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Totally Discrete Explicit and Semi-implicit Euler Methods for a Blow-up Problem in Several Space Dimensions

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Abstract

The equation $u_t = \Delta u + u^p$ with homogeneous Dirichlet boundary conditions has solutions with blow-up if $p > 1$. An adaptive time-step procedure is given to reproduce the asymptotic behavior of the solutions in the numerical approximations. We prove that the numerical methods reproduce the blow-up cases, the blow-up rate and the blow-up time. We also localize the numerical blow-up set.

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Keywords: Blow-up, adaptive numerical scheme, asymptotic behavior.

1. Introduction

We study the behavior of an adaptive time step procedure for the following parabolic problem

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \Omega \times [0, T), \\ u = 0 & \text{on } \partial\Omega \times [0, T), \\ u(x, 0) = u_0(x) > 0 & \text{on } \Omega. \end{cases} \tag{1.1}$$

Here p is superlinear ($p > 1$) in order to have solutions with blow-up. We assume u_0 is regular and $\Omega \subset \mathbb{R}^d$ is a bounded smooth domain in order to guarantee that $u \in C^{2,1}$. A remarkable fact in this problem is that solutions may develop singularities in finite time, no matter how smooth u_0 is. For many differential equations or systems the solutions can become unbounded in finite time (a phenomena that is known as blow-up). Typical examples where this happens are problems involving reaction terms in the equation like (1.1) where a reaction term of power type is present and so the blow-up phenomenon occurs in the sense that there exists a finite time T such that $\lim_{t \to T} \|u(\cdot, t)\|_\infty = +\infty$ if the initial data is large enough (see [23] and references therein). The blow-up set, which is defined as the set composed of points $x \in \Omega$ such that $u(x, t) \to +\infty$ as $t \to T$, is localized in thin regions of Ω, in [26] is proved that the $(d - 1)$ dimensional Hausdorff measure of the blow-up set is finite. The blow-up rate at these points is given by $u(x, t) \sim (T - t)^{-\frac{1}{p-1}}$, moreover

$$\lim_{t \to T} (T - t)^{\frac{1}{p-1}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = C_p, \qquad C_p = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}$$

(see [15, 16]).

Hereafter we use the notation $f(t) \sim g(t)$ to mean that there exist constants $c, C > 0$ independent of t such that

$$cg(t) \leq f(t) \leq Cg(t).$$

We remark that these results hold if p is subcritical ($p < \frac{d+2}{d-2}$ if $d \geq 3$). For supercritical p the solutions may present different behaviors. For that reason we assume $1 < p < \frac{d+2}{d-2}$ throughout the rest of the paper. However the asymptotic properties of the numerical schemes described above hold for every $p > 1$, this is a difference between the continuous solutions and their approximations.

Since the solution u may develop a singularity in finite time, it is relevant to study the asymptotic behavior (close to the blow-up time) of numerical approximations for this kind of problems.

The first works that address this topic are [21], [22], where the authors analyze finite differences schemes for problem (1.1) with $\Omega = (0, 1)$ and $p = 2$. They study totally discrete schemes with a uniform spatial mesh and they adapt the time step with an explicit Euler method. They prove that in case that both the numerical approximations and the continuous solution blow up, the numerical blow-up times converge to the continuous one as the parameter of the method goes to zero.

A similar result is proved in [11] for the more general case $p > 1$ and also a propagation result is shown: if the initial datum is symmetric and increasing in $[0, 1/2]$ then $x = 1/2$ is the only blow-up point. It is proved that if $x = 1/2$ is a point of the mesh then this is the only numerical blow-up point if $p > 2$, but if $p = 2$ the blow-up propagates to the adjacent nodes (the adjacent nodes also blow up).

An adaptive in space algorithm is developed in [7]. This method refines the mesh as time goes forward using the scale invariance of solutions to this equation. The authors use this scheme to conjecture a second term in the asymptotic expansion of the solution.

In [9], [8], [10], [19] the so called *moving mesh methods* are developed. They also make use of the scaling invariance. They use a spatial mesh that is modified as time goes forward. The nodes are moved according to a *moving mesh partial differential equation* in such a way that the mass is uniformly distributed at any time.

Semidiscrete schemes are considered in [1], [2] for $\Omega = (0, 1)$. The spatial variable is discretized while time remains continuous. The authors prove convergence of the method in regions where the solution is regular as well as conditions that ensure the presence of blow-up in the numerical scheme ($p > 1$ and some hypotheses on the initial data). They also prove convergence of the numerical blow-up times in some situations.

The same scheme is considered in [17], where the authors find the blow-up rate and the blow-up set of the numerical solutions and prove that they reproduce the theoretical ones. They also prove that the numerical solution blow up if the theoretical one does and if the parameter of the method is small enough. Convergence of the blow-up times is also proved without any further assumptions.

In [4], [14], the authors consider the heat equation in an interval with a nonlinear Neumann condition at the boundary. They find conditions that guarantee the presence of blow-up in the numerical approximations (that differs with the ones for the continuous problem) and convergence of the method and the numerical blow-up times. They consider semidiscrete schemes ([14]) and totally discrete schemes using Euler and Runge-Kutta methods ([4]).

Other works that deserve being mentioned are [5], [20], [25]. The survey [6] summarizes the results contained in most of these articles.

The development and analysis of numerical methods for this kind of problems in several space dimensions are much less developed than the one-dimensional case. In fact, numerical methods for this problem in dimension $d \geq 2$ with rigorous proofs of the their asymptotic properties are rare in the literature.

In this paper we introduce and analyze totally discrete explicit and semi-implicit methods for this problem in several space dimensions. For these methods we prove that

- they reproduce the blow-up cases: if the continuous solution blows up in finite time, the same occurs with the numerical solution for small choices of the parameters of the method;
- they have the correct blow-up rate;
- the numerical blow-up times converge to the theoretical one (we can only prove an iterated limit and just for the explicit scheme);
- the localization of the numerical blow-up set.

As a first step to introduce the method we propose a method of lines: we discretize the space variable but the time variable t remains continuous. In this stage, we consider a general method with adequate assumptions. More precisely, we assume that for every $h > 0$ small (h is the parameter of the method), there exists a set of nodes $\{x_1, \dots, x_N\} \subset \overline{\Omega}$ ($N = N(h)$), such that the numerical approximation of u at the nodes x_k, is given by

$$U(t) = (u_1(t), \dots, u_N(t)).$$

That is, $u_k(t)$ stands for an approximation of $u(x_k, t)$. We assume that U is the solution of the following ODE

$$MU'(t) = -AU(t) + MU^p, \tag{1.2}$$

with initial data given by $u_k(0) = u_0(x_k)$. In (1.2) and hereafter, all operations between vectors are understood coordinate by coordinate.

The precise assumptions on the matrices involved in the method are:

(P1) M is a diagonal matrix with positive entries m_k.
(P2) A is a nonnegative symmetric matrix, with non-positive coefficients off the diagonal, i.e., $a_{ki} \le 0$ if $k \neq i$) and $a_{kk} > 0$.
(P3) $\sum_{i=1}^{N} a_{ki} \ge 0$.

Taking into account (P1), the ODE (1.2) can be written as

$$m_k u_k'(t) = -\sum_{i=1}^{N} a_{ki} u_i(t) + m_k u_k^p(t), \;\; 1 \le k \le N,$$

with initial data $u_k(0) = u_0(x_k)$.

As an example, we can consider a linear finite element approximation of problem (1.1) on a regular acute triangulation of Ω (see [12]). Let V_h be the subspace of $H_0^1(\Omega)$ consisting of piecewise linear functions on the triangulation. The finite element approximation $u_h : [0, T_h) \to V_h$ verifies for each $t \in [0, T_h)$

$$\int_\Omega ((u_h)_t v)^I = -\int_\Omega \nabla u_h \nabla v + \int_\Omega ((u_h)^p v)^I,$$

for every $v \in V_h$. Here $(\cdot)^I$ stands for the linear Lagrange interpolate at the nodes of the mesh. The vector $U(t)$, the values of $u_h(\cdot, t)$ at the nodes x_k, verifies a system like (1.2). In this case M is the lumped mass matrix and A is the stiffness matrix. The assumptions on the matrices M and A hold as we are considering an acute regular mesh. We observe that in this case $u_h = U^I$.

As another example, if Ω is a cube, $\Omega = (0, 1)^d$, we can use a semidiscrete finite differences method to approximate the solution $u(x, t)$ obtaining an ODE system of the form (1.2).

We also need some kind of convergence result for the scheme. We will state the precise hypotheses concerning convergence in the statement of each theorem. Finally, in the Appendix we prove an L^∞ convergence theorem under the consistency assumption. The possible convergence assumptions are

(H1) For every $\tau > 0$ $\|u - u_h\|_{L^\infty(\Omega \times [0, T-\tau])} \to 0$ as $h \to 0$.
(H2) $\|u - u_h\|_{H_0^1(\Omega)}(t) \to 0$ as $h \to 0$ for a.e. t.

