Continuous and localized Riesz bases for $L^2$ spaces defined by Muckenhoupt weights

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Abstract
Let $w$ be an $A_{\infty}$-Muckenhoupt weight in $\mathbb{R}$. Let $L^2(wdx)$ denote the space of square integrable real functions with the measure $w(x)dx$ and the weighted scalar product $\langle f, g \rangle_w = \int_{\mathbb{R}} fg \, wdx$. By regularization of an unbalanced Haar system in $L^2(wdx)$ we construct absolutely continuous Riesz bases with supports as close to the dyadic intervals as desired. Also the Riesz bounds can be chosen as close to 1 as desired. The main tool used in the proof is Cotlar’s Lemma.

Keyword: Riesz bases, Haar wavelets, basis perturbations, Muckenhoupt weights, Cotlar’s Lemma.


1. Introduction and statement of the main result
A sequence $\{f_k, k \in \mathbb{Z}\}$ in a Hilbert space $H$ is said to be a Bessel sequence with bound $B$ if the inequality
$$\sum_{k \in \mathbb{Z}} |\langle f, f_k \rangle|^2 \leq B \|f\|^2_H$$
holds for every $f \in H$. If $\{f_k, k \in \mathbb{Z}\}$ is a Bessel sequence with bound $B$ and $\{e_k, k \in \mathbb{Z}\}$ is an orthonormal basis for the separable Hilbert space $H$, then the operator $T$ on $H$ defined by
$$Tf := \sum_{k \in \mathbb{Z}} \langle f, f_k \rangle e_k$$
is bounded on $H$ with bound $\sqrt{B}$. Conversely if $T$ is bounded on $H$, then $\{f_k, k \in \mathbb{Z}\}$ is a Bessel sequence with bound $\|T\|^2$.

When $\{f_k, k \in \mathbb{Z}\}$ itself is an orthonormal basis and $e_k = f_k$, $T$ is the identity. Of particular interest is the case of $H = L^2$ when the Bessel system and the orthonormal basis are built on scaling and translations of the underlying space. In such cases the operator $T$ has a natural decomposition as $T = \sum_{j \in \mathbb{Z}} T_j$. Sometimes the orthonormal basis can be chosen in such a way that the $T_j$’s become almost orthogonal in the sense of Cotlar. We aim to use Cotlar’s Lemma to produce smooth and localized Riesz bases for $L^2(\mathbb{R}, wdx)$ when $w$ is a Muckenhoupt weight.

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To introduce the problem let us start by some simple illustrations. Let \( \psi \) be a Daubechies compactly supported wavelet in \( \mathbb{R} \). Assume that \( \text{supp} \psi \subset [-N, N] \). The system \( \{ \psi^j_k(x) = 2^j \psi(2^j x - k) : j, k \in \mathbb{Z} \} \) is a compactly supported orthonormal basis for \( L^2(\mathbb{R}, 3x^2\,dx) \). More generally if \( w(x) \) is a non-negative locally integrable function in \( \mathbb{R} \) and \( W(x) = \int_0^x w(y)\,dy \), then the system \( \{ \psi^j_k(x) = 2^j \psi(2^j W(x) - k) \} \) is an orthonormal basis for \( L^2(w\,dx) \).

In fact, changing variables

\[
\int_{\mathbb{R}} \overline{\psi^j_k(x)} \psi^m(x) w(x)\,dx = 2^{j/2} \int_{\mathbb{R}} \psi(2^j W(x) - k) \psi(2^j W(x) - m) w(x)\,dx
\]

and we have the orthonormality of the system \( \{ \overline{\psi^j_k} : j \in \mathbb{Z}, k \in \mathbb{Z} \} \) in \( L^2(\mathbb{R}, w\,dx) \). As it is easy to verify in the case of \( w(x) = 3x^2 \), for \( j \) fixed the length of the supports of \( \overline{\psi^j_k} \) tend to zero as \( |k| \to +\infty \). On the other hand for \( k = 0 \) the scaling parameter is \( 2^{-\frac{j}{3}} \).

Notice also that if \( w \) is bounded above and below by positive constants the sequence \( \overline{\psi^j_k} \) is an orthonormal basis for \( L^2(w\,dx) \) with a metric control on the sizes of the supports provided by the scale.

A Riesz basis in \( L^2(w\,dx) \) is a Schauder basis \( \{ f_k \} \) such that there exist two constants \( A \) and \( B \) called the Riesz bounds of \( \{ f_k \} \) for which

\[
A \sum |c_k|^2 \leq \left\| \sum c_k f_k \right\|_{L^2(w\,dx)}^2 \leq B \sum |c_k|^2
\]

for every \( \{ c_k \} \) in \( l^2(\mathbb{R}) \), the space of square summable sequences of real numbers. In this note we aim to give sufficient conditions on a weight \( w \) defined on \( \mathbb{R} \) more general than \( 0 < c_1 \leq w(x) \leq c_2 < \infty \), in order to construct, for every \( \delta > 0 \), a system \( \Psi = \{ \psi_I(x), I \in \mathcal{D} \} \) ( \( \mathcal{D} \) are the dyadic intervals in \( \mathbb{R} \)) with the following properties,

(i) \( \Psi \) is a Riesz basis for \( L^2(w\,dx) \) with bounds \((1 - \delta)\) and \((1 + \delta)\),

(ii) each \( \psi_I^j \) is absolutely continuous,

(iii) for each \( I \), \( \psi_I \) is supported on a neighborhood \( I' \) of \( I \) such that

\[
0 < \frac{|I'|}{|I|} - 1 < \delta.
\]

As we have shown in the above example with \( w(x) = 3x^2 \), we have that \( \{ \overline{\psi^j_k} \} \) satisfies (i) and (ii) but not (iii).

