A CHARACTERIZATION OF THE EXTENDED BEST ϕ-APPROXIMATION OPERATOR

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Abstract. Given an non necessarily linear operator T defined from an Orlicz space $L^{\phi'}(\Omega, \mathcal{A}, \mu)$ into itself, where ϕ' denote the derivative of a strictly convex function ϕ, we give necessary and sufficient conditions on T assuring that this operator is an extended best ϕ-approximation operator given a suitable σ-lattice $\mathcal{L} \subseteq \mathcal{A}$.

1. Introduction and Notations

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathcal{M} = \mathcal{M}(\Omega, \mathcal{A}, \mu)$ be the set of all $\mathcal{A}$-measurable real valued functions defined on Ω. Given a C^1 strictly convex function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(0) = 0$, $\phi(t) > 0$ when $t > 0$ and $\phi'(0) = 0$, let $L^\phi = L^\phi(\Omega, \mathcal{A}, \mu)$ be the space of all functions $f \in \mathcal{M}$ such that

$$\int_\Omega \phi(\lambda|f|)\, d\mu < \infty, \tag{1.1}$$

for some $\lambda > 0$. According to [?] we say that a function ϕ is a Δ_2 function if there exists $k > 0$ such that $\phi(2t) \leq k\phi(t)$, for all $t > 0$, and in this case we write $\phi \in \Delta_2$. In this paper the function ϕ will be a Δ_2 function, so the space L^ϕ can be defined as the space of all functions $f \in \mathcal{M}$ where (??) holds for every positive number λ. The space $L^{\phi'}$ is analogously defined, where ϕ' is the derivative of the function ϕ. Besides observe that for a Δ_2 function ϕ it holds the next inequality

$$\phi(x) \leq x\phi'(x) \leq \phi(2x) \leq K\phi(x),$$

for all $x \geq 0$ and hence $L^\phi \subset L^{\phi'}$. It is well known that ϕ is a Δ_2 function if and only if ϕ' is a Δ_2 function. For more details about properties of Orlicz spaces we refer to [?].

We say, as stated in [?], that a collection $\mathcal{L}$ of sets in $\mathcal{A}$ is a σ-lattice if it is closed under countable unions and intersections and contains both $\emptyset$ and Ω. Given a σ-lattice $\mathcal{L}$ we denote by $\overline{\mathcal{L}}$ the σ-lattice of all the complementary sets of $\mathcal{L}$, i.e. $\overline{\mathcal{L}} = \{A^c : A \in \mathcal{L}\}$. A function f is called a $\mathcal{L}$-measurable function if $\{f > a\} \in \mathcal{L}$, for all real numbers a, and denoted by $L^\phi(\mathcal{L})$ for the set of all $\mathcal{L}$-measurable functions in L^ϕ.

According to [?] a set $C \subset L^\phi$ is called ϕ-closed if and only if $f_n \in C$ and $f_n \nearrow f \in L^\phi$ or $f_n \searrow f \in L^\phi$ then $f \in C$. Then $L^\phi(\mathcal{L})$ is a ϕ-closed convex set and is a lattice, that is closed for the maximum and minimum of functions. See [?].

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It is well known, see [?], that for every $f \in L^\phi$ there exists a unique element $f_{\mathcal{L}} \in L^\phi(\mathcal{L})$ such that

$$\int_\Omega \phi(|f - f_{\mathcal{L}}|)\, d\mu = \inf_{h \in L^\phi(\mathcal{L})} \int_\Omega \phi(|f - h|)\, d\mu. \tag{1.2}$$

The element $f_{\mathcal{L}}$ is called a best ϕ-approximation of f given $\mathcal{L}$, and we set $\mu_{\mathcal{L}}(.)$ for the mapping $f \to f_{\mathcal{L}}$ defined on L^φ which will be called the best approximation operator. In Section 2, Definition **??** the operator $\mu_{\mathcal{L}}(.)$ is extended to an operator from $L^{\phi'}$ to itself and it will be denoted by $\widetilde{\mu}_{\mathcal{L}}(.)$. In [?] another extension of the operator $\mu_{\mathcal{L}}(.)$ is given, and in Theorem **??**, it is proved that both extensions coincide. We refer to this extended operator as the extended best $\phi-$approximation operator.

