

The determinant of the distance matrix of graphs with at most two cycles

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Abstract

Let G be a connected graph on n vertices and $D(G)$ its distance matrix. The formula for computing the determinant of this matrix in terms of the number of vertices is known when the graph is either a tree or a unicyclic graph. In this work we generalize these results, obtaining the determinant of the distance matrix for all graphs in a class, including trees, unicyclic and bicyclic graphs. This class actually includes graphs with many cycles, provided that each block of the graph is at most bicyclic.

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1. Introduction

A graph $G = (V, E)$ consists of a set V of *vertices* and a set E of *edges*. We will consider graphs without multiple edges and without loops. Let G be a connected graph on n vertices with vertex set $V = \{v_1, \dots, v_n\}$. The *distance between vertices* v_i and v_j , denoted $d(v_i, v_j)$, is the number of edges of a shortest path from v_i to v_j . The *distance matrix* of G , denoted $D(G)$, is the $n \times n$ symmetric matrix having its (i, j) -entry equal to $d(v_i, v_j)$. We also use $d_{i,j}$ to denote $d(v_i, v_j)$.

The distance matrix has been widely studied in the literature. The interest in this matrix was motivated by the connection with a communication problem (see [3, 5] for more details). In an early article, [3], Graham and Pollack presented a remarkable result, proving that the determinant of the distance matrix of a tree T on n vertices only depends on n , being equal to $(-1)^{n-1}(n-1)2^{n-2}$. This result was generalized by Graham, Hoffman, and Hosoya in 1977 [4], who proved that, for any graph G , the determinant of $D(G)$ depends only on the blocks of G .

In 2005, more than 30 years after the result of Graham and Pollack on trees, Bapat, Kirkland and Neumann [1] exhibited a formula for the determinant of the distance matrix of a unicyclic graph. Specifically, they proved that the determinant is zero when its only cycle has an even number of edges, whereas if the graph has $2k+1+m$ vertices and a cycle with $2k+1$ edges, the determinant is equal to $(-2)^m \left[k(k+1) + \frac{2k+1}{2}m \right]$.

For a bicyclic graph, the determinant can be easily computed in the case where the cycles have no common edges, since its blocks are edges and cycles. In a conference article [2], we presented some advances for the remaining cases; i.e., when the cycles share at least one edge. Besides, we conjectured the formula for the remaining cases. In the present article, we completely solve these conjectures, extending the formula of the determinant of $D(G)$ to graphs G having bicyclic blocks as well as trees and unicyclic blocks.

This paper is organized as follows. In Section 2 we present some basic nota-

tions, preliminary results, and we briefly describe previous results in connection with the determinant of the distance matrix of a bicyclic graph. In Sections 3 we consider the determinant of the distance matrix of a θ -graph, a θ -graph plus a pendant vertex and a θ -graph attached to a path, where the definition of a θ -graph is stated in Section 2. In the last theorem, we present a formula for the determinant of a graph arised from a tree by the addition of at most two edges (graphs at most bicyclic).

2. Definitions and preliminary results

A *tree* is a connected acyclic graph. A *unicyclic graph* is a connected graph with as many edges as vertices. The path and the cycle on n vertices are denoted by C_n and P_n , respectively.

The determinant and the cofactor of the distance matrix of a cycle are known and they are given in the lemma below. We remember that the cofactor for any square matrix A , denoted by $\text{cof}(A)$, is the sum of the cofactors of A .

Lemma 1 ([1, 7]). *For each $n \geq 3$:*

- if n is odd, $\det D(C_n) = (n^2 - 1)/4$ and $\text{cof } D(C_n) = n$;
- if n is even, $\det D(C_n) = 0$ and $\text{cof } D(C_n) = 0$.

In [1] the determinant of $D(G)$ was obtained when G is a unicyclic graph.

Theorem 1 ([1]). *Let G be a unicyclic graph consisting of a cycle of length l plus m edges outside the cycle. If l is even, then $\det D(G) = 0$; otherwise:*

$$\det D(G) = (-2)^m \frac{l^2 + 2ml - 1}{4}$$

A *cut-vertex* of a connected graph is a vertex whose removal disconnects the graph. A *block of a graph* G is a maximal connected subgraph of G having no cut-vertices. A *block* is a connected graph having no cut-vertices.

In [4] it was proved that if the blocks of a graph G are G_1, G_2, \dots, G_k , then $\det D(G)$ depends only on the $\det D(G_1), \det D(G_2), \dots, \det D(G_k)$ and $\text{cof } D(G_1), \text{cof } D(G_2), \dots, \text{cof } D(G_k)$.

Theorem 2 ([4]). *If G is a connected graph whose blocks are G_1, G_2, \dots, G_k , then*

$$\det D(G) = \sum_{i=1}^k \det D(G_i) \prod_{j \in \{1, 2, \dots, k\} - \{i\}} \operatorname{cof} D(G_j)$$

and

$$\operatorname{cof} D(G) = \prod_{i=1}^k \operatorname{cof} D(G_i).$$

55 A *cactus* is a graph in which each two cycles have at most one vertex in common. By definition, every unicyclic graph is a cactus. Moreover, each block of a cactus on at least two vertices is either an edge or a cycle. As $\det D(G)$ depends only on the blocks of G and $\det D$ and $\operatorname{cof} D$ are known for an edge and for the cycles, we obtain the next corollary as an immediate consequence
60 of Lemma 1 and Theorem 2.