Once the ODE system is obtained, the next step is to discretize the time variable t. In [4], the authors suggest an adaptive in time step procedure to deal with the heat equation with a nonlinear boundary condition. They analyze explicit Euler and Runge-Kutta methods, however all these methods have to deal with restrictions in the time-step. In this work we first analyze an explicit Euler method and next we introduce a semi-implicit scheme in order to avoid the time-step restrictions. We use

$U^j = (u_1^j, \dots, u_N^j)$ for the values of the numerical approximation at time t_j, and $\tau_j = t_{j+1} - t_j$. When we consider the explicit scheme, U^j is the solution of

$$\begin{aligned} MU^{j+1} &= MU^j + \tau_j \left(-AU^j + M(U^j)^p\right) \\ U(0) &= u_0^I, \end{aligned} \tag{1.3}$$

or equivalently, if we denote $\partial u_k^{j+1} = \frac{1}{\tau_j}(u_k^{j+1} - u_k^j)$

$$\begin{aligned} m_k \partial u_k^{j+1} &= -\sum_{i=1}^{N} a_{ki} u_k^j + m_k (u_k^j)^p, \quad 1 \le i \le N \\ u_k^0 &= u_0(x_k), \quad 1 \le k \le N+1. \end{aligned} \tag{1.4}$$

While for the implicit scheme U^j is the solution of

$$\begin{aligned} MU^{j+1} &= MU^j + \tau_j \left(-AU^{j+1} + M(U^j)^p\right) \\ U(0) &= u_0^I. \end{aligned} \tag{1.5}$$

Observe that (P1)–(P3) ensure that $(M + \tau_j A)^{-1}$ is well defined.

Also note that the scheme is not totally implicit since the nonlinear source u^p is evaluated at time t^j while the stiffness matrix A is evaluated a time t^{j+1}. This mixture makes the scheme free of time-step restrictions while the explicit evaluation of $(U^j)^p$ avoids to solve a nonlinear system in each step that could lead to nonuniqunes or even to nonexistence (see [24]).

Now we choose the time steps $\tau_j = t_{j+1} - t_j$ in order to reproduce the asymptotic behavior. For different time-stepping strategies see [3, 5, 11, 21, 22, 24]. We fix λ small and take

$$\tau_j = \frac{\lambda}{(w^j)^p},$$

where

$$w^j = \sum_{k=1}^{N} m_k u_k^j .$$

This choice for the time step is inspired by [4]. In that work the authors develop an adaptive procedure that adapts the time step in a similar way but using the maximum (L^∞-norm) instead of w^j (L^1-norm). In their problem the maximum is fixed at the right boundary node (i.e.) $\|U^j\|_\infty = u_{N+1}^j$). In this problem, the maximum (the node k such that $u_k^j = \|U^j\|_\infty$) can move from one node to another as j increases. So the techniques used in [4] to study the behavior of $\|U^j\|_\infty$ do not apply here.

The motivation for this time-step is that, as will be shown, the behavior of w^j is given by

$$\partial w^{j+1} \sim (w^j)^p.$$

Hence, with our selection of τ_j we can obtain

$$w^{j+1} \sim w^j + \tau_j (w^j)^p = w^j + \lambda \sim w^0 + (j+1)\lambda,$$

and we obtain the asymptotic behavior of w^j, which is, as we will see, similar to the one for the continuous solution.

Then we study the asymptotic properties of the numerical schemes. We say that a solution of (1.3) (or (1.5)) blows up if

$$\lim_{j\to\infty} \|U^j\|_\infty = \infty, \quad \text{and} \quad T_{h,\lambda} := \sum_{j=1}^{\infty} \tau_j < \infty,$$

we call $T_{h,\lambda}$ the numerical blow-up time. Here $\|\cdot\|_\infty$ stands for the usual infinity norm in $\mathbb{R}^N$.

To describe when the blow-up phenomena occurs in the discrete problem we introduce the following functional $\Phi_h \colon \mathbb{R}^N \to \mathbb{R}$.

$$\Phi_h(U) \equiv \langle AU, U\rangle - \left\langle \frac{1}{p+1} MU^{p+1}, \mathbf{1} \right\rangle,$$

where $\mathbf{1} = (1, 1, \ldots, 1)^{\mathrm{T}}$. This functional is a discrete version of

$$\Phi(u)(t) \equiv \int_\Omega \frac{|\nabla u(s,t)|^2}{2}\, ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1}\, ds,$$

which characterizes solutions with blow-up in the continuous problem: in [13, 15] is proved that u blows up at time T if and only if $\Phi(u)(t) \to -\infty$ as $t \to T$. Here we prove a similar result for the discrete functional Φ_h and use this fact to prove that if the continuous solution has finite time blow-up, its numerical approximation also does when the parameters of the method are small enough.

Next we study the asymptotic behavior of the numerical approximations: if U^j is a numerical solution with blow-up at time $T_{h,\lambda}$ its behavior is given by

$$\|U^j\|_\infty \sim (T_{h,\lambda} - t^j)^{-1/(p-1)}.$$

Moreover, the numerical schemes reproduce the constant C_p in the sense that

$$\lim_{j\to\infty} \max_{1\le k\le N} u_k^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}.$$

The functional Φ_h is also useful to deal with the convergence of the numerical blow-up times. Unfortunately we can only prove the convergence of an iterated limit,

$$\lim_{h\to 0} \lim_{\lambda\to 0} T_{h,\lambda} = T.$$

By means of the numerical blow-up rate we observe a propagation property for blow-up points. We prove that the nodes adjacent to those that blow-up as fast as the maximum may also blow-up (opposite to the continuous problem), but they did it with a slower rate. The number of adjacent nodes that also blow up is determined only by p and is independent of h and λ.

In other words, if we call $B^*(U)$ the set of nodes k such that

$$\lim_{j\to\infty} u_k^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p,$$

then the number of blow-up points that do not lie in $B^*(U)$ depends (explicitly) only on the power p.

We split the paper in two parts, the first one deals with the explicit scheme, in the second one the analysis for the implicit method is developed.

2. The Explicit Scheme

The main tool in our proofs is a comparison argument, so we first prove a lemma which states that this comparison argument holds. Since we need restrictions in the time-step to prove this lemma, they are essential for every result in this section. That is not the case for the implicit scheme.

Hence, throughout this section we assume the hypotheses of Lemma 2.1 below.

Definition 2.1: *We say that* (Z^j) *is a supersolution* (*resp.: subsolution*) *for* (1.3) *if verifies the equation with an inequality* $\geq$ ($\leq$) *instead of an equality.*

Lemma 2.1: *Assume the time step verifies*

$$\tau_j < \min_{1\leq i\leq N} \frac{m_i}{a_{ii}}.$$

Let $(\overline{U}^j)$, $(\underline{U}^j)$ *a super and a subsolution respectively for* (1.3) *such that* $\underline{U}^0 < \overline{U}^0$, *then* $\underline{U}^j < \overline{U}^j$ *for every* j.

Proof: Let $Z^j = \overline{U}^j - \underline{U}^j$, by an approximation argument we can assume that we have strict inequalities in (1.3), then (Z^j) verifies

$$M\partial Z^{j+1} > -AZ^j + M((\overline{U}^j)^p - (\underline{U}^j)^p$$
$$Z^0 > 0.$$

If the statement of the Lemma is false, then there exists a first time t^{j+1} and a node x_k such that $z_k^{j+1} \leq 0$. At that time we have

$$z_k^{j+1} > \left(1 - \tau_j \frac{a_{kk}}{m_k}\right) z_k^j + \tau_j \left(-\sum_{i\neq k} \frac{a_{ki}}{m_k} z_i^j + (\overline{u}_k^j)^p - (\underline{u}_k^j)^p \right) \geq 0,$$

a contradiction. □

Remark 2.1: *Observe that assumption* (P1) *is essential in the proof of the above lemma. In fact the discrete maximum principle does not hold for a general mass matrix* M*, i.e., a symmetric positive definite matrix.*

Remark 2.2: *Since* $U(0) \geq 0$ *we have, as a consequence of the above lemma, that* $U^j \geq 0$ *for every* $j \geq 1$.

2.1. Blow-up in the Numerical Scheme

In this section, we find conditions to guarantee blow-up in (1.4). We begin with some lemmas.