The main objective of this paper is to give sufficient conditions on an operator $T : L^{\phi'} \to L^{\phi'}$ to ensure that T is the extended best ϕ-approximation operator, given a suitable σ-lattice $\mathcal{L} \subseteq \mathcal{A}$. This result is done in Theorem **??**, and it gives a characterization of the extended best ϕ-approximation operator. For some specific cases it is possible to define an extension of this best approximation operator, in a rather direct way. See [?].

This sort of characterization problem has been investigated for several authors in many cases. For the case of the classical conditional expectation, $\phi(t) = t^2$ and where $\mathcal{L}$ is a sub σ-algebra of $\mathcal{A}$, the first general results appear in [?], see Theorem 2.1. A similar characterization result was given by [?] for the operator T acting on the L^2 space. These results were also treated by, among others, [?], [?], [?] and [?].

A characterization for a non linear operator $T : L^2(\Omega) \to L^2(\Omega)$, as a conditional expectation given a σ-lattice, appears in [?]. A characterization of the projection operator from $L^p(\Omega, \mathcal{A})$ onto $L^p(\Omega, \mathcal{B})$, where $\mathcal{B}$ is a sub σ-algebra of $\mathcal{A}$ and $1 < p < \infty$, appeared in [?]. The best approximation operator $\mu_{\mathcal{L}}$ defined by equation (**??**) was characterized in [?], for a rather general function ϕ. For the special case $\phi(t) = t^p$ the same author characterized the extended best L^p approximation operator in [?]. This last operator is acting from $L^{p-1}(\Omega) \to L^{p-1}(\Omega)$.

2. Extended best ϕ-approximation operator.

In this section we present some basic notions about the extended best ϕ-approximation operator and in the next section we give a characterization of this operator.

We know that when ϕ is a strictly convex function, the best ϕ-approximation is unique. In [?] we gave an extension of the best ϕ-approximation operator to the space $L^{\phi'}$ in a monotone continuous way. In [?] we defined another extension of the best ϕ-approximation operator, even for the case when ϕ is a convex function. Here we prove that when ϕ is a strictly convex function both extensions are the same.

In order to introduce the extended best approximation operator we need the following definitions, according to [?].

Definition 2.1. Let ν be a signed measure on $\mathcal{A}$ and $\mathcal{L}$ be a σ-lattice contained in $\mathcal{A}$. We say that $P \in \mathcal{L}$ is a ν-positive set, if for all $D \in \overline{\mathcal{L}}$ we have $\nu(P \cap D) \geq 0$. A set $N \in \overline{\mathcal{L}}$ is called ν-negative if for all $C \in \mathcal{L}$ we have $\nu(N \cap C) \leq 0$.

Definition 2.2. Let $\{\nu_a\}_{a\in\mathbb{R}}$ be a family of measures on $\mathcal{A}$, and $\mathcal{L}$ be a σ-lattice contained in $\mathcal{A}$. A $\mathcal{L}$-measurable function g is called a Lebesgue-Radon-Nikodym function (LRN function) for $\{\nu_a\}$ given $\mathcal{L}$ if and only if the set $\{g > a\}$ is ν_a-positive for all $a \in \mathbb{R}$ and the set $\{g < a\}$ is ν_a-negative for all $a \in \mathbb{R}$.

Remark 2.3. We note that in Definition **??** it is sufficient to impose the conditions for all a in a dense set in $\mathbb{R}$. See page 588 of [**?**].

For $f \in L^\phi$, $g \in L^\phi(\mathcal{L})$ and $a \in \mathbb{R}$ we define the following measures on $\mathcal{A}$

$$\mu_g(A) = \int_A \underline{\phi}'(f-g)\, d\mu, \quad \mu_a(A) = \int_A \underline{\phi}'(f-a)\, d\mu, \tag{2.1}$$

where $\underline{\phi}'(x) = \phi'(|x|)\,\mathrm{sign}(x)$. Note that when $f \in L^{\phi'}$ and $g \in L^{\phi'}(\mathcal{L})$ the measure μ_g and μ_a are well defined.

The next theorem is a characterization of $\mu_{\mathcal{L}}(f)$, see Theorem 3.2 in [**?**].

