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Inequalities in L^{p-1} for the extended L^p best approximation operator

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ABSTRACT

The best polynomial approximation operator was recently extended by one of the authors from L^p to L^{p-1}. In this paper, we study weak and strong inequalities for maximal operators related with the extended best polynomial approximation operator. As an application, we obtain norm convergence of the coefficients of the best polynomial approximation.

1. Introduction and notation

Given a Lebesgue measurable set $A \subset \mathbb{R}^n$ we set $L^p = L^p(A), 0 < p < \infty$, for the equivalence class of Lebesgue measurable functions $f : A \to \mathbb{R}$ with $\|f\|_{L^p(A)} = \left(\int_A |f|^p d\mu\right)^{1/p} < \infty$, and $\|f\|_{L^\infty(A)} = \sup \operatorname{ess}\{|f(x)| : x \in A\}$. Let Π^m be the space of algebraic polynomials, defined on $\mathbb{R}^n$, of degree at most m. We write $|A|$ for the Lebesgue measure of the set A. If A is a measurable set with $|A| > 0$, and $f \in L^p(\mathbb{R}^n)$ we define $T_{A,p}(f)$ as the set of all $P \in \Pi^m$ minimizing

$$\int_A |f(t) - P(t)|^p \, dt.$$

It is well known that $T_{A,p}(f)$ is a single set if $1 < p < \infty$. Thus the best approximation polynomial $T_{A,p}$ is a mapping from $L^p(A)$ to Π^m, $1 < p < \infty$. In [1], this operator was extended from $L^p(A)$ to $L^{p-1}(A)$. More precisely, if A has finite measure then for $f \in L^{p-1}(A)$, $T_{A,p}(f)$ is the unique $P \in \Pi^m$ which verifies

$$\int_A |f(t) - P(t)|^{p-1} \operatorname{sgn}(f(t) - P(t)) Q(t) dt = 0, \tag{1.1}$$

for all $Q \in \Pi^m$. A more general class than Π^m was considered in [1]. The polynomial $T_{A,p}(f)$ is called the extended best polynomial approximation of f on A. Moreover, it is proved, in [1], that the operator $T_{A,p} : L^{p-1}(A) \to \Pi^m$ is the unique continuous extension to L^{p-1}. For the case $p = 2$, a detailed study of this operator appeared in [2].

Inequalities for the maximal functions associated with the extended best approximation operator have been extensively studied in the literature; see [3–9] and also [10].

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Set $B(x, \varepsilon)$ for the ball centered at x and radius $\varepsilon > 0$ and denote by $T_x^\varepsilon(f)$ for the extended best approximation operator $T_{B(x,\varepsilon),p}(f)$. This polynomial $T_x^\varepsilon(f)$ can be written as a polynomial in the variable t as $\sum_{|\alpha|\le m} a_\alpha(x, \varepsilon)(t - x)^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, $t^\alpha = t^{\alpha_1} \cdots t^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where the coefficients $a_\alpha(x, \varepsilon)$ are uniquely determined. It is well known, for $f \in C^m$, that the polynomial $T_x^\varepsilon(f)$ tends to the Taylor polynomial $T_x(f)(t) = \sum_{|\alpha|\le m} \frac{1}{\alpha!}\partial^\alpha f(x)(t-x)^\alpha$, as ε tends to 0. This result has a long standing history since [11] and for further results, see [12–14].

The main estimate for the operator $T_{A,p}(f)$ is given in Theorem 2.1. In the case $A = B(x, \varepsilon)$ we consider Theorems 2.8 and 2.13 as our principal results. We also point out that Corollary 2.7 gives a pointwise convergence result under a very general differentiability assumption on the function f. These results allow us to consider the L^p norm convergence of $a_\alpha(x, \varepsilon)$ to the function $\frac{1}{\alpha!}\partial^\alpha f(x)$ besides of the pointwise convergence previously studied. We should point out that the norm convergence results are known only for the cases $m = 0$, $p \ge 1$ (see [7,5]) and for $p = 2$, for any degree m, see [2].

We remark that to obtain the L^p convergence result

$$\left\| a_\alpha(\cdot, \varepsilon) - \frac{1}{\alpha!}\partial^\alpha f(\cdot) \right\|_{L^p(\mathbb{R}^n)} \to 0, \tag{1.2}$$

as ε tends to 0, even if the function f is in the original space L^p, we have to study inequalities for the extended best approximation operator defined on the space L^{p-1}.

