsto x \wedge z$ defines an MV_ω-isomorphism from $A/_{\theta_z}$ onto $[0,z]_\omega$.*
3. *$x \mapsto (x \wedge z, x \wedge \neg z)$ defines an MV_ω-isomorphism from A onto the direct product $[0,z]_\omega \times [0,\neg z]_\omega$.*

Proof: 1) Taking into account that A is a σ-complete MV-algebra, $[0,z]$ is closed by denumerable suprema and infima. Hence $[0,z]_{MV}$ is a σ-complete MV-algebra. If we define $\sum^{[0,z]} \vec{x} = \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} x_i$ for each sequence $\vec{x} = (x_i)_{i\in\mathbb{N}}$ in $[0,z]$ then, by Proposition 4.1, $[0,z]_\omega = \langle [0,z], \sum^{[0,z]}, \neg_z, 0\rangle$ is an MV_ω-algebra in which $\sum$ coincides with $\sum^{[0,z]}$ in $[0,z]$.

2) θ_z is an MV-congruence. Then, by Proposition 7.1, we need to prove that: if $(x_i, y_i) \in \theta_z$ for each $i \in \mathbb{N}$ then $(\bigvee_{i\in\mathbb{N}} x_i, \bigvee_{i\in\mathbb{N}} y_i) \in \theta_z$. Since $x_i \wedge z = y_i \wedge z$ then, $(\bigvee_{i\in\mathbb{N}} x_i) \wedge z = \bigvee_{i\in\mathbb{N}} (x_i \wedge z) = \bigvee_{i\in\mathbb{N}} (y_i \wedge z) = (\bigvee_{i\in\mathbb{N}} y_i) \wedge z$. Hence $(\bigvee_{i\in\mathbb{N}} a_i, \bigvee_{i\in\mathbb{N}} b_i) \in \theta_z$ and $\theta_z \in Con_\omega(A)$. Taking into account that $x/\theta_z \mapsto x \wedge z$ defines an MV-isomorphism from A/θ_z onto $[0,z]_{MV}$, it preserves denumerable suprema. Hence, by Proposition 4.3, it is an MV_ω-isomorphism.

3) Follows from the precedent items.

$\square$

Proposition 8.2 *Let A be an MV_ω-algebra. Then the following assertions are equivalent:*

1. *A is simple in $\mathcal{MV}_\omega$.*
2. *For each $x < 1$, $\square x = 0$.*

3. $B(A) = \{0, 1\}$.

4. *A is directly irreducible* $\mathcal{MV}_\omega$.

Proof: $1 \Rightarrow 2$) Let $x < 1$. By Proposition 7.4, $\langle x \rangle_\omega = [\Box x)$. Since A is simple in $\mathcal{MV}_\omega$, by Proposition 7.6, $\langle x \rangle_\omega = A$ and then $\Box x = 0$.

$2 \Rightarrow 3$) Let $z \in B(A)$. If $z < 1$ then $z = \Box z = 0$. Hence $B(A) = \{0, 1\}$.

$3 \Leftrightarrow 4$) Follows from Porposition 8.1.

$4 \Rightarrow 1$). Let F be a MV_ω-filter in A and $x \in F$ such that $x < 1$. By definition of MV_ω-filter, $\Box x \in F$ and $\Box x \in B(A) = \{0, 1\}$. Thus $F = A$ and A is a simple MV_ω-algebra.

□

Theorem 8.3 *Let A be an MV_ω-algebra. Then, A is simple in $\mathcal{MV}_\omega$ iff A is MV-isomorphic to $[0, 1]_{MV}$ or A is MV-isomorphic to $Ł_n$ for some $n \geq 2$.*

Proof: Suppose that A is a simple MV_ω-algebra. By Proposition 8.2, $B(A) = \{0, 1\}$. Let $x \in A$ such that $0 < x < 1$. We shall prove that x is nilpotent. Since $\langle A, \oplus, \neg, 0 \rangle$ is a σ-complete MV-algebra, it is semisimple and x is not a unity. Thus, there exists $n_0 \in \mathbb{N}$ such that $\neg x \to \bigodot_{i=1}^{n_0} x \neq 1$. By Proposition 5.1-9,

$$\begin{aligned} 1 &= \Box(\neg x \to \bigodot_{i=1}^{n_0} x) \vee \Box(\bigodot_{i=1}^{n_0} x \to \neg x) \\ &= \Box(\neg x \to \bigodot_{i=1}^{n_0} x) \vee \Box(\neg \bigodot_{i=1}^{n_0+1} x) \end{aligned}$$