An orthonormal basis in \( L^2(\mathbb{R}, w\,dx) \) satisfying (iii) but not (ii) when \( w \) is locally integrable is the following unbalanced version of the Haar system (see [12]). Let \( \mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}^j \) be the family of standard dyadic intervals in \( \mathbb{R} \). Each \( I \) in \( \mathcal{D}^j \) takes the form \( I = [k2^{-j}, (k + 1)2^{-j}) \) for some integer \( k \). For \( I \in \mathcal{D}^j \) we have that \( |I| = 2^{-j} \). We shall frequently use \( a_I \) and \( b_I \) to denote the left and right points of \( I \) respectively, for each \( I \in \mathcal{D} \), define

\[
h^j_I(x) = \frac{1}{\sqrt{w(I)}} \left\{ \sqrt{\frac{w(I)}{w(I_1)}} \chi_{I_1}(x) - \sqrt{\frac{w(I)}{w(I_2)}} \chi_{I_2}(x) \right\}
\]

where \( w(E) = \int_E w\,dx \), \( I_1 \) is the left half of \( I \) and \( I_2 \) is its right half. Notice that with the above notation \( h^j_I \) is the standard Haar basis \( h_I \) for \( L^2(\mathbb{R}) \) when \( w = 1 \).
The real numbers with the usual distance and measure $d\mu = wdx$ with $w$ a Mucken- 
houpt weight, is a space of homogeneous type. Some constructions of wavelet type bases 
on spaces of homogeneous type are contained in [2] and [3]. Those in [2] are not regular 
and those in [3] are not compactly supported.

In this note we prove that the $A_\infty$ Muckenhoupt condition on a weight $w$ is sufficient 
for building a Riesz basis in $L^2(wdx)$ satisfying (i), (ii), and (iii).

Aside from Cotlar’s Lemma, other fundamental tools we shall use are the basic 
properties of Muckenhoupt weights and a result due to Favier and Zalik [8] on small Bessel 
perturbations of Riesz bases.

system to produce a regular and compactly supported Riesz basis with bounds as close to 
one as desired and supported on small neighborhoods of the dyadic intervals. In [11] the 
same type of result is obtained via regularizing by convolution. In both cases the main 
tool is contained in Theorem 5 in [8].

Let $1 < p < \infty$. A locally integrable nonnegative function $w$ defined on $\mathbb{R}$ is said to be 
an $A_p$ Muckenhoupt weight if there exists $C > 0$ such that

$$\left( \int_J wdx \right) \left( \int_J w^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C |J|^p,$$

for every interval $J$. The class $A_\infty$ is defined by $A_\infty = \cup_{1 < p < \infty} A_p$.

The typical nontrivial examples of $A_\infty$ weights are the powers of the distance to a fixed 
point. In particular $w(x) = |x|^\alpha$ belongs to $A_\infty$ for every $\alpha > -1$. For the general 
theory of Muckenhoupt weights, introduced by B. Muckenhoupt in [11], see the book [9].

A simple and well known result for $A_\infty$ weights that implies the doubling condition for 
the measure $w(x)dx$, due to B. Muckenhoupt, is the inequality

$$\left( \frac{|E|}{|J|} \right)^p \leq C_{\mathcal{w}}(E) / w(J)$$

which holds for some constant $C$ and every measurable subset $E$ of any interval $J$, provided 
that $w \in A_p$. From (1.2) it follows easily that the function $W(x) = \int_0^x w(y)dy$ defines a 
one to one and onto change of variables on $\mathbb{R}$ with Jacobian $w$. Set $W^{-1}$ to denote the 
inverse function of $W$.

In order to produce a regularization of the system $h^u_I$ given by (1.1) we first use the 
change of variables defined by $W^{-1}$ to obtain another orthonormal basis $\{H^u_I\}$ in the spaces 
$L^2$ with respect to the translation invariant measure $dx$. Next we regularize by convolution 
with a smooth and compactly supported function \(\varphi\) the functions $H^u_I$ to produce a Riesz 
_basis for $L^2(\mathbb{R}, dx)$ which we shall denote by $\{H^{u, \varphi}_I\}$. Finally in order to obtain the desired 
regularization $h^{u, \varphi}_I$ of $\{h^u_I\}$ we go back to $L^2(\mathbb{R}, wdx)$ by reversing the change of variables 
induced by $W^{-1}$. Since the regularizing function $\varphi$ can be assumed to be as smooth as 
desired, the regularity of each $h^{u, \varphi}_I$ is only limited by the regularity of $W(x)$ which is at 
least locally absolutely continuous. Let us precisely define the three families $\{H^u_I\}$, $\{H^{u, \varphi}_I\}$ 
and $\{h^{u, \varphi}_I\}$.

For each $I \in \mathcal{D}$ set $H^u_I = h^u_I \circ W^{-1}$. Notice that

$$H^u_I(x) = \frac{1}{\sqrt{|I'|}} \left\{ \frac{|H'|}{|I'|} \chi_{I'}(x) - \sqrt{\frac{|H'|}{|I'|}} \chi_{I'}(x) \right\}$$

where $I' = \{W(y), y \in I\}$. Now take a function $\varphi$ to be $C^\infty$, nonnegative, non-increasing 
to the right of 0, even and supported in $(-1, 1)$ with $\int_\mathbb{R} \varphi = 1$. With the standard notation
\[ \varphi_t(x) = \frac{1}{t} \varphi\left(\frac{x}{t}\right), \quad t > 0, \text{define} \]
\[ H^w_t(x) = \left( \varphi_{ew(t)} * H^w_t \right)(x). \]  
(1.4)

Finally, set \( h^w_t(x) = (H^w_t \circ W)(x) \) for \( \epsilon \) positive small enough.