2. Inequalities in L^{p-1} and its applications to convergence results

From now on and throughout the remainder of the paper p will be a fixed real number bigger than 1. We begin with the following theorem.

Theorem 2.1. *Let $p > 1$ and set $K = 2^{p+1}$ if $p \ge 2$, and $K = 8$ if $1 < p \le 2$. Then we have*

$$\int_A |T_{A,p}(f)(t)|^{p-1}|Q(t)|dt \le K \int_A |f(t)|^{p-1}|Q(t)|dt \tag{2.1}$$

for any measurable set $A \subset \mathbb{R}^n$, $0 < |A| < \infty$, $f \in L^{p-1}(A)$, for all $Q \in \Pi^m$ satisfying $sgn(Q(t)T_{A,p}(f)(t)) = (-1)^\eta$, for any $t \in A$ such that $Q(t)T_{A,p}(t) \ne 0$, where $\eta = 0$ or $\eta = 1$.

Proof. Let $A \subset \mathbb{R}^n$ be with $0 < |A| < \infty$ and let $f \in L^{p-1}(A)$. We write $P = T_{A,p}(f)$. Clearly, only the case $\eta = 0$ it must be considered. Let $Q \in \Pi^m$ with $Q(t)P(t) > 0$ for all $t \in A$ such that $Q(t)P(t) \ne 0$. It is easy to show that there are constants $B = B(p)$ and $C = C(p)$ with $0 < B \le 1 \le C$ such that

$$B(a^{p-1} + b^{p-1}) \le (a+b)^{p-1} \le C(a^{p-1} + b^{p-1}), \quad a \ge 0, b \ge 0. \tag{2.2}$$

We consider the following sets

$$N = \{t \in A : f(t) > P(t)\} \quad \text{and} \quad L = \{t \in A : f(t) \le P(t)\}, \tag{2.3}$$

and set

$$\begin{aligned} &U_1 = N \cap \{t \in A : P(t) \ge 0\}, \qquad U_2 = N \cap \{t \in A : P(t) < 0\}, \\ &U_3 = L \cap \{t \in A : P(t) \ge 0\} \quad \text{and} \quad U_4 = L \cap \{t \in A : P(t) < 0\}. \end{aligned} \tag{2.4}$$

We write $H(t) = |P(t) - f(t)|^{p-1}Q(t)$. By (1.1) we have

$$\int_N H(t)dt = \int_L H(t)dt,$$

which is equivalent to

$$\int_{U_3} H(t)dt - \int_{U_2} H(t)dt = \int_{U_1} H(t)dt - \int_{U_4} H(t)dt. \tag{2.5}$$

From (2.2) and (2.4) it follows that

$$\begin{aligned} \int_A |P(t)|^{p-1}|Q(t)|dt &= \int_A |P(t) - f(t) + f(t)|^{p-1}|Q(t)|dt \\ &\le C\sum_{i=1}^4 \int_{U_i} |H(t)|\,dt + C\sum_{i=1}^4 \int_{U_i} |f(t)|^{p-1}|Q(t)|dt = C(I_1 + I_2). \end{aligned} \tag{2.6}$$

Now we only have to deal with $I_1 = \sum_{i=1}^4 \int_{U_i} |H(t)|\,dt$.

Clearly

$$\int_{U_1\cup U_4} |H(t)|\,dt \le \int_{U_1\cup U_4} |f(t)|^{p-1}|Q(t)|dt \le 2\int_A |f(t)|^{p-1}|Q(t)|dt. \tag{2.7}$$

Since $\operatorname{sgn} P = \operatorname{sgn} Q$ we have that

$$\int_{U_2\cup U_3} |H(t)|\,dt = \int_{U_3} H(t)\,dt - \int_{U_2} H(t)\,dt. \tag{2.8}$$

Next we use (2.5) and we get

$$\int_{U_3} H(t)\,dt - \int_{U_2} |H(t)|\,dt = \int_{U_1} H(t)\,dt - \int_{U_4} H(t)\,dt = \int_{U_1\cup U_4} |H(t)|\,dt, \tag{2.9}$$

and by (2.7) the above integral is less than or equal to $2\int_A |f(t)|^{p-1}|Q(t)|\,dt$. $\square$

We have the following corollary of Theorem 2.1.