Since $\Box(\neg x \to \bigodot_{i=1}^{n_0} x) \in B(A) = \{0, 1\}$ and taking into account that $\Box(\neg x \to \bigodot_{i=1}^{n_0} x) \leq \neg x \to \bigodot_{i=1}^{n_0} x < 1$, $\Box(\neg x \to \bigodot_{i=1}^{n_0} x) = 0$. This implies that $1 = \Box(\neg \bigodot_{i=1}^{n_0+1} x) \leq \neg \bigodot_{i=1}^{n_0+1} x$ and x is nilpotent in A. Thus, $\langle A, \oplus, \neg, 0 \rangle$ is a σ-complete simple MV-algebra. Hence A is MV-isomorphic to $[0, 1]_{MV}$ or A is MV-isomorphic to $Ł_n$ for some $n \geq 2$.

□

Corollary 8.4 *Simple algebras in $\mathcal{MV}_\omega$ are rigid algebras.*

Proof: Since $[0, 1]_{MV}$ and all the sub-algebras are rigid algebras, the proof follows from the fact that MV_ω-homomorphisms are MV-homomorphisms.

□

Corollary 8.5 *The only totally ordered MV_ω-algebras, up to isomorphisms, are $[0,1]_{MV}$ or $Ł_n$ for each $n \geq 2$.*

Proof: If A is a totaly ordered MV_ω-algebra then $B(A) = \{0,1\}$. Hence, by Proposition 8.2 and Theorem 8.3, A is MV-isomorphic to $[0,1]_{MV}$ or A is MV-isomorphic to $Ł_n$ for some $n \geq 2$.

$\square$

Proposition 8.6 *Let A be an MV_ω-algebra and $F \subseteq A$ be a non-empty set. Then the following conditions are equivalent:*

1. *F is a prime MV_ω-filter,*
2. *F is maximal MV-filter closed by denumerable infima,*
3. *For each $x \in A$, $x \notin F$ iff $\neg\Box x \in F$.*
4. *F is maximal in $Filt_{MV_\omega}$.*
5. *F is a MV_ω-filter and $A/_F$ is a simple MV_ω-algebra.*

Proof: $1 \Longrightarrow 2$) See [1, Proposition 6].

$2 \Longrightarrow 3$) F is a maximal MV-filter closed by denumerable infima. If $x \notin F$ then $\langle \{x\} \cup F \rangle_{MV_\omega} = A$. Therefore there exists a sequence $(x_i)_{i\in\mathbb{N}}$ in F such that $\Box x \odot \bigodot_{i\in\mathbb{N}} x_i = 0$ and $\bigodot_{i\in\mathbb{N}}(x_i) \leq \neg\Box x$. By Proposition 7.3, $\bigodot_{i\in\mathbb{N}} x_i \in F$. Hence $\neg\Box x \in F$. Conversely, if $x \in F$ then $\Box x \in F$. Since $\Box x \odot \neg\Box x = 0$ and F is proper, $\neg\Box x \notin F$.

$3 \Longrightarrow 4$) Let $K \neq F$ be a MV_ω-filter of A such that $F \subseteq K$. Suppose that $x \in K$ and $x \notin F$. By hypothesis we must have $\neg\Box x \in F$. Hence $0 = \Box x \odot \neg\Box x \in K$ and $K = A$.

$4 \Longrightarrow 5$) Let us assume that F is maximal in $Filt_{MV_\omega}$ and let us consider an element $x_F \in A/_F$ such that $x_F \neq 1_F$. Then $x \notin F$ and $\langle \{x\} \cup F \rangle_{MV_\omega} = A$. Therefore, there exists a sequence $(x_i)_{i\in\mathbb{N}}$ in F such that $\Box x \odot \bigodot_{i\in\mathbb{N}} x_i = 0$ and $\bigodot_{i\in\mathbb{N}}(x_i) \leq \neg\Box x$. Since $\bigodot_{i\in\mathbb{N}}(x_i) \in F$, $\neg\Box x \in F$ and then $\neg\Box x_F = 1_F$. Hence $\Box x_F = 0_F$ and, by Proposition 8.2, $A/_F$ is a simple MV_ω-algebra.