Corollary 1. *Let G be a connected cactus having precisely c cycles whose lengths are l_1, l_2, \dots, l_c plus m other edges outside these cycles.*

- *If some of l_1, l_2, \dots, l_c is even, then $\det D(G) = 0$.*
- *Otherwise (i.e., if all of l_1, l_2, \dots, l_c are odd),*

$$\det D(G) = (-2)^m \left(\prod_{i=1}^c l_i \right) \left(\frac{m}{2} + \sum_{i=1}^c \frac{l_i^2 - 1}{4l_i} \right).$$

A *bicyclic graph* is a graph obtained by adding an edge to a unicyclic graph.
65 The special case of $c = 2$ in the formula of the above corollary was also obtained in [6] by alternative means, corresponding to a special class of bicyclic graphs.

As $\det D$ for all cacti is known, in order to find $\det D$ for all bicyclic graphs, it is enough to find $\det D$ and $\operatorname{cof} D$ for bicyclic blocks.

Definition 1. *Let $P_{l+1}, P_{p+1}, P_{q+1}$ be three vertex disjoint paths, $l \geq 1$ and
70 $p, q \geq 2$, each of them having endpoints, $v_1^l, v_2^l, v_1^p, v_2^p, v_1^q, v_2^q$, respectively. We denote by $\theta(l, p, q)$ -graph, or simply θ -graph, the graph obtained by identifying the vertices v_1^l, v_1^p, v_1^q as one vertex, and proceeding in the same way for v_2^l, v_2^p, v_2^q .*

Note that $\theta(l, p, q)$ -graph is a bicyclic graph, with no pendant edge, whose cycles share at least one edge. In [2], we proved the following results:

Proposition 1 ([2, Lemma 3.1]). For every positive integer k ,

$$\det D(\theta(2, 2, 2k + 1)) = 4(k^2 + k - 1).$$

75 **Proposition 2** ([2, Lemma 3.2]). Let G be one of the graphs bellow:

- $\theta(1, 2k - 1, 2k - 1)$, for $k \geq 2$;
- $\theta(2, 2, 2k - 2)$, for $k \geq 3$;
- $\theta(l, p, q)$, for $l \geq 2$, $p \geq 3$, and $q \geq 3$.

Then, $\det D(G) = 0$.

80 3. Bicyclic graphs

The next theorem gives the determinant of $D(G)$ when $G = \theta(l, p, q)$, completing the remaining cases in [2].

Theorem 3. The following assertions hold:

- (a) If $G = \theta(1, p, q)$ for even integers p and q , then $\det D(G) = \frac{-(p+q)^2}{4}$.
- 85 (b) If $G = \theta(2, 2, 2)$, then $\det D(G) = -16$.
- (c) If $G = \theta(2, 2, q)$ for some odd integer $q > 1$, then $\det D(G) = q^2 - 5$.
- (d) Otherwise, $\det D(G) = 0$.

Proof. Items (c) and (d) have been proven in [2] and correspond to Proposition 1 and Proposition 2 respectively. Case (b) can be easily computed. The proof
90 of case (a) will be divided in the following 2 cases:

Case 1:

Let $G = \theta(1, 2, 2k)$, for some $k \geq 1$, with its vertices labeled as in Figure 1.

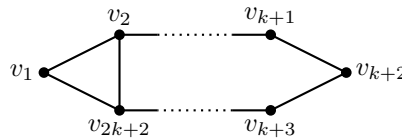


Figure 1: $\theta(1, 2, 2k)$

The distance matrix of $\theta(1, 2, 2k)$ is

$$D(\theta(1, 2, 2k)) = \begin{pmatrix} 0 & v^t \\ v & D(C_{2k+1}) \end{pmatrix},$$

where $D(C_{2k+1})$ is the distance matrix of the cycle induced by the vertices v_2, \dots, v_{2k+2} and $v^t = (1, 2, \dots, k, k+1, k, \dots, 2, 1)$.