Corollary 2.2. *Let $1 < p < \infty$ and $A \subset \mathbb{R}^n$, $0 < |A| < \infty$. Then*

(a) *There is a constant $K = K(p)$ such that*

$$\|T_{A,p}(f)\|^p_{L^p(A)} \le K\|f\|^{p-1}_{L^{p-1}(A)}\|T_{A,p}(f)\|_{L^\infty(A)}. \tag{2.10}$$

(b) *There exists a constant $K = K(A, p)$ such that*

$$\|T_{A,p}(f)\|_{L^{p-1}(A)} \le K\|f\|_{L^{p-1}(A)}, \tag{2.11}$$

for all $f \in L^{p-1}(A)$.

Proof. (a) follows from Theorem 2.1. Since the norms $\|\cdot\|_{L^p(A)}$, $\|\cdot\|_{L^\infty(A)}$, and $\|\cdot\|_{L^{p-1}(A)}$ are equivalent on the finite dimensional space Π^m, from (a) we get (b). $\square$

Inequality (2.11) is unknown for us, even if $f \in L^p$. In that case $T_{A,p}(f)$ is the best $\|\cdot\|_{L^p(A)}$-approximation to f from Π^m, so we have $\|T_{A,p}(f)\|_{L^p(A)} \le 2\|f\|_{L^p(A)}$. However, we cannot obtain (2.11) starting of the last inequality. For a different approximation class this inequality was obtained in [15].

Let $B(x, \varepsilon)$ be the ball in $\mathbb{R}^n$ of center at x and radius $\varepsilon > 0$. Henceforward, we denote $T^\varepsilon_x(f) = T_{B(x,\varepsilon),p}(f)$, for $f \in L^{p-1}$. We will need the following lemma; see Lemma 2.6 in [16].

Lemma 2.3. *Given an integer m, $m \ge 0$ there exists a function $\phi(t)$ infinitely differentiable with support in $\{|x| \le 1\}$, such that for every $\varepsilon > 0$ and every polynomial $P \in \Pi^m$*

$$P(x) = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n}\phi\left(\frac{x-y}{\varepsilon}\right)P(y)\,dy, \tag{2.12}$$

holds.

The following inequalities for polynomials will be used in the sequel.

Lemma 2.4. *For all $P \in \Pi^m$ over $\mathbb{R}^n$ we have*

(i)

$$\left(\int_{B(x,\varepsilon)} |P(t)|^{q_1}\frac{dt}{|B(x,\varepsilon)|}\right)^{\frac{1}{q_1}} \le \left(\int_{B(x,\varepsilon)} |P(t)|^{q_2}\frac{dt}{|B(x,\varepsilon)|}\right)^{\frac{1}{q_2}} \le \|P\|_{L^\infty(B(x,\varepsilon))}, \tag{2.13}$$

for $0 < q_1 < q_2$.

(ii) *There exists $C > 0$, depending only on n, m and q, $0 < q < \infty$, such that*

$$C\|P\|_{L^\infty(B(x,\varepsilon))} \le \left(\int_{B(x,\varepsilon)} |P(t)|^q\frac{dt}{|B(x,\varepsilon)|}\right)^{\frac{1}{q}}. \tag{2.14}$$

(iii) *There exists a constant $C > 0$ depending only on n and m such that*

$$\frac{1}{C}\|P\|_{L^\infty(B(x,\varepsilon))} \le \max_{|\alpha|\le m}\varepsilon^{|\alpha|}|a_\alpha| \le C\|P\|_{L^\infty(B(x,\varepsilon))}, \tag{2.15}$$

when the polynomial P is written as $P(t) = \sum_{|\alpha|\le m} a_\alpha(x)(t-x)^\alpha$.

Proof. Inequality (2.13) follows using Jensen's inequality. The same arguments used to prove the equivalence of norms in finite dimensional normed spaces ensure that there exists $C > 0$, depending only on n, m and q, $0 < q < \infty$, such that

$$\max_{|y|\le 1} |Q(y)| \le C \left(\int_{|y|\le 1} |Q(y)|^q \, dy \right)^{\frac{1}{q}}, \tag{2.16}$$

for any polynomial $Q \in \Pi^m$. Now, given a polynomial $P \in \Pi^m$ set $Q(y) = P(x + \varepsilon y)$ in Eq. (2.16) and, after a change of variable, we get (2.14).