$5 \Longrightarrow 1$) If $A/_F$ is a simple MV_ω-algebra, by Proposition 8.3, $A/_F$ is a totally ordered set. Hence F is a prime MV-filter.

$\square$

Proposition 8.7 *Let A be an MV_ω-algebra, B be a sub-MV_ω-algebra of A and F be a MV_ω-filter. Then:*

1. *If F is a maximal MV_ω-filter then $F \cap B$ is a maximal MV_ω-filter of B.*

2. *F is a maximal MV_ω-filter of A iff $F \cap B(A)$ is a maximal Boolean filter of $B(A)$ closed by denumerable infima.*

Proof: 1) By Proposition 7.5-4, $F \cap B$ is an MV_ω-filter of B. By Proposition 8.6, if $x, y \in B$ then $x \to y \in F \cap B$ or $y \to x \in F \cap B$. Thus $F \cap B$ is a prime MV_ω-filter of B and then it is a maximal MV_ω-filter of B.

2) By item 2 we only need to prove that if $F \cap B(A)$ is a maximal Boolean filter of $B(A)$, closed by denumerable infima, then F is a maximal MV_ω-filter of A. Let us consider two elements $x, y \in A$. By Proposition 5.1-9 we have that $\Box(x \to y) \vee \Box(y \to x) = 1 \in F \cap B(A)$. Since $F \cap B(A)$ is a maximal Boolean filter of $B(A)$, $\Box(x \to y) \in F \cap B(A)$ or $\Box(y \to x) \in F \cap B(A)$. Then, $\Box(x \to y) \leq x \to y \in F$ or $\Box(y \to x) \leq y \to x \in F$ i.e., F is a prime filter. Hence, by Proposition 8.6, F is a maximal MV_ω-filter of A.

$\Box$

9 Standard completeness for MV_ω-algebras

We have seen that the structure of the MV_ω-algebra is a good abstraction for Łukasiewicz tribes. However, the class of all Łukasiewicz tribes is not large enough to represent every MV_ω-algebra. Despite this, they play a crucial role in the study of MV_ω-equations. In fact, using the Loomis-Sikorski theorem for MV-algebras, we will establish an standard completeness theorem for MV_ω-equations respect to $[0,1]_{MV_\omega}$.

The famous Loomis-Sikorski theorem for σ-complete Boolean algebras was generalized independently by Mundici [23] and Dvurečenskij [10] to σ-complete MV-algebras in the following way:

Theorem 9.1 *Let A be a σ-complete MV-algebra A. Then there exist a Łukasiewicz tribe T and a surjective σ-homomorphism $f : T \to A$.*

$\Box$

The latter theorem together with Theorem 4.4 allows to establish the following standard completeness result for MV_ω-algebras:

Theorem 9.2 (Standard completeness) *Let $p(\vec{x}) = q(\vec{x})$ be an equation of type $\langle \Sigma, \neg, 0 \rangle$. Then:*

$$\mathcal{MV}_\omega \models p(\vec{x}) = q(\vec{x}) \quad iff \quad [0,1]_{MV_\omega} \models p(\vec{x}) = q(\vec{x})$$

Proof: As regard to the non-trivial direction assume that $[0,1]_{MV_\omega} \models p(\vec{x}) = q(\vec{x})$. Since each Łukasiewicz tribe T is a σ-complete MV-algebra that can be embedded into a direct product $\prod_X [0,1]_{MV_\omega}$ preserving denumerable suprema, by Proposition 6.1, $T \models p(\vec{x}) = q(\vec{x})$. Therefore, for each Łukasiewicz tribe T, we have that $T \models p(\vec{x}) = q(\vec{x})$. Let A be an MV_ω-algebra and $\vec{a}$ be a sequence in A. By Theorem 9.1, there exists a Łukasiewicz tribe T and a surjective σ-homomorphism $f : T \to A$. By Proposition 4.3, f is an MV_ω-homomorphism. Since f is surjective, there exists a sequence $\vec{m}$ in T such that $f(\vec{m}) = \vec{a}$. Since $p^T(\vec{m}) = q^T(\vec{m})$ then $p^A(\vec{a}) = f(p^T(\vec{m})) = f(q^T(\vec{m})) = q^A(\vec{a})$. Hence $A \models p(\vec{x}) = q(\vec{x})$ and the equation holds in $\mathcal{MV}_\omega$.