From [1], we know that

$$D(C_{2k+1})^{-1} = -2I - C^k - C^{k+1} + \frac{2k+1}{k(k+1)}J, \quad (1)$$

and $\det D(C_{2k+1}) = k(k+1)$, where J is the all ones matrix, with appropriate size, and C is the cyclic permutation matrix of order $2k+1$ having $C_{i,i+1} = 1$ for $i = 1, \dots, 2k+1$, taking indices modulo $2k+1$. Therefore, we have that

$$D(\theta(1, 2, 2k))^{-1} = M_1^t M_2 M_1, \quad (2)$$

95 where

$$M_1 = \begin{pmatrix} 1 & -v^t D(C_{2k+1})^{-1} \\ 0 & I \end{pmatrix},$$

$$M_2 = \begin{pmatrix} (-v^t D(C_{2k+1})^{-1} v)^{-1} & 0 \\ 0 & D(C_{2k+1})^{-1} \end{pmatrix},$$

and

$$\begin{aligned} \det D(\theta(1, 2, 2k)) &= \det M_2^{-1} = -v^t D(C_{2k+1})^{-1} v \det D(C_{2k+1}) \\ &= -v^t D(C_{2k+1})^{-1} v k(k+1). \end{aligned} \quad (3)$$

Now we will calculate $v^t D(C_{2k+1})^{-1} v$, using equation (1) we obtain

$$\begin{aligned} v^t D(C_{2k+1})^{-1} v &= -2v^t v - v^t C^k v - v^t C^{k+1} v + \frac{2k+1}{k(k+1)} v^t J v \\ &= -4 \sum_{i=1}^k i^2 - 2(k+1)^2 - 2 \sum_{i=1}^k i(k+1-i) \\ &\quad - 2 \sum_{i=1}^{k+1} i(k+2-i) + \frac{2k+1}{k(k+1)} (k+1)^4 \\ &= -2 \sum_{i=1}^k i(k+1) - 2 \sum_{i=1}^{k+1} i(k+2) + \frac{(2k+1)(k+1)^3}{k} \\ &= -k(k+1)^2 - (k+1)(k+2)^2 + \frac{(2k+1)(k+1)^3}{k} = \frac{k+1}{k}. \end{aligned} \quad (4)$$

Combining this result with (3), we deduce that

$$\det D(\theta(1, 2, 2k)) = -(k+1)^2 = -\frac{(2k+2)^2}{4} = -n^2(-2)^{-2}, \quad (5)$$

with $n = p + q$, where $p = 2$ and $q = 2k$.

Case 2:

100 Let $H = \theta(1, 2s, 2k)$ and $G = \theta(1, 2(s-1), 2(k+1))$, for some $k \geq 2$ and $s \geq 2$, with its vertices labeled as in Figure 2 and Figure 3 respectively.

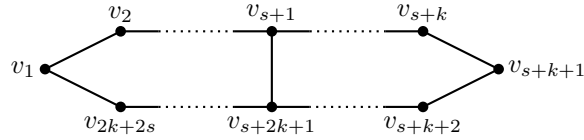


Figure 2: $\theta(1, 2s, 2k)$

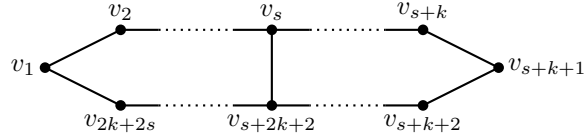


Figure 3: $\theta(1, 2(s-1), 2(k+1))$

The distance matrices of G and H are

$$D(G) = \begin{pmatrix} P & A^t \\ A & P \end{pmatrix} \quad \text{and} \quad D(H) = \begin{pmatrix} P & B^t \\ B & P \end{pmatrix},$$

where

$$P = \sum_{i=1}^{k+s} \sum_{j=1}^{k+s} |i-j| e_i e_j^t, \quad (6)$$

is the distance matrix of P_{k+s} (the path on $k + s$ vertices), and e_i denotes a vector having an entry equal to 1 on the i -th coordinate and 0's in the remaining

105 coordinates. Moreover,

$$\begin{aligned}
B^t &= \sum_{j=1}^{k+s} (k+s+1-j)e_1 e_j^t + \sum_{i=2}^{k+s} (k+s+1-i)e_i e_1^t \\
&+ \sum_{i=2}^{s+1} \sum_{j=2}^{k+1} (s+k+3-j-i)e_i e_j^t + \sum_{i=s+2}^{s+k} \sum_{j=k+2}^{k+s} (j+i-s-k-1)e_i e_j^t \\
&+ \sum_{i=3}^{s+1} s e_i e_{i+k-1}^t + \sum_{i=2}^{s+1} \sum_{\substack{j=k+2 \\ j \neq i+k-1}}^{k+s} |r_{2s+1}(1-k+j-i) - s - 1| e_i e_j^t \\
&+ \sum_{i=s+2}^{s+k} k e_i e_{i-s}^t + \sum_{i=s+2}^{s+k} \sum_{\substack{j=2 \\ j \neq i-s}}^{k+1} |r_{2k+1}(s+j-i) - k - 1| e_i e_j^t \tag{7}
\end{aligned}$$

and

$$\begin{aligned}
A^t &= \sum_{j=1}^{k+s} (k+s+1-j)e_1 e_j^t + \sum_{i=2}^{k+s} (k+s+1-i)e_i e_1^t \\
&+ \sum_{i=2}^s \sum_{j=2}^{k+2} (s+k+3-j-i)e_i e_j^t + \sum_{i=s+1}^{s+k} \sum_{j=k+3}^{k+s} (j+i-s-k-1)e_i e_j^t \\
&+ \sum_{i=3}^s (s-1) e_i e_{i+k}^t + \sum_{i=2}^s \sum_{\substack{j=k+3 \\ j \neq i+k}}^{k+s} |r_{2s-1}(j-k-i) - s| e_i e_j^t \\
&+ \sum_{i=s+1}^{s+k} (k+1) e_i e_{i-s+1}^t + \sum_{i=s+1}^{s+k} \sum_{\substack{j=2 \\ j \neq i-s+1}}^{k+2} |r_{2k+3}(s+j-i-1) - k - 2| e_i e_j^t, \tag{8}
\end{aligned}$$

where $r_\alpha(\beta)$ represent the remainder when integer β is divided by α .