Since the matrix A is symmetric (property (P2)), there exists a basis of eigenvectors for the following eigenvalue problem

$$A\phi_i = \lambda_i M\phi_i.$$

We call $\eta = \eta(h)$ the greatest eigenvalue of this problem, that is

$$0 \leq \lambda_i \leq \eta(h), \quad 1 \leq i \leq N.$$

Lemma 2.2: *For every* $y \in \mathbb{R}^N$ *there holds*

$$\langle Ay, y\rangle \leq \eta(h)\langle My, y\rangle.$$

Proof: As the matrix M defines a scalar product in $\mathbb{R}^N$, we can assume that the eigenvectors ϕ_i are normalized such that

$$\langle M\phi_i, \phi_j\rangle = \delta_{ij}.$$

Let $y \in \mathbb{R}^N$, $y = \sum_{i=1}^N \alpha_i\phi_i$, then

$$\begin{aligned}\langle Ay, y\rangle &= \left\langle \sum_{i=1}^N \alpha_i\lambda_i M\phi_i, \sum_{j=1}^N \alpha_j\phi_j \right\rangle \\ &= \sum_{i=1}^N \alpha_i^2\lambda_i\langle M\phi_i, \phi_i\rangle \\ &\leq \eta(h)\langle My, y\rangle.\end{aligned}$$

□

Lemma 2.3: *There exists a positive constant* κ *that depends only on* h *such that* $U^{j_0} \geq \kappa$ *for some* j_0 *implies that* $(U^j)_{j\geq 1}$ *blows up in finite time.*

Remark 2.3: *The constant* κ *can be computed, in fact*

$$\kappa = \frac{(2\sum_{i,k=1}^N a_{ki})^{\frac{1}{p-1}}(\min_k m_k)^{\frac{p-2}{p-1}}}{(\sum_{k=1}^N m_k)^{p-1}}.$$

Proof: Recall the definition of $w^j = \sum_{k=1}^N m_k u_k^j$ and that $u_k^j \geq 0$ for every k, j (Remark 2.2). Hence w^j verifies

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{k=1}^N \sum_{i=1}^N a_{ki} u_i^j + \tau_j \sum_{k=1}^N m_k (u_k^j)^p \\ &\geq w^j - \tau_j \sum_{i=1}^N u_i^j \sum_{k=1}^N a_{ki} + \tau_j \left(\sum_{k=1}^N m_k \right)^{1-p} (w^j)^p \\ &\geq w^j - \tau_j \|U^j\|_\infty \sum_{i,k=1}^N a_{ki} + \tau_j \left(\sum_{k=1}^N m_k \right)^{1-p} (w^j)^p \\ &= w^j + \tau_j \left(-c_1 w^j + c_2 (w^j)^p \right), \end{aligned}$$

where $c_1 = \frac{\sum_{i,k=1}^N a_{ki}}{\min m_k}$.

So if $w^{j_0} \geq \left(\frac{2c_1}{c_2} \right)^{1/(p-1)}$ we have

$$\begin{aligned} w^{j_0+1} &\geq w^{j_0} + \frac{c_2}{2} \tau_j (w^{j_0})^p \\ &= w^{j_0} + \frac{c_2}{2} \lambda. \end{aligned}$$

Applying this inequality inductively we obtain for $j \geq j_0$

$$w^j \geq w^{j_0} + c\lambda (j - j_0).$$

Hereafter c, c_i, C, C_i, etc. are constants that may depend on h but do not depend on λ or the time variables. They may change from one line to another in the course of the proofs.

We have proved $w^j \to \infty$ as $j \to \infty$. It remains to check that $\sum \tau_j < \infty$. To do so, we observe that

$$\tau_j = \frac{\lambda}{(w^j)^p} \leq \frac{\lambda}{(w^{j_0} + c\lambda (j - j_0))^p},$$

and

$$\sum_{j=j_0}^\infty \frac{\lambda}{(w^{j_0} + c\lambda (j - j_0))^p} \leq \int_0^\infty \frac{\lambda}{(w^{j_0} + c\lambda s)^p} \, ds < \infty.$$

This completes the proof. □

Remark 2.4: *In the course of the proof of the above lemma we also proved $w^j \geq cj$ for j large enough.*

Now we are going to prove the reverse inequality to obtain the asymptotic behavior of $\|U^j\|_\infty$.

Lemma 2.4: *If* (U^j) *is unbounded then*

$$\|U^j\|_\infty \sim w^j \sim j.$$

Proof: The relation $\|U^j\|_\infty \sim w^j$ holds since they define equivalent norms in $\mathbb{R}^N$. So we just have to prove $w^j \leq Cj$. Observe that

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{i=1}^N \sum_{k=1}^N a_{ki} u_i^j + \tau_j \sum_{i=1}^N m_k (u_k^j)^p \\ &\leq w^j + \tau_j \sum_{k=1}^N m_k (u_k^j)^p \\ &\leq w^j + C\tau_j (w^j)^p \\ &= w^j + C\lambda. \end{aligned}$$

Proceeding as before we get $w^j \leq w^0 + C\lambda j \leq Cj$, as we wanted to prove. □

Theorem 2.1: *Assume the time step* τ_j *verifies the restriction*

$$\tau_j < \frac{2}{p(w^{j+1})^{p-1} + \eta(h)}. \tag{2.1}$$

Then positive solutions of (1.3) *blow up if there exists* j_0 *such that* $\Phi_h(U^{j_0}) < 0$.

Remark 2.5: *We remark that the condition* $\Phi_h(U^{j_0}) < 0$ *is similar to the one for the blow-up phenomena in the continuous problem, in fact for the continuous problem* u *blows up if and only if* $\Phi(u)(t_0) < 0$ *for some* $t_0 \geq 0$.

Proof: First we observe that $\Phi_h(U^j)$ is decreasing in j, in order to do that we take inner product of (1.3) with $U^{j+1} - U^j$ to obtain

$$\begin{aligned} 0 &= \left\langle \frac{1}{\tau_j} M(U^{j+1} - U^j) + AU^j - M(U^j)^p, U^{j+1} - U^j \right\rangle \\ &= \tau_j \langle M\partial U^{j+1}, \partial U^{j+1} \rangle + \Phi_h(U^{j+1}) - \Phi_h(U^j) - \tfrac{1}{2}\langle AU^{j+1}, U^{j+1} \rangle \\ &\quad + \langle AU^j, U^{j+1} \rangle - \tfrac{1}{2}\langle AU^j, U^j \rangle - \tfrac{1}{2}\langle Mp(\xi^j)^{p-1}, (U^{j+1} - U^j)^2 \rangle. \end{aligned}$$

Hence we get,

$$\begin{aligned}
&\Phi_h(U^{j+1}) - \Phi_h(U^j) \\
&\quad \le \tau_j \left(\tau_j \frac{p(w^{j+1})^{p-1}}{2} - 1 \right) \langle M\partial U^{j+1}, \partial U^{j+1} \rangle + \frac{\tau_j^2}{2} \langle A\partial U^{j+1}, \partial U^{j+1} \rangle \\
&\quad \le \tau_j \left(\tau_j \frac{p(w^{j+1})^{p-1}}{2} + \frac{\eta(h)\tau_j}{2} - 1 \right) \langle M\partial U^{j+1}, \partial U^{j+1} \rangle \le 0,
\end{aligned}$$

due to Lemma 2.2 and the restriction in the time step τ_j (2.1). Actually $\Phi_h(U^{j+1}) < \Phi_h(U^j)$ unless (U^j) is independent of j. So, Φ_h is a Lyapunov functional for (1.3). Next we observe that the steady states of (1.3) have positive energy. Let $(W^j) = W$ be a stationary solution of (1.3), then

$$0 = -AW + MW^p.$$

Multiplying by $W/2$ we obtain,

$$\begin{aligned}
0 &= -\frac{1}{2}\langle AW, W \rangle + \frac{p+1}{2}\frac{1}{p+1}\langle MW^p, W \rangle \\
&\ge -\Phi_h(W).
\end{aligned}$$

Assume $(U^j)_{j\ge 1}$ is a bounded solution of (1.3), then there exists a subsequence that we still denote (U^j) that converges to a steady state W with positive energy.

As $\Phi_h(U^j)$ decreases and there exists j_0 such that $\Phi_h(U^{j_0}) < 0$, then $\Phi_h(W) < 0$, a contradiction. We conclude that (U^j) is unbounded and (Lemma 2.3) has finite time blow-up. □

Corollary 2.1: *Assume the time-step restriction of the above theorem and the convergence hypotheses* (H1), (H2)*. Let u_0 an initial data for* (1.1) *such that u blows up in finite time T. Then (U^j) blows up in finite time $T_{h,\lambda}$ for every h, $\lambda = \lambda(h)$ small enough. Moreover*

$$\lim_{h\to 0}\lim_{\lambda\to 0} T_{h,\lambda} = T.$$

Remark 2.6: *If the fully-discrete method converges in $H_0^1(\Omega)$ a.e. t then λ can be chosen independent of h.*

Proof: If u blows up in finite time T then (see [13], [15])

$$\Phi(u)(t) \equiv \int_\Omega \frac{|\nabla u(s,t)|^2}{2}\,ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1}\,ds \to -\infty \qquad (t \nearrow T).$$

Hence we have $\Phi(u)(t_0) < 0$ for some $t_0 < T$. Let $j_0 = j_0(h,\lambda)$ be the first j such that $t^{j_0} \ge t_0$. Note that the existence of j_0 is guaranteed by the convergence theorem for small h, λ (see the Appendix). Now we use the convergence of (U^j) to u_h in $[0, t_0]$ and (H1) to see that

$$\lim_{h\to 0}\lim_{\lambda\to 0}\Phi_h(U^{j_0}) = \Phi(u)(t_0).$$

So for $h, \lambda = \lambda(h)$ small enough we get $\Phi_h(U^{j_0}) < 0$ and so, by the above theorem (U^j) blows up.