The main result in this note is contained in the following statement.

**Theorem 1.1.** Let \( w \) be a weight in \( A_\infty(\mathbb{R}) \). Then there exists \( \epsilon_0 > 0 \) depending only on \( w \) such that

a) for each positive \( \epsilon < \epsilon_0 \), the system \( \{h^w_t, I \in \mathcal{D}\} \) is a Riesz basis for \( L^2(w dx) \) of absolutely continuous functions,

b) the Riesz bounds of \( \{h^w_t, I \in \mathcal{D}\} \) can be taken as close to one as desired by taking \( \epsilon \) small enough,

c) for each dyadic interval \( I = [a_I, b_I] \) the support of \( h^w_t \) is an interval \( I^\epsilon = [a_I^\epsilon, b_I^\epsilon] \) with \( a_I^\epsilon \nearrow a_I, b_I^\epsilon \searrow b_I \) when \( \epsilon \to 0 \) and for some constant \( C \), \( 0 < |I^\epsilon| = |I| \leq 1 < C\epsilon^{\frac{1}{p}} \) if \( w \in A_p \).

Let us point out that the regularity of each \( h^w_t \) can be better than absolute continuity if \( w \) is smooth. In particular, when \( w \equiv 1 \) the functions \( h^w_t \) are C\( \infty \). In other words we get a basis for \( L^2(dx) \) with full regularity and small supports. To get simultaneously these two properties we have to pay loosing orthogonality.

In Section 2 we give the basic result used in Section 3 in order to prove Theorem 1.1.

2. Preliminaries and basic results

In this section we introduce three basic results from functional and harmonic analysis which we shall use in Section 3 to prove Theorem 1.1. We shall refer to them as Coifman-Fefferman inequality, Cotlar’s Lemma and Favor-Zalk stability, respectively.

Aside from (1.2) another important property of \( A_\infty \) weights that we shall use in the proof Theorem 1.1 is contained in the next statement which is proved as Theorem 2.9 page 401 in [9] and originally proved in [5].

**Coifman-Fefferman.** If \( w \in A_p, 1 < p < \infty \) then there exist positive and finite constants \( C, \gamma \) such that the inequality

\[ \frac{w(E)}{w(J)} \leq C \left( \frac{|E|}{|J|} \right)^\gamma \]  
(2.1)

holds for every interval \( J \) and every measurable subset \( E \) of \( J \).

The original proof of Cotlar’s Lemma is contained in [6]. For more easily available proofs see [7] or [12].

**Cotlar’s Lemma.** Let \( \{T_i : i \in \mathbb{Z}\} \) be a sequence of bounded operators in a Hilbert space \( H \). Assume that they are almost orthogonal in the sense that there exists a sequence \( s : \mathbb{Z} \to (0, \infty) \) with \( \sum_{k \in \mathbb{Z}} \sqrt{s(k)} = A < \infty \) such that

\[ \|T_i^*T_j\| + \|T_iT_j^*\| \leq s(i - j) \]

for every \( i, j \in \mathbb{Z} \). Then

\[ \left\| \sum_{i=-N}^{N} T_i \right\| \leq A \]
for every positive integer \( N \).

The third result, due to S. Favier and R. Zalik, deals with the perturbation of Riesz bases and is contained in Theorem 5 of [8]. A basis \( \{ f_n \} \) for a Hilbert space \( H \) is said to be a Riesz basis with bounds \( A \) and \( B \) if and only if the inequalities

\[
A \| f \|^2 \leq \sum |\langle f_n, f \rangle|^2 \leq B \| f \|^2
\]

hold for every \( f \in H \) (see, for example, Theorem 6.1.1 in [4]).

**Favier-Zalik stability.** Let \( \{ f_n \} \) be a Riesz basis for a Hilbert space \( \mathcal{H} \) with bounds \( A \) and \( B \). Let \( \{ g_n \} \) be a sequence in \( \mathcal{H} \) such that \( \{ f_n - g_n \} \) is a Bessel sequence with bound \( M < A \). Then \( \{ g_n \} \) is a Riesz basis with bound \( \left[ 1 - \left( \frac{M}{A} \right)^2 \right] A \) and \( \left[ 1 - \left( \frac{M}{B} \right)^2 \right] B \).

The next lemma is a consequence of (1.2). It will be crucial in the proof of Theorem 1.1.

**Lemma 2.1.** Let \( w \) be a weight in \( A_p \). For a given dyadic interval \( I \), set \( a_I, b_I, c_I \) to denote the left endpoint of \( I \), the right endpoint of \( I \) and the center of \( I \) respectively. As before \( I_l \) and \( I_r \) denote the left and right halves of \( I \). Then

a) with \( C \) the constant in (1.2) and \( \epsilon < \left( \frac{1}{2} \right)^p \frac{1}{2C} \) we have that \( 2\epsilon w(I) < w(I_l) \) and \( 2\epsilon w(I) < w(I_r) \);

b) with \( C \) as above and \( \epsilon < \frac{1}{C^2} \frac{1}{2} \) we also have that \( \sum_{I \in D^j} \chi_{W^\epsilon(I)}(x) \leq 2 \) for every \( j \in \mathbb{Z} \), where \( W^\epsilon(I) \) is the \( \epsilon w(I) \) neighborhood of the interval \( W(I) \), in other words \( W^\epsilon(I) = \left( W(a_I) - \epsilon w(I) \right) \cup \left( W(b_I) + \epsilon w(I) \right) \).