It is easy to see that P is invertible and

$$\begin{aligned}
P^{-1} &= -\frac{k+s-2}{2(k+s-1)} e_1 e_1^t - \frac{k+s-2}{2(k+s-1)} e_{k+s} e_{k+s}^t - \sum_{i=2}^{k+s-1} e_i e_i^t \\
&+ \sum_{i=1}^{k+s-1} \frac{1}{2} e_i e_{i+1}^t + \sum_{i=2}^{k+s} \frac{1}{2} e_i e_{i-1}^t \\
&+ \frac{1}{2(k+s-1)} e_1 e_{k+s}^t + \frac{1}{2(k+s-1)} e_{k+s} e_1^t.
\end{aligned}$$

We define

$$N := \begin{pmatrix} I & 0 \\ (A - MB)P^{-1} & M \end{pmatrix},$$

where

$$M := e_1 e_1^t + e_2 e_{k+1}^t - e_2 e_{k+s}^t + \sum_{i=2}^{k+s} e_i e_{i-1}^t.$$

We claim that

$$D(G) = N \cdot D(H) \cdot N^t. \tag{9}$$

Indeed, it is easy to see that

$$N \cdot D(H) \cdot N^t = \begin{pmatrix} P & A^t \\ A & \widehat{P} \end{pmatrix}.$$

where

$$\widehat{P} = A P^{-1} (A^t - B^t M^t) + (A - MB) P^{-1} B^t M^t + M P M^t.$$

Hence, it is sufficient to prove that $\widehat{P} = P$. We first compute $M P M^t$. Since

$$M^t = e_1 e_1^t + e_{k+1} e_2^t - e_{k+s} e_2^t + \sum_{i=2}^{k+s} e_{i-1} e_i^t,$$

we have

$$M P = \sum_{j=1}^{k+s} (j-1) e_1 e_j^t + \sum_{j=1}^{k+s} (|k+1-j| + 2j - 1 - k - s) e_2 e_j^t + \sum_{i=3}^{k+s} \sum_{j=1}^{k+s} |i-1-j| e_i e_j^t$$

and

$$\begin{aligned} M P M^t &= (1-s) e_2 e_1^t + (1-s) e_1 e_2^t + 4(1-s) e_2 e_2^t & (10) \\ &+ \sum_{i=3}^{k+s} (i-2) e_i e_1^t + \sum_{j=3}^{k+s} (j-2) e_1 e_j^t \\ &+ \sum_{j=3}^{k+s} (|k+2-j| + 2j - 3 - k - s) e_2 e_j^t \\ &+ \sum_{i=3}^{k+s} (|k+2-i| + 2i - 3 - k - s) e_i e_2^t \\ &+ \sum_{i=3}^{k+s} \sum_{j=3}^{k+s} |i-j| e_i e_j^t. \end{aligned}$$

110 We continue obtaining $A^t - B^t M^t$, multiplying B^t with M^t we have

$$\begin{aligned}
B^t M^t &= B^t e_1 e_1^t + B^t e_{k+1} e_2^t - B^t e_{k+s} e_2^t + \sum_{i=2}^{k+s} B^t e_{i-1} e_i^t \\
&= \sum_{i=1}^{k+s} (k+s+1-i) e_i e_1^t + (k+2s-1) e_1 e_2^t + \sum_{j=3}^{k+s} (k+s+2-j) e_1 e_j^t \\
&\quad + \sum_{i=2}^s (k+2s+3-3i) e_i e_2^t + \sum_{i=s+1}^{k+s} (k+2-i) e_i e_2^t \\
&\quad + \sum_{i=2}^{s+1} \sum_{j=3}^{k+2} (s+k+4-j-i) e_i e_j^t + \sum_{i=s+2}^{s+k} \sum_{j=k+3}^{k+s} (j+i-s-k-2) e_i e_j^t \\
&\quad + \sum_{i=3}^s s e_i e_{i+k}^t + \sum_{i=2}^{s+1} \sum_{\substack{j=k+3 \\ j \neq i+k}}^{k+s} |r_{2s+1}(j-i-k) - s - 1| e_i e_j^t \\
&\quad + \sum_{i=s+2}^{s+k} k e_i e_{i-s+1}^t + \sum_{i=s+2}^{s+k} \sum_{\substack{j=3 \\ j \neq i-s+1}}^{k+2} |r_{2k+1}(s+j-i-1) - k - 1| e_i e_j^t.
\end{aligned}$$