To prove (iii) first we observe that $\|P\|_{L^\infty(B(x,\varepsilon))} \le C_{m,n} \max_{|\alpha|\le m} \varepsilon^{|\alpha|} |a_\alpha|$. To prove the right hand side of Eq. (2.15) we use Lemma 2.3. In fact Eq. (2.12) implies

$$\partial^\alpha P(x) = \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} \frac{1}{\varepsilon^{|\alpha|}} (\partial^\alpha \phi) \left(\frac{x-y}{\varepsilon} \right) P(y) \, dy. \tag{2.17}$$

Hence

$$\varepsilon^{|\alpha|} \partial^\alpha P(x) = \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^n} (\partial^\alpha \phi) \left(\frac{x-y}{\varepsilon} \right) P(y) \, dy. \tag{2.18}$$

Then

$$\begin{aligned} \varepsilon^{|\alpha|} |\partial^\alpha P(x)| &\le \|P\|_{L^\infty(B(x,\varepsilon))} \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^n} \left| (\partial^\alpha \phi) \left(\frac{x-y}{\varepsilon} \right) \right| dy \\ &\le \|P\|_{L^\infty(B(x,\varepsilon))} \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} \left| (\partial^\alpha \phi) \left(\frac{x-y}{\varepsilon} \right) \right| dy = \|P\|_{L^\infty(B(x,\varepsilon))} \int_{|y|\le 1} |(\partial^\alpha \phi)(y)| \, dy. \quad \square \end{aligned} \tag{2.19}$$

Corollary 2.5. *Let f be in $L^{p-1}_{loc}(\mathbb{R}^n)$; then*

$$\left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} |T_x^\varepsilon(f)(t)|^p \, dt \right)^{\frac{1}{p}} \le M \left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} |f(t)|^{p-1} \, dt \right)^{\frac{1}{p-1}}, \tag{2.20}$$

for every $\varepsilon > 0$ and $x \in \mathbb{R}^n$. Where the constant M depends only on m, n and p.

Proof. Using (2.14) in (2.10) we have

$$\|T_x^\varepsilon(f)\|^p_{L^p(B(x,\varepsilon))} \le \frac{K}{C} \|f\|^{p-1}_{L^{p-1}(B(x,\varepsilon))} \|T_x^\varepsilon(f)\|_{L^p(B(x,\varepsilon))},$$

and thus (2.20) follows with $M = (\frac{K}{C})^{\frac{1}{p-1}}$. $\square$

Henceforward, the constants M or C may not be the same in each occurrence.

Note that, from the statement of the last corollary and its proof, we also have

$$\|T_x^\varepsilon(f)\|_{L^\infty(B(x,\varepsilon))} \le C \left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} |f(t)|^{p-1} \, dt \right)^{\frac{1}{p-1}}, \tag{2.21}$$

for some constant C depending only on n, m and p.

Let $T_x^\varepsilon(f)(t) = \sum_{|\alpha|\le m} a_\alpha(x,\varepsilon)(t-x)^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $t^\alpha = t^{\alpha_1} \cdots t^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Corollary 2.6. *Let f be in $L^{p-1}_{loc}(\mathbb{R}^n)$; then*

$$\varepsilon^{|\alpha|} |a_\alpha(x,\varepsilon)| \le C \left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} |f(t)|^{p-1} \, dt \right)^{\frac{1}{p-1}}, \tag{2.22}$$

for every $\varepsilon > 0$, $|\alpha| \le m$ and $x \in \mathbb{R}^n$. Where the constant C depends only on n, m and p.

Proof. Using (2.14) and (2.15) we have

$$\max_{|\alpha|\le m} \varepsilon^{|\alpha|} |a_\alpha(x,\varepsilon)| \le C \left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} |T_x^\varepsilon(f)(t)|^p \, dt \right)^{\frac{1}{p}}.$$

Now (2.22) follows using (2.20). $\square$

We point out that the estimate given by the above corollary is useful only for the case $a_\alpha(x,\varepsilon)$ with $\alpha = 0$. For the remainder values of α we shall prove sharper estimates. To be more specific we shall write $a_\alpha(f)(x,\varepsilon)$ for $a_\alpha(x,\varepsilon)$. We set $T_x(f)$ for the Taylor polynomial of degree m of the function f at the point x, that is

$$T_x(f)(t) = \sum_{|\alpha|\le m} \frac{\partial^\alpha f(x)}{\alpha!}(t-x)^\alpha. \tag{2.23}$$

Now we recall the following pointwise smoothness condition which was introduced by Calderón and Zygmund in [16]. The function $f \in t_m^{p-1}(x)$ if there exists a polynomial $T_x(f)(t)$ such that

$$\left(\frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} |f(t) - T_x(f)(t)|^{p-1}\,dt\right)^{\frac{1}{p-1}} = \circ(\varepsilon^m), \tag{2.24}$$

as $\varepsilon \to 0$. It is easy to see that condition (2.24) uniquely determines the polynomial $T_x(f)$, and its coefficients, multiply by $\alpha!$, are called the partial derivative denoted by $\partial^\alpha f(x)$. Indeed, by (2.15) and taking into account that $\|P\|_{L^\infty(B(x,\varepsilon))}$ is equivalent to $\max_{|\alpha|\le m} \varepsilon^{|\alpha|}|a_\alpha|$, we have that a polynomial in Π^m that satisfies (2.24) must be the zero polynomial.