**Proof.** a) Using (1.2) with \( J = I, E = I_l \) we obtain

\[
\frac{w(I_l)}{w(I)} \geq \frac{1}{C} \left( \frac{|I|}{|I_l|} \right)^p = \frac{1}{C^2} > 2\epsilon.
\]

The same inequality is true for \( I_r \) instead of \( I_l \).

b) Let us consider \( \overline{I}, \overline{K} \) and \( J \) three consecutive intervals in \( D^j \) with \( b_I = a_K \) and \( b_K = a_J \). Let \( M \) be the interval obtained as the union of \( I, J \) and \( K \). From (1.2) we see that

\[
\epsilon < \frac{1}{C} \frac{1}{3^p} = \frac{1}{C} \left( \frac{|K|}{|M|} \right)^p \leq \frac{w(K)}{w(M)}
\]

Hence \( \epsilon (w(I) + w(J)) \leq w(M) < w(K) = W(a_J) - W(b_I) \), so that \( W(b_I) + \epsilon w(I) < W(a_J) - \epsilon w(J) \). Then, no point \( x \in \mathbb{R} \) can belong to more than two of the intervals \( W^\epsilon(I) \).

\[\square\]

3. **Proof of Theorem 1.1**

Throughout this section \( w \) is a weight in \( A_p(\mathbb{R}) \) for some \( 1 < p < \infty \). We shall use the standard inner product notation \( \langle \cdot, \cdot \rangle \) for the scalar product in \( L^2(dx) \). We shall write \( \langle \cdot, \cdot \rangle_w \) to denote the inner product in \( L^2(wdx) \).

Notice first that \( \{ h^I : I \in D \} \) defined in (1.1) is an orthonormal basis for \( L^2(\mathbb{R}, wdx) \). For \( j \in \mathbb{Z} \), set

\[
\mathcal{V}_j = \{ f \in L^2(wdx) : f \text{ is constant on each } I \in D^j \},
\]

and observe that \( \bigcup_{j \in \mathbb{Z}} \mathcal{V}_j \) is dense in \( L^2(wdx) \). By (2.1) \( wdx \) is doubling and hence \( \int_{\mathbb{R}} w = \infty \). Thus, we have \( \bigcap_{j \in \mathbb{Z}} \mathcal{V}_j = \{0\} \). For \( I \in D \) fixed, the two dimensional vector space
of those functions $f$ defined on $I$ which are constant on each half $I_l$ and $I_r$ of $I$ has 
$$ \{ \frac{1}{\sqrt{w(I)}} h^w_I, \ h^w_I \} $$ as an orthonormal basis with the $L^2(\omega dx)$ inner product. For $j \in \mathbb{Z}$, we
define $W_j$ as the $L^2(\omega dx)$ orthogonal complement of $V_j$ in $V_{j+1}$. In other words, as usual,
$V_{j+1} = V_j \oplus W_j$.

From the above mentioned properties of the multiresolution $\{ V_j : j \in \mathbb{Z} \}$ we see that
$$ L^2(\omega dx) = \bigoplus_{j \in \mathbb{Z}} W_j. $$

Since, for $j \in \mathbb{Z}$ fixed, the family $\{ h^w_I : I \in \mathcal{D} \}$ is an orthonormal basis of $W_j$ we get that
$\{ h^w_I : I \in \mathcal{D} \}$ is an orthonormal basis for $L^2(\omega dx)$.

Given a set $E \subset \mathbb{R}$ we shall write $E'$ to denote the image of $E$ by $W$. In other words
$E' = \{ W(x), \ x \in E \}$. We write $\mathcal{D}' = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j'$ to denote the family of all the images $I'$ of
intervals $I \in \mathcal{D}$ through $W$, here $\mathcal{D}$ denote the family of all dyadic intervals in $\mathbb{R}$ defined
above. Notice that $|I'| = w(I)$.

For each $I \in \mathcal{D}$ we shall use $H^w_I$ to denote the composition $h^w_I \circ W^{-1}$. It is easy
to see that
$$ H^w_I(x) = \frac{1}{\sqrt{|I'|}} \left\{ \frac{|E|}{|I'|} \chi_{E}(x) - \frac{|E|}{|I'|} \chi_{I'}(x) \right\} $$
and that $\{ H^w_I, \ I \in \mathcal{D} \}$ is an orthonormal basis of $L^2(\mathbb{R}, \omega dx)$. In fact, for $f \in L^2(\omega dx)$ we have $\langle f, H^w_I \rangle = \langle f \circ W, h^w_I \rangle_w$
for every $I \in \mathcal{D}$. Moreover
$$ \sum_{I \in \mathcal{D}} |\langle f, H^w_I \rangle|^2 = \sum_{I \in \mathcal{D}} |\langle f \circ W, h^w_I \rangle_w|^2 = \| f \circ W \|^2_{L^2(\omega dx)} = \| f \|^2_{L^2(\omega dx)}. $$

Next we regularize by convolution the function $H^w_I$ for $I \in \mathcal{D}$ in order to get $H^w_{I, \epsilon}$, defined by
$H^w_{I, \epsilon} = \varphi_{\epsilon w(I)} \ast H^w_I$. Here $I \in \mathcal{D}$, $\varphi$ is as described in the introduction, and $\epsilon$ is
as in Lemma 2.1.