□

10 σ-complete Boolean Algebras

In this section we shall study the class of σ-complete Boolean algebras as an equationally definable subclass of $\mathcal{MV}_\omega$.

As shown by Chang [3], Boolean algebras coincide with MV-algebras satisfying the equation $x \oplus x = x$. In this case the operation $\oplus$ coincides with $\vee$ and the operation $\odot$ coincides with $\wedge$. Let $\sigma\mathcal{B}$ be the category whose objects are σ-complete Boolean algebras and whose arrows (called $\sigma\mathcal{B}$-homomorphims) are Boolean homomorphisms preserving denumerable suprema and consequently, denumerable infima. Then, by Theorem 4.4,

$$\sigma\mathcal{B} = \mathcal{MV}_\omega + \{x \oplus x = x\}$$

Hence $\sigma\mathcal{B}$ is equationally definable as a class of ω-algebras.

In what follows we reformulate the equational base for $\sigma\mathcal{B}$ in the language of Boolean algebras, Let A be a Boolean algebra viewed as an MV-algebra and $\vec{x} = (x_i)_{i\in\mathbb{N}}$ be a sequence in A. If A is σ-complete, by Proposition 4.1, we have:

$$\sum \vec{x} = \bigvee_{i\in\mathbb{N}} x_i = \bigvee \vec{x}$$

Thus $\sum$ and $\bigvee_{i\in\mathbb{N}}$ coincide as ω-ary operations on A and the operation $\vee$ becomes a definable operation in the following way: $x \vee y = \bigvee(x, y, 0, 0 \ldots)$. Note that $Sup_i(\vec{x}) = Sum_i(\vec{x})$ for each $i \geq 0$ and the distance function $d(x,y)$ is the symmetric difference $x \triangle y = (x \wedge \neg y) \vee (y \wedge \neg x)$. In this way one obtains the following equivalent equational base for σ-complete Boolean algebras:

Definition 10.1 A σ-complete Boolean algebra is an ω-algebra $\langle A, \bigvee, \vee, \wedge, \neg, 0, 1\rangle$ of type $\langle \omega, 2, 2, 1, 0, 0\rangle$ such that satisfies for each $\vec{x} = (x_i)_{i\in\mathbb{N}}$:

1. $\langle A, \bigvee, 0\rangle$ is an ω-monoid,
2. $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra,
3. $x \vee y = \bigvee(x, y, 0, 0, \ldots)$,
4. $\bigvee \vec{x} = \bigvee_{i\in\mathbb{N}}(Sup_i(\vec{x}) \triangle Sup_{i-1}(\vec{x})) = x_1 \vee \bigvee_{i\geq 2} x_i$,
5. $(\bigvee_{i\in\mathbb{N}}(Sup_i(\vec{x}) \wedge y) \triangle (Sup_{i-1}(\vec{x}) \wedge y)) \to y = 1$.

Let A be a σ-complete Boolean algebra. By Theorem 7.6, ω-congruences in A ($\sigma\mathcal{B}$-congruences) are identified with Boolean filters in A closed by denumerable infima ($\sigma\mathcal{B}$-filters). Note that $\mathbf{2} = \{0, 1\}$ is the unique simple and directly irreducible algebra in $\sigma\mathcal{B}$. Observe that, the unary operation $\Box$ is the *discrete quantifier* in the sense of Halmos [13].

The concept of tribe is a direct generalization of a σ-field of sets. By a *σ-field of sets* over a non-empty set X we mean a σ-complete Boolean algebra of $\mathbf{2}$-valued functions over X, where countable suprema are given by pointwise countable suprema. Using the Loomis-Sikorski theorem for σ-complete Boolean algebras and *sigma*-field of sets, we can also establish an standard completeness theorem for $\sigma\mathcal{B}$-equations respect to $\mathbf{2}$. The famous Loomis-Sikorski Theorem, proved independently by Loomis [18] and Sikorski [26] reads:

Theorem 10.2 *Let A be a σ-complete Boolean algebra. Then there exist a σ-field of sets T and a surjective $\sigma\mathcal{B}$-homomorphism $f : T \to A$.* $\Box$