Now we turn our attention to the blow-up times. In [17] it is proved that the blow-up time of the semi-discrete solutions (solutions of (1.2)), that we denote T_h, converges to T as $h \to 0$. That work deals just with $\Omega = (0, 1)$ and a finite element method but the arguments used there can be extended with no difficulty to our case. We sketch the proof for the sake of completeness.

Using similar arguments to the ones above, it can be seen that if the continuous solution blows up then for every h small enough the semidiscrete solution $U(t)$ also does. Hence we can assume that the semidiscrete solution $U(t)$ is large enough in order to verify

$$\begin{aligned}
\frac{d}{dt}\langle MU(t), U(t)\rangle &= 2\langle MU'(t), U(t)\rangle\\
&= 2\langle -AU(t), U(t)\rangle + 2\langle MU^p(t), U(t)\rangle\\
&= -4\Phi_h(U(t)) + \frac{2(p-1)}{p+1}\langle MU^p(t), U(t)\rangle\\
&\geq 4|\Phi_h(U(t))| + \frac{2(p-1)}{p+1}(\langle MU(t), U(t)\rangle)^{\frac{p+1}{2}}.
\end{aligned}$$

Integrating between t_0 and T_h we obtain

$$(T_h - t_0) \leq \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+1}}}.$$

Here C depends only on p.

Given $\varepsilon > 0$, we can choose L large enough to ensure that

$$\left(\frac{C}{L^{\frac{p-1}{p+1}}}\right) \leq \frac{\varepsilon}{2}.$$

As u blows up at time T we can choose $\tau < \frac{\varepsilon}{2}$ such that

$$-\Phi(u(\cdot, T-\tau) \geq 2L.$$

If h is small enough,

$$-\Phi_h(U(T-\tau)) \geq L,$$

and hence

$$T_h - (T-\tau) \le \left(\frac{C}{(-\Phi_h(U(T-\tau)))^{\frac{p-1}{p+1}}} \right) \le \left(\frac{C}{L^{\frac{p-1}{p+1}}} \right) \le \frac{\varepsilon}{2}.$$

Therefore,

$$|T_h - T| \le |T_h - (T-\tau)| + |\tau| < \varepsilon.$$

We have proved $\lim_{h\to 0} T_h = T$, so we just have to prove that for fixed h

$$\lim_{\lambda\to 0} T_{h,\lambda} = T_h.$$

To do that we observe that from Lemma 2.3 there exists j_0, that does not depend on λ such that for $j \ge j_0$

$$w^j \ge w^{j_0} + c\lambda(j - j_0),$$

hence

$$\begin{aligned} T_{h,\lambda} - t^j &= \sum_{l=j+1}^{\infty} \tau_l = \sum_{l=j+1}^{\infty} \frac{\lambda}{(w^l)^p} \\ &\le \sum_{l=j+1}^{\infty} \frac{\lambda}{(w^{j_0} + c\lambda(l-j_0))^p} \le \int_j^{\infty} \frac{\lambda}{(w^{j_0} + c\lambda(s-j_0))^p}\, ds \\ &= \frac{1}{c} \int_{w^{j_0}+c\lambda(j-j_0)}^{\infty} \frac{1}{s^p}\, ds \le \frac{1}{c} \int_{w^{j_0}}^{\infty} \frac{1}{s^p}\, ds. \end{aligned}$$

This holds for any j_0 large enough and for every $j \ge j_0$. In particular we get

$$T_{h,\lambda} - t^j \le \frac{1}{c} \int_{w^j}^{\infty} \frac{1}{s^p}\, ds.$$

This inequality is very useful since gives a bound (independent of λ) for the distance to the numerical blow-up time in terms of w^j. Hence, given $\varepsilon > 0$ we can choose K large enough in order to have

$$\frac{1}{c} \int_K^{\infty} \frac{1}{s^p}\, ds \le \frac{\varepsilon}{3}, \qquad K^{-p} < \frac{\varepsilon}{3}.$$

Next we choose $\tau < \frac{\varepsilon}{3}$ such that $\sum m_k u_k(T_h - 2\tau) \ge 2K$ (recall that the vector $(u_1(t), \dots, u_N(t))$ is the solution of the semidiscrete scheme). For $\lambda = \lambda(h,\tau)$ small enough we get, from (H2), that $w^j \ge K$ for every j such that $T_h - 2\tau \le t^j \le T_h - \tau$. We choose one of those j and compute

$$\begin{aligned} |T_{h,\lambda} - T_h| &\le |T_{h,\lambda} - t^j| + |t^j - T_h| \\ &\le \frac{1}{c} \int_K^{\infty} \frac{1}{s^p}\, ds + 2\tau \\ &\le \varepsilon \end{aligned}$$

This completes the proof. □

2.2. Blow-up Rate

In this section, we study the asymptotic behavior of numerical solutions with blow-up.

Theorem 2.2: *Let $u_{h,\lambda}$ a discrete solution with numerical blow-up at time $T_{h,\lambda}$, then*

$$\|U^j\|_\infty \sim (T_{h,\lambda} - t^j)^{-1/(p-1)}.$$

Furthermore

$$\lim_{j\to\infty} \|U^j\|_\infty (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p = \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

We want to remark that this is the behavior of the continuous solutions with blow-up.

Proof: We know from Lemma 2.3 that $w^j = \sum m_k u_k^j$ verifies

$$w^{j+1} \geq w^j + c\lambda,$$

so we have

$$\begin{aligned}(T_{h,\lambda} - t^j) &= \sum_{k=j+1}^{\infty} \tau_j = \sum_{k=j+1}^{\infty} \frac{\lambda}{(w^j)^p} \\ &\leq \int_j^\infty \frac{\lambda}{(w(s))^p}\, ds.\end{aligned}$$

Here $w(s)$ is the linear interpolant of w^j $(w(j) = w^j)$, hence for $j < s < j+1$ we have $w'(s) = w^{j+1} - w^j \geq c\lambda$, and so

$$\int_j^\infty \frac{\lambda}{(w(s))^p}\, ds \leq \int_{w^j}^\infty \frac{\lambda}{cv^p\lambda}\, dv \leq \frac{1}{c(p-1)}\left(\frac{1}{w^j}\right)^{p-1},$$

or equivalently

$$\|U^j\|_\infty \leq Cw^j \leq C(T_{h,\lambda} - t^j)^{-1/(p-1)}.$$

The reverse inequalities can be handled in a similar way to obtain

$$\|U^j\|_\infty \sim w^j \sim (T_{h,\lambda} - t^j)^{-1/(p-1)}.$$

Next we look for the constant C_p in the asymptotic behavior of the numerical solution, to do that we change variables. Let (Y^j) be defined by

$$y_k^j = u_k^j (T_{h,\lambda} - t^j)^{1/(p-1)} \qquad 1 \leq k \leq N.$$

In the sequel of the proof we will use Δy_k^{j+1} to denote

$$\frac{y_k^{j+1} - y_k^j}{\tau_j/(T_{h,\lambda} - t^j)},$$

This can be thought as $\tau_j/(T_{h,\lambda} - t^j)$ to be the time step in the new variables. With this notation the new variables verify

$$\begin{aligned} m_k \Delta y_k^{j+1} &= -\frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} (T_{h,\lambda} - t^j) \sum_{i=1}^N a_{ki} y_i^j \\ &\quad + m_k \frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} (y_k^j)^p \\ &\quad + \frac{(T_{h,\lambda} - t^j) m_k u_k^j}{\tau_j} \left((T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}} - (T_{h,\lambda} - t^j)^{\frac{1}{p-1}} \right), \\ y_k^0 &= (T_{h,\lambda})^{1/(p-1)} u_0(x_k), \qquad 1 \le k \le N+1. \end{aligned}$$

We want to prove that $\|Y^j\|_\infty \to C_p$. To do that, we first observe that

$$\lim_{j\to\infty} \frac{T_{h,\lambda} - t^j}{T_{h,\lambda} - t^{j+1}} = 1,$$

since

$$1 \le \frac{T_{h,\lambda} - t^j}{T_{h,\lambda} - t^{j+1}} = \frac{\sum_{k=j+1}^\infty \tau_k}{\sum_{k=j+2}^\infty \tau_k} = 1 + \frac{\tau_{j+1}}{\sum_{k=j+2}^\infty \tau_k} \le 1 + \frac{\lambda/(w^{j+1})^p}{C/(w^{j+1})^{p-1}} \to 1.$$

Now assume there exists $\varepsilon > 0$ and a subsequence that we still denote (y_k^j) such that $y_k^j > C_p + \varepsilon$ for some $k = k(j)$. Then for those y_k^j we have

$$(y_k^j)^p - \frac{1}{p-1} y_k^j > \frac{3\delta}{m_k}.$$

We also know from the blow-up rate that the new variables y_i^j are bounded and so, we obtain for j large enough

$$\begin{aligned} m_k \Delta y_k^{j+1} &\ge -\delta + m_k \left((y_k^j)^p - \frac{1}{p-1} y_k^j \right) + m_k (y_k^j)^p \left(\frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} - 1 \right) \\ &\ge \delta. \end{aligned} \tag{2.2}$$

This means that actually $y_k^j > C_p + \varepsilon$ for every j large and consequently (2.2) is verified for all those j. So y_k^j is unbounded, a contradiction.