Theorem 2.4. *Let $f \in L^\phi$, $\mathcal{L} \subset \mathcal{A}$ be a σ-lattice and $g \in L^\phi(\mathcal{L})$. Then the following statements are equivalent.*

(1) $g = \mu_{\mathcal{L}}(f)$.

(2) a) the set $\{g > a\}$ is μ_g-positive for all $a \in \mathbb{R}$ and
b) the set $\{g < a\}$ is μ_g-negative for all $a \in \mathbb{R}$.

(3) g is a LRN function for the family $\{\mu_a\}_{a\in\mathbb{R}}$ given $\mathcal{L}$.

The following definition was given in [**?**].

Definition 2.5. Let $\mathcal{L}$ be a σ-lattice and let $f \in L^{\phi'}$. Then g is called an extended best ϕ-approximation if and only if $g \in L^{\phi'}(\mathcal{L})$ and

i) the set $\{g > a\}$ is μ_g-positive for all $a \in \mathbb{R}$,
ii) the set $\{g < a\}$ is μ_g-negative for all $a \in \mathbb{R}$.

For $f \in L^{\phi'}$ we denote by $\widetilde{\mu}_{\mathcal{L}}(f)$ the set of all extended best ϕ-approximation functions. In [**?**] it was proved that $\widetilde{\mu}_{\mathcal{L}}(f)$ is a non-empty set.

The conditions (2) and (3) are equivalent even if we assume $f \in L^{\phi'}$ and $g \in L^{\phi'}(\mathcal{L})$. In fact the proof given in Theorem 3.2 in [**?**] holds in this situation. Thus we have the next Remark.

Remark 2.6. Given $f \in L^{\phi'}$ and $g \in L^{\phi'}(\mathcal{L})$ the following statements are equivalent

(1) $g \in \widetilde{\mu}_{\mathcal{L}}(f)$.

(2) g is a LRN function for the family $\{\mu_a\}_{a\in\mathbb{R}}$ given $\mathcal{L}$.

In the next theorem we prove the uniqueness of the extended best ϕ-approximation operator when ϕ is a strictly convex function.

Theorem 2.7. *If ϕ is a strictly convex function then the extended best ϕ-approximation is unique.*

Proof. Given $g_1,\ g_2 \in \widetilde{\mu}_{\mathcal{L}}(f)$ such that $g_1 \neq g_2$, it is enough to verify that

$$A = \{x \in \Omega\ :\ g_1(x) < a < b < g_2(x)\} = \{g_1 < a\} \cap \{g_2 > b\}$$

is a μ-null set for $a, b \in \mathbb{R}$ such that $a < b$. Since $\underline{\phi}'(\cdot)$ is a strictly increasing function, and assuming $\mu(A) > 0$ we have

$$\int_{\{g_1<a\}\cap\{g_2>b\}} \underline{\phi}'(f-b)\, d\mu < \int_{\{g_1<a\}\cap\{g_2>b\}} \underline{\phi}'(f-a)\, d\mu. \tag{2.2}$$

Since $\{g_1 < a\} \in \overline{\mathcal{L}}$ and using Remark **??** we have $0 \leq \int_{\{g_1<a\}\cap\{g_2>b\}} \underline{\phi}'(f-b)\, d\mu$. Similarly the second integral is less than, or equal to, 0. Which is a contradiction.

□

In [**?**], where ϕ is a strictly convex function, the operator of best ϕ-approximation is extended in a monotone continuous way to the space $L^{\phi'}$. We denote this extension by $\mu^*_{\mathcal{L}}(f)$ and it was defined as follows. For $f \in L^{\phi'}$ and for every $m \in \mathbb{N}$ set $f_m = \lim_{n\to\infty} \mu_{\mathcal{L}}((f \vee (-m)) \wedge n)$ and then $\mu^*_{\mathcal{L}}(f) = \lim_{m\to\infty}(f_m)$.

The next theorem and its corollary appeared in [**?**] and both of them characterize the extended best ϕ-approximation $\mu^*_{\mathcal{L}}(f)$.

Theorem 2.8. *Let $f \in L^{\phi'}$ and let ℓ be a strictly increasing function such that $\ell(\mu^*_{\mathcal{L}}(f))$ is bounded. Then $g = \mu^*_{\mathcal{L}}(f)$ if and only if $g \in L^{\phi'}(\mathcal{L})$, $\ell(g)$ is bounded and*

(1) $\int_\Omega \underline{\phi}'(f-g)h\, d\mu \leq 0$, for all bounded $\mathcal{L}$-measurable function h,
(2) $\int_\Omega \underline{\phi}'(f-g)\ell(g)\, d\mu = 0$.