From this, we deduce that

$$A^t - B^t M^t = \sum_{i=1}^s (2i-2-s) e_i e_2^t + \sum_{i=s+1}^{s+k} s e_i e_2^t + \sum_{i=s+1}^{s+k} \sum_{j=3}^{k+s} e_i e_j^t - \sum_{i=1}^s \sum_{j=3}^{k+s} e_i e_j^t.$$

It follows that

$$\begin{aligned}
P^{-1}(A^t - B^t M^t) &= \left(-\frac{k+s-2}{2(k+s-1)} e_1 e_1^t + \frac{1}{2} e_1 e_2^t + \frac{1}{2(k+s-1)} e_1 e_{k+s}^t \right. \\
&\quad + \sum_{i=2}^{k+s-1} \frac{1}{2} e_i e_{i+1}^t - \sum_{i=2}^{k+s-1} e_i e_i^t + \sum_{i=2}^{k+s-1} \frac{1}{2} e_i e_{i-1}^t \\
&\quad + \frac{1}{2(k+s-1)} e_{k+s} e_1^t + \frac{1}{2} e_{k+s} e_{k+s-1}^t - \frac{k+s-2}{2(k+s-1)} e_{k+s} e_{k+s}^t \left. \right) \\
&\quad \cdot \left(\sum_{i=1}^s (2i-2-s) e_i e_2^t + \sum_{i=s+1}^{s+k} s e_i e_2^t + \sum_{i=s+1}^{s+k} \sum_{j=3}^{k+s} e_i e_j^t - \sum_{i=1}^s \sum_{j=3}^{k+s} e_i e_j^t \right) \\
&= e_1 e_2^t + \sum_{i=3}^{k+s} e_s e_i^t - \sum_{i=2}^{k+s} e_{s+1} e_i^t.
\end{aligned}$$

Finally, we see that

$$\begin{aligned}
AP^{-1}(A^t - B^t M^t) &= A \left(e_1 e_2^t + \sum_{i=3}^{k+s} e_s e_i^t - \sum_{i=2}^{k+s} e_{s+1} e_i^t \right) \\
&= s e_1 e_2^t + (s-2) e_2 e_2^t \\
&\quad + \sum_{i=3}^{k+1} (s-3) e_i e_2^t + \sum_{i=k+2}^{k+s} (s+2k+1-2i) e_i e_2^t \\
&\quad + \sum_{j=3}^{k+s} e_1 e_j^t - \sum_{i=3}^{k+s} \sum_{j=3}^{k+s} e_i e_j^t,
\end{aligned}$$

and

$$\begin{aligned}
(A - MB)P^{-1}B^tM^t &= \left(e_2e_1^t + \sum_{i=3}^{k+s} e_i e_s^t - \sum_{i=2}^{k+s} e_i e_{s+1}^t \right) B^t M^t \\
&= s e_2 e_1^t + (3s - 2) e_2 e_2^t \\
&\quad + \sum_{j=3}^{k+1} (s - 1) e_2 e_j^t + \sum_{j=k+2}^{k+s} (s + 2k + 3 - 2j) e_2 e_j^t \\
&\quad + \sum_{i=3}^{k+s} e_i e_1^t + \sum_{i=3}^{k+s} 2e_i e_2^t + \sum_{i=3}^{k+s} \sum_{j=3}^{k+s} e_i e_j^t.
\end{aligned}$$

115 Thus

$$\begin{aligned}
AP^{-1}(A^t - B^tM^t) + (A - MB)P^{-1}B^tM^t &= s e_1 e_2^t + s e_2 e_1^t + 4(s - 1) e_2 e_2^t \\
&\quad + \sum_{j=3}^{k+s} e_1 e_j^t + \sum_{i=3}^{k+s} e_i e_1^t \\
&\quad + \sum_{i=3}^{k+s} (s + k + 1 - i - |k + 2 - i|) e_i e_2^t \\
&\quad + \sum_{j=3}^{k+s} (s + k + 1 - j - |k + 2 - j|) e_2 e_j^t.
\end{aligned}$$

Therefore, by (10), we obtain

$$\widehat{P} = AP^{-1}(A^t - B^tM^t) + (A - MB)P^{-1}B^tM^t + MPM^t = P.$$

This complete the proof of (9). Futhermore, since $\det N \cdot \det N^t = 1$, we deduce that

$$\det D(G) = \det D(H).$$

Combining this with (5), using an inductive argument, we have

$$\det D(\theta(1, 2s, 2k)) = -n^2(-2)^{-2},$$

with $n = p + q$, where $p = 2s$ and $q = 2k$. □

In order to compute $\text{cof } D(\theta(l, p, q))$, we need firstly compute the determinant of the graphs defined as follows.