If we assume $y_k^j < C_p - \varepsilon$ arguing along the same lines we obtain that y_k^j verifies

$$
\begin{aligned}
m_k \Delta y_k^{j+1} &\leq \delta + m_k \frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} \left((y_k^j)^p \frac{1}{p-1} y_k^j \right) \\
&\quad + \frac{m_k}{p-1} y_k^j \left(\frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} - \frac{(T_{h,\lambda} - \xi^j)^{\frac{1}{p-1}-1}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}-1}} \right) \\
&\leq 2\delta + C \left((y_k^j)^p - \frac{1}{p-1} y_k^j \right).
\end{aligned}
$$

This shows that either $y_k^j \to 0$ as $j \to \infty$ or $m_k \Delta y_k^{j+1} < -\delta$, which means that y_k^j is not bounded from below (this is not possible).

We conclude that if y_k^j does not converge to zero, then converges to C_p. As the blow-up rate implies that for every j

$$\|Y^j\|_\infty \geq c,$$

we have

$$\lim_{j\to\infty} \|Y^j\|_\infty = \lim_{j\to\infty} \|U^j\|_\infty (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p,$$

as we wanted to prove. □

2.3. Blow-up Set

Now we turn our attention to the blow-up set. We consider the set $B^*(U)$ composed of those nodes that blow-up like $\|U^j\|_\infty$, that is

$$B^*(U) = \left\{ k : \lim_{j\to\infty} u_k^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p \right\}.$$

We study the behavior of the nodes adjacent to this set, then we repeat the procedure with these last nodes.

Definition 2.2: *We define $\mathcal{G}$ to be the graph with vertices in the nodes and we say that two different nodes are connected if and only if $a_{ij} \neq 0$. We consider the usual distance between nodes measured as a graph, see* [18]. *Finally, we denote by $d(k)$ the distance of the node x_k to $B^*(U)$.*

We prove that u_k^j blows up if and only if $d(k) \leq K$, where K depends only on p.

Theorem 2.3: *Assume the time step verifies $\tau_j < \frac{m_i}{a_{ii}}$, $(1 \leq i \leq N)$. Then the blow-up propagates outside $B^*(U)$ in the following way: let $K = \left[\frac{1}{p-1}\right]$, the solution of* (1.3) *blows up exactly at K nodes near $B^*(U)$. More precisely,*

$$u_k^j \to +\infty \qquad \Longleftrightarrow \qquad d(k) \leq K.$$

Moreover, if $d(k) \le K$, the asymptotic behavior of u_k^j is given by

$$u_k^j \sim (T_{h,\lambda} - t^j)^{-\frac{1}{p-1}+d(k)},$$

if $d(k) \neq \frac{1}{p-1}$ and if $d(k) = \frac{1}{p-1}$

$$u_k^j \sim \ln(T_{h,\lambda} - t^j).$$

Proof of Theorem 1.4: We want to show that the blow-up propagates K nodes around $B^*(U)$, we begin with a node x_k such that $d(k) = 1$. We claim that the behavior of u_k^j is given by

$$u_k^j \sim \begin{cases} j^{-p+2} & \text{if } p < 2 \\ \ln j & \text{if } p = 2, \\ C & \text{if } p > 2, \end{cases}$$

to prove that we will show that

$$w_A^j = A \sum_{s=1}^{j} s\tau_{s-1},$$

which has the behavior described above, can be used as super and subsolution for an equation verified by u_k^j choosing A appropriately.

We observe that u_k^j satisfies

$$\begin{aligned} m_k \partial u_k^{j+1} &= -\sum_{i=1}^{N} a_{ki} u_i^j + m_k (u_k^j)^p \\ &\sim \Gamma_1 \|U^j\|_\infty - \frac{a_{kk}}{m_k} u_k^j + \Gamma_2 (u_k^j)^p, \end{aligned}$$

for some constants Γ_i. In other words, there exists constants $c_i, C_i > 0$, $i = 1, 2$ such that for j large enough

$$\partial u_k^{j+1} \le C_1 j - \frac{a_{kk}}{m_k} u_k^j + C_2 (u_k^j)^p \tag{2.3}$$

and

$$\partial u_k^{j+1} \ge c_1 j - \frac{a_{kk}}{m_k} u_k^j + c_2 (u_k^j)^p. \tag{2.4}$$

Now we observe that if A and j are large enough, w_A^j verifies

$$\begin{aligned} \partial w_A^j &= A(j+1) \\ &\ge C_1 j - \frac{a_{kk}}{m_k} w_A^j + C_2 (w_A^j)^p, \end{aligned}$$

since $(w_A^j)^p/j \to 0$ as j goes to infinity. Hence w_A^j is a supersolution for (2.3) and so

$$u_k^j \le w_A^j$$

(a comparison principle like Lemma 2.2 holds for this equation and can be proved in the same way) On the other hand if we choose A small we get

$$\begin{aligned} \partial w_A^j &= A(j+1) \\ &\le c_1 j - \frac{a_{kk}}{m_k} w_A^j + c_2 (w_A^j)^p, \end{aligned}$$

Hence now we can use w_A^j as a subsolution for (2.4) to handle the other inequality. Therefore

$$u_k^j \sim w_A^j.$$

We observe that if $p < 2$ the node x_k is a blow-up node and we also have the blow-up rate for this node ($u_k^j \sim j^{-p+2}$). If $p > 2$ this node is bounded. Next we assume $p < 2$ (if $p > 2$ it is easy to prove that every node k with $d(k) \ge 1$ is bounded) and we are going to find the behavior of a node, that we still denote k, such that $d(k) = 2$. That is, it is not adjacent to $B^*(U)$ and it is adjacent to a node which has the behavior j^{-p+2}.

Now let

$$w_A^j = A \sum_{s=1}^{j} \tau_s s^{-p+2},$$

and observe that u_k^j verifies

$$m_k \partial u_k^{j+1} = -\sum_{i=1}^{N} a_{ki} u_i^j + m_k (u_k^j)^p \sim \Gamma_1 j^{-p+2} - \frac{a_{kk}}{m_k} u_k^j + \Gamma_2 (u_k^j)^p.$$

That is, for j large we have

$$\partial u_k^{j+1} \le C_1 j^{-p+2} - \frac{a_{kk}}{m_k} u_k^j + C_2 (u_k^j)^p \tag{2.5}$$

and

$$\partial u_k^{j+1} \ge c_1 j^{-p+2} - \frac{a_{kk}}{m_k} u_k^j + c_2 (u_k^j)^p$$

Now for A and j large, w_A^j verifies

$$\begin{aligned} \partial w_A^j &= A(j+1)^{-p+2} \\ &\ge C_1 j^{-p+2} - C_2 w_A^j + C_3 (w_A^j)^p. \end{aligned}$$

Hence w_A^j is a supersolution for (2.5) and so

$$u_k^j \leq w_A^j.$$

On the other hand if we choose A small we get

$$\begin{aligned} \partial w_A^j &= A(j+1)^{-p+2} \\ &\leq c_1 j^{-p+2} - c_2 w_A^j + c_3 (w_A^j)^p, \end{aligned}$$

Now we can use w_A^j as a subsolution for (2.4) to handle the other inequality. So

$$u_k^j \sim w_A^j.$$

If $p < 3/2$ the node x_k is a blow-up node and we also have the blow-up rate for this node ($u_k^j \sim j^{-2p+3}$). If $p > 3/2$ this node is bounded. In the case $p < 3/2$ we repeat this procedure inductively to obtain the theorem. □

3. The Implicit Scheme

In order to avoid the time step restrictions we now consider the semi-implicit scheme (1.5) and prove that similar properties can be observed in this procedure.