From now on, we will write $a_\alpha(f)(x,\varepsilon)$ for $a_\alpha(x,\varepsilon)$ in order to emphasize the dependence on the function f for the coefficients of the polynomial $T_x^\varepsilon(f)$. That is $T_x^\varepsilon(f)(t) = \sum_{|\alpha|\le m} a_\alpha(f)(x,\varepsilon)(t-x)^\alpha$.

Corollary 2.7. *Let $f \in L^{p-1}_{loc}(\mathbb{R}^n)$ and assume that $f \in t_m^{p-1}(x)$ in a fixed point x. Then $T_x^\varepsilon(f)(t) \to T_x(f)(t)$, for each t, as $\varepsilon \to 0$.*

Proof. Since $T_x^\varepsilon(f - T_x(f)) = T_x^\varepsilon(f) - T_x(f)$, [1], we have

$$a_\alpha(f - T_x(f))(x,\varepsilon) = a_\alpha(f)(x,\varepsilon) - \frac{\partial^\alpha f(x)}{\alpha!}. \tag{2.25}$$

Now by Corollary 2.6 applied to $f - T_x(f)$ we have

$$\varepsilon^{|\alpha|}\left|a_\alpha(f)(x,\varepsilon) - \frac{\partial^\alpha f(x)}{\alpha!}\right| \le C\left(\frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} |f(t) - T_x(f)(t)|^{p-1}\,dt\right)^{\frac{1}{p-1}}. \tag{2.26}$$

Thus, by (2.24), it follows that $a_\alpha(f)(x,\varepsilon)$ tends to $\frac{\partial^\alpha f(x)}{\alpha!}$, as ε tends to 0, for $|\alpha| \le m$. □

We point out that this pointwise convergence result is new for $f \in L^{p-1}$. If $f \in L^p$ and the differentiability assumption (2.24) is assumed in L^p instead of L^{p-1} the convergence result is well known and it has a direct proof. For our case we have used Corollary 2.6. Also even if $f \in L^p$ we may assume the weaker condition $f \in t_m^{p-1}(x)$ and still obtain the pointwise convergence.

Theorem 2.8. *Let f be in $L^{p-1}_{loc}(\mathbb{R}^n)$ and $T_x^\varepsilon(f)(t) = \sum_{|\alpha|\le m} a_\alpha(f)(x,\varepsilon)(t-x)^\alpha$; then we have*

$$|a_\beta(f)(x,\varepsilon)| \le \frac{M}{\varepsilon^l}\left(\frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)}\left|f(t) - \sum_{|\alpha|\le l-1}\frac{\partial^\alpha f(x)}{\alpha!}(t-x)^\alpha\right|^{p-1} dt\right)^{\frac{1}{p-1}}, \tag{2.27}$$

for any $|\beta| = l,\ l \le m$.

Proof. Since $T_x^\varepsilon(f - f(x)) = T_x^\varepsilon(f) - f(x)$, we have $a_0(f - f(x)) = a_0(f) - f(x)$, but $a_\alpha(f) = a_\alpha(f - f(x))$, for $|\alpha| > 0$. Then, by Corollary 2.6,

$$|a_\alpha(f)(x,\varepsilon)| \le \frac{M}{\varepsilon}\left(\frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} |f(t) - f(x)|^{p-1}\,dt\right)^{\frac{1}{p-1}}, \tag{2.28}$$

for any $|\alpha| = 1$. Thus (2.27) holds in this case.