We prove a) in Theorem 1.1 by applying the Favier-Zalik stability result. We shall
estimate the Bessel bound in $L^2(\omega dx)$ for the difference $b^j_I = H^w_I - H^w_{I, \epsilon}$ between the basic
element $H^w_I$ and its regularization $H^w_{I, \epsilon}$.

We use the strategy described in the introduction, taking as $\{ f_k \}$ the sequence $\{ b^j_I \}$ and
as the orthonormal basis $\{ e_k \}$ the sequence $H^w_I$. Precisely, define
$$ T_j f = \sum_{I \in \mathcal{D}} \langle f, b^j_I \rangle H^w_I $$
and $T_j f = \sum_{J \in \mathcal{D}_j} \langle f, b^j_I \rangle H^w_I$, thus $T_j = \sum_j T_j$. To prove that $\{ b^j_I : I \in \mathcal{D} \}$ is a Bessel
sequence with small bound, we apply Cotlar’s Lemma to the sequence $\{ T_j \}$ of operators
in $L^2(\mathbb{R})$. We begin by estimating $\| T_j^\ast T_j \|$ and $\| T_j T_j^\ast \|$ where $T_j^\ast$ is the adjoint of $T_j$,
$$ T_j^\ast f = \sum_{J \in \mathcal{D}_j} \langle f, H^w_{j, \epsilon} \rangle b^j_I. $$

Since the family $\{ H^w_I, \ I \in \mathcal{D} \}$ is orthonormal, for $i \neq j$ we have
$$ T_i^* T_j f = \sum_{J \in \mathcal{D}_j, \ I \in \mathcal{D}_i} \langle f, b^j_I \rangle \langle H^w_{j, \epsilon}, H^w_I \rangle b^i_I = 0. $$
On the other hand, for $i = j$, \(\| T_j^\ast T_j \| = \| T_j \|^2 \) and \(\| T_j f \|^2 = \sum_{I \in \mathcal{D}_j} |\langle f, b^j_I \rangle|^2 \).

Since $H^w_I$ is piecewise constant, for $\epsilon$ small enough the support of $b^j_I$ splits into three
intervals, each of them centered at the images through $W$ of the two endpoints $a_j, b_J$ of $J$ and
of its center $c_J$. All of them have the same length $2 \epsilon w(J)$. Precisely, with $S^j_I = \text{supp } b^j_I$ we have that $S^j_I = \bigcup_{m=1}^3 S^j_{I, m}$, where $S^j_{I, m} = (W(a_j) - w(J) \epsilon, W(a_j) + w(J) \epsilon)$.,
$S^2_j = (W(c_j) - w(J)\epsilon, W(c_j) + w(J)\epsilon)$ and $S^3_j = (W(b_j) - w(J)\epsilon, W(b_j) + w(J)\epsilon)$. Now, from Schwartz inequality we have that

$$|\langle f, b_j' \rangle|^2 \leq \left( \int_{S^2_j} |f|^2 \right) \left( \int |b_j'|^2 \right).$$

In order to estimate $\int |b_j'|^2$, let us first notice that $|b_j'| \leq |H^p_j| + |H^w_{j'}| \leq 2 |H^p_j| \leq \frac{2}{\sqrt{w(I)}} \max \left\{ \sqrt{\frac{w(I)}{w(I)}}, \sqrt{\frac{w(I)}{w(I)}} \right\}$, which is bounded by a constant $C$, depending only on $w$, times $w(I)^{-\frac{1}{2}}$. Then $\int |b_j'|^2 \leq \frac{C^2}{w(I)} |S^2_j| = 6C^2\epsilon$.

Then, from $b$ in Lemma 2.1 we have

$$\|T_j f\|^2 \leq 6C^2\epsilon \sum_{J \in \mathcal{D}^j} \int_{S^2_j} |f|^2 \leq 6C^2\epsilon \sum_{J \in \mathcal{D}^j} \int_{w^*(J)} |f|^2 \leq 6C^2\epsilon \int_R \left( \sum_{J \in \mathcal{D}^j} \chi_{w^*(J)} \right) |f|^2 \leq 12C^2\epsilon \|f\|^2_2.$$

Hence $\|T_j^* T_j\| = \|T_j\|^2 \leq 12C^2\epsilon$, and since $\|T_j^* T_j\| = 0$ for $i \neq j$, any $s(k)$ with $s(0) \leq 12C^2\epsilon$ and $s(k) > 0$ for $k \neq 0$ is admissible for the estimate $\|T_i^* T_j\| \leq s(i - j)$ required by Cotlar’s Lemma.

The behavior of the sequence $\|T_i^* T_j\|$ is more subtle since $T_i^* T_j f = \sum_{l \in \mathcal{D}^j} \sum_{J \in \mathcal{D}^j} \langle f, H^p_J \rangle \langle b'_l, b'_j \rangle H^p_J$ and now the functions $b'_j$ are not orthogonal. In this case the Lipschitz smoothness of each $b'_j$ away from its points of discontinuity, and its mean vanishing properties will play essential roles. These two properties are made precise in the following claims, which we proof later.

**Claim 1.** For each $I \in \mathcal{D}$ with $I = [a, b)$ centered at $c_I$, on each one of the segments $\sigma_1 = (-\infty, W(a))$, $\sigma_2 = (W(a), W(c_I))$, $\sigma_3 = (W(c_I), W(b))$ and $\sigma_4 = (W(b), \infty)$ the function $b'_j$ is Lipschitz with norm bounded by a constant times $(cw(I))^{-\frac{1}{2}}$.