Lemma 3.1: *Let* $(\overline{U}^j)$, $(\underline{U}^j)$ *a super and a subsolution respectively for* (1.5) *such that* $\underline{U}^0 < \overline{U}^0$, *then* $\underline{U}^j < \overline{U}^j$ *for every* j.

Proof: Let $Z^j = \overline{U}^j - \underline{U}^j$, we assume that we have strict inequalities in (1.5), then (Z^j) verifies

$$\begin{aligned} M\partial Z^{j+1} &> -AZ^{j+1} + M((\overline{U}^j)^p - (\underline{U}^j)^p), \\ Z^0 &> 0. \end{aligned} \tag{3.1}$$

If the statement of the Lemma is false, then there exists a first time t^{j+1} and a node x_k with $z_k^{j+1} = \min_{1\leq i\leq N} z_i^{j+1} \leq 0$ that verifies

$$z_k^{j+1} > z_k^j - \tau_j \sum_{i=1}^{N} \frac{a_{ki}}{m_k} z_i^{j+1} + \tau_j \big((\overline{u}_k^j)^p - (\underline{u}_k^j)^p\big) \geq z_k^j - \frac{\tau_j z_k^{j+1}}{m_k} \sum_{i=1}^{N} a_{ki} \geq 0,$$

a contradiction.

Remark 3.1: *As in the explicit scheme, this lemma does not hold for a general mass matrix M. Assumption* (P1) *can not be relaxed.*

3.1. Blow-up in the Numerical Scheme

Lemma 3.2: *There exists a constant κ such that $U^{j_0} \geq \kappa$ implies that U^j blows up in finite time. Furthermore*

$$\|U^j\|_\infty \sim w^j \sim j$$

Remark 3.2: *Unfortunately in this case we are not able to prove that the constant κ does not depend on λ and hence we can not prove the convergence of the numerical blow-up times.*

Proof:

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{k=1}^{N} \sum_{i=1}^{N} a_{ki} u_i^{j+1} + \tau_j \sum_{k=1}^{N} m_k (u_k^j)^p \\ &\leq w^j + \tau_j \sum_{k=1}^{N} m_k (u_k^j)^p \\ &\leq w^j + C\tau_j (w^j)^p \\ &= w^j + C\lambda. \end{aligned}$$

Hence $w^j \leq Cj$. To prove the reverse inequality we observe that

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{k=1}^{N} \sum_{i=1}^{N} a_{ki} u_i^{j+1} + \tau_j \sum_{k=1}^{N} m_k (u_k^j)^p \\ &\geq w^j - \tau_j C_1 w^{j+1} + \tau_j C_2 (w^j)^p, \end{aligned}$$

where

$$C_1 = \max_{1 \leq i \leq N} \frac{\sum_k a_{ki}}{m_i}, \qquad C_2 = \left(\sum_{k=1}^{N} m_k \right)^{1-p}.$$

Hence

$$(1 + C_1 \tau_j) w^{j+1} \geq w^j + C_2 \tau_j (w^j)^p. \tag{3.2}$$

Now we look for a subsolution of (3.2) of the form $z^j = \Gamma j$. This sequence verifies

$$(1 + C_1 \tau_j) z^{j+1} = z^j + \Gamma C_1 \tau_j j + \Gamma(1 + C_1 \tau_j) \leq z^j + C_2 \tau_j (z^j)^p$$

if $\Gamma \geq (\frac{\lambda C_1 + C_1 + 1}{\lambda C2})^{1/(p-1)}$. As the discrete maximum principle holds for this equation we obtain that if $w^{j_0} > \Gamma$ then for every $j \geq j_0$

$$w^j \geq z^j = \Gamma j.$$

This completes the proof. □

Theorem 3.1: *If $\Phi_h(U^{j_0}) < 0$ for some j_0, then $(U^j)_{j\geq 1}$ blows up.*

Proof: We first observe that also for this scheme $\Phi_h(U^j)$ is decreasing in j. We take inner product of (1.5) with $U^{j+1} - U^j$

$$\begin{aligned}0 &= \left\langle \frac{1}{\tau_j} M(U^{j+1} - U^j) + AU^{j+1} - M(U^j)^p, U^{j+1} - U^j \right\rangle \\ &= \tau_j \langle M\partial U^{j+1}, \partial U^{j+1}\rangle + \Phi_h(U^{j+1}) - \Phi_h(U^j) + \tfrac{1}{2}\langle AU^{j+1}, U^{j+1}\rangle \\ &\quad -\langle AU^j, U^{j+1}\rangle + \tfrac{1}{2}\langle AU^j, U^j\rangle + \frac{p}{2}\langle M(\xi^j)^{p-1}, (U^{j+1} - U^j)^2\rangle.\end{aligned}$$

Hence we obtain,

$$\begin{aligned}\Phi_h(U^{j+1}) - \Phi_h(U^j) &= -\tau_j \langle M\partial U^{j+1}, \partial U^{j+1}\rangle - \frac{\tau_j^2}{2}\langle A\partial U^{j+1}, \partial U^{j+1}\rangle \\ &\quad -\frac{p}{2}\langle M(\xi^j)^{p-1}, (U^{j+1} - U^j)^2\rangle \\ &\leq 0.\end{aligned}$$

The steady states of (1.5) are the same of the ones for (1.3), so they have positive energy. Now, assume (U^j) is a bounded solution of (1.5), then it has a convergent subsequece. Its limit W is a steady state with positive energy.

As $\Phi_h(U^j)$ decreases and there exists j_0 with $\Phi_h(U^{j_0}) < 0$ then $\Phi_h(W) < 0$, a contradiction. We conclude that (U^j) is unbounded and by Lemma 3.2 has finite time blow-up. □

Corollary 3.1: *Assume the convergence hypotheses* (H1), (H2). *Let u_0 an initial data for* (1.1) *such that u blows up in finite time T. Then $u_{h,\lambda}$ blows up in finite time $T_{h,\lambda}$ for every h, $\lambda = \lambda(h)$ small enough.*

Proof: If u blows up in finite time T then

$$\Phi(u)(t) \equiv \int_\Omega \frac{|\nabla u(s,t)|^2}{2}\, ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1}\, ds \to -\infty \qquad (t \nearrow T).$$

Hence there exists a time $t_0 < T$ with $\Phi(u)(t_0) < 0$. Let $j_0 = \inf\{j : t^j \geq t_0\}$. We use the convergence hypothesis (H1) and the convergence of (U^j) to u_h in $[0, t_0]$ to see that

$$\lim_{h\to 0}\lim_{\lambda\to 0} \Phi_h(U^{j_0}) = \Phi(u)(t_0).$$

So for $h, \lambda(h)$ small enough we get $\Phi_h(U^{j_0}) < 0$ and so (U^j) blows up. □

Next we turn our attention to the blow-up rate of the discrete solutions.

3.2. Blow-up Rate

Theorem 3.2: *Let $u_{h,\lambda}$ be a solution with blow-up at time $T_{h,\lambda}$, then*

$$\max_{1\le k\le N} u_k^j \sim (T_{h,\lambda} - t^j)^{-1/(p-1)}.$$

Moreover

$$\lim_{j\to\infty} \max_{1\le k\le N} u_k^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p = \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

Proof: The first part of the proof is the same as the one for the explicit scheme so we assume we have proved

$$\|U^j\|_\infty \sim (T_{h,\lambda} - t^j)^{-\frac{1}{p-1}},$$

and we are going to prove the convergence of the self-similar variables (Y^j) to C_p. Let (Y^j), Δy_k^{j+1} be defined as in the previous section. In the semi-implicit scheme, these variables verify

$$\begin{aligned} m_k \Delta y_k^{j+1} &= -(T_{h,\lambda} - t^j) \sum_{i=1}^N a_{ki} y_i^{j+1} + m_k \frac{(T_{h,\lambda} - t^{j+1})^{1/(p-1)}}{(T_{h,\lambda} - t^j)^{1/(p-1)}} (y_k^j)^p \\ &\quad - m_k u_k^j ((T_{h,\lambda} - t^j)^{1/(p-1)} - (T_{h,\lambda} - t^{j+1})^{1/(p-1)}), \\ y_k^0 &= T_{h,\lambda}^{1/(p-1)} u_0(x_k). \end{aligned}$$

If we assume the existence of $\varepsilon > 0$ and a subsequence such that $y_k^j > C_p + \varepsilon$ for some $k = k(j)$. Then for those y_k^j, as they are bounded, we have for j large

$$\begin{aligned} \Delta y_k^{j+1} &\ge -\delta + m_k \frac{(T_{h,\lambda} - t^{j+1})^{1/(p-1)}}{(T_{h,\lambda} - t^j)^{1/(p-1)}} \left((y_k^j)^p - \frac{1}{p-1} y_k^j \right) \\ &\quad + \frac{1}{p-1} y_k^j \left[1 - \frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}-1}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}-1}} \right] \\ &\ge \delta. \end{aligned} \tag{3.3}$$

Hence $y_k^j > C_p + \varepsilon$ for every j large enough and consequently (3.3) is verified for all those j. So y_k^j is unbounded, a contradiction.