Set $T_x^l(f)(t) = \sum_{|\alpha|\le l} \frac{\partial^\alpha f(x)}{\alpha!}(t-x)^\alpha$. Now for any $2 \le l \le m$ we have

$$T_x^\varepsilon(f - T_x^{l-1}(f)) = T_x^\varepsilon(f) - T_x^{l-1}(f),$$

and then $a_\beta(f - T_x^{l-1}(f))(x,\varepsilon)$ is equal to $a_\beta(f)(x,\varepsilon) - \frac{\partial^\beta(f)(x)}{\beta!}$, when $|\beta| \le l-1$ and $a_\beta(f - T_x^{l-1}(f))(x,\varepsilon) = a_\beta(f)(x,\varepsilon)$, for $|\beta| = l$. Now, by Corollary 2.6, we have (2.27). □

For $f \in L^q_{loc}(\mathbb{R}^n)$ and $q > 0$ we denote by $M_q(f)$ the following Hardy–Littlewood maximal function

$$M_q(f) := \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} |f(y)|^q \, dy \right)^{\frac{1}{q}}. \tag{2.29}$$

Now we introduce the following polynomial maximal functions associated to the extended approximation polynomial operator.

Definition 2.9. Let $f \in L^{p-1}_{loc}(\mathbb{R}^n)$. The polynomial maximal function $T^*_p(f)$ on $\mathbb{R}^n$ is given by

$$T^*_p(f)(x) = \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} |T^\varepsilon_x(f)(t)|^p \, dt \right)^{\frac{1}{p}}, \tag{2.30}$$

where $T^\varepsilon_x(f)(t) = \sum_{|\alpha|\le m} a_\alpha(x,\varepsilon)(t-x)^\alpha$.

Definition 2.10. Let $f \in L^{p-1}_{loc}(\mathbb{R}^n)$. We define the next maximal function $a^*_\alpha(f)$, $0 \le |\alpha| \le m$, on $\mathbb{R}^n$ as

$$a^*_\alpha(f)(x) = \sup_{\varepsilon>0} |a_\alpha(x,\varepsilon)|. \tag{2.31}$$

Note that even $a^*_\alpha(f)$ depends on p, for simplicity, we have omitted p in the notations of this operator.

Theorem 2.11. *Let $f \in L^{p-1}_{loc}(\mathbb{R}^n)$. For any $|\alpha| \le m$ the polynomial maximal functions $T^*_p(f)$ and $a^*_\alpha(f)$ are measurable.*

Proof. Given $\varepsilon > 0$ we will prove that $T^\varepsilon_x(f)(t)$ is a continuous function of x. Let $x_k \to x$ as $k \to \infty$, and assume $|x_k - x| < \varepsilon$. We have that

$$\begin{aligned} \|T^\varepsilon_{x_k}(f)\|_{L^\infty(B(x,\varepsilon))} &\le \|T^\varepsilon_{x_k}(f)\|_{L^\infty(B(x_k,2\varepsilon))} \le C \left(\frac{1}{\varepsilon^n} \int_{B(x_k,2\varepsilon)} |f(t)|^{p-1} \, dt \right)^{\frac{1}{p-1}} \\ &\le C \left(\frac{1}{\varepsilon^n} \int_{B(x,3\varepsilon)} |f(t)|^{p-1} \, dt \right)^{\frac{1}{p-1}}, \end{aligned} \tag{2.32}$$

where we have used (2.21) to obtain the second inequality.

Thus, the polynomials $\{T^\epsilon_{x_k}(f)\}$ are uniformly bounded. Suppose that $\{T^\epsilon_{x_{k_j}}(f)\}$ is a subsequence which converges to a polynomial P of degree m. For any polynomial Q of degree less than or equal to m, we denote $g_j(t) = |f - T^\epsilon_{x_{k_j}}(f)(t)|^{p-1}\mathrm{sgn}(f - T^\epsilon_{x_{k_j}}(f)(t))Q(t)$ and $g(t) = |f - P(t)|^{p-1}\mathrm{sgn}(f - P(t))Q(t)$. We have

$$\begin{aligned} \left| \int_{B(x_{k_j},\epsilon)} g_j(t)dt - \int_{B(x,\epsilon)} g(t)dt \right| &\le \left| \int_{B(x_{k_j},\epsilon)} g_j(t)dt - \int_{B(x,\epsilon)} g_j(t)dt \right| + \left| \int_{B(x,\epsilon)} g_j(t)dt - \int_{B(x,\epsilon)} g(t)dt \right| \\ &\le \int_{B(x_{k_j},\epsilon)\Delta B(x,\epsilon)} |g_j(t)|dt + \int_{B(x,\epsilon)} \left|(g_j(t) - g(t))\right| dt. \end{aligned} \tag{2.33}$$

Since each term in the last member of inequality (2.33) tends to zero, as $j \to \infty$, we get that

$$\begin{aligned} 0 &= \int_{B(x_{k_j},\epsilon)} |f - T^\epsilon_{x_{k_j}}(f)(t)|^{p-1}\mathrm{sgn}(f - T^\epsilon_{x_{k_j}}(f)(t))Q(t)dt \\ &\to \int_{B(x,\epsilon)} |f - P(t)|^{p-1}\mathrm{sgn}(f - P(t))Q(t)dt, \end{aligned} \tag{2.34}$$

for any polynomial Q of degree m. Thus $P = T^\epsilon_x(f)$. Then, $\{T^\epsilon_{x_k}(f)\}$ converges uniformly to $T^\epsilon_x(f)$.