Corollary 2.9. *Let $f \in L^{\phi'}$. Then $g = \mu^*_{\mathcal{L}}(f)$ if and only if $g \in L^{\phi'}(\mathcal{L})$ and*

i)

$$\int_C \underline{\phi}'(f-g)\, d\mu \leq 0 \textit{ for all } C \in \mathcal{L}. \tag{2.3}$$

ii)

$$\int_{\{g\geq a\}} \underline{\phi}'(f-g)\, d\mu = 0 \textit{ for all } a \in \mathbb{R}. \tag{2.4}$$

Using this Corollary we will prove the next result.

Theorem 2.10. *If ϕ is a strictly convex function, then $\widetilde{\mu}_{\mathcal{L}} = \mu^*_{\mathcal{L}}$ on $L^{\phi'}$.*

Proof. Set $g = \widetilde{\mu}_{\mathcal{L}}(f)$, by Definition **??** we have

$$\int_{\{g>a\}\cap D} \underline{\phi}'(f-g)\, d\mu \geq 0 \text{ for all } \; D \in \overline{\mathcal{L}} \textit{ and} \tag{2.5}$$

$$\int_{\{g<a\}\cap C} \underline{\phi}'(f-g)\, d\mu \leq 0 \text{ for all } C \in \mathcal{L}, \tag{2.6}$$

for all $a \in \mathbb{R}$. As $f,\ g \in L^{\phi'}$ we can use Lebesgue's Theorem when $a \to \infty$ in (**??**) and we obtain

$$\int_C \underline{\phi}'(f-g)\, d\mu \leq 0 \text{ for all } C \in \mathcal{L}. \tag{2.7}$$

From (**??**) we also can obtain

$$\int_{\{g\geq a\}} \underline{\phi}'(f-g)\, d\mu \geq 0,$$

for all $a \in \mathbb{R}$. The condition (**??**) holds in particular for $C = \{g \geq a\}$ and we get

$$\int_{\{g\geq a\}} \underline{\phi}'(f-g)\, d\mu = 0. \tag{2.8}$$

Then by (**??**), (**??**) and Corollary **??** we obtain that $g = \mu^*_{\mathcal{L}}(f)$. $\square$

3. Characterization of the extended best ϕ-approximation operator.

Given an operator $T : L^{\phi'} \to L^{\phi'}$ we impose conditions on T in order to insure that T is an extended best ϕ-approximation operator. We begin with some definitions and auxiliary results.

Definition 3.1. An operator $T : L^{\phi'} \to L^{\phi'}$ is called monotone continuous if $f_n \nearrow f$ or $f_n \searrow f$ where $f_n, f \in L^{\phi'}$ for all $n \in \mathbb{N}$ then $Tf_n \nearrow Tf$ or $Tf_n \searrow Tf$.

Definition 3.2. An operator $T : L^{\phi'} \to L^{\phi'}$ is called ϕ-expectation invariant if for all $f \in L^{\phi'}$ we have that

$$\int_\Omega \underline{\phi}'(f - Tf)\, d\mu = 0.$$

Note that the operator $\widetilde{\mu}_{\mathcal{L}}(f)$, introduced in Section 2 is both a monotone continuous operator and a monotone operator. In [?] the properties above were proved for the operator $\mu^*_{\mathcal{L}}$ and by Theorem **??** $\mu^*_{\mathcal{L}} = \widetilde{\mu}_{\mathcal{L}}$. Also $\widetilde{\mu}_{\mathcal{L}}$ is a ϕ-expectation invariant operator by Theorem **??**. The properties as translation invariant, $T(f+c) = T(f) + c,\ c \in \mathbb{R}$, and idempotent for the operator $\widetilde{\mu}_{\mathcal{L}}(f)$ follow directly from the definition.

Observe also that $\widetilde{\mu}_{\mathcal{L}}(f \pm \frac{1}{2}\widetilde{\mu}_{\mathcal{L}}(f)) = (1 \pm \frac{1}{2})\widetilde{\mu}_{\mathcal{L}}(f)$ using Corollary **??**.