**Claim 2.** On each one of the three connected components $S_i^m$ of its support we have $\int_{S^m_i} b'_j = 0$, $m = 1, 2, 3$.

Let us assume Claims 1 and 2 and continue the proof.

To estimate $\|T_i^* T_j\|$, observe that, since $\{H^p_I, I \in \mathcal{D}\}$ is an orthonormal basis, we have

$$\|T_i^* T_j f\|^2 = \sum_{l \in \mathcal{D}^j} \left( \sum_{J \in \mathcal{D}^j} \langle f, H^p_J \rangle \langle b'_l, b'_j \rangle \right)^2.$$  \hspace{1cm} (3.1)

Assume first that $j > i$. For a fixed $I \in \mathcal{D}^j$, we consider the partition of $\mathcal{D}^j$ provided by the three sets, $A(I) = \{ J \in \mathcal{D}^j : S_j^j \cap S_j' = \emptyset \}; B(I) = \{ J \in \mathcal{D}^j \setminus A(I) : b'_j$ is continuous and not identically zero on $S_j' \}$ and $C(I) = \mathcal{D}^j \setminus (A(I) \cup B(I))$. Since for $J \in \mathcal{A}(I)$ we have that $\langle b'_j, b'_j \rangle = 0$, then

$$\|T_i^* T_j f\|^2 = \sum_{l \in \mathcal{D}^j} \left( \sum_{J \in B(I) \cup C(I)} \langle f, H^p_J \rangle \langle b'_l, b'_j \rangle \right)^2.$$
\begin{align*}
&\leq \sum_{I \in D^v} \left( \sum_{J \in B(I) \cup C(I)} |\langle f, \mathcal{H}_J^w \rangle|^2 \right) \left( \sum_{J \in B(I) \cup C(I)} |\langle b_I^t, b_J^i \rangle|^2 \right) \\
&= \sum_{I \in D^v} \left( \sum_{J \in B(I) \cup C(I)} |\langle f, \mathcal{H}_J^w \rangle|^2 \right) \left( \sum_{J \in C(I)} |\langle b_I^t, b_J^i \rangle|^2 \right) \\
&\quad + \sum_{I \in D^v} \left( \sum_{J \in B(I) \cup C(I)} |\langle f, \mathcal{H}_J^w \rangle|^2 \right) \left( \sum_{J \in B(I)} |\langle b_I^t, b_J^i \rangle|^2 \right) \\
&= I_1 + I_2.
\end{align*}

In order to estimate \( I_1 \) notice that \( C(I) \) has at most six elements. On the other hand, from (2.1)

\[ |\langle b_I^t, b_J^i \rangle| \leq \int_{S_J^i} |b_I^t(x)| |b_J^i(x)| \, dx \]

\[ \leq C \frac{\epsilon w(J)}{(w(I)w(J))^2} \leq C \frac{1}{2^{(j-i)^2}}, \]

hence

\[ I_1 \leq C \varepsilon^2 2^{-\gamma(j-i)} \sum_{I \in D^v} \sum_{J \in B(I) \cup C(I)} |\langle f, \mathcal{H}_J^w \rangle|^2 \]

\[ \leq C \varepsilon^2 2^{-\gamma(j-i)} \sum_{J \in D^v} |\langle f, \mathcal{H}_J^w \rangle|^2 \chi\{I \in D^v : J \notin \mathcal{A}(I)\} \leq C \varepsilon^2 2^{-\gamma(j-i)} \|f\|_2^2, \]

which has again the desired form to apply Cotlar's Lemma with \( s(j - i) = C \varepsilon^2 \frac{1}{2^{(j-i)}} \).

For a given interval \( I \), set \( \tilde{I} \) to denote the concentric with \( I \) and twice its length. Since for \( J \in B(I) \) the function \( b_J^i \) is Lipschitz on the support of \( b_J^i \), if \( x_J^i \) is the center of the \( m \)-th connected component of the support of \( b_J^i \), from Claims 2 and 1 and applying again (2.1) we get

\[ \sum_{J \in B(I)} |\langle b_I^t, b_J^i \rangle|^2 = \sum_{J \in B(I)} \left( \sum_{m=1}^3 \int_{S_J^m} |b_J^i(x)| (b_I^t(x) - b_I^t(x_J^i)) \, dx \right)^2 \]

\[ \leq \sum_{J \in B(I)} \frac{C}{(\epsilon w(J))^3} \left( \sum_{m=1}^3 \int_{S_J^m} |b_J^i(x)| |x - x_J^i| \, dx \right)^2 \]

\[ \leq C \sum_{J \in B(I)} \frac{1}{\epsilon^3 w(I)^3} |S_J^i|^2 \frac{1}{w(J)} \epsilon^2 w(J)^2 \]

\[ \leq C \varepsilon \sum_{J \in B(I)} \left( \frac{w(J)}{w(I)} \right)^2 \frac{w(J)}{w(I)} \]

\[ \leq C \varepsilon \sum_{J \in B(I)} \left( \frac{|J|}{|I|} \right)^{2\gamma} \frac{1}{w(I)} \int_J w(x) \, dx \]

\[ \leq C \varepsilon \left( \frac{1}{2} \right)^{2(j-i)\gamma} \frac{1}{w(I)} \int_R \sum_{J \in B(I)} \chi_J(x) w(x) \, dx \]