Finally, if $f_\epsilon(x) = \frac{1}{|B(x,\epsilon)|}\left(\int_{B(x,\epsilon)} |T^\epsilon_x(f)(t)|^p dt\right)^{\frac{1}{p}}$, clearly $f_\epsilon(x)$ is a continuous function for all $\epsilon > 0$, and $a_\alpha(x,\epsilon)$ is a continuous function for all $\epsilon > 0$, $|\alpha| \le m$. Then it follows that $T^*_p(f)$ and $a^*_\alpha(f)$ are lower semi-continuous functions. Hence they are measurable functions. $\square$

Theorem 2.12. *Let* $f \in L^{p-1}_{loc}(\mathbb{R}^n)$; *then*

$$T^*_p(f)(x) \le M_{p-1}(f)(x). \tag{2.35}$$

Thus, for any $f \in L^{p-1}(\mathbb{R}^n)$ *we have*

$$|\{x \in \mathbb{R}^n : T^*_p(f)(x) > \lambda\}| \le K \frac{1}{\lambda^{p-1}} \int_{\mathbb{R}^n} |f(x)|^{p-1}\, dx, \tag{2.36}$$

where the constant K *depends only on* p, n *and* m.

Proof. Eq. (2.35) follows by (2.20) and (2.36) follows by the well known property for the Hardy–Littlewood maximal function; see [17]. □

As an application of the above theorem, we prove the following result.

Theorem 2.13. *Let* $f \in L^{p-1}_{loc}(\mathbb{R}^n)$ *with continuous derivatives* $\partial^\beta f$, $|\beta| \le m$. *Then*

$$a^*_\beta(f) \le CM_1(\partial^\beta f)(x), \tag{2.37}$$

for $1 \le |\beta| \le m$ *and* $p - 1 < 1$,

$$a^*_\beta(f) \le CM_{p-1}(\partial^\beta f)(x), \tag{2.38}$$

for $1 \le |\beta| \le m$ *and* $p - 1 \ge 1$.

Also we have

$$a^*_0(f) \le CM_{p-1}(f)(x). \tag{2.39}$$

Thus, for any $f \in L^{p-1}(\mathbb{R}^n)$ *we have*

$$|\{x \in \mathbb{R}^n : a^*_0(f)(x) > \lambda\}| \le C \frac{1}{\lambda^{p-1}} \int_{\{x\in\mathbb{R}^n : |f(x)| > \frac{\lambda}{2}\}} |f(x)|^{p-1}\, dx, \tag{2.40}$$

and

$$|\{x \in \mathbb{R}^n : a^*_\beta(f)(x) > \lambda\}| \le C \frac{1}{\lambda} \int_{\{x\in\mathbb{R}^n : |\partial^\beta f(x)| > \frac{\lambda}{2}\}} |\partial^\beta f(x)|\, dx, \tag{2.41}$$

for $1 \le |\beta| \le m$, *and the constant* C *depends only on* p, n *and* m.

Proof. Eq. (2.39) follows by (2.22). Now by Theorem 2.8 we have, for $|\beta| = l$,

$$|a_\beta(f)(x,\varepsilon)| \le \frac{C}{\varepsilon^l} \left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \left| f(t) - \sum_{|\alpha|\le l-1} \frac{\partial^\alpha f(x)}{\alpha!}(t-x)^\alpha \right|^{p-1} dt \right)^{\frac{1}{p-1}}. \tag{2.42}$$

Now, using the integral form for the remainder in the Taylor expansion we have that $|a_\beta(f)(x,\varepsilon)|$ is bounded by

$$C\frac{1}{\varepsilon^l} \left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \left| \sum_{|\alpha|=l} \frac{l}{\alpha!}(t-x)^\alpha \int_0^1 \partial^\alpha f(x + s(t-x))(1-s)^l\, ds \right|^{p-1} dt \right)^{\frac{1}{p-1}}. \tag{2.43}$$