Definition 2. For each positive integers l, p, q such that at most one of them is 1, we denote by $\theta'(l, p, q)$ any graph that arises from $\theta(l, p, q)$ by adding a pendant edge (see Figures 4 and 5).
120

Since every graph $\theta'(l, p, q)$ has as blocks $\theta(l, p, q)$ and one edge, it follows from Theorem 2 that $\det D(\theta'(l, p, q))$ and $\text{cof } D(\theta'(l, p, q))$ are well defined (i.e., they are independent of the vertex of $\theta(l, p, q)$ to which the pendant edge is attached in order to obtain $\theta'(l, p, q)$). Moreover, from Theorem 2, it follows that

$$\text{cof } D(\theta(l, p, q)) = -2 \det D(\theta(l, p, q)) - \det D(\theta'(l, p, q)).$$

Theorem 4. *Let $G = \theta'(l, p, q)$, for integers l, p, q such that at most one of them is 1. Then, the following assertions hold:*

- (a) $G = \theta'(1, p, q)$ for some even integers p and q , then $\det D(G) = \frac{(1+p+q)^2-1}{2}$
- 125 (b) If $G = \theta'(2, 2, 2)$, then $\det D(G) = -16$.
- (c) If $G = \theta'(2, 2, q)$ for some odd integer $q > 1$, then $\det D(G) = -2(q^2 + 2q - 9)$.
- (d) Otherwise, $\det D(G) = 0$.

Proof. Once more, items (c) and (d) have already been proven in [2] and (b) can
 130 be computed directly. The proof of case (a) will be divided in the following 2 cases. All along this proof, $\theta'(l, p, q)$ denotes the graph that arises from $\theta(l, p, q)$ by adding a pendant edge incident precisely to the midpoint of the path of length p joining the two vertices of degree 3 of $\theta(l, p, q)$. Notice that in Figures 4 and 5 such midpoint is the vertex v_1 .

135 **Case 1:**

Let $G = \theta'(1, 2, 2k)$, for some $k \geq 1$, with its vertices labeled as in Figure 4.

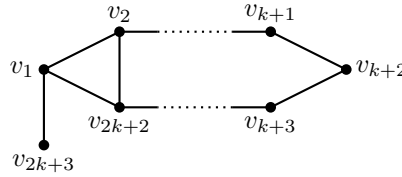


Figure 4: $\theta'(1, 2, 2k)$

The distance matrix of $\theta'(1, 2, 2k)$ is

$$D(\theta'(1, 2, 2k)) = \begin{pmatrix} 0 & v^t & 1 \\ v & D(C_{2k+1}) & v + \mathbf{1} \\ 1 & v^t + \mathbf{1}^t & 0 \end{pmatrix},$$

where $D(C_{2k+1})$ is the distance matrix of the cycle induced by the vertices v_2, \dots, v_{2k+2} and $v^t = (1, 2, \dots, k, k+1, k, \dots, 2, 1)$. By (2), we have that

$$\begin{pmatrix} 0 & v^t \\ v & D(C_{2k+1}) \end{pmatrix}^{-1} = M_1^t M_2 M_1.$$

If we define

$$M_3 := \begin{pmatrix} I & 0 \\ -w^t M_1^t M_2 M_1 & 1 \end{pmatrix},$$

$$M_4 := \begin{pmatrix} M_1^t M_2 M_1 & 0 \\ 0 & (-w^t M_1^t M_2 M_1 w)^{-1} \end{pmatrix},$$

with $w^t := (1, v^t + \mathbf{1}^t)$, then

$$D(G)^{-1} = M_3^t M_4 M_3$$

and

$$\det D(G) = \det M_4^{-1} = -w^t M_1^t M_2 M_1 w \det \begin{pmatrix} 0 & v^t \\ v & D(C_{2k+1}) \end{pmatrix}.$$

Combining this result with (5), we conclude that

$$\det D(G) = w^t M_1^t M_2 M_1 w (k+1)^2.$$

Now we will calculate $w^t M_1^t M_2 M_1 w$, by (1) we obtain

$$\begin{aligned}
w^t M_1^t M_2 M_1 w &= (0, v^t) M_1^t M_2 M_1 \begin{pmatrix} 0 \\ v \end{pmatrix} + 2 (0, v^t) M_1^t M_2 M_1 \begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} \\
&\quad + (1, \mathbf{1}^t) M_1^t M_2 M_1 \begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} \\
&= (0, v^t) \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} + 2 (1, \mathbf{0}^t) \begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} + (1, \mathbf{1}^t) M_1^t M_2 M_1 \begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} \\
&= 2 + (1, \mathbf{1}^t) M_1^t M_2 M_1 \begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} \\
&= 2 + \frac{v^t D(C_{2k+1})^{-1} v \mathbf{1}^t D(C_{2k+1})^{-1} \mathbf{1} - (v^t D(C_{2k+1})^{-1} \mathbf{1} - 1)^2}{v^t D(C_{2k+1})^{-1} v}.
\end{aligned}$$

By (1) we have that

$$\begin{aligned}
v^t D(C_{2k+1})^{-1} \mathbf{1} &= -2v^t \mathbf{1} - v^t C^k \mathbf{1} - v^t C^{k+1} \mathbf{1} + \frac{2k+1}{k(k+1)} v^t J \mathbf{1} \\
&= -8 \sum_{i=1}^k i - 4(k+1) + \frac{2k+1}{k(k+1)} (2k+1) (2 \sum_{i=1}^k i + (k+1)) \\
&= -4(k+1)^2 + \frac{(2k+1)^2}{k} (k+1) = \frac{k+1}{k},
\end{aligned}$$