The next Lemma plays the role of Lemma 7.3, of [?] and its proof follows the pattern given there, even if we deal with an non necessarily positive homogeneous operator T.

Lemma 3.3. *Let $T : L^{\phi'} \to L^{\phi'}$ be an operator which is translation invariant, monotone, ϕ-expectation invariant, $T(0) = 0$ and $T(f \pm \frac{1}{2}Tf) = (1 \pm \frac{1}{2})Tf$. Then*

(1) T is a monotone continuous operator.
(2) $\mathcal{L} = \{A \in \mathcal{A} \,:\, T\chi_A = \chi_A\}$ is a σ-lattice.
(3) If $H = \{f \in L^{\phi'} \,:\, Tf = f\}$ then
a) For all $b \in \mathbb{R}$, $H + b \subset H$.
b) For all $n \in \mathbb{N}$, $(\frac{3}{2})^n H \subset H$.
c) H is ϕ'-close.
d) H is a lattice.
e) $H = L^{\phi'}(\mathcal{L})$.

Proof.

(1) Let $f_n, f \in L^{\phi'}$ for all $n \in \mathbb{N}$, such that $f_n \nearrow f$ (or $f_n \searrow f$), we show that $Tf_n \nearrow Tf$ ($Tf_n \searrow Tf$). We prove only the increasing case, the decreasing case runs analogously. Since T is a monotone operator we have $Tf_n \leq Tf$ and by the ϕ-expectation invariant property

$$\int_\Omega \underline{\phi}'(f_n - Tf_n)\, d\mu = 0.$$

We define $F = \lim_{n\to\infty} Tf_n$ then $F \leq Tf$. Since $\underline{\phi}'(t)$ is a strictly increasing function then

$$\underline{\phi}'(f_1 - Tf) \leq \underline{\phi}'(f_n - Tf_n) \leq \underline{\phi}'(f - Tf_1).$$

Using the Lebesgue's Theorem we obtain

$$\int_\Omega \underline{\phi}'(f - F)\, d\mu = \lim_{n\to\infty} \int_\Omega \underline{\phi}'(f_n - Tf_n)\, d\mu = 0. \tag{3.1}$$

Using the ϕ-expectation invariant property for $f \in L^{\phi'}$ we have that

$$\int_\Omega \underline{\phi}'(f - Tf)\, d\mu = 0. \tag{3.2}$$

Since $F \leq Tf$ and $\underline{\phi}'$ is a strictly increasing function, (**??**) and (**??**) imply that $F = Tf$ *a.e.* Then we conclude that T is a monotone continuous operator.

(2) Since $T(0) = 0$ we have $\emptyset \in \mathcal{L}$ and using the translation invariant property we have $T(1) = T(0) + 1$ and then $\Omega \in \mathcal{L}$.

Let $\{A_n\}_{n\in\mathbb{N}}$ be a set sequence of $\mathcal{L}$. We can define for each $n \in \mathbb{N}$ the following functions $f_n = \chi_{\bigcup_{i=1}^n A_i}$. We observe that the sequence $\{f_n\}$ is increasing and $f_n \nearrow f$ where $f = \chi_{\bigcup_{i\in\mathbb{N}} A_i}$ when $n \to \infty$.

We will prove that $T(\chi_{\bigcup_{i=1}^n A_i}) = \chi_{\bigcup_{i=1}^n A_i}$ for all $n \in \mathbb{N}$.
Since T is a monotone operator we have

$$T(\chi_{\bigcup_{i=1}^n A_i}) = T(\bigvee_{i=1}^n \chi_{A_i}) \geq \bigvee_{i=1}^n T(\chi_{A_i}) = \bigvee_{i=1}^n \chi_{A_i}. \tag{3.3}$$

As T is a ϕ-expectation invariant operator we have

$$\int_\Omega \underline{\phi}'(\chi_{\cup_{i=1}^n A_i} - T(\chi_{\cup_{i=1}^n A_i}))\, d\mu = 0, \tag{3.4}$$

then $T(\chi_{\cup_{i=1}^n A_i}) = \chi_{\cup_{i=1}^n A_i}$.
As T is a monotone continuous operator we have

$$T(\chi_{\bigcup_{i\in\mathbb{N}} A_i}) = \chi_{\bigcup_{i\in\mathbb{N}} A_i}.$$

Then in a similar way we can prove that $\mathcal{L}$ is closed for countable intersection of sets in $\mathcal{L}$.