With the same argument used in Theorem 9.2, we can apply Theorem 10.2 to obtain the following standard completeness for σ-complete Boolean algebras:

Theorem 10.3 (Standard completeness) *Let $p(\vec{x}) = q(\vec{x})$ be an equation of type $\langle \bigvee, \vee, \wedge, \neg, 0, 1\rangle$. Then:*

$$\sigma\mathcal{B} \models p(\vec{x}) = q(\vec{x}) \quad iff \quad \mathbf{2} \models p(\vec{x}) = q(\vec{x})$$

$\Box$

11 σ-complete product MV-algebras

In this section we shall study the class of σ-complete product MV-algebras as an equationally definable class of ω-algebras. A *product MV-algebra* [20, 21, 24] (for short: PMV-algebra) is an algebra $\langle A, \oplus, \bullet, \neg, 0\rangle$ of type $\langle 2, 2, 1, 0\rangle$ satisfying the following:

1 $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra,

2 $\langle A, \bullet, 1\rangle$ is an abelian monoid,

3 $x \bullet (y \odot \neg z) = (x \bullet y) \odot \neg(x \bullet z)$.

The terminology product MV-algebra conflicts with terminology of [9], according to which product in a product MV-algebra needs not be commutative and needs not have unit. PMV-algebras correspond to commutative product MV-algebras in [9] satisfying the equation $1 \bullet x = x$.

An important example of PMV-algebra is $[0, 1]_{MV}$ equipped with the usual multiplication (called *product t-norm*). This algebra is denoted by $[0, 1]_{PMV}$. Note that every Boolean algebra becomes a PMV-algebra by letting the product operation coincide with the infimum operation.

Remark 11.1 It is shown in [24, Theorem 3.1.4] that the ordinary product in $[0, 1]$ is the only binary operation satisfying the conditions of definition of PMV-algebra. Hence Ł$_2$ is the unique finite sub-algebra of $[0, 1]_{PMV}$ which admits product.

The following are almost immediate consequences of the definition of PMV-algebras:

Lemma 11.2 *In each PMV-algebra we have:*

1. $0 \bullet x = 0$,

2. If $a \leq b$ *then* $a \bullet x \leq b \bullet x$,

3. $x \odot y \leq x \bullet y \leq x \wedge y$.

□

Lemma 11.3 [20, Lemma 2.11]. *A PMV-algebra and the underlying MV-algebra have the same congruences.* □

Definition 11.4 A *product MV_ω-algebra*, (PMV_ω-algebra for short) is an ω-algebra $\langle A, \sum, \bullet, \neg, 0\rangle$ of type $\langle \omega, 2, 1, 0\rangle$ such that satisfies:

1. $\langle A, \sum, \neg, 0\rangle$ is aN MV_ω-algebra,
2. $\langle A, \oplus, \bullet, \neg, 0\rangle$ is a PMV-algebra.

We denote by $\mathcal{PMV}_\omega$ the category whose objects are PMV_ω-algebras and whose arrows are MV_ω-homomorphisms preserving the operation $\bullet$. Note that $[0,1]_{MV_\omega}$ equipped with the usual multiplication is a PMV_ω-algebra denoted by $[0,1]_{PMV_\omega}$ and called *standard PMV_ω-algebra*. By Remark 11.1, Ł$_2$ is the unique finite sub-PMV_ω-algebra of $[0,1]_{PMV_\omega}$.

Proposition 11.5 *A PMV_ω-algebra and the underlying MV_ω-algebra have the same congruences.*

Proof: Let A be PMV_ω-algebra and θ be an ω-congruence of the underlying MV_ω-structure. Since θ is an MV-congruence, by Lemma 11.3, θ is compatible with the operation $\bullet$. Therefore θ is an ω-congruence on A. Thus A and the underlying MV_ω-algebra have the same congruences.

$\square$

Theorem 11.6 *Let A be a PMV_ω-algebra. Then the following assertions are equivalent:*

1. *A is simple in $\mathcal{PMV}_\omega$.*
2. *A is PMV-isomorphic to $[0,1]_{PMV}$ or A is PMV-isomorphic to Ł$_2$.*
3. *A is directly irreducible in $\mathcal{PMV}_\omega$.*

Proof: $1 \Longleftrightarrow 2$) By Proposition 11.5 A is simple in $\mathcal{PMV}_\omega$ iff A is simple as MV_ω-algebra. Hence, by Theorem 8.3 and Remark 11.1, A is simple in $\mathcal{PMV}_\omega$ iff A is PMV-isomorphic to $[0,1]_{PMV}$ or A is PMV-isomorphic to Ł$_2$.