140 and

$$\begin{aligned}
\mathbf{1}^t D(C_{2k+1})^{-1} \mathbf{1} &= -2\mathbf{1}^t \mathbf{1} - \mathbf{1}^t C^k \mathbf{1} - \mathbf{1}^t C^{k+1} \mathbf{1} + \frac{2k+1}{k(k+1)} \mathbf{1}^t J \mathbf{1} \\
&= -4(2k+1) + \frac{2k+1}{k(k+1)} (2k+1)^2 \\
&= \frac{2k+1}{k(k+1)}
\end{aligned}$$

Thus, by (4), we deduce that

$$w^t M_1^t M_2 M_1 w = 2 + \frac{\frac{k+1}{k} \frac{2k+1}{k(k+1)} - \left(\frac{k+1}{k} - 1\right)^2}{\frac{k+1}{k}} = \frac{2k+4}{k+1}.$$

Finally, we obtain

$$\det D(\theta'(1, 2, 2k)) = (2k+4)(k+1) = -n(n+2m)(-2)^{m-2}, \quad (11)$$

with $n = p + q$ and $m = 1$, where $p = 2$ and $q = 2s$.

Case 2:

Let $\widehat{H} = \theta'(1, 2s, 2k)$ and $\widehat{G} = \theta'(1, 2(s-1), 2(k+1), 1)$ be the graphs with its vertices labeled as in Figure 5 and Figure 6, respectively, for some $k \geq 2$

145 $s \geq 2$.

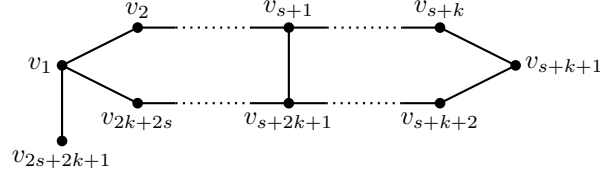


Figure 5: $\theta'(1, 2s, 2k)$

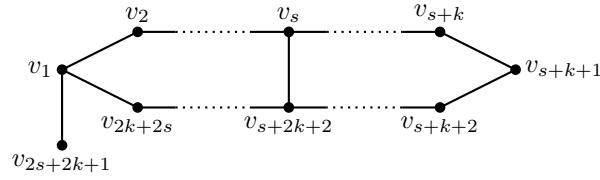


Figure 6: $\theta'(1, 2(s-1), 2(k+1))$

The distance matrices of \widehat{G} and \widehat{H} are

$$D(\widehat{G}) = \begin{pmatrix} P & A^t & v + \mathbf{1} \\ A & P & w_1 + \mathbf{1} \\ v^t + \mathbf{1}^t & w_1^t + \mathbf{1}^t & 0 \end{pmatrix},$$

$$D(\widehat{H}) = \begin{pmatrix} P & B^t & v + \mathbf{1} \\ B & P & w_2 + \mathbf{1} \\ v^t + \mathbf{1}^t & w_2^t + \mathbf{1}^t & 0 \end{pmatrix},$$

where P , A and B are the matrices defined in 6, 7 and 8, respectively, v is the first column of P , w_1 is the first column of A and w_2 is the first column of B .

We claim that

$$D(\widehat{G}) = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} D(\widehat{H}) \begin{pmatrix} N^t & 0 \\ 0 & 1 \end{pmatrix}, \quad (12)$$

where

$$N = \begin{pmatrix} I & 0 \\ AP^{-1} - MBP^{-1} & M \end{pmatrix}.$$

Indeed, by (9), we have

$$\begin{pmatrix} P & A^t \\ A & P \end{pmatrix} = N \begin{pmatrix} P & B^t \\ B & P \end{pmatrix} N^t.$$

Hence, it is sufficient to prove that

$$N \begin{pmatrix} v + \mathbf{1} \\ w_2 + \mathbf{1} \end{pmatrix} = \begin{pmatrix} v + \mathbf{1} \\ w_1 + \mathbf{1} \end{pmatrix}.$$

It is easy to check that

$$N \begin{pmatrix} v + \mathbf{1} \\ w_2 + \mathbf{1} \end{pmatrix} = \begin{pmatrix} v + \mathbf{1} \\ (A - MB)P^{-1}(v + \mathbf{1}) + M(w_2 + \mathbf{1}) \end{pmatrix}.$$

Since v is the first column of P , we obtain

$$\begin{aligned} (A - MB)P^{-1}v + Mw_2 &= (A - MB)e_1 + Mw_2 \\ &= w_1 - Mw_2 + Mw_2 = w_1. \end{aligned}$$