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\[
\leq C \varepsilon \left( \frac{1}{2} \right)^{2\gamma(j-i)} \frac{w(\tilde{I})}{w(I)}
\]

\[
\leq C \varepsilon \left( \frac{1}{2} \right)^{2\gamma(j-i)}
\]

So that, for \( j > i \)

\[
\sum_{J \in \mathcal{B}(I)} |\langle b_j^f, b_j^g \rangle|^2 \leq C \varepsilon 2^{-2(j-i)\gamma}
\]

(3.2)

hence

\[
I_2 \leq C \varepsilon 2^{-2(j-i)\gamma} \sum_{I \in \mathcal{D}^i} \sum_{J \in \mathcal{B}(I) \cup \mathcal{C}(I)} |\langle f, H^{y_j}_j \rangle|^2
\]

\[
\leq C \varepsilon 2^{-2(j-i)\gamma} \|f\|^2_2,
\]

finally

\[
\|T_i^* T_j f\|_2^2 \leq I_1 + I_2 \leq C \varepsilon 2^{-\gamma(j-i)} \|f\|_2^2.
\]

Hence, for \( j > i \) taking \( s(j-i) = C \varepsilon \frac{1}{2} 2^{-\frac{2}{2}(j-i)} \) we have a good sequence in order to use Cotlar’s Lemma.

For \( i \geq j \), with the above notation for \( J \in \mathcal{D}^j \) given, we have the three classes \( \mathcal{A}(J) \), \( \mathcal{B}(J) \) and \( \mathcal{C}(J) \),

\[
\|T_i^* T_j f\|_2^2 \leq C \sum_{I \in \mathcal{D}^i} \left( \sum_{\{J \in \mathcal{D}^j \mid S^*_i \cap S^*_j \neq \emptyset\}} |\langle f, H^{y_j}_j \rangle|^2 \left| \langle b_j^f, b_j^g \rangle \right|^2 \right)
\]

\[
\leq C \sum_{J \in \mathcal{D}^j} |\langle f, H^{y_j}_j \rangle|^2 \left( \sum_{I \in \mathcal{C}(J) \cup \mathcal{B}(J)} \left| \langle b_j^f, b_j^g \rangle \right|^2 \right)
\]

\[
\leq C \sum_{J \in \mathcal{D}^j} |\langle f, H^{y_j}_j \rangle|^2 \left( \sum_{I \in \mathcal{C}(J)} \left| \langle b_j^f, b_j^g \rangle \right|^2 \right) + C \sum_{J \in \mathcal{D}^j} |\langle f, H^{y_j}_j \rangle|^2 \left( \sum_{I \in \mathcal{B}(J)} \left| \langle b_j^f, b_j^g \rangle \right|^2 \right).
\]

For the first term, notice that if \( I \in \mathcal{C}(J) \), we obtain from (2.1) as before

\[
|\langle b_j^f, b_j^g \rangle| \leq \int_{S^*_j} |b_j^f(x)| |b_j^g(x)| \, dx
\]

\[
\leq C \frac{\epsilon w(I)}{w(J)^{1/2} w(I)^{1/2}} \leq C \varepsilon 2^{-(i-j)\frac{1}{2}},
\]

since the number of elements in \( \mathcal{C}(J) \) is bounded we get that

\[
\sum_{J \in \mathcal{D}^j} |\langle f, H^{y_j}_j \rangle|^2 \left( \sum_{I \in \mathcal{C}(J)} \left| \langle b_j^f, b_j^g \rangle \right|^2 \right) \leq C \varepsilon 2^{-\gamma(i-j)} \|f\|_2^2.
\]

For the second term observe that if \( I \in \mathcal{B}(J) \) and \( y^m_j \) is the center of the interval \( S^*_j \), since the integral of \( b_j^f \) vanishes on each connected component \( S^*_j \), we have

\[
|\langle b_j^f, b_j^g \rangle| \leq \left( \sum_{m=1}^{3} \int_{S^*_j} b_j^f(y) (b_j^f(y) - b_j^g(y^m_j)) \, dy \right)^2,
\]

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then, from Claim 1,

\[
|\langle b_f^*, b_f^\epsilon \rangle|^2 \leq \left( \frac{C}{\epsilon^2 w(J) \frac{3}{2}} \sum_{m=1}^{\frac{3}{2}} \int_{S_j^m} |b_f^*(y)| |y - y_j^m| \, dy \right)^2 \leq \left( \frac{3Cw(I) |S_j^m|}{\epsilon^2 w(J) \frac{3}{2} w(I) \frac{3}{2}} \right)^2 \leq C \epsilon \left( \frac{w(I)}{w(J)} \right)^3.
\]

Hence

\[
\|T_i T_j f\|_2^2 \leq C \epsilon^2 2^{-\gamma(i-j)} \|f\|_2^2 + C \epsilon^2 2^{-\gamma(i-j)} 2^\gamma \sum_{J \in D_0^i} |\langle f, H_j^\gamma \rangle|^2 \left( \frac{1}{w(J)} \sum_{I \in B(J)} w(I) \right)^{\frac{1}{2}} \leq C \epsilon^2 2^{-\gamma(i-j)} \|f\|_2^2 + C \epsilon^2 2^{-\gamma(i-j)} \|f\|_2^2.
\]

Then \(\|T_i T_j\| \leq C \epsilon^2 2^{-\frac{\gamma}{2}(i-j)},\) for \(i \geq j.\)