(3) a) For all $b \in \mathbb{R}$, $H + b \subset H$. In fact, since T is a translation invariant operator, $T(h + b) = T(h) + b$, and the property follows.

b) By hypothesis we have

$$(\frac{3}{2})^n H \subset H, \tag{3.5}$$

for $n = 1$. Now, for all $n \in \mathbb{N}$ it follows by induction.

c) This follows since T is a monotone continuous operator.

d) For $f,\ g \in H$ we are going to prove that $f \vee g,\ f \wedge g, \in H$. By the monotone property of T we have $T(f \vee g) \geq Tf \vee Tg = f \vee g$. Furthermore, by the ϕ-invariant expectation property on T we have

$$\int_\Omega \underline{\phi}'(f \vee g - T(f \vee g))\, d\mu = 0.$$

Using that $\underline{\phi}'$ is a strictly increasing function we have $T(f \vee g) = f \vee g$. Similarly we can prove that $T(f \wedge g) = f \wedge g$.

(4) We will prove that $H = L^{\phi'}(\mathcal{L})$.
Let $f \in H$ and for all $b \in \mathbb{R}$, we prove that $\{f > b\} \in \mathcal{L}$. For each $n \in \mathbb{N}$ we can define $f_n = ((\frac{3}{2})^n(f - b) \vee 0) \wedge 1$. By (??), (??) and since H is a lattice we have that $f_n \in H$. As $f_n \nearrow \chi_{\{f>b\}}$, and H is ϕ'-closed, then $\chi_{\{f>b\}} \in H$. Thus $\{f > b\} \in \mathcal{L}$ for all $b \in \mathbb{R}$, that is $f \in L^{\phi'}(\mathcal{L})$.

To prove that $L^{\phi'}(\mathcal{L}) \subset H$, observe that without loss of generality we can assume that $f \geq 0$. In fact we can define for each $m \in \mathbb{N}$, $f_m = f \vee (-m) + m \geq 0$ and $f_m - m \searrow f$.

Given $C \in \mathcal{L}$ we have that $\chi_C \in H$, and now we prove $a\chi_C \in H$ for all $a \geq 0$. If $a > 1$ we know that $(\frac{3}{2})^n \chi_C \wedge a \in H$ and $(\frac{3}{2})^n \chi_C \wedge a \nearrow a\chi_C$ then $a\chi_C \in H$. The case $a = 0$ is trivial. For $0 < a < 1$ we have $a\chi_C \leq a$ and $a\chi_C \leq \chi_C$ then by the monotone property of T we have that $T(a\chi_C) \leq a \wedge \chi_c = a\chi_C$. Now, using the ϕ-invariant expectation property on T, we have $a\chi_C = T(a\chi_C)$.

Given $f \in L^{\phi'}(\mathcal{L})$ and $f \geq 0$ we define

$$g_n = \sup_{v=0,\dots,n2^n} \{\frac{v}{2^n}\ \chi_{\{f \geq \frac{v}{2^n}\}}\}.$$

We have proved that $g_n \in H$ and it is well known that $g_n \nearrow f$, and then we have $f \in H$.

$\square$

Remark 3.4. If $T : L^{\phi'} \to L^{\phi'}$ is an operator that satisfies $T(T(0)) = T(0)$ and $T(f \pm \frac{1}{2}Tf) = (1 \pm \frac{1}{2})Tf$, for all $f \in L^{\phi'}$, then $T(0) = 0$.

Proof. Using the hypothesis we have $T(\frac{1}{2}T(0)) = T(0 + \frac{1}{2}T(0)) = \frac{3}{2}T(0)$. And also $T(\frac{1}{2}T(0)) = T(T(0) - \frac{1}{2}T(0)) = T(T(0) - \frac{1}{2}T(T(0))) = \frac{1}{2}T(0)$. Thus it follows $T(0) = 0$. □

Now we can set the main result.