The case where there exists an infinite number of (j, k) with $y_k^j < C_p - \varepsilon$ can be handled in the same way to conclude that as $j \to \infty$ either $y_k^j \to 0$ or $y_k^j \to C_p$. Now we use the blow-up rate to obtain

$$\lim_{j\to\infty} \max_{1\le k\le N} y_k^j = \lim_{j\to\infty} \max_{1\le k\le N} (T_{h,\lambda} - t^j)^{1/(p-1)} u_k^j = C_p,$$

as we wanted to prove. □

3.3. Blow-up Set

The propagation property for the blow-up nodes holds for the implicit scheme and its proof is very similar. We do not include it.

4. Numerical Experiments

In this section, we include some numerical experiments to illustrate the theoretical results proved in the previous sections. We show solutions to (1.3) for $p = 4$ (Fig. 1)

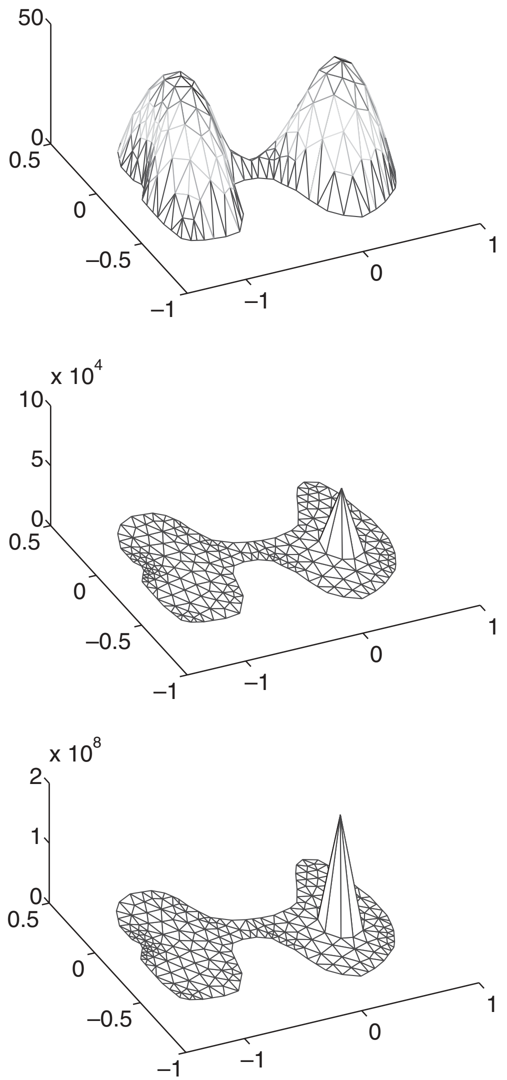

Fig. 1. Single point blow-up ($p = 4$). Numerical solution at time $t = 0$ (above), $t = 2.562955143779549 \times 10^{-4}$ (middle) and $t = 2.562955143780117 \times 10^{-4}$ (below)

and $p = 1.5$ (Fig. 2). The matrices M and A were obtained by the Finite Element Method with mass lumping (see the Introduction). For the initial datum we have used the (numerical) solution to

$$\begin{cases} \Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The constant μ was chosen large enough in order to get a solution with blow-up.

In Fig. 1 single point blow-up can be observed (i.e., there is just one node that blows up and the rest of the nodes seems to be be bounded) as proved in Theorem 2.3.

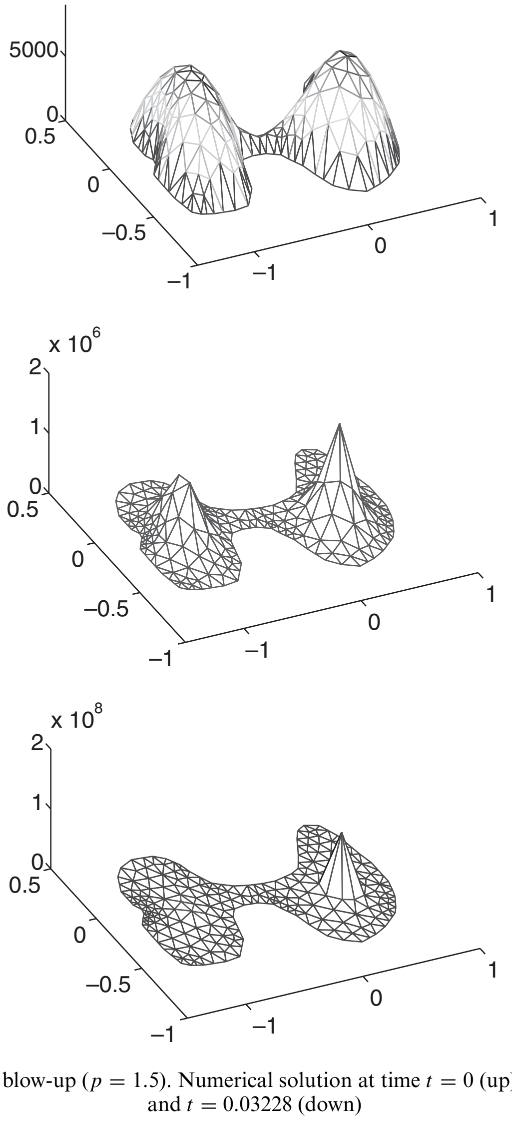

Fig. 2. Propagation of blow-up ($p = 1.5$). Numerical solution at time $t = 0$ (up), $t = 0.03078$ (middle) and $t = 0.03228$ (down)

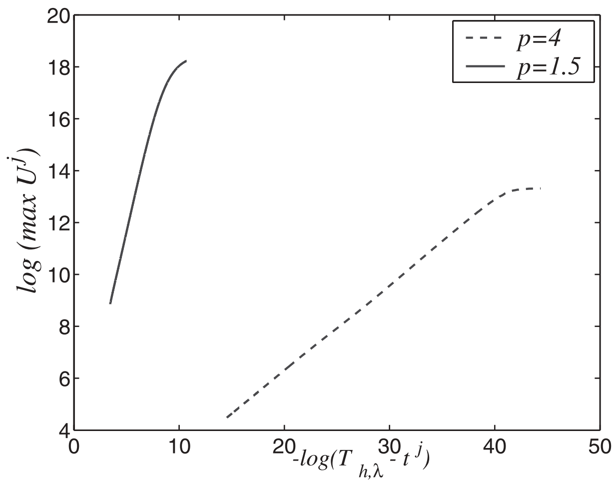

Fig. 3. Blow-up rates

We compare these results with the case $p = 1.5$ in Fig. 2, where more than one node blow up, although they do with different rates.

Finally, Fig. 3 shows the blow-up rates in both cases, it can be appreciated that the slopes of the curves are approximately $1/3$ for $p = 4$ and 2 for $p = 1.5$, showing that

$$\|U^j\|_\infty \sim (T_{h,\lambda} - t^j)^{-1/(p-1)},$$

as proved in Theorem 2.2. This blow-up rate coincides with the one of the continuous solution.

5. Conclusion

Nonlinear parabolic equations like (1.1) may develop singularities in finite time. In the analysis of numerical solutions to these problems standard techniques do not apply, mostly due to the fact that convergence theorems do not include cases with singularities as the one studied here.

We have introduced totally discrete schemes to deal with the solutions with blow-up of (1.1) and we have proved that these schemes reproduce the blow-up cases, the blow-up rate and the blow-up set. In addition, we proved that the numerical blow-up times converge to the theoretical one (just an iterated limit).

Appendix

In this appendix, we prove that if the general method considered for the space discretization is consistent (see below) then the totally discrete method converges in the L^∞ norm. We perform the proofs for the explicit scheme, they can be extended to the implicit one.

Definition 6.1: *Let w be a regular solution of*

$$\begin{aligned} w_t &= \Delta w + f(x,t) && \text{in } \Omega \times (0,T), \\ w &= 0 && \text{on } \partial\Omega \times (0,T). \end{aligned}$$

We say that the scheme (1.2) *is consistent if for any $t \in (0, T-\tau)$ it holds*

$$m_k w_t(x_k, t) = -\sum_{i=1}^{N} a_{ki} w(x_k, t) + m_k f(x_k, t) + \rho_{k,h}(t), \tag{6.1}$$

and there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\max_k \frac{|\rho_{k,h}(t)|}{m_k} \le \rho(h), \qquad \text{for every } t \in (0, T-\tau),$$

with $\rho(h) \to 0$ if $h \to 0$. The function ρ is called the modulus of consistency of the method.

If we consider for example a finite differences scheme in a cube $\Omega = (0,1)^d \subset \mathbb{R}^d$. Then the modulus of consistency can be taken as $\rho(h) = Ch^2$.