As $|(t-x)^\alpha| \le \varepsilon^{|\alpha|}$ in the above integral and using (2.2) we have

$$|a_\beta(f)(x,\varepsilon)| \le C \sum_{|\alpha|=l} \left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \left(\int_0^1 |\partial^\alpha f(x + s(t-x))|\, ds \right)^{p-1} dt \right)^{\frac{1}{p-1}}. \tag{2.44}$$

Now for $p - 1 > 1$ we use the Minkovski integral inequality

$$\left(\int_{B(x,\varepsilon)} \left(\int_0^1 |\partial^\alpha f(x + s(t-x))|\, ds \right)^{p-1} \frac{dt}{\varepsilon^n} \right)^{\frac{1}{p-1}} \le \int_0^1 \left(\int_{B(x,\varepsilon)} |\partial^\alpha f(x + s(t-x))|^{p-1} \frac{dt}{\varepsilon^n} \right)^{\frac{1}{p-1}} ds,$$

in (2.44) and we get

$$|a_\beta(f)(x,\varepsilon)| \le C \sum_{|\alpha|=l} \int_0^1 \left(\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} |\partial^\alpha f(x+s(t-x))|^{p-1}\, dt \right)^{\frac{1}{p-1}} ds$$

$$= C \sum_{|\alpha|=l} \int_0^1 \left(\frac{1}{(s\varepsilon)^n} \int_{B(x,s\varepsilon)} |\partial^\alpha f(y)|^{p-1}\, dy \right)^{\frac{1}{p-1}} ds \le C \sum_{|\alpha|=l} M_{p-1}\left(|\partial^\alpha f|\right)(x), \tag{2.45}$$

for $|\beta| = l$. Thus we have proved (2.38). Now for $p-1<1$ we use Jensen's inequality in (2.44) and we get

$$|a_\beta(f)(x,\varepsilon)| \le C \sum_{|\alpha|=l} \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \int_0^1 |\partial^\alpha f(x+s(t-x))|\, ds\, dt. \tag{2.46}$$

Now we proceed as we did in (2.45) and we have (2.37).

The inequalities (2.40) and (2.41) follow by the standard properties of the Hardy–Littlewood maximal function. □

We point out that in the previous theorems we obtained the inequalities for the polynomial maximal functions for $f \in L^{p-1}_{loc}$, $1<p$. Additional inequalities can be obtained when f is also in $L^q(\mathbb{R}^n)$, $q>p-1$.

Corollary 2.14. *For every $f \in L^q(\mathbb{R}^n)$ we have*

$$\|a_0^*(f)\|_q \le C_q \|f\|_q, \tag{2.47}$$

for $q>p-1$.

Beside if for any $1 \le |\alpha| \le m$ the derivatives $\partial^\alpha f$ are in $L^q(\mathbb{R}^n)$, we have

$$\|a_\alpha^*(f)\|_q \le C_q \|\partial^\alpha f\|_q, \tag{2.48}$$

for every $q>1$ if $p-1 \le 1$ and $q>p-1$ if $p-1 \ge 1$.

Proof. The inequalities (2.47) and (2.48) follow using the arguments of the standard Marcinkiewicz interpolation theorem; see [17]. □

Note that in the case $p-1>1$ we obtain Eq. (2.48) for the right range of q, i.e. $p-1<q$. On the other hand, we do not know if inequality (2.37) can be improved for the case $0<p-1<1$, as follows $a_\alpha^*(f) \le CM_{p-1}(\partial^\alpha f)(x)$, in order to obtain the same range of q.

By Corollary 2.7 for a smooth function f we have $|a_\alpha(x,\varepsilon) - \frac{1}{\alpha!}\partial^\alpha f(x)|$ tends to 0 for every $x \in \mathbb{R}^n$ as ε tends to 0. Now, by Corollary 2.14, we can use Lebesgue's dominated convergence theorem to obtain the next remark.

Remark 2.15. For every $f \in L^q(\mathbb{R}^n)$ such that the derivatives $\partial^\alpha f$ are in $L^q(\mathbb{R}^n)$, $|\alpha| \le m$, we have

$$\left\| a_\alpha(\cdot,\varepsilon) - \frac{1}{\alpha!}\partial^\alpha f(\cdot) \right\|_{L^q(\mathbb{R}^n)} \to 0, \tag{2.49}$$

as ε tends to 0, for $1 \le |\alpha| \le m$ and for every $q>1$ if $p-1 \le 1$ and $q>p-1$ if $p-1 \ge 1$. And for the case $\alpha = 0$ we always consider $q>p-1$.

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