Theorem 3.5. *Let $T : L^{\phi'} \to L^{\phi'}$ be an operator which is*

i) translation invariant;
ii) idempotent;
iii) monotone;
iv) $T(f \pm \frac{1}{2}Tf) = (1 \pm \frac{1}{2})Tf$;
v) ϕ-expectation invariant ;
vi) weak ϕ-monotonic at 0; i.e., $\int_\Omega \underline{\phi}'(f)Tf \geq 0$ for $f \in L^\infty$ with $Tf \geq 0$.

Then there exists a σ-lattice $\mathcal{L} \subset \mathcal{A}$ such that $Tf = \widetilde{\mu}_{\mathcal{L}}(f)$, for all $f \in L^{\phi'}$.

Proof. According to Lemma **??** we have $L^{\phi'}(\mathcal{L}) = \{f \in L^{\phi'} : Tf = f\}$, for $\mathcal{L} = \{A \in \mathcal{A} : T\chi_A = \chi_A\}$ and T is monotone continuous operator. Besides, by (**??**), $Tf \in L^{\phi'}(\mathcal{L})$, for $f \in L^{\phi'}$.

First we suppose that f is a bounded function. Since the operator T is translation invariant we assume $0 \leq f \leq a$, then, using Remark **??**, we have $0 \leq Tf \leq a$ and we prove that $Tf = \widetilde{\mu}_{\mathcal{L}}(f)$. For each $g \in L^\infty(\mathcal{L})$ we will prove that

A) $\int_\Omega \underline{\phi}'(f - Tf)g \, d\mu \leq 0$ and
B) $\int_\Omega \underline{\phi}'(f - Tf)Tf \, d\mu = 0$.

First we will prove A). Let $g = \chi_C$ where $C \in \mathcal{L}$. Since $\phi'(0) = 0$ then $\underline{\phi}'(f - Tf)\chi_C = \underline{\phi}'((f - Tf)\chi_C)$.

Now, by (**??**), we have $T(\frac{3}{2}f) = \frac{3}{2}f$, for all $f \in L^{\phi'}(\mathcal{L})$ and since $\frac{3}{2}f \in L^{\phi'}(\mathcal{L})$ we have that $T((\frac{3}{2})^n f) = (\frac{3}{2})^n f$, for $f \in L^{\phi'}(\mathcal{L})$. We choose $(\frac{3}{2})^n > a$, thus $0 \leq Tf \leq (\frac{3}{2})^n$.

By (**??**) $T(f\chi_C) \leq Tf$ and $T(f\chi_C) \leq T((\frac{3}{2})^n \chi_C) = (\frac{3}{2})^n \chi_C$ then $T(f\chi_C) \leq Tf \wedge (\frac{3}{2})^n \chi_C = (Tf)\chi_C$.

Hence $\underline{\phi}'(f - Tf)\chi_C \leq \underline{\phi}'(f\chi_C - T(f\chi_C))$ and by (**??**)

$$\int_\Omega \underline{\phi}'(f - Tf)\chi_C \, d\mu \leq \int_\Omega \underline{\phi}'(f\chi_C - T(f\chi_C)) \, d\mu = 0. \tag{3.6}$$

Let g be a non negative simple function, then by (**??**) we have that

$$\int_\Omega \underline{\phi}'(f - Tf)g \, d\mu \leq 0. \tag{3.7}$$

For a non negative bounded function g, there exists an increasing sequence of non negative simple functions that converge to g. Thus (**??**) holds for these functions g.

If $g \in L^\infty(\mathcal{L})$ there exists $M > 0$ such that $g + M > 0$ and then we have

$$\begin{aligned} 0 &\geq \int_\Omega \underline{\phi}'(f - Tf)(g + M)\ d\mu = \int_\Omega \underline{\phi}'(f - Tf)g\ d\mu + M\ \int_\Omega \underline{\phi}'(f - Tf)\ d\mu \\ &= \int_\Omega \underline{\phi}'(f - Tf)g\ d\mu. \end{aligned}$$

thus the condition (**??**) holds for all $g \in L^\infty(\mathcal{L})$.

Now we prove condition (**??**). By (**??**) we have

$$\int_\Omega \underline{\phi}'(f - Tf)Tf\ d\mu \leq 0. \tag{3.8}$$

We have to prove

$$\int_\Omega \underline{\phi}'(f - Tf)Tf\ d\mu \geq 0.$$

We define the following sequence of real number

$$\gamma_1 = \frac{1}{2}, \ldots, \gamma_{n+1} = \frac{1 + \gamma_n}{2}.$$

Since $0 \leq \gamma_n \leq 1$ we have $\{\gamma_n\}_{n\in\mathbb{N}}$ converge to 1 when $n \to \infty$.