Theorem 6.1: *Let u be a regular solution of* (1.1) ($u \in C^{2,1}(\overline{\Omega} \times [0, T-\tau])$ *and*) $(U^j)_{j\ge 1}$ *the numerical approximation given by* (1.4). *If the scheme* (1.2) *is consistent with modulus of consistency ρ, then there exists positive constants C, h_0, λ_0 depending on $\|u\|$ in $C^{2,1}(\overline{\Omega} \times [0, T-\tau])$ such that for every $h < h_0$, $\lambda < \lambda_0$ holds*

$$\max_j \max_{1\le k\le N} |u_k^j - u(x_k, t_j)| \le C(\rho(h) + \lambda).$$

Proof: We define the error functions

$$e_k^j = u(x_k, t_j) - u_k^j.$$

By (6.1) and (1.4), these functions verify

$$m_k \partial e_k^{j+1} \le -\sum_{i=1}^{N} a_{ki} e_i^j + m_k (u^p(x_k, t_j) - (u_k^j)^p) + \rho_k(h) + C m_k \lambda,$$

where C is a bound for $\|u_{tt}\|_{L^\infty(\Omega\times[0,T-\tau])}$. Let

$$t_0 = \max\left\{ t : \ t < T - \tau, \ \max_i \max_{t_j < t} |e_i^j| \le 1 \right\}.$$

We will see by the end of the proof that $t_0 = T - \tau$ for h, λ small enough. In $[0, t_0]$ we have

$$m_k \partial e_k^{j+1} = -\sum_{i=1}^{N} a_{ki} e_i^j + m_k p(\xi_k^j)^{p-1} e_k^j + \rho_k(h) + C m_k \lambda,$$

hence, in $[0, t_0]$, $E^j = (e_1^j, \ldots, e_N^j)$ satisfies

$$\begin{aligned} M\partial E^{j+1} &\le -AE^j + KME^j + (\rho(h) + C\lambda)M\mathbf{1}^t, \\ E(0) &= 0. \end{aligned} \tag{6.2}$$

Here K is the Lipchitz constant for $f(u) = u^p$ in $[0, \|u(\cdot, T - \tau)\|_{L^\infty}]$. Let us now define $W^j = (w_1^j, \ldots, w_N(t))$, which will be used as a supersolution.

$$w_i^j = e^{(2K+1)t_j}(\rho(h) + C\lambda).$$

It is easy to check that W^j verifies

$$M\partial W^{j+1} > -AW^j + KMW^j + (\rho(h) + C\lambda)M\mathbf{1}^t,$$

Hence W^j is a supersolution for (6.2), and by Lemma 2.1 we get

$$e_k^j \le e^{(2K+1)t_j}(\rho(h) + C\lambda), \qquad t_j \in [0, t_0].$$

Arguing along the same lines with $-E^j$, we obtain

$$|e_k^j| \le e^{(2K+1)T}(\rho(h) + C\lambda) \le C(\rho(h) + \lambda), \qquad t_j \in [0, t_0].$$

Using this fact, since $\rho(h)$ goes to zero, we get that $|e_k^j| \le 1$ for every $t_j \in [0, T - \tau]$ for every h, λ small enough. Therefore $t_0 = T - \tau$ for h, λ small enough. This proves the convergence of the scheme. In fact we have that for every $h < h_0, \lambda < \lambda_0$

$$\max_j \max_{1 \le k \le N} |u_k^j - u(x_k, t_j)| \le C(\rho(h) + \lambda). \qquad \square$$

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References

[1] Abia, L. M., López-Marcos, J. C., Martínez, J.: Blow-up for semidiscretizations of reaction-diffusion equations. Appl. Numer. Math. *20(1–2)*, 145–156 (1996). Workshop on the Method of Lines for Time-dependent Problems. Lexington, KY, 1995.

[2] Abia, L. M., López-Marcos, J. C., Martínez, J.: On the blow-up time convergence of semidiscretizations of reaction-diffusion equations. Appl. Numer. Math. *26(4)*, 399–414 (1998).

[3] Abia, L. M., López-Marcos, J. C., Martínez, J.: The Euler method in the numerical integration of reaction-diffusion problems with blow-up. Appl. Numer. Math. *38(3)*, 287–313 (2001).

[4] Acosta, G., Durán, R. G., Rossi, J. D.: An adaptive time step procedure for a parabolic problem with blow-up. Computing *68(4)*, 343–373 (2002).

[5] Bandle, C., Brunner, H.: Numerical analysis of semilinear parabolic problems with blow-up solutions. Rev. Real Acad. Cienc. Exact. Fí s. Natur. Madrid *88(2–3)*, 203–222 (1994).

[6] Bandle, C., Brunner, H.: Blow-up in diffusion equations: a survey. J. Comput. Appl. Math. *97(1–2)*, 3–22 (1998).
[7] Berger, M., Kohn, R. V.: A rescaling algorithm for the numerical calculation of blowing-up solutions. Comm. Pure Appl. Math. *41(6)*, 841–863 (1988).
[8] Budd, C. J., Chen, S., Russell, R. D.: New self-similar solutions of the nonlinear Schrödinger equation with moving mesh computations. J. Comput. Phys. *152(2)*, 756–789 (1999).
[9] Budd, C. J., Collins, G. J.: An invariant moving mesh scheme for the nonlinear diffusion equation. In: Proc. Int. Centre for Math. Sci. Conf. on Grid Adaptation in Computational PDEs: Theory and Applications (Edinburgh, 1996), vol. *26*, pp. 23–39 (1998).
[10] Budd, C. J., Huang, W., Russell, R. D.: Moving mesh methods for problems with blow-up. SIAM J. Sci. Comput. *17(2)*, 305–327 (1996).
[11] Chen, Y. G.: Asymptotic behaviours of blowing-up solutions for finite difference analogue of $u_t = u_{xx} + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. IA Math. *33(3)*, 541–574 (1986).
[12] Ciarlet, P. G.: The finite element method for elliptic problems. Studies in Mathematics and its applications, vol. 4. Amsterdam: North-Holland 1978.
[13] Cortázar, C., del Pino, M., Elgueta, M.: The problem of uniqueness of the limit in a semilinear heat equation. Comm. Partial Differ. Eq. *24(11–12)*, 2147–2172 (1999).
[14] Duran, R. G., Etcheverry, J. I., Rossi, J. D.: Numerical approximation of a parabolic problem with a nonlinear boundary condition. Discrete Contin. Dyn. Sys. *4(3)*, 497–506 (1998).
[15] Giga, Y., Kohn, R. V.: Characterizing blow-up using similarity variables. Indiana Univ. Math. J. *36(1)*, 1–40 (1987).
[16] Giga, Y., Kohn, R. V.: Nondegeneracy of blow-up for semilinear heat equations. Comm. Pure Appl. Math. *42(6)*, 845–884 (1989).
[17] Groisman, P., Rossi, J. D.: Asymptotic behaviour for a numerical approximation of a parabolic problem with blowing up solutions. J. Comput. Appl. Math. *135(1)*, 135–155 (2001).
[18] Harary, F.: Graph theory. Reading, MA/Menlo Park, CA/London: Addison-Wesley 1969.
[19] Huang, W., Ren, Y., Russell, R. D.: Moving mesh partial differential equations (MMPDES) based on the equidistribution principle. SIAM J. Numer. Anal. *31(3)*, 709–730 (1994).
[20] Le Roux, M.-N.: Semidiscretization in time of nonlinear parabolic equations with blow-up of the solution. SIAM J. Numer. Anal. *31(1)*, 170–195 (1994).
[21] Nakagawa, T.: Blowing up of a finite difference solution to $u_t = u_{xx} + u_2$. Appl. Math. Optim. *2(4)*, 337–350 (1975/76).
[22] Nakagawa, T., Ushijima, T.: Finite element analysis of the semi-linear heat equation of blow-up type. In J. J. H. Miller (ed.), Topics in numerical analysis, III. Published for the Royal Irish Academy by: London New York: Academic Press, pp. 275–291 (1977).
[23] Samarskii, A. A., Galaktionov, V. A., Kurdyumov, S. P., Mikhailov, A. P.: Blow-up in quasilinear parabolic equations. Gruyter Expositions in Mathematics, vol. 19. Berlin: de Gruyter 1995. Translated from the 1987 Russian original by Michael Grinfeld and revised by the authors.
[24] Stuart, A. M., Floater, M. S.: On the computation of blow-up. Europ. vol. 19. Eur. J. Appl. Math. *1(1)*, 47–71 (1990).
[25] Ushijima, T. K.: On the approximation of blow-up time for solutions of nonlinear parabolic equations. Publ. Res. Inst. Math. Sci. *36(5)*, 613–640 (2000).
[26] Velázquez, J. J. L.: Classification of singularities for blowing up solutions in higher dimensions. Trans. Amer. Math. Soc. *338(1)*, 441–464 (1993).

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