From the proof of Theorem 3, Case 2, we have that

$$(A - MB)P^{-1} = e_2e_1^t + \sum_{i=3}^{k+s} e_i e_s^t - \sum_{i=2}^{k+s} e_i e_{s+1}^t,$$

combining this with the definition of M , we see that

$$(A - MB)P^{-1}\mathbf{1} + M\mathbf{1} = 0 + \mathbf{1} = \mathbf{1}.$$

This completes the proof of (12). Futhermore, since

$$\det \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \cdot \det \begin{pmatrix} N^t & 0 \\ 0 & 1 \end{pmatrix} = 1,$$

we deduce that

$$\det D(\widehat{G}) = \det D(\widehat{H}).$$

Combining this with (11), using an inductive argument, we have

$$\det D(\theta'(1, 2s, 2k)) = -n(n + 2m)(-2)^{m-2},$$

with $n = p + q$ y $m = 1$, where $p = 2s$ and $q = 2k$. □

150 We now consider the case when a path is attached to a vertex of $\theta(l, p, q)$.
 We denote by $\theta'_m(l, p, q)$ the graph obtained from $\theta(l, p, q)$ by identifying one
 vertex of degree three of $\theta(l, p, q)$ with one vertex of degree one of a path of
 length $m \geq 0$.

The next proposition investigates the determinant of these graphs, when p
 155 and q are even.

Proposition 3. *If p and q are even integers, then*

$$\det D(\theta'_m(1, p, q)) = -n(n + 2m)(-2)^{m-2},$$

where $n = p + q$ and $m \geq 0$.

Proof. Let $G_m = \theta'_m(1, p, q)$, $V(G_m) = \{1, \dots, p+q, \dots, p+q+m\}$ such that the
 vertices $\{1, \dots, p+q\}$ induce $\theta(1, 2s, 2k)$ and the vertices $\{p+q, \dots, p+q+m\}$
 induce P_{m+1} , where $p = 2s$, $q = 2k$ and $m \geq 0$ for some $k \geq 1$ and $s \geq 1$.
 Arguing as in [6, Theorem 3.2], we obtain

$$\det D(G_m) = -4 \det D(G_{m-1}) - 4 \det D(G_{m-2}),$$

for $m \geq 2$. Combining this identity with the results of Theorem 3, case (a) and
 Theorem 4, case (a), we deduce that

$$\det D(G_m) = -n(n + 2m)(-2)^{m-2},$$

where $n = p + q$ and $m \geq 0$. □

As we already know the determinant of a θ -graph and θ' -graph, we obtain
 the values of $\text{cof } D(G)$, for $G = \theta(l, p, q)$.

160 **Corollary 2.** *The following assertions hold:*

- If $G = \theta(1, p, q)$ for some even integers p and q , then $\text{cof } D(G) = -(p+q)$.
- If $G = \theta(2, 2, 2)$, then $\text{cof } D(G) = -16$.
- If $G = \theta(2, 2, q)$ for some odd integer $q > 1$, then $\text{cof } D(G) = 4q - 8$.

- Otherwise, $\text{cof } D(G) = 0$.

165 **Remark 1.** A graph is said to be *at most bicyclic* if it arises from a tree by the addition of at most two edges. The blocks that are at most bicyclic graphs having at least two vertices are: edge blocks, cycles, and θ -graphs. The values of $\det D(G)$ and $\text{cof } D(G)$ where already known in the first two cases. Now, we have obtained the values of $\det D(G)$ and $\text{cof } D(G)$ for the last case.

170 From the results above, by applying Theorem 2, we present in the sequence a formula for $\det D(G)$ for all graphs having at most bicyclic blocks. Notice that this class generalizes the class of cacti (which are graphs having at most unicyclic blocks).

Theorem 5. Let G be a connected graph having at most bicyclic blocks. If $G =$
175 K_1 or any block of G is an even cycle or a $\theta(l, p, q)$ with $\det D(\theta(l, p, q)) = 0$, then $\det D(G) = \text{cof } D(G) = 0$. Otherwise, if the blocks of G are:

- m edge blocks,
- c odd cycles of lengths l_1, l_2, \dots, l_c ,
- r graphs $\theta(1, p_1, q_1), \theta(1, p_2, q_2), \dots, \theta(1, p_r, q_r)$ for even integers $p_1, q_1, \dots, p_r, q_r$,
- 180 • s graphs $\theta(2, 2, 2)$, and
- t graphs $\theta(2, 2, q_1), \theta(2, 2, q_2), \dots, \theta(2, 2, q_t)$ for odd integers $q_1, q_2, \dots, q_t >$
1,

then

$$\det D(G) = \left(\frac{m}{2} + \sum_{h=1}^c \frac{l_h^2 - 1}{4l_h} + \sum_{i=1}^r \frac{p_i + q_i}{4} + s + \sum_{j=1}^t \frac{q_j^2 - 5}{4q_j - 8} \right) \text{cof } D(G),$$

where

$$\text{cof } D(G) = (-2)^m (-1)^r (-16)^s \left(\prod_{h=1}^c l_h \right) \left(\prod_{i=1}^r (p_i + q_i) \right) \prod_{j=1}^t (4q_j - 8).$$

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