$1 \Longleftrightarrow 3$) Follows by Proposition 11.5 and Proposition 8.2.

$\square$

We denote by $\sigma\mathcal{PMV}$ the category whose objects are σ-complete PMV-algebras and whose arrows are PMV-homomorphisms preserving denumerable suprema and infima. By Theorem 4.4 we obtain:

Theorem 11.7 *$\sigma\mathcal{PMV} = \mathcal{PMV}_\omega$ i.e., they have the same objects and the same arrows, resulting $\sigma\mathcal{PMV}$ an equationally definable class of ω-algebras.*

$\square$

Now we present the Loomis-Sikorski theorem for PMV-algebras. A *product tribe* is a tribe which is closed under the pointwise usual product. Observe that each product tribe is a PMV_ω-algebra.

Theorem 11.8 [11, 23] *Let A be a σ-complete PMV-algebra. Then there exist a product tribe T and a surjective σPMV-homomorphism $f : T \to A$.*

$\square$

With the same argument used in Theorem 9.2, we can apply Theorem 11.8 to obtain the following standard completeness theorem for equations in the language of $\mathcal{PMV}_\omega$:

Theorem 11.9 (Standard completeness) *Let $p(\vec{x}) = q(\vec{x})$ be an equation of type $\langle \sum, \bullet, \neg, 0 \rangle$. Then:*

$$\mathcal{PMV}_\omega \models p(\vec{x}) = q(\vec{x}) \quad iff \quad [0,1]_{PMV_\omega} \models p(\vec{x}) = q(\vec{x})$$

$\square$

It is well known that the axiomatization of all identities in the language of $\mathcal{PMV}$, which are valid in the PMV-algebra arising from the real interval $[0,1]$, is an open problem [17, 20]. In our case we have provided a completeness theorem for $\mathcal{PMV}_\omega$-equations with respect to the standard PMV_ω-algebra.

12 Injectives in $\sigma\mathcal{B}$, $\mathcal{MV}_\omega$ and $\mathcal{PMV}_\omega$

Halmos [12] raised the question concerning to the existence of nontrivial injective objects in $\sigma\mathcal{B}$. Consider the category $m\mathcal{B}$ whose objects are m-complete Boolean algebras where m is a an infinite cardinal and whose arrows are Boolean-homomorphisms preserving m-suprema. In [19] Monk proved the following theorem:

Theorem 12.1 *$m\mathcal{B}$ has only trivial injectives.* $\square$

Consequently $\sigma\mathcal{B}$ has only trivial injectives. An interesting application of the last theorem is the characterization of injectives in $\mathcal{MV}_\omega$ and $\mathcal{PMV}_\omega$.

Proposition 12.2 *$\mathcal{MV}_\omega$ and $\mathcal{PMV}_\omega$ have only trivial injectives.*

Proof: Suppose that $\mathcal{A}$ is either $\mathcal{MV}_\omega$ or $\mathcal{PMV}_\omega$. Let A be an injective in $\mathcal{A}$. We shall prove that $B(A)$ is injective in $\sigma\mathcal{B}$. Let $g : B \to B(A)$ be a $\sigma\mathcal{B}$-homomorphism and $f : B \to C$ be a $\sigma\mathcal{B}$-monomorphism. Since $\sigma\mathcal{B}$ is a full subcategory of $\mathcal{A}$ and A is injective in $\mathcal{A}$, there exists an $\mathcal{MV}_\omega$-homomorphism $h : C \to A$ such that $g = h \circ f$. Taking into account that an $\mathcal{MV}_\omega$-homomorphism maps Boolean elements into boolean elements, $Imag(h) \subseteq B(A)$ and the following diagram is commutative:

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\equiv"] & B(A) \\
C \arrow[ur, "h"'] & {}
\end{tikzcd}$$

Thus $B(A)$ is injective in $\sigma\mathcal{B}$. By Theorem 12.1, $B(A)$ is trivial and then A is trivial. Hence $\mathcal{MV}_\omega$ and $\mathcal{PMV}_\omega$ have only trivial injectives.

□

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