By induction we prove that $T(f - \gamma_n Tf) = (1 - \gamma_n)Tf$. For $n = 1$ it is the property (**??**). We assume that the property is satisfied by n and we will prove that it is satisfied by $n + 1$.

$$\begin{aligned} T(f - \gamma_{n+1}Tf) &= T(f - \frac{(1 + \gamma_n)}{2}Tf) \\ &= T(f - \frac{1}{2}Tf - \frac{\gamma_n}{2}Tf + \gamma_n Tf - \gamma_n Tf) \\ &= T(f - \gamma_n Tf - \frac{1}{2}(1 - \gamma_n)Tf), \end{aligned}$$

Now

$$\begin{aligned} T(f - \gamma_n Tf - \frac{1}{2}(1 - \gamma_n)Tf) &= T(f - \gamma_n Tf - \frac{1}{2}T(f - \gamma_n Tf)) \\ &= \frac{1}{2}T(f - \gamma_n Tf) \\ &= \frac{1}{2}(1 - \gamma_n)Tf = (1 - \gamma_{n+1})Tf. \end{aligned}$$

Now as $f \geq 0$ and $(1 - \gamma_n)Tf \geq 0$ then $T(f - \gamma_n Tf) \geq 0$. By (**??**) we have

$$\int_\Omega \underline{\phi}'(f - \gamma_n Tf)T(f - \gamma_n Tf)\ d\mu \geq 0.$$

Then

$$0 \leq \int_{\Omega} \underline{\phi}'(f - \gamma_n Tf)(1-\gamma_n)Tf \, d\mu =$$
$$= (1-\gamma_n) \int_{\Omega} \underline{\phi}'(f - \gamma_n Tf)Tf \, d\mu$$
$$\leq \int_{\Omega} \underline{\phi}'(f - \gamma_n Tf)Tf \, d\mu. \tag{3.9}$$

And by taking limit in (**??**) we have that

$$\int_{\Omega} \underline{\phi}'(f - Tf)Tf \, d\mu \geq 0, \tag{3.10}$$

for all non negative bounded functions f. For $f \in L^{\infty}$ set $f_n = f \vee (-n) + n$, thus $f_n - n \searrow f$, and $f_n \geq 0$. By (**??**), property (**??**) and (**??**) we have that

$$0 \leq \int_{\Omega} \underline{\phi}'((f_n - n) - T(f_n - n))T(f_n - n) \, d\mu = \int_{\Omega} \underline{\phi}'(f_n - Tf_n)(Tf_n - n) \, d\mu = \tag{3.11}$$
$$\int_{\Omega} \underline{\phi}'(f_n - Tf_n)Tf_n \, d\mu$$

Now if $n \to \infty$ in (**??**) we obtain (**??**) for f bounded. Then, using (**??**), we obtain (**??**). By Theorem **??** we have that $T(f) = \widetilde{\mu}_{\mathcal{L}}(f)$ when f is a bounded function. Again, for these bounded functions, by Theorem **??** we have

(1) $\int \underline{\phi}'(f - Tf)g \, d\mu \leq 0$, for all bounded $\mathcal{L}$-measurable function g,
(2) $\int \underline{\phi}'(f - Tf)\ell(Tf) \, d\mu = 0$,

where ℓ is strictly increasing function from $\mathbb{R}$ to $(-1, 1)$.

Now, for a general $f \in L^{\phi'}$ we set $f_{m,n} = (f \vee (-m)) \wedge n$ and $f_m = \lim_{n\to\infty} f_{m,n}$. Since T is a monotone continuous operator, and using Lebesgue Theorem, we have (1) and (2) for f_m and Tf_m. Again, for m tending to ∞ we have (1) and (2) for $f \in L^{\phi'}$ and the Theorem follows by Theorem **??**. □

Finally we point out that conditions (i) to (v) of Theorem 1, given in [**?**], imply the conditions (i) to (vi) in Theorem **??** for $f \in L^{\phi}$.

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