So far we have the hypotheses of Cotlar’s Lemma for the sequence \(\{T_j\}\) with \(s(k) = C \epsilon^2 2^{-\frac{\gamma}{2}|k|}, k \in \mathbb{Z}.\) Then \(\|T_i\| \leq C \epsilon^i, 0 < \epsilon < \epsilon_0 = \min \left\{ \frac{2^p}{16}, \frac{3^p}{16} \right\}\) where \(C\) is the constant in (1.2). Now from the Favier-Zalkin stability Lemma, we get that \(\{h_I^w, \epsilon : I \in D\}\) is a Riesz basis for \(L^2(\mathbb{R}, dx)\) with bounds \((1 - \sqrt{C \epsilon^i})^2\) and \((1 + \sqrt{C \epsilon^i})^2\). Since \(h_I^w, \epsilon = H_I^w, \epsilon \circ W\) and for \(f \in L^2wdx\) we have the identity

\[
\sum_{I \in D} \langle f, h_I^w, \epsilon \rangle_w^2 = \sum_{I \in D} \langle f \circ W^{-1}, H_I^w, \epsilon \rangle^2
\]

we immediately see that \(\{h_I^w, \epsilon : I \in D\}\) is a Riesz basis for \(L^2(\mathbb{R}, wdx)\) with bounds \((1 \pm \sqrt{C \epsilon^i})^2\). This proves a).

The absolute continuity of each \(h_I^w, \epsilon\) follows from the regularity of \(H_I^w, \epsilon\) and the absolute continuity of \(W\). Part b) in the statement of Theorem 1.1 follows directly from the Riesz bounds for \(\{h_I^w, \epsilon : I \in D\}\) obtained before.

Let us prove c). With \(a_I\) and \(b_I\) the left and right endpoint of \(I\) we have that the support of \(h_I^w, \epsilon\) is the interval \(I_e = [W^{-1}(W(a_I) - \epsilon w(I)), W^{-1}(W(b_I) + \epsilon w(I))] = [a_I^e, b_I]\) containing \(I.\) Notice that since \(W(a_I) - W(a_I^e) = \epsilon w(I)\) and \(W(b_I^e) - W(b_I) = \epsilon w(I),\) from the continuity of \(W^{-1}\) it follows that \(a_J^e\to a_I\) and \(b_J^e\to b_I\) when \(\epsilon\to 0.\) A more quantitative estimate of the rate of approximation can be obtained using again (1.2). In fact, set \(I^*\) to denote the interval concentric with \(I\) with three times its length. Let \(J\) be the interval \([a_I^e, a_{I^*}]\), then from (1.2)

\[
\frac{a_I - a_{I^*}}{3 |I|} = \frac{|J|}{|I^*|} \leq C \left( \frac{w(J)}{w(I^*)} \right)^{\frac{1}{p}} = C \left( \frac{w(I)}{w(I^*)} \right)^{\frac{1}{p}} \leq C \epsilon^{\frac{1}{p}}.
\]

In a similar way \(b_{I^*} - b_{I^*} \leq C \epsilon^{\frac{1}{p}}.\) Hence \(\frac{|I_e|}{|I|} = 1 + \frac{a_I^e - a_I}{|I|} + \frac{b_I^e - b_I}{|I|}\) and \(0 < \frac{|I_e|}{|I|} - 1 < C \epsilon^{\frac{1}{p}}\) where \(C\) depends on the \(A_p\) constant of \(w.\) Notice that the rate of approximation is better as \(p\) tends to 1.

Let us finally prove Claims 1 and 2.
Proof of Claim 1. Since for $x, y \in \sigma_i$, $i = 1, \ldots, 4$ we have that $H^w_{\ell}(x) = H^w_{\ell}(y)$, then

$$|b^\ell_f(x) - b^\ell_f(y)| = H^w_{\ell} \ast \varphi_{w(I)}(x) - H^w_{\ell} \ast \varphi_{w(I)}(y) = \left| \int_{\mathbb{R}} \frac{H^p_{\ell}(z)}{w(I)} \left( \frac{x - z}{w(I)} - \frac{y - z}{w(I)} \right) dz \right|.$$  

Since $\varphi$ is smooth, applying the mean value theorem we get that

$$|b^\ell_f(x) - b^\ell_f(y)| \leq \frac{\|\varphi^\prime\|_{\infty}}{w(I)^2} |x - y| \int_{\{x - z \in \omega(I) \cup \{y - z \in \omega(I)\}} |H^p_{\ell}(z)| dz \leq \frac{c \|\varphi^\prime\|_{\infty}}{(w(I))^2} |x - y|$$

as desired. \hfill \box

Proof of Claim 2. It is easy to see that $\int b^\ell_f dx = 0$. In fact, we can see from (1.3)

$$\sqrt{|I|} \int_I H^w_{\ell}(x) dx = \sqrt{|I|} \int_I \chi_{H^w_{\ell}}(x) dx - \sqrt{|I|} \int_I \chi_{H^w_{\ell}}(x) dx = \sqrt{|I|} \int_I \chi_{H^w_{\ell}}(x) dx = 0.$$  

On the other hand, since $\int \varphi(z) dz = 1$, we also have that $\int H^w_{\ell} dx = 0$.

Notice that, after normalization, $\int_{S^r_I} b^\ell_f dx = 0$ since $\int_{-\delta}^{\delta} |x(z, 0, \infty) - (\chi_{(0, \infty)} \ast \varphi_{\delta})| dx = 0$ for $\delta > 0$. Since a similar argument proves that $\int_{S^r_I} b^\ell_f dx = 0$ and $\int b^\ell_f = 0$, we also have $\int_{S^r_I} b^\ell_f dx = 0$. \hfill \box

References


